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METRIC PROPERTIES OF ENGEL SERIES EXPANSIONS OF LAURENT SERIES

Peter J. Grabner and Arnold Knopfmacher

Abstract

We derive metric properties of the polynomial digits occuring in certain series expansions for Laurent series, analogous to the Engel series representation for real numbers. In particular, we obtain limiting distributions for the degrees of the digit polynomials and the order of approximation by the partial sums of the series.

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1. Introduction

Recently A. Knopfmacher and J. Knopfmacher [8] introduced and studied some properties of various unique expansions of formal Laurent series over a field F, as the sums of reciprocals of polynomials, involving "digits" $a_1, a_2, \ldots$ lying in a polynomial ring $F[X]$ over F. In particular, one of these expansions (described below) was constructed to be analogous to the so-called Engel expansion of a real number, discussed in Perron [15] Chapter 4.

Previously, Artin [1] and Magnus [11,12] had studied a Laurent series analogue of simple *continued fractions* of real numbers, involving "digits" $x_1, x_2, \ldots$ in a polynomial ring $F[X]$. In addition to sketching elementary properties of an n-dimensional "Jacobi-Perron" variant of this, Paysant-Leroux and Dubois [13,14] also briefly outlined certain "metric" theorems analogous to some of Khintchine [7] for real continued fractions, in the case when F is a finite field. The main aim of this paper is to derive similar metric results for the Laurent series Engel-type expansion referred to above. (For analogous results concerning Engel expansions of *real* numbers, see Erdős, Renyi, Szüsz [2] and Renyi [16], and Galambos [5].)

In the corresponding case of Lüroth type expansions for Laurent series ergodic and other metric properties have recently been investigated by J. Knopfmacher [10] and extended by A. and J. Knopfmacher in [9]. For both the continued fraction and Lüroth expansions of Laurent series, ergodicity of the corresponding transformations were used to derive the results. However, in the case of Engel expansions the underlying transformation is not ergodic. The growth conditions satisfied by the polynomial digits suggest that an approach via Markov chains could be used. For the corresponding ideas in the case of Engel, Lüroth and more generally Oppenheim expansions for real numbers we refer to Galambos' book [5].

In order to explain the conclusions, we first fix some notation and describe the inverse-polynomial Engel-type representation to be considered:

Let $\mathcal{L} = F\big((z^{-1})\big)$ denote the field of all *formal Laurent series* $A = \sum_{n=v}^{\infty} c_n z^{-n}$ in an indeterminate z, with coefficients c_n all lying in a given field F. (We consider $F\big((z^{-1})\big)$ rather than $F\big((z)\big)$ as in [8,9] since it turns out to be more convenient for stating our results.)

We also consider the ring $F[z]$ of *polynomials* in z, and the field $F(z)$ of *rational* *functions* in z, with coefficients in F.

If $c_v \neq 0$ we call $v = v(A)$ the order of A above, and define the *norm* (or *valuation*) of A to be $||A|| = q^{-v(A)}$, where initially $q > 1$ may be an arbitrary constant, but *later will be chosen* as $q = \text{card}(F)$, if F is finite. Letting $v(0) = +\infty, \quad ||0|| = 0$, one then has (cf. Jones and Thron [6] Chapter 5):

$$\begin{cases} ||A|| \geqq 0 \text{ with } ||A|| = 0 \text{ iff } A = 0\ , \\ ||AB|| = ||A|| \cdot ||B||, \text{ and} \\ ||\alpha A + \beta B|| \leqq \max\big(||A||, ||B||\big) \text{ for non-zero } \alpha,\ \beta \in F, \\ \text{with equality when } ||A|| \neq ||B||\ . \end{cases} \tag{1.1}$$

By (1.1), the norm $||\quad||$ is non-Archimedean, and it is well known that $\mathcal{L}$ forms the *completion* of $F(z)$ at infinity in the same way that $\mathbb{R}$ is the completion at infinity of the rational numbers $\mathbb{Q}$.

