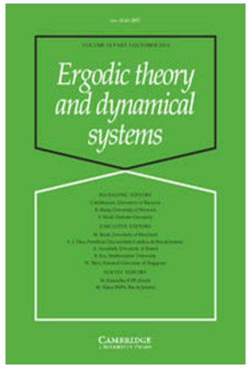

Combinatorial and probabilistic properties of systems of numeration

GUY BARAT and PETER GRABNER

Ergodic Theory and Dynamical Systems / *FirstView* Article / September 2014, pp 1 - 36
DOI: 10.1017/etds.2014.69, Published online: 16 September 2014

Ergod. Th. & Dynam. Sys. (First published online 2014), page 1 of 36*
doi:10.1017/etds.2014.69
*Provisional—final page numbers to be inserted when paper edition is published

Combinatorial and probabilistic properties of systems of numeration

GUY BARAT† and PETER GRABNER‡

† *Institut de Mathématiques de Marseille, Case 907, Université d'Aix-Marseille, 163 avenue de Luminy, 13288 Marseille Cedex 9, France*
(e-mail: guy.barat@tugraz.at)
‡ *Institut für Analysis und Computational Number Theory, Technische Universität Graz, NAWI Graz, Steyrergasse 30, 8010 Graz, Austria*
(e-mail: peter.grabner@tugraz.at)

(*Received* 11 *August* 2013 *and accepted in revised form* 1 *June* 2014)

Dedicated to the memory of Pierre Liardet, our friend and teacher, who introduced us to the subject

Abstract. Let $G = (G_n)_n$ be a strictly increasing sequence of positive integers with $G_0 = 1$. We study the system of numeration defined by this sequence by looking at the corresponding compactification $\mathcal{K}_G$ of $\mathbb{N}$ and the extension of the addition-by-one map τ on $\mathcal{K}_G$ (the 'odometer'). We give sufficient conditions for the existence and uniqueness of τ-invariant measures on $\mathcal{K}_G$ in terms of combinatorial properties of G.

1. *Introduction and notation*

1.1. *Systems of numeration.* Throughout the paper, $\mathbb{N}$ denotes the set of non-negative integers, that is $\{0, 1, 2, \ldots\}$.

Let $(A, \leqslant)$ be a totally ordered set, called an 'alphabet'; in most cases the alphabet is a subset of $\mathbb{N}$. We denote by A^* the set of finite words over A, $A^* = \{a_0a_1 \cdots a_n \mid a_i \in A\} \cup \{\epsilon\}$, where ϵ denotes the empty word. The basic idea of 'numeration' is to give a bijection between $\mathbb{N}$ and a set of words $\mathcal{W} \subset A^*$, which respects a suitable ordering on $\mathcal{W}$ inherited from the ordering on A. Before we give the precise definitions of this bijection and the set $\mathcal{W}$, we introduce some basic notation about words. For more information on word combinatorics we refer to the book [**27**].

For finite words v and w we write vw for their concatenation and $v^{(k)}$ for the concatenation of k times the word v; we write the exponent in parentheses to avoid confusion with powers of numbers. Absolute values denote the length of a word. We have $|v^{(k)}| = k|v|$.

We shall use two different orders on $\mathcal{W}$: first we write $v \overset{\text{pref}}{\preccurlyeq} w$ if v is a prefix of w, i.e. if there exists a word u such that $vu = w$. In particular, for $|x| \geqslant m$, we denote by $x[m]$

the prefix of x of length m. By convention, $x[0]=\epsilon$. Secondly, we consider the 'reverse genealogical order' on $\mathcal{W}$ (inherited from the ordering on A): for $v,\, w \in \mathcal{W}$ with $|v|=|w|$ we have $v \overset{\text{gen}}{\leqslant} w$ if $v=w$ or if there exists some index $0 \leqslant k < |v|$ such that $v_k < w_k$ and $v_j = w_j$ for all $|v| > j > k$. If $|v| < |w|$, then $v \overset{\text{gen}}{\leqslant} w$. As opposed to the lexicographical order, this order takes the length of the words into account. The word 'reverse' refers to the fact that we read the digits from right to left.

A system of numeration is given by an increasing sequence of integers $(G_n)_n$ with $G_0 = 1$, called the *scale*. In this system, any non-negative integer n can be represented as a (finite) sum

$$n = \sum_{k \geqslant 0} \varepsilon_k(n) G_k. \tag{1.1}$$

The digits $\varepsilon_k(n)$ are computed by the so-called greedy algorithm: if $n=0$, then $\varepsilon_k(n)=0$ for all k. If $n \neq 0$, let $L(n)$ be the smallest k such that $n < G_k$ and proceed with the Euclidean division $n = \varepsilon_{L(n)-1}(n) G_{L(n)-1} + n'$. Apply the same procedure to n' and repeat it recursively. After finitely many steps, the remainder of the Euclidean division is zero. Finally, set $\varepsilon_k(n) = 0$ for all those k that did not occur during the process. With the computed values, the relation (1.1) holds. Obviously, for an arbitrary n, one has that

$$\text{for all } m \in \mathbb{N} : \sum_{k<m} \varepsilon_k(n) G_k < G_m; \tag{1.2}$$

notice that these conditions imply $\varepsilon_k(n) < (G_{k+1}/G_k)$. These inequalities are called 'Yaglom conditions' (cf. [**13**]); they characterise the expansions and provide a necessary and sufficient condition for their uniqueness in the following sense: if $n = \sum x_k G_k$ with $\sum_{k<m} x_k G_k < G_m$ for all m, then $x_k = \varepsilon_k(n)$ for all k (cf. [**35**]). In other words, the greedy algorithm provides a bijection between $\mathbb{N}$ and the set of expansions. Let $\mathcal{W}$ be the set of *admissible words*

$$\mathcal{W} = \left\{ w = w_0 w_1 \cdots w_m \mid m \geqslant 0,\, w_m \neq 0,\, \forall k \leqslant m+1,\, \sum_{j<k} w_j G_j < G_k \right\} \cup \{\epsilon\}.$$

We introduce the two maps '`rep`resentation' and '`val`ue'

$$\begin{array}{llll} \texttt{rep}: & \mathbb{N} & \to & \mathcal{W} \\ & n & \mapsto & \varepsilon_0(n) \cdots \varepsilon_{L(n)-1}(n) \end{array} \qquad \begin{array}{llll} \texttt{val}: & \mathcal{W} & \to & \mathbb{N} \\ & w_0 \cdots w_m & \mapsto & \displaystyle\sum_{i=0}^{m} w_i G_i \end{array}$$

with $\texttt{rep}(0) = \epsilon$. The order $\overset{\text{gen}}{\leqslant}$ is defined so that the maps `rep` and `val` are mutually inverse order preserving bijections between $(\mathbb{N}, \leqslant)$ and $(\mathcal{W}, \overset{\text{gen}}{\leqslant})$. Such representations of the positive integers for various base sequences have been investigated from different points of view. As a general reference for systems of numeration in this generality, we refer to [**13**]. In the general case of an increasing base sequence $(G_n)_n$ the corresponding language does not have any nice structural properties, like recognisability or even factoriality.

For sequences $(G_n)_n$ with bounded sequence of quotients $(G_{n+1}/G_n)_n$ the alphabet A is finite; for such sequences the recognisability of the language $\mathcal{W}$ by a finite automaton and the representability of arithmetic operations by finite automata are natural questions. The most prominent and best investigated sequences of this type are linear recursive sequences; i.e. sequences satisfying a recurrence relation

$$G_{n+d} = a_{d-1}G_{n+d-1} + \cdots + a_1 G_{n+1} + a_0 G_n \quad \text{for } n \geqslant 0$$

with fixed non-negative integer coefficients $a_0, \ldots, a_{d-1}$. Numeration based on linear recurrent scales have been studied in [**14–20**] with respect to the representability of the addition-by-one map by a finite automaton. In the context of linear recursive base sequences the case when the characteristic polynomial

$$x^d - a_{d-1}x^{d-1} - \cdots - a_1 x - a_0$$

is the minimal polynomial of a Pisot-number (i.e. an algebraic integer all of whose conjugates have modulus < 1) plays a special role. The underlying language $\mathcal{W}$ can then be related to β-expansions (cf. [**29**]) of real numbers. In this case the language $\mathcal{W}$ is recognised by a finite automaton (cf. [**18**]). For a more detailed discussion of recognisability of $\mathcal{W}$ in the context of linear recursive base sequence $(G_n)_n$ we refer to [**18**].

The special case of the sequence $G_n = (n+1)^2$ and some properties of the according representations have been studied in [**28**].

A different approach to number representation is based on substitutions over a finite alphabet (cf. [**11**, **12**]). In this context the non-negative integer n is represented as the prefix of length n of the fixed point of a substitution on a finite alphabet. This prefix is expressed in a unique way as a concatenation of iterates of the substitution applied to certain prefixes of the fixed point. This finite number of prefixes plays the role of digits in this numeration. We remark here that numeration systems with linear recurrent base sequence can be seen as special cases of this type of numeration.

In a similar vein, let $\mathcal{L}$ be a regular language endowed with the genealogical order originating from an ordered alphabet. Then the integer n can be represented as the nth word in $\mathcal{L}$. This language- and automata-theoretic approach has been used in [**26**, **31**] to define abstract numeration systems. Again, numeration systems with linear recurrent scale are a special case.