We shall make frequent use of the polynomial $[A] = \sum\limits_{0 \leq n \leq v} c_n z^n \in F[z]$, and refer to $[A]$ as the *integral part* of $A \in \mathcal{L}$. Then $v = -v(A)$ is the *degree* $\deg[A]$ of $[A]$ relative to z, and the same function [.] was used by Artin [1] and Magnus [11,12] for their continued fractions.

Given $A \in \mathcal{L}$, now note that $[A] = a_0 \in F[z]$ iff $v(A_1) \geq 1$ where $A_1 = A - a_0$. As in [8], if $A_n \neq 0 (n > 0)$ is already defined, we then let $a_n = \left[\frac{1}{A_n}\right]$ and put $A_{n+1} = (a_n A_n - 1)$. If some $A_m = 0$ or $a_n = 0$, this recursive process stops. It was shown in [8] that this algorithm leads to a finite or convergent (relative to ρ) Engel-type series expansion

$$A = a_0 + \frac{1}{a_1} + \sum_{r \geq 2} \frac{1}{a_1 \cdots a_r}\ , \tag{1.2}$$

where $a_r \in F[z]$, $a_0 = [A]$, and $\deg(a_{r+1}) \geq \deg(a_r) + 1$ for $r \geq 1$. Furthermore this expansion is *unique* for A subject to the preceding conditions on the "digits" a_r. For notational convenience we set

$$\frac{p_n}{q_n} = a_0 + \sum_{r=1}^{n} \frac{1}{a_1 \cdots a_r}, \quad \text{where} \quad q_n = a_1 \ldots a_n\ .$$

From now on we assume that $F = \mathbb{F}_q$ is a finite field with exactly q elements. Let I denote the valuation ideal $z^{-1}F\big[[z^{-1}]\big]$ in the ring of formal power series $F\big[[z^{-1}]\big]$ and let $\mathbb{P}$ denote probability with respect to the Haar measure on $\mathcal{L}$ normalised by $\mathbb{P}(I) = 1$. The

Haar measure on I is the product measure on $\prod_{n=1}^{\infty} \mathbb{F}_q$ defined by $\mathbb{P}(\{x\}) = q^{-1}$ for each factor and any element $x \in \mathbb{F}_q$.

We now state our main results

Theorem 1.

(i)
$$\lim_{n\to\infty} \mathbb{P}\left(x \in I : \frac{\deg a_n - \frac{q}{q-1}\, n}{\sqrt{n\,q}\Big/(q-1)} < t \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du\ .$$

(ii) For almost all $x \in I$,
$$\limsup_{n\to\infty} \frac{\deg a_{n+1}(x) - \deg a_n(x)}{\log_q n} = 1\ ,$$
and
$$\liminf_{n\to\infty} \deg a_{n+1}(x) - \deg a_n(x) = 1\ .$$

(iii) For almost all $x \in I$,
$$\left|\left| x - \frac{p_n}{q_n} \right|\right| = q^{-\left(\frac{q}{q-1}\ \frac{n^2}{2}\left(1+o(1)\right)\right)}, \quad as \quad n \to \infty\ .$$

More precisely
$$\lim_{n\to\infty} \mathbb{P}\left(x \in I : \frac{v\left(x - \frac{p_n}{q_n}\right) - \frac{q}{q-1}\ \frac{(n+1)(n+2)}{2}}{\sqrt{\mathrm{Var}\ (\deg q_{n+1})}} < t \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} du\ .$$

where $\mathrm{Var}\ (\deg q_{n+1}) = \frac{(n+1)(n+2)(2n+3)}{6} \left(\frac{q}{(q-1)^2}\right)$.

In particular we see from (i) that for almost all $x \in I$, $||a_n||^{1/n} \to q^{q/q-1}$, as $n \to \infty$. Regarding (i) and (iii) above we note the similar but weaker results shown in [8] holding for all x in I,
$$\deg(a_n) \geq n$$
and
$$\left|\left| x - \frac{p_n}{q_n} \right|\right| \leq q^{\frac{-(n+1)(n+2)}{2}}\ , \qquad n = 1, 2, 3, \ldots\ .$$

Furthermore, we consider the random variables $\left|\left| \frac{a_{r+1}(x)}{a_r(x)} \right|\right| \equiv q^{\Delta_r}$, $r = 1, 2, 3, \ldots$. These are independent and identically distributed with infinite expectation. However, the following result holds.

Theorem 2. *For any fixed $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left\{x \in I : \frac{1}{n\log_q n}\left|\sum_{r=1}^{n}\left\|\frac{a_{r+1}(x)}{a_r(x)}\right\| - (q-1)\right| > \varepsilon\right\} = 0 ,$$

i.e. $\frac{1}{n\log_q n}\sum_{r=1}^{n}\left\|\frac{a_{r+1}(x)}{a_r(x)}\right\| \to q-1$ *in probability over I.*

Remark. Since a theorem in Galambos [5, p.46], implies that *either*

$$\limsup_{n\to\infty} \frac{1}{n\log_q n}\sum_{r=1}^{n}\left\|\frac{a_{r+1}(x)}{a_r(x)}\right\| = \infty \ a.e.$$

or

$$\liminf_{n\to\infty} \frac{1}{n\log_q n}\sum\left\|\frac{a_{r+1}(x)}{a_r(x)}\right\| = 0 \ a.e.,$$

the conclusion of Theorem 2 does not carry over to validity with probability one.

The paper is organized into sections, which split the proofs of the theorems. Section 2 gives some elementary probabilities, which will be used in the proofs, Section 3 gives the proof of Theorem 1 and Section 4 gives the proof of Theorem 2.

2. Basic Probabilities

We begin by deriving some basic probabilistic results concerning the digits in Engel expansions of Laurent series.

Lemma 1.

The digits $a_n \in F[z]$ from a Markov chain with initial probabilities

$$\mathbb{P}(\deg a_1 = j) = (q-1)q^{-j} \ . \tag{2.1}$$

and transition probabilities

$$\mathbb{P}(\deg a_{n+1} = k \big| \deg a_n = j) = \begin{cases} (q-1)q^{j-k} & k > j \\ 0 & \text{else} \ . \end{cases} \tag{2.2}$$

Proof: Firstly by the Engel algorithm $A_1 = x \in I$. Then using the definition of Haar measure $\mathbb{P}\big(v(A_1) < -j\big) = \mathbb{P}(\deg a_1 > j) = q^{-j}$. Thus $\mathbb{P}(\deg a_1 = j) = \mathbb{P}(\deg a_1 > j-1) - \mathbb{P}(\deg a_1 > j) = (q-1)q^{-j}$.

Next, the coefficients of A_2 are obtained from those of A_1 by a system of linear equations arising from the relation $A_2 = a_1 A_1 - 1$. From this it follows that A_2 is uniformly distributed in $z^{-j}I$ where $j = \deg a_1$. Inductively if $\deg a_n = j$ then A_{n+1} is uniformly distributed in $z^{-j}I$ for all $n > 1$. Since the event $\deg a_{n+1} > k$ under the condition that $\deg a_n = j$ is described by $k - j$ linear equations arising from equating coefficients of z^{-m} in $a_n A_n$ equal to zero we conclude that

$$\mathbb{P}(\deg a_{n+1} > k \big| \deg a_n = j) = q^{j-k}$$

and (2.2) follows immediately. □

Remark. Since the probability in (2.2) depends only on the difference $k - j$ this implies that the random variables $\deg a_{n+1} - \deg a_n$ are independent and identically distributed. Thus for

$$n_1 < n_2 < \cdots < n_j \quad \text{and} \quad k_i \geq 1, \ i = 1, 2, \ldots j \ ,$$

$$\begin{aligned} \mathbb{P}(\deg a_{n_j+1} = \deg a_{n_j} + k_j, \ \deg a_{n_{j-1}+1} = \deg a_{n_{j-1}} + k_{j-1} \ , \ldots \\ \deg a_{n_1+1} = \deg a_{n_1} + k_1) = (q-1)^j q^{-(k_1+\cdots+k_j)}. \end{aligned} \tag{2.3}$$

Corollary 1. Let $\Delta_n = \Delta_n(x)$ denote the random variable $\deg a_{n+1} - \deg a_n$, with $\Delta_0 = \deg a_1$. Then

$$\mathbb{P}\Big(\#\{1 \leq \ell \leq n | \Delta(\ell) = 1\} = k\Big) = \binom{n}{k} \left(1 - \frac{1}{q}\right)^k q^{k-n} \ .$$

Thus the number of times that degrees of consecutive digits increase by 1 has a binomial distribution with mean value $n\left(1 - \frac{1}{q}\right)$ and variance $n \, \frac{q-1}{q^2}$.