1.2. *Odometer.* For convenience, we introduce a further notation of $\mathtt{rep}^*(n) = \mathtt{rep}(n)0^{(\infty)}$. We introduce the compact set

$$\mathcal{K}_G = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_n \{0, 1, \ldots, \lceil G_{n+1}/G_n \rceil - 1\} \mid \forall m \in \mathbb{N} : \sum_{k<m} x_k G_k < G_m \right\}, \quad (1.3)$$

which is the closure in $\prod_n \{0, 1, \ldots, \lceil G_{n+1}/G_n \rceil - 1\}$ of the set of infinite expansions $\mathtt{rep}^*(\mathbb{N})$ with respect to the product topology. For an infinite word $x = x_0x_1x_2\ldots$, we denote by $x[m] = x_0x_1\cdots x_{m-1}$ the prefix of x of length m, extending the previous notation. Furthermore, we extend concatenation vw for infinite words w as well. Notice that $v \in \mathcal{W}$ and $w \in \mathcal{K}_G$ does not imply that $vw \in \mathcal{K}_G$. The function $\mathtt{rep}^*$ is the embedding of $\mathbb{N}$ into $\mathcal{K}_G$, and the image of $\mathbb{N}$ is dense in $\mathcal{K}_G$.

In order to describe cylinders in $\mathcal{K}_G$, we first define

$$\mathcal{W}' = \{w \in A^* \mid \exists x \in \mathcal{K}_G,\, w \overset{\text{pref}}{\preccurlyeq} x\},$$

the set of prefixes of elements of $\mathcal{K}_G$. By the properties of greedy numeration this set satisfies

$$\mathcal{W}' = \{w0^{(k)} \mid w \in \mathcal{W},\, k \in \mathbb{N}\} = \{w \in A^* \mid \exists v \in \mathcal{W},\, w \overset{\text{pref}}{\preccurlyeq} v\} \supset \mathcal{W}.$$

The map $\mathtt{val}$ as defined on $\mathcal{W}$ extends readily to $\mathcal{W}'$; notice that $\mathtt{val}$ is no longer injective on $\mathcal{W}'$, but it is still injective on words of fixed length. For a finite word $w \in \mathcal{W}'$, we denote by $[w]$ the cylinder associated with w, that is $[w] = \{x \in \mathcal{K}_G \mid w \overset{\text{pref}}{\preccurlyeq} x\}$; by convention we set $[\epsilon] = \mathcal{K}_G$. Let $w \in \mathcal{W}'$, $w \neq 0^{|w|}$. The word w^- is defined to be the word $v \in \mathcal{W}'$ satisfying $|v| = |w|$ and $\mathtt{val}(v) = \mathtt{val}(w) - 1$; by the above discussion such a word exists and is unique. It is the predecessor of w for the reverse genealogical order among the elements of length $|w|$ in $\mathcal{W}'$.

As proved in [**2**, **21**], the operation $\tau : n \mapsto n + 1$ can be extended from $\mathbb{N}$ to $\mathcal{K}_G$ by

$$\tau(x) = \lim_{n\to\infty} \mathtt{rep}^*(\mathtt{val}(x[n]) + 1).$$

Notice that the addition of 1 either affects only the digit x_0, which yields $\tau(x) = (x_0 + 1)$ $x_1x_2\ldots$, or there occurs a *carry*, which means that the sequence $(x_i)_i$ has a prefix of the form $\mathtt{rep}(G_m - 1)$. If m is the largest value with this property, then the addition of 1 gives $\tau(x) = (0^{(m)}, x_m + 1, x_{m+1}, \ldots)$; if there are infinitely many m with this property, then $\tau(x) = 0^{(\infty)}$. That yields a dynamical system $(\mathcal{K}_G, \tau)$, called an *odometer*, the subject of this paper. We emphasise that the odometer need not be continuous; the set of discontinuity points has been determined in [**2**] as

$$\mathrm{Disc}(\tau) = \{\mathtt{rep}^*(G_n - 1) \mid n \in \mathbb{N}\}' \setminus \tau^{-1}(\{0\}),$$

the set of accumulation points of the sequence $(\mathtt{rep}^*(G_n - 1))_n$, which are not mapped to 0 by τ. The set of accumulation points of $(\mathtt{rep}^*(G_n - 1))_n$ can even be the whole space; an example of such an odometer is given in [**2**, Example 15]. In that example the odometer is even uniquely ergodic. Notice that the set $\tau^{-1}(\{0\})$ has measure 0 for any invariant probability measure as a consequence of Proposition 3. This lack of continuity makes the existence of invariant measures of $(\mathcal{K}_G, \tau)$ an important question in the context of odometers. For instance, Example 7 in §5 provides a family of examples of odometers, which do not admit an invariant measure.

In the classical case of b-adic numeration, $\mathcal{K}_G$ is $\mathbb{Z}_b$, the topological group of b-adic integers. Addition by one is then a group rotation. The general theory has been developed in [**21**]. The paper [**6**] *inter alia* gives an example of an odometer with continuous spectrum, disproving a conjecture stated in [**21**]. For systems of numeration arising from regular languages as in [**26**], the according odometer has been introduced and studied in [**5**]. A different approach related to Bratelli diagrams and Markov compacta has been pursued in [**34**] (see also [**33**]). For a survey on the subject we refer to [**9**]. For a general survey on dynamics and numeration we refer to [**1**].

A further classical example of a numeration system, which has been studied intensively in the context of Diophantine approximation, is Ostrowski numeration. Given an irrational real number $0 < \alpha < \frac{1}{2}$ and its continued fraction expansion $\alpha = [0; a_1, a_2, \ldots]$ the denominators $(q_n)_n$ of the convergents satisfy the recursion

$$q_{n+1} = a_{n+1} q_n + q_{n-1}, \quad q_0 = 1, \quad q_1 = a_1.$$

Taking $(q_n)_n$ as the base sequence gives the Ostrowski numeration system. This has been studied from the dynamical point of view in [**4**]. In particular, it has been shown that the system $(\mathcal{K}_G, \tau)$ is conjugate to the rotation by α on $\mathbb{R}/\mathbb{Z}$ and is therefore uniquely ergodic.

1.3. *Content of the paper.* The subject of the present paper is a deeper understanding of the dynamical properties of the odometer based on the combinatorial structure of the underlying base sequence $(G_n)_n$. Section 2 will define an infinite triangular array of coefficients that we call *greedy recurrence coefficients*. These coefficients are obtained from the greedy expansion of the numbers $G_n - 1$ and provide a linear recursive expression of G_n in terms of its preceding values $G_0, \ldots, G_{n-1}$. Most of our results will be formulated in terms of these coefficients. Our approach is more direct than the one chosen in [**3**]: there, instead of considering the expansions of the non-negative integers and studying the associated compactification $\mathcal{K}_G$, the *G-valuation*

$$v_G(n) = \min\{k \in \mathbb{N} \mid \varepsilon_k(n) > 0\} \in \mathbb{N} \cup \{\infty\}$$

is introduced. The *valumeter* is then the dynamical system $((\mathbb{N} \cup \{\infty\})^{\mathbb{N}}, \sigma)$ obtained as the orbit closure of the sequence $(v_G(n))_n$ under the shift σ in the compact space $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$. The valumeter is then shown to be conjugate to the odometer. Results on the existence and uniqueness of invariant measures are then obtained for the valumeter and translated to the odometer. For instance, it is shown that the convergence of the series $\sum_n G_n^{-1}$ is sufficient for the existence of an invariant measure on $(\mathcal{K}_G, \tau)$, whereas the relation $\limsup_n G_n/n = \infty$ provides a necessary condition. Furthermore, it is shown that $\lim_n (G_{n+1} - G_n) = \infty$ plus boundedness of the sequence $(G_n \sum_{k \geqslant n} G_k^{-1})_n$ is sufficient for unique ergodicity.

In §3 we discuss in more detail a condition for the existence and uniqueness of an invariant measure on $(\mathcal{K}_G, \tau)$ given in [**3**, Théorème 8], namely the boundedness of the sequence $(G_n \sum_{k \geqslant n} G_k^{-1})_n$. We relate this condition to combinatorial properties of the greedy recurrence coefficients, namely a weak non-lacunarity of these coefficients. Furthermore, we show that this condition implies exponential growth of the base sequence $(G_n)_n$ in the sense that

$$\liminf_{n \to \infty} \frac{\log G_n}{n} > 0.$$

Section 4 is the heart of the paper. Here we relate the existence of an invariant measure to the existence of a solution of an infinite linear system of equations (Theorem 1). These equations are built from the greedy recurrence coefficients introduced in §2. In §4.2 we give a sufficient condition for the existence of invariant measures (Corollary 2) and describe all invariant measures in terms of cluster points of certain combinatorially defined measures (Theorem 2). In §4.3 we derive an explicit and computable sufficient condition for the uniqueness of the invariant measure (Theorem 3). Our approach of using the greedy recurrence coefficients allows less restrictive conditions (compared to [**3**, Théorème 8]) which ensure uniqueness of the invariant measure, especially as our conditions for unique ergodicity do not imply exponential growth of the base sequence $(G_n)_n$.

Several simple examples are given throughout the paper to illustrate notation and theorems. The last section, §5, is devoted to families of more elaborate examples, which show that the sufficient conditions given in the present paper are weaker than those given in [**3**], but that they are still not necessary. Furthermore, we give examples of odometers which are not uniquely ergodic. The examples are worked out using the greedy recurrence coefficients associated to the base sequence and computing the invariant measures as solutions of the linear system of equations derived in §4.1. This allows for a very direct and explicit understanding of the combinatorial structure of the odometer and the invariant measure.

2. *Greedy recurrence coefficients*

It will turn out that the expansions of the integers $G_m - 1$ play an important role in the description of the invariant measures on the odometer. The coefficients $a_{k,m}$ we introduce below are related to these expansions; they consist in non-admissible expansions of the G_n, which can be read as a recurrence relation for the base sequence $(G_n)_n$. They will be used in §4 to build τ-invariant measures. Those coefficients are constructed from a so-called *descent function* introduced in [**2**] as

$$T(m) = \max\{k < m \mid \texttt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} \texttt{rep}(G_m - 1)\}, \quad \text{for } m \geqslant 1. \tag{2.1}$$

In particular, $T(m) = 0$, if $\texttt{rep}(G_m - 1)$ has no non-empty prefix of the form $\texttt{rep}(G_k - 1)$. The motivation for this definition is that the addition of 1 to $x \in \mathcal{K}_G$ produces a carry, if and only if there exists a $k > 0$ such that $\texttt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} x$, as mentioned in the definition of τ above. We will provide examples illustrating this definition in the end of this section.

Remark 1. In [**2**], a *tree of carries* has been introduced as follows. The set of the vertices is $\mathbb{N}$; the edges are given by the relations $T(m) = n$. Then, it turns out that the transformation τ on the odometer is continuous if and only if the tree is of finite type, that is all vertices have finite degree (cf. [**2**, Théorème 5]).

For $m \geqslant 1$, using the definition of T, the expansion of $G_m - 1$ can be written as

$$G_m - 1 = \sum_{j=0}^{T(m)-1} \varepsilon_j(G_{T(m)} - 1)G_j + \sum_{j=T(m)}^{m-1} \varepsilon_j(G_m - 1)G_j, \tag{2.2}$$

which can be understood in two ways:

$$\texttt{rep}(G_m - 1) = \texttt{rep}(G_{T(m)} - 1)\varepsilon_{T(m)}(G_m - 1) \cdots \varepsilon_{m-1}(G_m - 1) \tag{2.3}$$

as a concatenation of expansions, and as numerical equality. We now define coefficients $a_{k,m}$ by

$$a_{k,m} = \begin{cases} \varepsilon_k(G_m - 1) & \text{if } T(m) + 1 \leqslant k \leqslant m - 1, \\ \varepsilon_{T(m)}(G_m - 1) + 1 & \text{if } k = T(m), \\ 0 & \text{if } k < T(m) \text{ or } k \geqslant m. \end{cases} \tag{2.4}$$

We call them the *greedy recurrence coefficients* of the scale $(G_n)_n$; they will be used throughout the paper to describe the combinatorial structure of the system of numeration. Some special values of the $a_{k,m}$ are $a_{m-1,m} \neq 0$, $a_{T(m),m} \neq 0$, $a_{0,1} = G_1$. The following equations are equivalent to (2.4)

$$\sum_{j=0}^{k-1} a_{j,m} G_j = \begin{cases} 0 & \text{if } k \leqslant T(m), \\ \mathtt{val}(\mathtt{rep}(G_m - 1)[k]) + 1 & \text{if } k > T(m). \end{cases} \tag{2.5}$$

In particular, this allows us to rewrite (2.2) as

$$G_m = \sum_{k=0}^{m-1} a_{k,m} G_k = \sum_{k=T(m)}^{m-1} a_{k,m} G_k, \tag{2.6}$$

a 'recurrence relation' for G_m in terms of the preceding values. The expression (2.6) can be obtained as the greedy expansion of G_m in terms of the values $G_0, G_1, \ldots, G_{m-1}$, which is not the expansion of G_m with respect to the scale G, since obviously $\mathtt{rep}(G_m) = 0^{(m)}1$. Then $T(m)$ is the smallest index of a non-zero digit in this expansion of G_m. That remark shall be used frequently in §4.1.

Reinterpreting (1.2), we have

$$x = x_0 x_1 x_2 \cdots \in \mathcal{K}_G \iff \text{for all } m \geqslant 1 : x[m] \overset{\text{gen}}{<} a_{0,m} a_{1,m} \cdots a_{m-1,m}. \tag{2.7}$$

Later we will need the inequality

$$\text{for all } m > k \geqslant 1 : \sum_{j=0}^{k-1} a_{j,m} G_j < G_k, \tag{2.8}$$

which we prove by using (2.5): either the left-hand side vanishes, then the inequality is trivial, or it equals $\mathtt{val}(\mathtt{rep}(G_m - 1)[k]) + 1 \leqslant G_k$. Assume now that $k < m$ and $\mathtt{val}(\mathtt{rep}(G_m - 1)[k]) + 1 = G_k$. This can occur if and only if $k = T^s(m)$ for some $s \geqslant 1$, i.e. k and m belong to the same branch of the tree of carries. But then $k \leqslant T(m)$ and hence the left-hand side of (2.8) vanishes.

In order to make the definition of the $a_{k,m}$ more transparent, we present three simple examples.

Example 1. Consider the usual d-adic expansion, where $d \geqslant 2$ is an integer. We have $G_n = d^n$ and $\mathtt{rep}(G_n - 1) = (d-1)^{(n)}$. Therefore, for all $k < m$, we have $\mathtt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)$. For fixed m, the greatest $k < m$ such that $\mathtt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)$ is then $m - 1$ and we have $T(m) = m - 1$. We may write

$$G_m - 1 = \sum_{j=0}^{m-1} (d-1) G_j = d G_{m-1} - 1, \tag{2.9}$$

$$G_m = d G_{m-1}. \tag{2.10}$$

Expansion (2.9) is the admissible expansion of $G_m - 1$, since $(d-1)^{(m)} \in \mathcal{W}$. Expansion (2.10) is not admissible and corresponds to (2.6).

We have $\mathcal{K}_G = \{0, 1, \ldots, d-1\}^\infty$ and its elements can be characterised by

$$x \in \mathcal{K}_G \iff \forall m \geqslant 0 : x[m] \overset{\text{gen}}{<} 0^{(m-1)} d.$$

Example 2. Consider the so-called Tribonacci sequence, given by the linear recurrence

$$G_0 = 1, \quad G_1 = 2, \quad G_2 = 4, \quad \text{for all } n \geqslant 0 : G_{n+3} = G_{n+2} + G_{n+1} + G_n.$$

It was shown in [**7**, **8**] that

$$\mathcal{K}_G = \{(x_n)_n \in \{0, 1\}^{\mathbb{N}} \mid \forall n \in \mathbb{N} : x_n x_{n+1} x_{n+2} \neq 111\}.$$

Then we have

$$\mathtt{rep}(G_{3n+r} - 1) = \begin{cases} (011)^{(n)} & \text{if } r = 0, \\ 1(011)^{(n)} & \text{if } r = 1, \\ 11(011)^{(n)} & \text{if } r = 2, \end{cases}$$

and $T(n) = n - 3$ for $n \geqslant 3$, and $T(1) = 0$, $T(2) = 1$. Equation (2.6) reads as

$$G_1 = 2G_0, \quad G_2 = 2G_1, \quad G_n = G_{n-1} + G_{n-2} + G_{n-3} \quad \text{for } n \geqslant 3. \tag{2.11}$$

G. Rauzy initiated the study of dynamical systems related to substitutions and numeration, as well as related fractals (nowadays called 'Rauzy fractals') in his seminal paper [**30**] on the Tribonacci numeration system.

Example 3. Consider the scale of slowest possible growth. It is given by $G_n = n + 1$. We have $G_n = G_{n-1} + G_0$ and $T(n) = 0$ for all n. Furthermore,

$$\mathcal{K}_G = \{(x_n)_n \in \{0, 1\}^{\mathbb{N}} \mid \#\{n \mid x_n = 1\} \leqslant 1\}.$$

3. *Scales with locally slow growth*

The following condition for the uniqueness of the invariant measure on $(\mathcal{K}_G, \tau)$ was given in [**3**, Théorème 8]

$$\text{there exists } M : \text{for all } n \in \mathbb{N} : G_n \sum_{k \geqslant n} \frac{1}{G_k} \leqslant M \quad \text{and} \quad \lim_{n\to\infty} (G_{n+1} - G_n) = \infty. \tag{3.1}$$

These conditions are concerned with the order of growth of the sequence $(G_n)_n$. We now investigate in detail the boundedness of the sequence $(G_n \sum_{k>n} G_k^{-1})_n$, giving equivalent formulations, which shed more light on the growth conditions on the scale behind (3.1). For that purpose, we introduce a further notation. Using (2.2), set

$$e(m) = \begin{cases} 0 & \text{if } a_{m-1,m} \geqslant 2, \\ \ell & \text{if } a_{m-1,m} = 1,\ a_{m-2,m} = \cdots = a_{m-\ell,m} = 0,\ a_{m-\ell-1,m} \neq 0. \end{cases} \tag{3.2}$$

Since $G_m > G_{m-1}$, $e(m)$ is well defined for all $m \geqslant 1$. As a motivation for this definition, we note that

$$\begin{aligned} e(m) = 0 &\Leftrightarrow G_m \geqslant 2G_{m-1} \\ e(m) \leqslant \ell &\Leftrightarrow G_m \geqslant G_{m-1} + G_{m-\ell-1}; \end{aligned} \tag{3.3}$$

the larger values $e(m)$ attains, the slower the sequence $(G_m)_m$ grows. This is made precise in the following proposition.