In particular the lim inf result of part (ii) of Theorem 1 follows immediately.

Corollary 2. The random variables Δ_n have mean value and variance

$$E(\Delta_n) = \frac{q}{q-1}$$

and

$$\mathrm{Var}(\Delta_n) = \frac{q}{(q-1)^2} \ .$$

Proof: By Lemma 1

$$E(\Delta_n) = \sum_{\ell=1}^{\infty} \ell\, \mathbb{P}(\deg a_{n+1} - \deg a_n = \ell) = (q-1) \sum_{\ell=1}^{\infty} \ell q^{-\ell} = \frac{q}{q-1} \; .$$

Similarly

$$E(\Delta_n^2) = (q-1) \sum_{\ell=1}^{\infty} \ell^2 q^{-\ell} = \frac{q}{q-1} + 2\, \frac{q}{(q-1)^2}$$

from which the formula for $\mathrm{Var}(\Delta_n)$ is immediate.

Lemma 2.

(i) $\mathbb{P}(\deg a_n = t) = (q-1)^n q^{-t} \binom{t-1}{n-1}$ (2.4)

and therefore

$\mathbb{P}(\exists n : \deg a_n = t) = 1 - \frac{1}{q}$. (2.5)

(ii) $\mathbb{P}(\deg a_{n+m} = t) \big| \deg a_n = s) = (q-1)^m q^{s-t} \binom{t-s-1}{m-1}$.

Proof:

(i) Since the sequence of degrees of the digits $a_1, a_2, \ldots$ is strictly increasing we have by Lemma 1,

$$\begin{aligned}
&\mathbb{P}(\deg a_n = t) = \sum_{1 \le j_1 < j_2 < \cdots < j_{n-1} < t} \mathbb{P}(\deg a_n = t \big| \deg a_{n-1} = j_{n-1}) \times \\
&\mathbb{P}(\deg a_{n-1} = j_{n-1} \big| \deg a_{n-2} = j_{n-1}) \cdots \mathbb{P}(\deg a_2 = j_2 \big| \deg a_1 = j_1) \mathbb{P}(\deg a_1 = j_1) \\
&= (q-1)^n \sum_{1 \le j_1 < j_2 < \cdots < j_{n-1} < t} q^{j_{n-1} - t} q^{j_{n-2} - j_{n-1}} \ldots q^{j_1 - j_2} q^{-j_1} \; , \\
&= (q-1)^n q^{-t} \sum_{1 \le j_1 < j_2 < \cdots < j_{n-1} < t} 1 \\
&= (q-1)^n q^{-t} \binom{t-1}{n-1} \; .
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{P}(\exists n : \deg a_n = t) &= \sum_{n=1}^{\infty} (q-1)^n q^{-t} \binom{t-1}{n-1} \\
&= (q-1) q^{-t} \sum_{\ell=0}^{t-1} (1-1)^{\ell} \binom{t-1}{\ell} = 1 - \frac{1}{q} \; .
\end{aligned}$$

(ii)

$$\begin{aligned}\mathbb{P}(\deg a_{n+m} = t \big| \deg a_n = s) = \sum_{s<j_1<j_2<\cdots<j_{m-1}<t} \mathbb{P}(\deg a_{n+m} = t \big| \deg a_{n+m-1} = j_{m-1})\\ \cdots \mathbb{P}(\deg a_{n+2} = j_2 \big| \deg a_{n+1} = j_1)\mathbb{P}(\deg a_{n+1} = j_1 \big| \deg a_n = s)\\ = (q-1)^m q^{s-t} \sum_{s<j_1<j_2<\cdots<j_{m-1}<t} 1\\ = (q-1)^m q^{s-t} \binom{t-s-1}{m-1} . \quad \square\end{aligned}$$