PROPOSITION 1. *Let $(G_n)_n$ be a system of numeration.*

For any $\ell \geqslant 1$, let

$$A_\ell = \inf_n \frac{G_{n+\ell}}{G_n} \quad \textit{and} \quad B_\ell = \liminf_{n\to\infty} \frac{G_{n+\ell}}{G_n}.$$

Then both sequences $(A_\ell)_\ell$ and $(B_\ell)_\ell$ satisfy

$$\text{for all } k,\, \ell \in \mathbb{N} : A_{k+\ell} \geqslant A_k A_\ell, \quad B_{k+\ell} \geqslant B_k B_\ell$$

and we have that

$$\lim A_\ell = \lim B_\ell \in \{1, \infty\}. \tag{3.4}$$

Furthermore, the following properties are equivalent:

(i) $\lim_{\ell\to\infty} A_\ell = \infty$;

(ii) there exists $\ell \geqslant 1 : A_\ell > 1$;

(iii) there exists $\ell \geqslant 1 : \sup_n \min\{e(n), e(n+1), \ldots, e(n+\ell-1)\} < \infty$;

(iv) *the sequence $(G_n \sum_{k\geqslant n} G_k^{-1})_n$ is bounded.*

If one of these properties holds, then

$$\liminf_{n\to\infty} \frac{\log G_n}{n} > 0. \tag{3.5}$$

Proof. We have for all $\ell \geqslant 1$ and all $n \geqslant 0$

$$1 \leqslant A_\ell = \inf_n \frac{G_{n+\ell}}{G_n} \leqslant \liminf_{n\to\infty} \frac{G_{n+\ell}}{G_n} = B_\ell.$$

Since the sequence $(G_n)_n$ is increasing, both $(A_\ell)_\ell$ and $(B_\ell)_\ell$ are non-decreasing, hence the existence of their limits. Furthermore, taking the infimum (respectively the inferior limit, both with respect to n) in

$$\frac{G_{n+k+\ell}}{G_n} = \frac{G_{n+k+\ell}}{G_{n+k}} \frac{G_{n+k}}{G_n} \tag{3.6}$$

yields $A_{k+\ell} \geqslant A_\ell A_k$ (respectively $B_{k+\ell} \geqslant B_\ell B_k$). Then, Fekete's lemma yields

$$\lim \frac{\log A_\ell}{\ell} = \sup \frac{\log A_\ell}{\ell}. \tag{3.7}$$

Then, we either have $A_\ell = 1$ for all ℓ, or there exist a $C > 0$ and ℓ_0 such that $\log A_\ell \geqslant C\ell$ for $\ell \geqslant \ell_0$ (this will be used later). On the other hand, since $G_{n+\ell} > G_n$ for all n, it follows from $A_\ell = 1$ that $B_\ell = 1$ as well. We thus have proved both (3.4) and the equivalence between (i) and (ii).

Assume that (iii) is true and choose $m \geqslant 1$ and $r \geqslant 0$ such that

$$\text{for all } n \in \mathbb{N} : \min\{e(n), e(n+1), \ldots, e(n+m-1)\} \leqslant r.$$

Given n, by (3.3) there exists $0 \leqslant k \leqslant m-1$ such that $G_{n+k} \geqslant G_{n+k-1} + G_{n+k-1-r}$. From this inequality and the fact that the sequence $(G_n)_n$ is increasing, it follows that

$$G_{n+m-1} \geqslant G_{n+k} \geqslant G_{n+k-1} + G_{n+k-r-1} > 2G_{n-r-1}$$

holds, hence (ii) holds for $\ell = m + r$.

Assume conversely that (iii) is false. Take ℓ arbitrary. Then there exist infinitely many n such that $e(n+k) \geqslant \ell$ for all $1 \leqslant k \leqslant \ell$, hence again by (3.3) we have $G_{n+k} - G_{n+k-1} \leqslant G_{n+k-1-\ell}$. Summing up these relations yields

$$G_{n+\ell} \leqslant G_n + \ell G_{n-1} \leqslant (\ell+1) G_n, \tag{3.8}$$

hence $A_\ell \leqslant \ell + 1$ for all $\ell \geqslant 1$. Then we have $\lim_\ell ((\log A_\ell)/\ell) = 0$ and therefore $A_\ell = 1$ for all ℓ. Hence (ii) does not hold.

Assume now that (ii) holds with some $A_\ell > 1$. Then

$$\begin{aligned} G_n \sum_{k \geqslant n} \frac{1}{G_k} &= G_n \sum_{k=0}^{\ell-1} \sum_{s=0}^{\infty} \frac{1}{G_{n+s\ell+k}} \leqslant G_n \sum_{k=0}^{\ell-1} \frac{1}{G_{n+k}} \sum_{s=0}^{\infty} \frac{1}{A_\ell^s} \\ &= G_n \sum_{k=0}^{\ell-1} \frac{1}{G_{n+k}} \frac{A_\ell}{A_\ell - 1} \leqslant \frac{\ell A_\ell}{A_\ell - 1}. \end{aligned}$$

This proves that (iv) follows from (ii).

Conversely, one has that

$$G_n \sum_{k \geqslant n} \frac{1}{G_k} \geqslant G_n \sum_{k=n}^{n+\ell} \frac{1}{G_k} \geqslant (\ell+1) \frac{G_n}{G_{n+\ell}}.$$

Assume that (ii) does not hold. Then, for all ℓ, we have

$$\limsup_{n \to \infty} \left(G_n \sum_{k \geqslant n} \frac{1}{G_k} \right) \geqslant (\ell+1) \frac{1}{B_\ell} = \ell + 1.$$

Then (iv) does not hold either.

Finally, if (ii) holds, then by the trivial inequality $G_n = G_n/G_0 \geqslant A_n$ we have

$$\liminf_{n \to \infty} \frac{\log G_n}{n} \geqslant \liminf_{n \to \infty} \frac{\log A_n}{n} > 0. \tag{3.9}$$

□

Remark 2. By Proposition 1 the condition for unique ergodicity given in (3.1) implies exponential growth of the scale $(G_n)_n$ in the sense of (3.5). Our approach, especially Theorem 3, will allow us to show unique ergodicity for scales which increase much more slowly. We call sequences which do not satisfy the conditions of Proposition 1 *scales of locally slow growth* to indicate that we can especially allow linear growth in arbitrarily long intervals (see §5 for examples of this type). However, in contrary to condition (3.1), the condition (4.34) we give in Theorem 3 is not a pure growth condition, but also takes into account the combinatorics of the scale.

Remark 3. The conditions (3.1) are satisfied, if $G_n \sim C\alpha^n$ for $C > 0$ and $\alpha > 1$. Thus an exponential asymptotic growth of $(G_n)_n$ implies unique ergodicity of the odometer.

4. *Invariant measures on the odometer*

4.1. *Equations.* Let μ be an invariant measure on $(\mathcal{K}_G, \tau)$. We use the invariance of μ to get relations between the measures of the cylinders. We denote a disjoint union by $\uplus$. We first recall that by the definition of $T(n)$ in (2.1) we have $\mathtt{rep}(G_m - 1) \overset{\mathrm{pref}}{\preccurlyeq} \mathtt{rep}(G_n - 1)$ if and only if $m \leqslant n$ and n and m belong to the same branch of the tree of carries, that is if there exists r such that $T^r(n) = m$. This observation and the definition of the addition of 1 yield

$$\tau^{-1}([0^{(k)}]) = \bigcup_{m \geqslant k} [\mathtt{rep}(G_m - 1)] = \biguplus_{\substack{m \geqslant k \\ T(m) < k}} [\mathtt{rep}(G_m - 1)]. \tag{4.1}$$

Observe that $m = k$ always occurs in the disjoint union above. This yields

$$\tau^{-1}([0^{(k)}] \setminus [0^{(k+1)}]) = [\mathtt{rep}(G_k - 1)] \setminus \biguplus_{T(m) = k} [\mathtt{rep}(G_m - 1)], \tag{4.2}$$

where the disjoint union is a subset of $[\mathtt{rep}(G_k - 1)]$, since for $T(m) = k$, $\mathtt{rep}(G_k - 1) \overset{\mathrm{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)$ by the definition of $T(m)$. Furthermore, let $w \in \mathcal{W}'$ be a word of length k, $w \neq 0^{(k)}$, then we have

$$\tau^{-1}([w]) = [w^-] \setminus \bigcup_{m \geqslant k} [\mathtt{rep}(G_m - 1)] = [w^-] \setminus \biguplus_{\substack{m \geqslant k, T(m) < k \\ w^- \overset{\mathrm{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)}} [\mathtt{rep}(G_m - 1)], \tag{4.3}$$

where the disjoint union is a subset of $[w^-]$.

In the following we will use Iverson's notation: $[\![P]\!]$ is defined to be 1 if condition P is satisfied, and 0 otherwise. From (4.1) and (4.3) we derive

$$\mu([0^{(k)}]) = \sum_{m \geqslant k} \mu([\mathtt{rep}(G_m - 1)])[\![T(m) < k]\!] \tag{4.4}$$

and for $|v| = k$, $v \neq 0^{(k)}$,

$$\mu([v]) - \mu([v^-]) = -\sum_{m \geqslant k, T(m) < k} \mu([\mathtt{rep}(G_m - 1)])[\![v^- \overset{\mathrm{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)]\!]. \tag{4.5}$$

Summing this equation for all $v \in \mathcal{W}'$ of length k with $0 < \mathtt{val}(v) \leqslant \mathtt{val}(w)$ we obtain

$$\begin{aligned}
&\mu([w]) - \mu([0^{(k)}]) \\
&\quad = -\sum_{|v| = k} [\![0 < \mathtt{val}(v) \leqslant \mathtt{val}(w)]\!] \sum_{m \geqslant k, T(m) < k} \mu([\mathtt{rep}(G_m - 1)])[\![v^- \overset{\mathrm{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)]\!] \\
&\quad = -\sum_{|v| = k} [\![\mathtt{val}(v) < \mathtt{val}(w)]\!] \sum_{m \geqslant k, T(m) < k} \mu([\mathtt{rep}(G_m - 1)])[\![v \overset{\mathrm{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)]\!].
\end{aligned}$$

Inserting (4.4), rearranging the double sum and using (2.5) in the second line we get

$$\begin{aligned}
\mu([w]) &= \sum_{m \geqslant k, T(m) < k} \mu([\mathtt{rep}(G_m - 1)])[\![\mathtt{val}(\mathtt{rep}(G_m - 1)[k]) \geqslant \mathtt{val}(w)]\!] \\
&= \sum_{m \geqslant k} \mu([\mathtt{rep}(G_m - 1)])[\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(w)]\!].
\end{aligned} \tag{4.6}$$

Equation (4.6) is even true for $w = 0^{(k)}$, since it then coincides with (4.4). Furthermore, equation (4.5) shows that $\mu([w]) \leqslant \mu([w^-])$. Therefore, we have that

$$\mu([\mathtt{rep}(G_k - 1)]) \leqslant \mu([\mathtt{rep}^*(G_k - 2)[k]]) \leqslant \cdots \leqslant \mu([\mathtt{rep}^*(1)[k]]) \leqslant \mu([0^{(k)}]). \tag{4.7}$$

In particular, since $\sum_{w\in\mathcal{W}',|w|=k} \mu([w]) = 1$, we have

$$\mu([0^{(k)}]) \geqslant \frac{1}{G_k} \quad \text{and} \quad \mu([\mathtt{rep}(G_k - 1)]) \leqslant \frac{1}{G_k}. \tag{4.8}$$

To get a relation between the numbers $\mu([\mathtt{rep}(G_k - 1)])$, we use

$$[\mathtt{rep}(G_k - 1)] = \biguplus_{\substack{|w|=k+1\\ \mathtt{rep}(G_k-1) \overset{\text{pref}}{\preccurlyeq} w}} [w]$$

and (4.6) to write

$$\begin{aligned}
&\mu([\mathtt{rep}(G_k - 1)])\\
&\quad= \sum_{|w|=k+1} \mu([w])[\![\mathtt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} w]\!]\\
&\quad= \sum_{|w|=k+1} [\![\mathtt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} w]\!] \sum_{m\geqslant k+1} \mu([\mathtt{rep}(G_m - 1)])[\![\textstyle\sum_{j=0}^{k} a_{j,m}G_j > \mathtt{val}(w)]\!]\\
&\quad= \sum_{m\geqslant k+1} \mu([\mathtt{rep}(G_m - 1)])\\
&\qquad \times \#\left\{ w \in \mathcal{W}' \;\middle|\; |w| = k + 1,\ \mathtt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} w,\ \text{and} \sum_{j=0}^{k} a_{j,m}G_j > \mathtt{val}(w)\right\}.
\end{aligned} \tag{4.9}$$

The words to be counted in the last line are those w satisfying

$$\mathtt{val}(w) + 1 = (\eta + 1)G_k \leqslant \sum_{j=0}^{k} a_{j,m}G_j.$$

By (2.8) the possible choices for η are exactly $0, \ldots, a_{k,m} - 1$. Hence we get

$$\begin{aligned}
\mu([\mathtt{rep}(G_k - 1)]) &= \sum_{m\geqslant k+1, T(m)<k+1} a_{k,m}\mu([\mathtt{rep}(G_m - 1)])\\
&= \sum_{m\geqslant k+1} a_{k,m}\mu([\mathtt{rep}(G_m - 1)]),
\end{aligned} \tag{4.10}$$

where the last expression comes from the fact that $a_{k,m} = 0$ whenever $k < T(m)$ by (2.4). We can now summarise the results.

PROPOSITION 2. *Let $(G_n)_n$ be a system of numeration and $a_{k,m}$ be given by (2.4). If μ is an invariant probability measure on the odometer $(\mathcal{K}_G, \tau)$, then μ has the following* *properties:*

$$\text{for all } w \in \mathcal{W},\ |w| = k : \mu([w]) = \sum_{m \geqslant k} \mu([\mathtt{rep}(G_m - 1)])[\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(w)]\!], \quad (4.6)$$

$$\text{for all } k \geqslant 0 : \mu([\mathtt{rep}(G_k - 1)]) = \sum_{m \geqslant k+1} a_{k,m} \mu([\mathtt{rep}(G_m - 1)]). \quad (4.10)$$

For an invariant probability measure μ on the odometer (if there exists any), we set $\alpha_m = \mu([\mathtt{rep}(G_m - 1)])$ with the convention $\alpha_0 = 1$. By (4.8) $\lim_m \alpha_m = 0$ holds.

It turns out that equation (4.10) characterises invariant measures on the odometer.

THEOREM 1. *Let $(G_n)_n$ be a system of numeration and $a_{k,m}$ be given by (2.4). For any finite non-negative invariant measure μ on the corresponding odometer $\mathcal{K}_G$, let*

$$\mathfrak{A}(\mu) = (\alpha_m)_{m \geqslant 0} = (\mu([\mathtt{rep}(G_m - 1)]))_{m \geqslant 0}.$$

Then $\mathfrak{A}$ realises a homeomorphism between the set of τ-invariant non-negative measures on the odometer endowed with the weak topology and the set of the non-negative solutions $(\alpha_m)_{m \geqslant 0}$ of the infinite system of equations

$$\alpha_k = \sum_{m \geqslant k+1} a_{k,m} \alpha_m \quad (k \geqslant 0), \quad (4.11)$$

endowed with the product topology.

Proof. If μ is an invariant non-negative measure and $\alpha_m = \mu([\mathtt{rep}(G_m - 1)])$ for all m, it follows from (4.10) that $(\alpha_m)_{m \geqslant 0}$ satisfies (4.11). Obviously, $\alpha_m \geqslant 0$ and $\mu(\mathcal{K}_G) = \alpha_0$. Moreover, $\mathfrak{A}$ is continuous by definition of both weak convergence and product topology.

On the other hand, let $(\alpha_m)_{m \geqslant 0}$ be a non-negative solution of (4.11). We first define a function $\mathfrak{m}$ on the cylinders as follows: for $w \in \mathcal{W}'$, $|w| = k \geqslant 1$, set

$$\mathfrak{m}([w]) = \sum_{m \geqslant k} \alpha_m [\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(w)]\!] \quad (4.12)$$

and $\mathfrak{m}([\epsilon]) = \alpha_0$. By (4.6), if μ is such that $\mathfrak{A}(\mu) = (\alpha_m)_m$, then μ and $\mathfrak{m}$ coincide on all cylinders. Since the set of cylinders is stable under finite intersection and generates the Borel σ-algebra, there exists at most one measure μ with this property. The existence of μ will follow from Kolmogorov's consistency theorem if we prove that $\mathfrak{m}$ defines a consistent system.

For $k \geqslant 1$, let $\mathcal{F}_k = \sigma(\{[w] \mid w \in \mathcal{W}',\ |w| = k\})$. On $(\mathcal{K}_G, \mathcal{F}_k)$, the set-function $\mathfrak{m}$ induces a non-negative measure $\mathbb{P}_k$. To prove the consistency of the system $(\mathcal{K}_G, \mathcal{F}_k, \mathbb{P}_k)_k$, it is enough to show that for any $w \in \mathcal{W}'$, the equality

$$\begin{aligned} \mathbb{P}_{k+1}([w]) &= \sum_{|v|=|w|+1} \mathbb{P}_{k+1}([v])[\![w \overset{\text{pref}}{\preccurlyeq} v]\!] \\ &= \sum_{|v|=|w|+1} \mathfrak{m}([v])[\![w \overset{\text{pref}}{\preccurlyeq} v]\!] = \mathfrak{m}([w]) = \mathbb{P}_k([w]) \end{aligned}$$

holds. Indeed, for fixed $w \in \mathcal{W}'$, $|w| = k$, we have

$$\begin{aligned}
&\sum_{|v|=k+1} \mathfrak{m}([v]) [\![w \overset{\text{pref}}{\preccurlyeq} v]\!] \\
&\quad = \sum_{|v|=k+1} [\![w \overset{\text{pref}}{\preccurlyeq} v]\!] \sum_{m \geqslant k+1} \alpha_m [\![\textstyle\sum_{j=0}^{k} a_{j,m} G_j > \mathtt{val}(v)]\!] \\
&\quad = \sum_{m \geqslant k+1} \alpha_m \# \left\{ v \,\middle|\, |v| = k+1,\ w \overset{\text{pref}}{\preccurlyeq} v \text{ and } \sum_{j=0}^{k} a_{j,m} G_j > \mathtt{val}(v) \right\}.
\end{aligned}$$

Thus we have to count the number of words $v = w\eta \in \mathcal{W}'$ such that

$$\mathtt{val}(v) = \eta G_k + \mathtt{val}(w) < \sum_{j=0}^{k} a_{j,m} G_j = \sum_{j=T(m)}^{k} a_{j,m} G_j.$$