Remark. From the proof of (i) we can also deduce the joint probability distribution

$$\mathbb{P}(\deg a_1 = j_1, \ldots, \deg a_n = j_n) = (q-1)^n q^{-j_n} .$$

3. Proof of Theorem 1.

Since we can write $\deg a_n$ as the sum of independent random variables

$$\deg a_n = \sum_{i=1}^{n-1} (\deg a_{i+1} - \deg a_i) + \deg a_1 = \sum_{i=0}^{n-1} \Delta_i ,$$

it follows from Corollary 2 that $\deg a_n$ has mean and variance

$$E(\deg a_n) = \frac{q}{q-1}\, n .$$

and

$$\operatorname{Var}(\deg a_n) = n\, \frac{q}{(q-1)^2} ,$$

respectively.

Hence by the central limit theorem (see e.g. Feller [4, p.253]) part (i) of Theorem 1 follows.

(ii) The events $\deg a_{n+1} - \deg a_n > k(n)$ are independent with probabilities $\mathbb{P}\big(\Delta_n > k(n)\big) = q^{-k(n)}$. The Borel-Cantelli lemmas then yield

$$\mathbb{P}\big(\Delta_n > k(n) \text{ for infinitely many } n\big)$$

$$= \begin{cases} 0, & \sum_{n=1}^{\infty} q^{-k(n)} \quad \text{converges} \\ 1 & \sum_{n=1}^{\infty} q^{-k(n)} \quad \text{diverges.} \end{cases}$$

By choosing $k(n) = c\log_q n$ we see that with probability 1 the events $(\deg a_{n+1} - \deg a_n)\big/\log_q n > c$ occur infinitely often if $c \leq 1$ and only finitely often if $c > 1$. The lim sup result then follows. The corresponding lim inf result was already shown in Section 2.

(iii) We first compute the mean and variance of $\|x - \frac{p_n}{q_n}\|$. In [8] it is shown that

$$\left\|A - \frac{p_n}{q_n}\right\| = q^{-\deg q_{n+1}} .$$

Now

$$E(\deg q_{n+1}) = \sum_{r=1}^{n+1} E(\deg a_n) = \frac{q}{q-1}\,\frac{(n+1)(n+2)}{2} .$$

To compute the variance we make use of the fact that

$$\begin{aligned}\deg q_{n+1} &= \sum_{r=1}^{n+1} a_r = \sum_{r=1}^{n+1}\sum_{l=0}^{r-1}\Delta_l \\ &= \sum_{l=0}^{n}\Delta_l(n+1-l).\end{aligned}$$

We now remark that the last sum has the same distribution as the sum

$$\sum_{l=0}^{n}(l+1)\Delta_l.$$

Thus we have for the variance

$$\mathrm{Var}\ (\deg q_{n+1}) = \sum_{l=0}^{n}(l+1)^2\mathrm{Var}\ \Delta_l = \frac{(n+1)(n+2)(2n+3)}{6}\left(\frac{q}{(q-1)^2}\right).$$

Now we check that the random variables $(l+1)\Delta_l$ satisfy Lindeberg's condition (cf. [4, p.256]): since $s_n^2 = \mathrm{Var}\ (\deg q_{n+1})$ is of order of magnitude n^3, we have to compute the integrals

$$\int\limits_{|y|\geq tn^{3/2}} y^2\,dF_k(y) = (k+1)^2\int_{|x|\geq\frac{tn^{3/2}}{k+1}} x^2\,dF(x) \leq (k+1)^2\int_{|x|\geq\frac{t}{2}\sqrt{n}} x^2\,dF(x),$$

where F_k is the distribution function of $(k+1)(\Delta_k - \frac{q}{q-1})$ and $F = F_0$. Thus the last integral is equal to the sum

$$\sum_{k\geq\frac{q}{q-1}+\frac{t}{2}\sqrt{n}}\left(k - \frac{q}{q-1}\right)^2 q^{-k} = O\left(nq^{-\frac{t}{2}\sqrt{n}}\right)$$

for n sufficiently large, and we have

$$\frac{1}{s_n^2}\sum_{k=0}^{n}\int\limits_{|y|\geq ts_n} y^2\,dF_k(y)=O\left(\frac{1}{n}q^{-\frac{t}{2}\sqrt{n}}\right)\to 0$$