That inequality is automatically satisfied if $\eta < a_{k,m}$; for $\eta = a_{k,m}$ it is satisfied if and only if

$$\mathtt{val}(w) < \sum_{j=0}^{k-1} a_{j,m} G_j.$$

Therefore, we can write

$$\begin{aligned}
\sum_{|v|=k+1} \mathfrak{m}([v]) [\![w \overset{\text{pref}}{\preccurlyeq} v]\!] &= \sum_{m \geqslant k+1} \alpha_m \sum_{v = wa} [\![\textstyle\sum_{j=0}^{k} a_{j,m} G_j > \mathtt{val}(w) + a G_k]\!] \\
&= \sum_{m \geqslant k+1} a_{k,m} \alpha_m + \sum_{m \geqslant k+1} \alpha_m [\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(w)]\!] \\
&= \alpha_k + \sum_{m \geqslant k+1} \alpha_m [\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(w)]\!] \\
&= \sum_{m \geqslant k} \alpha_m [\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(w)]\!] = \mathfrak{m}([w]),
\end{aligned}$$

by (4.11) since $G_k = \sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(w)$. For $|w| = 0$ we get for all $k \geqslant 1$

$$\begin{aligned}
\mathbb{P}_k(\mathcal{K}_G) = \sum_{|v|=1} \mathfrak{m}([v]) &= \sum_{|v|=1} \sum_{m \geqslant 1} \alpha_m [\![a_{0,m} > \mathtt{val}(v)]\!] \\
&= \sum_{m \geqslant 1} \alpha_m \sum_{|v|=1} [\![a_{0,m} > \mathtt{val}(v)]\!] = \sum_{m \geqslant 1} a_{0,m} \alpha_m = \alpha_0.
\end{aligned}$$

Let μ be the extension of $\mathfrak{m}$ on $\mathcal{B}(\mathcal{K}_G)$. We check τ-invariance of μ. Equations (4.1) and (4.12) give $\mu(\tau^{-1}([0^{(k)}])) = \mu([0^{(k)}])$; for $v \neq 0^{(k)}$ we apply μ to (4.3) to obtain

$$\mu(\tau^{-1}([v])) = \mu([v^-]) - \sum_{m \geqslant k, T(m) < k} \mu([\mathtt{rep}(G_m - 1)]) [\![v^- \overset{\text{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)]\!].$$

On the other hand, we obtain from (4.12)

$$\begin{aligned}
\mu([v^-]) - \mu([v]) &= \sum_{m \geqslant k} \alpha_m \left([\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(v) - 1]\!] - [\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j > \mathtt{val}(v)]\!] \right) \\
&= \sum_{m \geqslant k} \alpha_m [\![\textstyle\sum_{j=0}^{k-1} a_{j,m} G_j = \mathtt{val}(v)]\!].
\end{aligned}$$

For $v \in \mathcal{W}'$ with $|v| = k$ and $v \neq 0^{(k)}$ the equality $\sum_{j=0}^{k-1} a_{j,m} G_j = \mathtt{val}(v)$ holds, if and only if the conditions $v^- \overset{\text{pref}}{\preccurlyeq} \mathtt{rep}(G_m - 1)$ and $T(m) < k$ are satisfied. Thus we have proved $\mu(\tau^{-1}([v])) = \mu([v])$. It follows that μ is τ-invariant. Hence we have proved that any non-negative solution of (4.11) defines a unique invariant Borel measure on $\mathcal{K}_G$.

It remains to prove that $\mathfrak{A}^{-1}$ is continuous. For convenience, we define $\ell_m([w]) = [\![\sum_{j=0}^{|w|-1} a_{j,m} G_j > \mathtt{val}(w)]\!]$ if $m \geqslant |w|$ and 0 otherwise. Every open set $\mathcal{O}$ can be written (not uniquely) as a countable disjoint union of cylinders

$$\mathcal{O} = \biguplus_{w \in \mathcal{A}(\mathcal{O})} [w].$$

Given this decomposition of $\mathcal{O}$ we have for $\mu = \mathfrak{A}^{-1}((\alpha_m)_m)$

$$\mu(\mathcal{O}) = \sum_{w \in \mathcal{A}(\mathcal{O})} \mu([w]) = \sum_{m=1}^{\infty} \alpha_m \sum_{w \in \mathcal{A}(\mathcal{O})} \ell_m([w]).$$

Let $(\alpha^{(n)})$ be a sequence of solutions of (4.11) converging to $(\alpha_m)_m$. Set $\mu_n = \mathfrak{A}^{-1}((\alpha_m^{(n)})_m)$ and $\mu = \mathfrak{A}^{-1}((\alpha_m)_m)$ be the corresponding measures. For any finite M we have

$$\mu_n(\mathcal{O}) \geqslant \sum_{m=1}^{M} \alpha_m^{(n)} \sum_{w \in \mathcal{A}(\mathcal{O})} \ell_m([w]),$$

from which we conclude $\liminf_{n\to\infty} \mu_n(\mathcal{O}) \geqslant \mu(\mathcal{O})$.

Therefore, $\mu(\mathcal{O}) \leqslant \liminf \mu_n(\mathcal{O})$ for every open set $\mathcal{O}$, which by regularity of the measure implies that $\lim \mu_n = \mu$ weakly. For $K > 0$, the set of invariant measures with $\mu(\mathcal{K}_G) \leqslant K$ is compact. Therefore, $\mathfrak{A}$ induces on it a homeomorphism onto the set of solutions with $\alpha_0 \leqslant K$. It follows that $\mathfrak{A}$ realises itself a homeomorphism and the theorem is proved. □

Remark 4. The above theorem ensures that the set of invariant probability measures is compact. It was not clear without any topological assumption on τ that τ-invariance is stable under weak convergence.

From now on we restrict ourselves to the study of invariant probability measures, i.e. those such that $\alpha_0 = 1$. We denote by $\mathtt{Inv}(\mathcal{K}_G)$ the set of τ-invariant probability measures; the set of ergodic probability measures is denoted by $\mathtt{Erg}(\mathcal{K}_G)$. These are the extremal points of $\mathtt{Inv}(\mathcal{K}_G)$. With the introduced machinery, we easily retrieve a result of [**3**].

PROPOSITION 3. *Any non-zero invariant probability measure on the odometer is continuous (i.e. all countable sets have zero measure). Furthermore, every such measure charges every open subset of $\mathcal{K}_G$. In other words, $\alpha_k \neq 0$ for any k.*

Proof. Assume that $\mu(\{x\}) > 0$ for some $x \in \mathcal{K}_G$. Then

$$0 < \mu(\{x\}) = \lim_{m\to\infty} \mu([x[m]]).$$

Since by (4.7) $\mu(x[m]) \leqslant \mu([0^{(m)}])$, this implies that $\mu(\{0\}) > 0$. By τ-invariance it then follows that $\mu(\{\mathtt{rep}^*(n)\}) \geqslant \mu(\{0\}) > 0$ for all $n \in \mathbb{N}$, which would assign infinite measure to $\mathcal{K}_G$.

Assume that there is an invariant measure μ on $\mathcal{K}_G$ and a word $w \in \mathcal{W}'$ with $\mu([w]) = 0$. By (4.7), we have $\alpha_{|w|} = \mu([\mathtt{rep}(G_{|w|} - 1)]) = 0$. Inserting this into (4.11) for $k = |w|$ and using that $a_{k,k+1} \neq 0$, we get $\alpha_{k+1} = 0$, hence $\alpha_\ell = 0$ for all $\ell \geqslant |w|$ by immediate induction. Then (4.6) shows that all cylinders of length $|w|$ have measure 0, hence $\mu(\mathcal{K}_G) = 0$. □

We illustrate this section by continuing to investigate the examples of scales given in §2.

Example 1 (continued). We have already seen in (2.10) that $a_{m-1,m} = d$ and $a_{k,m} = 0$ otherwise. Hence equations (4.11) become $\alpha_k = d\alpha_{k+1}$ for $k \geqslant 0$. The system has a unique solution such that $\alpha_0 = 1$, namely $\alpha_k = d^{-k}$, and (4.6) becomes $\mu([w]) = d^{-k}$ for any word w of length k. On $\mathcal{K}_G = \{0, 1, \dots, d-1\}^{\mathbb{N}}$ we get the product measure of the uniform probability measure on $\{0, 1, \dots, d-1\}$. Alternatively, it is also easy to extend the addition on $\mathcal{K}_G$ by $x \dot{+} y = \lim \mathtt{rep}^*(\mathtt{val}(x[k]) + \mathtt{val}(y[k]))$, defining the compact group $(\mathbb{Z}_d, \dot{+})$ of d-adic integers. Existence, uniqueness and the description of μ of course also follow from the properties of the Haar measure.

Example 2 (continued). From (2.11) and taking $\alpha_0 = 1$, we get

$$1 = 2\alpha_1 + \alpha_3, \quad \alpha_1 = 2\alpha_2 + \alpha_3 + \alpha_4, \quad \alpha_k = \alpha_{k+1} + \alpha_{k+2} + \alpha_{k+3} (k \geqslant 2). \tag{4.13}$$

The polynomial $X^3 + X^2 + X - 1$ has a real root $\rho \simeq 0.543\,689$ and two complex conjugate roots of modulus greater than 1. Since the solutions $(\alpha_k)_k$ of (4.13) we are interested in are bounded, they have to be of the form $\alpha_k = C\rho^k$ for $k \geqslant 2$. Inserting this into the first two equations of (4.13) yields $C = \rho$, hence $\alpha_1 = 1 - \rho$ and $\alpha_k = \rho^{k+1}$ for all $k \geqslant 2$. This shows unique ergodicity of the odometer. Furthermore, we get $\mu([0^{(k)}]) = \rho^k = \mu([\mathtt{rep}(G_{k-1} - 1)])$. We notice that in this case the odometer does not come from a topological group as in the previous example. This can be seen, for instance, by observing that $\tau^{-1}(\{0^\infty\}) = \{(101)^\infty, (011)^\infty, (110)^\infty\}$.

Example 3 (continued). Equations (4.11) become

$$1 = \sum_{k \geqslant 1} \alpha_k, \quad \alpha_k = \alpha_{k+1} (k \geqslant 1),$$

which obviously does not have any solution. Therefore, there is no invariant probability measure on this odometer.

The following proposition describes the case of equality in (4.7) for one fixed length j.

PROPOSITION 4.

(1) *Let $(\alpha_n)_{n \in \mathbb{N}}$ be a solution of (4.11). Let j be a non-negative integer. Then the following statements are equivalent:*
   (i) $\alpha_j G_j = 1$ *holds,*
   (ii) *for all $k < j$, for all $m > j$: $a_{k,m} = 0$,*
   (iii) *for all $m > j$: $G_j \mid G_m$,*
   (iv) *for all $m > j$: $T(m) \geqslant j$.*

(2) *Let $G = (G_n)_n$ be a numeration system as above. Define $G'_n = G_{n+j}/G_j$ for all $n \geqslant 0$ and consider the numeration system $G' = (G'_n)_n$. Let p and q be the projections $p(x) = (x_j, x_{j+1}, \ldots)$ and $q(x) = (x_0, \ldots, x_{j-1})$ on $\mathcal{K}_G$. Furthermore introduce a function φ on $\mathbb{Z}/G_j\mathbb{Z}$ by $\varphi(-1) = \tau_{G'}$ and $\varphi(a) = \mathrm{id}_{\mathcal{K}_{G'}}$ if $a \neq -1$. On the product $\mathcal{K}_{G'} \times \mathbb{Z}/G_j\mathbb{Z}$, we define a transformation $T(x, a) = (\varphi(a)(x), a+1)$. Then $\Phi\colon x \mapsto (p(x), q(x))$ gives a homeomorphism between $\mathcal{K}_G$ and $\mathcal{K}_{G'} \times \mathbb{Z}/G_j\mathbb{Z}$ which commutes with τ_G and T (if $(\mathcal{K}_{G'}, \tau_{G'})$ is a group this is a skew product). Furthermore, $\mathtt{Inv}(\mathcal{K}_G)$ and $\mathtt{Inv}(\mathcal{K}_{G'})$ are homeomorphic through the transformation $\mu_{G'} \mapsto \mu_{G'} \otimes h_{\mathbb{Z}/G_j\mathbb{Z}}$, where h denotes the Haar measure.*

Proof. (1) Assume that (i) holds. From equation (4.11) for k from 0 to $j-1$ and (2.6), we deduce

$$\sum_{k=0}^{j-1} \alpha_k G_k = \sum_{k=0}^{j-1} G_k \sum_{m=k+1}^{\infty} a_{k,m}\alpha_m = \sum_{m=1}^{j} \alpha_m \sum_{k=0}^{m-1} a_{k,m} G_k + \sum_{m=j+1}^{\infty} \alpha_m \sum_{k=0}^{j-1} a_{k,m} G_k$$
$$= \sum_{m=1}^{j} \alpha_m G_m + \sum_{m=j+1}^{\infty} \alpha_m \sum_{k=0}^{j-1} a_{k,m} G_k.$$

After simplification, we get

$$1 = \alpha_0 = \alpha_j G_j + \sum_{m=j+1}^{\infty} \alpha_m \sum_{k=0}^{j-1} a_{k,m} G_k. \tag{4.14}$$

Since $\alpha_j G_j = 1$, the sum has to vanish. Moreover, all α_m are strictly positive, which implies (ii).

If (ii) holds, then $a_{k,m} = 0$ for $0 \leqslant k < j < m$. By (2.6), this implies

$$G_{j+1} = a_{j,j+1} G_j \quad \text{hence } G_j \mid G_{j+1}.$$

The assertion (iii) then follows by induction using (2.6) again.

It follows from (2.6) that $G_j \mid G_{j+1}$ if and only if $T(j+1) = j$. By induction on p, one shows using (2.6) that $G_j \mid G_{j+r}$ for all r from 1 to p if, and only if, $T(j+r) \geqslant j$ for all r from 1 to p, hence the equivalence between (iii) and (iv).

If (iv) holds, then, again by (2.6), (ii) holds as well, which implies that the sum in (4.14) vanishes, hence $\alpha_j G_j = 1$.

(2) This statement relates the structure of the odometer for which $\alpha_j G_j = 1$ for some j to the odometer generated by the sequence $G' = (G_{n+j}/G_j)_n$. The proof is straightforward. □

Remark 5. The situation described in Proposition 4(2) is equivalent to the fact that the sequence $(\tau^n(x)[j])_n$ formed by the first j digits of any orbit is purely periodic. Furthermore, all cylinders $[w]$ ($w \in \mathcal{W}'$) with $|w| = j$ have the same measure.

COROLLARY 1. *The sequence $(\alpha_j G_j)_j$ is constant equal to 1, if and only if $G_j \mid G_{j+1}$ for all j. In this case $\mathcal{K}_G$ is the group of $\mathbf{a}$-adic integers with $\mathbf{a} = (G_{j+1}/G_j)_j$ as described in* [**23**].

4.2. *Invariant measures as cluster points.* For m a positive integer and $w \in \mathcal{W}'$ we define

$$C_m^k = \#\{0 \leqslant n < G_m \mid \mathtt{rep}^*(n) \in [\mathtt{rep}(G_k - 1)]\}, \tag{4.15}$$
$$D_m(w) = \#\{0 \leqslant n < G_m \mid \mathtt{rep}^*(n) \in [w]\}. \tag{4.16}$$

The initial values are $C_m^k = 0$ for $m < k$ and $C_k^k = 1$. For $m \leqslant |w|$, $D_m(w) = [\![G_m < \mathtt{val}(w)]\!]$; in other words, for $w = w_1 0^{(s)}$ with $w_1 \in \mathcal{W}$, we have $D_0(w) = \cdots = D_{|w_1|-1}(w) = 0$ and $D_{|w_1|}(w) = \cdots = D_{|w|}(w) = 1$. Notice that $C_m^k = D_m(\mathtt{rep}(G_k - 1))$.

LEMMA 1. *The quantities C_m^k satisfy the recursion*

$$C_m^k = \sum_{\ell=0}^{m-1} a_{\ell,m} C_\ell^k + \delta_{m,k} \text{ and } C_0^k = \cdots = C_{k-1}^k = 0. \tag{4.17}$$

For $m \geqslant |w|$ the quantities $D_m(w)$ can be expressed in terms of C_m^k in the following way

$$D_m(w) = \sum_{\ell=|w|}^{\infty} C_m^\ell [\![\textstyle\sum_{j=0}^{|w|-1} a_{j,\ell} G_j > \mathtt{val}(w)]\!]. \tag{4.18}$$

Proof. We define
$$N_{\ell,m} = \sum_{j=\ell}^{m-1} a_{j,m} G_j \quad (0 \leqslant \ell \leqslant m). \tag{4.19}$$

Notice that $N_{\ell,m}$ is non-decreasing in the index ℓ, with $N_{m,m} = 0$ and $N_{0,m} = G_m$ by (2.6).

For $m \in \mathbb{N}$ and $w \in \mathcal{W}'$, we have

$$\begin{aligned}
D_m(w) &= \sum_{\ell=0}^{m-1} \sum_{N_{\ell+1,m} \leqslant n < N_{\ell,m}} \mathbb{1}_{[w]}(n) = \sum_{\ell=0}^{m-1} \sum_{n < a_{\ell,m} G_\ell} \mathbb{1}_{[w]}(n + N_{\ell+1,m}) \\
&= \sum_{\ell=|w|}^{m-1} \sum_{n < a_{\ell,m} G_\ell} \mathbb{1}_{[w]}(n) + \sum_{n < G_m - N_{|w|,m}} \mathbb{1}_{[w]}(n + N_{|w|,m}) \\
&= \sum_{\ell=|w|}^{m-1} a_{\ell,m} D_\ell(w) + [\![G_m > N_{|w|,m} + \mathtt{val}(w)]\!] \\
&= \sum_{\ell=|w|}^{m-1} a_{\ell,m} D_\ell(w) + [\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m} G_j > \mathtt{val}(w)]\!].
\end{aligned} \tag{4.20}$$

In particular, for $w = \mathtt{rep}(G_k - 1)$, we get (4.17), and, for $w = \epsilon$, we get (2.6).

We prove (4.18) by induction on m. For $m = |w|$ the formula gives $D_{|w|}(w) = 1$ and is therefore correct. Assume that the formula holds for $|w| \leqslant \ell < m$ and insert (4.18) into the right-hand side of (4.20) to obtain

$$
\begin{aligned}
D_m(w) &= \sum_{\ell=|w|}^{m-1} a_{\ell,m} \sum_{k=|w|}^{\infty} C_\ell^k [\![\textstyle\sum_{j=0}^{|w|-1} a_{j,k} G_j > \mathtt{val}(w)]\!] + [\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m} G_j > \mathtt{val}(w)]\!] \\
&= \sum_{k=|w|}^{\infty} [\![\textstyle\sum_{j=0}^{|w|-1} a_{j,k} G_j > \mathtt{val}(w)]\!] \underbrace{\sum_{\ell=|w|}^{m-1} a_{\ell,m} C_\ell^k}_{C_m^k - \delta_{m,k}} + [\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m} G_j > \mathtt{val}(w)]\!] \\
&= \sum_{k=|w|}^{\infty} [\![\textstyle\sum_{j=0}^{|w|-1} a_{j,k} G_j > \mathtt{val}(w)]\!] C_m^k. \qquad \square
\end{aligned}
$$

LEMMA 2. *The quantities C_m^k satisfy the equation*

$$
C_m^k = \sum_{\ell=k+1}^{m} a_{k,\ell} C_m^\ell + \delta_{m,k} = \sum_{\ell \geqslant k+1} a_{k,\ell} C_m^\ell + \delta_{m,k} \tag{4.21}
$$

for all m and k.