for any $t>0$. Thus

$$\frac{\deg q_{n+1}-\frac{q}{q-1}\frac{(n+1)(n+2)}{2}}{\sqrt{\mathrm{Var}\,(\deg q_{n+1})}}$$

has asymptotically normal distribution and the proof is completed. $\square$

4. Proof of Theorem 2.

We first notice that by Lemma 1 the random variables $\|\frac{a_{r+1}(x)}{a_r(x)}\|\equiv q^{\Delta_r}$ are independent and identically distributed with infinite expectation. We write $s=\log_q y$ iff $y=q^s$ and use the truncation method of Feller [3], Chapter 10, §2, applied to the random variables $U_r, V_r (r\leq n)$ defined by

$$\begin{aligned}
&U_r(x)=\big\|a_{r+1}/a_r(x)\big\|\,, && V_r(x)=0 && \text{if } \big\|a_{r+1}/a_r(x)\big\|\leq \log_q n\,,\\
&U_r(x)=0\,, && V_r(x)=\big\|a_{r+1}/a_r(x)\big\| && \text{if } \big\|a_{r+1}/a_r(x)\big\|> \log_q n\,,
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{P}&\left\{x\in I:\frac{1}{n\log_q n}\left|\sum_{r=1}^{n}\left\|\frac{a_{r+1}(x)}{a_r(x)}\right\|-(q-1)\right|>\varepsilon\right\}\\
&\leq \mathbb{P}\left\{x:\Big|U_1+\cdots+U_n-(q-1)n\log_q n\Big|>\varepsilon n\log_q n\right\}+\\
&\qquad+\mathbb{P}\big\{x:V_1+\cdots+V_n\neq 0\big\}\,,
\end{aligned}$$

and using Lemma 1,

$$\begin{aligned}
\mathbb{P}\Big\{x:V_1+\cdots+V_n\neq 0\Big\}&\leq n\mathbb{P}\left\{\left\|\frac{a_2(x)}{a_1(x)}\right\|>n\log_q n\right\}\\
&=n\sum_{\substack{k\\ q^k>n\log_q n}}(q-1)q^{-k}\ll\frac{1}{\log_q n}=o(1).
\end{aligned}$$

Now note that

$$E(U_1+\cdots+U_n)=nE(U_1),\ \ \mathrm{Var}\,(U_1+\cdots+U_n)=n\,\mathrm{Var}\,(U_1)\,,$$

where

$$\begin{aligned}
E(U_1)&=\sum_{\|\frac{a_2(x)}{a_1(x)}\|\leq n\log_q n} q^k\mathbb{P}(\Delta_1=k)=\sum_{q^k\leq n\log_q n}q^{-k}(q-1)q^k\\
&=(q-1)\log_q\big([n\log_q n]\big)\,,
\end{aligned}$$

and

$$\mathrm{Var}\,(U_1) < E(U_1^2) = \sum_{q^k \le n \log_q n} (q-1)q^k < qn \log_q n \,.$$

Chebyshev's inequality then yields

$$\begin{aligned} &\mathbb{P}\Big\{x : \big|U_1 + \cdots + U_n - nE(U_1)\big| > \varepsilon n E(U_1)\Big\} \\ &\quad \le \frac{n\,\mathrm{Var}\,(U_1)}{\big(\varepsilon n E(U_1)\big)^2} < \frac{qn^2 \log_q n}{\Big(\varepsilon (q-1) n \log\big([n \log_q n]\big)\Big)^2} = o(1)\,. \end{aligned}$$

Since $E(U_1) \sim (q-1)\log_q n$ as $n \to \infty$, Theorem 2 follows. $\square$

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Authors' Addresses

P.J. GRABNER
Institut für Mathematik
Technische Universität Graz
Steyrergasse 30
8010 Graz, Austria
email: grabner@weyl.math.tu-graz.ac.at

A. KNOPFMACHER
Dept. of Computational & Applied Mathematics
University of the Witwatersrand
Private Bag 3
WITS 2050, South Africa
email:arnoldk@gauss.cam.wits.ac.za