Proof. By the definition of $D_m(w)$ we have

$$
C_m^k = \sum_{|w|=k+1} D_m(w) [\![\mathtt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} w]\!].
$$

If $k \geqslant m$, (4.21) is obviously fulfilled. If $k < m$ we may apply (4.18) which gives

$$
\begin{aligned}
C_m^k &= \sum_{|w|=k+1} [\![\mathtt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} w]\!] \sum_{\ell \geqslant k+1} C_m^\ell [\![\textstyle\sum_{j=0}^{k} a_{j,\ell} G_j > \mathtt{val}(w)]\!] \\
&= \sum_{\ell \geqslant k+1} C_m^\ell \# \left\{ w \in \mathcal{W}' \,\middle|\, |w| = k+1,\ \mathtt{rep}(G_k - 1) \overset{\text{pref}}{\preccurlyeq} w \text{ and } \sum_{j=0}^{k} a_{j,\ell} G_j > \mathtt{val}(w) \right\}.
\end{aligned}
$$

By the argument given after (4.9) the cardinality in the last line equals $a_{k,\ell}$. This gives the desired equation. $\square$

LEMMA 3. *The following inequalities hold for the quantities C_m^k for all k and m*

$$
C_m^k \leqslant \frac{G_m}{G_k}. \tag{4.22}
$$

Proof. The inequality is obviously true for $m = 0, \ldots, k$. Then induction using the recursion (4.17) for the values C_m^k gives the inequality for all m. $\square$

In the following, we are interested in a description of the set of invariant measures. For that purpose, let us introduce the measures

$$
\nu_k = \frac{1}{k} \sum_{j<k} \delta_j, \tag{4.23}
$$

where k is a positive integer and δ_j denotes for short the Dirac measure at point $\mathtt{rep}^*(j)$ in $\mathcal{K}_G$. Let

$$
\mathcal{M} = \{\nu_k \mid k \geqslant 1\} \quad \text{and} \quad \mathcal{M}_G = \{\nu_{G_n} \mid n \geqslant 0\}. \tag{4.24}
$$

THEOREM 2. *Let $(G_n)_n$ be a system of numeration, $a_{k,m}$ be given by (2.4), ν_k be the measures defined in (4.23), and $\mathcal{M}$ and $\mathcal{M}_G$ the sets of these measures given by (4.24). Then the following assertions hold.*

(i) *The sets of cluster points of the sequences*

$$\left(\sum_{k=1}^{n} \pi(k,n)\nu_k\right)_n \quad \text{with for all } n: \sum_{k=1}^{n} \pi(k,n) = 1$$
$$\text{and for all } k: \lim_{n\to\infty} \pi(k,n) = 0 \tag{4.25}$$

as well as those of the sequences

$$\left(\sum_{k=0}^{n} \xi(k,n)\nu_{G_k}\right)_n \quad \text{with for all } n: \sum_{k=0}^{n} \xi(k,n) = 1$$
$$\text{and for all } k: \lim_{n\to\infty} \xi(k,n) = 0 \tag{4.26}$$

coincide.

(ii) *All invariant measures on $\mathcal{K}_G$ are cluster points of sequences (4.26).*

(iii) *If the series*

$$\sum_{m\geqslant k+1} \frac{a_{k,m}}{G_m} \tag{4.27}$$

converge for all $k \geqslant 0$, then $\mathtt{Inv}(\mathcal{K}_G)$ is the set of cluster points of sequences (4.26). More precisely, if $(\mu_n)_n$ is a sequence of measures as in (4.26), and if $(\mu_n([\mathtt{rep}(G_k-1)]))_n$ converges for all $k \geqslant 0$, then $(\mu_n)_n$ converges weakly to a τ-invariant measure on $\mathcal{K}_G$. Moreover, the set $\mathtt{Inv}(\mathcal{K}_G)$ is not empty, and $\mathtt{Erg}(\mathcal{K}_G)$ is the set of extremal points of the weak closures of both $\mathcal{M}$ and $\mathcal{M}_G$.

Proof of (i). Let $w \in \mathcal{W}'$ be a word of length at most m and take a positive integer $N > G_m$. According to (1.1), we define

$$N = \sum_{j=0}^{L(N)-1} \varepsilon_j(N)G_j, \quad N_k = \sum_{j=k}^{L(N)-1} \varepsilon_j(N)G_j, \quad N_{L(N)} = 0. \tag{4.28}$$

Setting the summation variable $n = N_{k+1} + r$ in the second line below we obtain

$$\begin{aligned}
\nu_N([w]) &= \frac{1}{N} \sum_{k=0}^{L(N)-1} \sum_{N_{k+1}\leqslant n<N_k} \delta_n([w]) \\
&= \frac{1}{N} \sum_{k=m}^{L(N)-1} \sum_{r<\varepsilon_k(N)G_k} \delta_r([w]) + \mathcal{O}\left(\frac{G_m}{N}\right) \\
&= \sum_{k=m}^{L(N)-1} \frac{\varepsilon_k(N)G_k}{N} \nu_{G_k}([w]) + \mathcal{O}\left(\frac{G_m}{N}\right),
\end{aligned} \tag{4.29}$$

where the implied constant in the $\mathcal{O}$-notation can be taken as 1. Let $(\mu_K)_K$ be a sequence of probability measures on $\mathcal{K}_G$ as in (4.25) (without loss of generality the number of summands can be assumed to be some G_K by completing if necessary with zero-weights). We can write

$$\mu_K = \sum_{n=1}^{G_K} \pi(n, K)\nu_n \quad \text{with} \sum_{n=0}^{G_K} \pi(n, K) = 1 \text{ and for all } n \leqslant G_K : \lim_{K\to\infty} \pi(n, K) = 0.$$

Assume that $(\mu_K)_K$ converges weakly to μ, say. We shall show that the sequence $(\xi_K)_K$ converges to μ as well, where

$$\xi_K = \sum_{n=1}^{G_K} \pi(n, K) \sum_{k=0}^{L(n)-1} \frac{\varepsilon_k(n) G_k}{n} \nu_{G_k} = \sum_{k=0}^{K} \rho(k, K)\nu_{G_k}.$$

By construction, we get $\sum_{k=0}^{K} \rho(k, K) = 1$. To prove that the sequence $(\xi_K)_K$ converges to μ, let $w \in \mathcal{W}'$ be a word of $\mathcal{K}_G$ with length $m < N < K$. Then, using $n = \sum_k \varepsilon_k(n) G_k$, we have

$$\begin{aligned}
|\xi_K([w]) - \mu_K([w])| &\leqslant \sum_{n=1}^{G_K} \pi(n, K) \left| \sum_{k=0}^{L(n)-1} \frac{\varepsilon_k(n) G_k}{n} \nu_{G_k}([w]) - \nu_n([w]) \right| \\
&\leqslant \sum_{n=1}^{G_N - 1} \pi(n, K) \left| \sum_{k=0}^{L(n)-1} \frac{\varepsilon_k(n) G_k}{n} (\nu_{G_k}([w]) - \nu_n([w])) \right| \\
&\quad + \sum_{n=G_N}^{G_K} \pi(n, K) \left| \sum_{k=0}^{L(n)-1} \frac{\varepsilon_k(n) G_k}{n} (\nu_{G_k}([w]) - \nu_n([w])) \right| \\
&\leqslant \sum_{n=1}^{G_N - 1} \pi(n, K) + \sum_{n=G_N}^{G_K} \pi(n, K) \frac{G_m}{n} \leqslant \sum_{n=1}^{G_N - 1} \pi(n, K) + \frac{G_m}{G_N},
\end{aligned}$$

where we have applied (4.29) for estimating the second sum in the middle line. We first let K tend to infinity; then letting N tend to infinity shows that $\lim_{K\to\infty} |\xi_K([w]) - \mu_K([w])| = 0$.

Proof of (ii). We first prove that $\mathtt{Inv}(\mathcal{K}_G) \subset \overline{\mathrm{Conv}(\mathcal{M})}$ (the closed convex hull of $\mathcal{M}$). Assume that $\mathtt{Inv}(\mathcal{K}_G) \neq \emptyset$ and let $\mu \in \mathtt{Erg}(\mathcal{K}_G)$. By the ergodic theorem, there exists a generic point x for the measure μ: for any cylinder $[w]$, we have

$$\frac{1}{N} \sum_{n<N} \mathbb{1}_{[w]}(\tau^n x) \xrightarrow[N\to\infty]{} \mu([w]). \tag{4.30}$$

Take a positive integer m, that we fix for a while. Let $w \in \mathcal{W}'$ be a word of length m. Let $k_1, k_2, \ldots$ be the sequence of return times of x to $[0^{(m)}]$ under the action of τ; we set $k_0 = 0$ for convenience. We have $k_{j+1} - k_j \leqslant G_m$ for all j and

$$\text{for all } j \geqslant 1, \text{ for all } i \in \{0, 1, \ldots, k_{j+1} - k_j - 1\} : \tau^{k_j + i}(x)[m] = \mathtt{rep}^*(i)[m]. \tag{4.31}$$

We split the counting in (4.30) according to the k_j. Setting $s = \max\{j \mid k_j < N\}$ and using (4.31) we get

$$\begin{aligned}\sum_{n<N} \mathbb{1}_{[w]}(\tau^n x) &= \sum_{n<k_1} \mathbb{1}_{[w]}(\tau^n x) + \sum_{j=1}^{s-1} \sum_{i=0}^{k_{j+1}-k_j-1} \mathbb{1}_{[w]}(\tau^{k_j+i}(x)) + \sum_{n=k_s}^{N-1} \mathbb{1}_{[w]}(\tau^n x)\\ &= \mathcal{O}(G_m) + \sum_{j=1}^{s-1} \sum_{i=0}^{k_{j+1}-k_j-1} \mathbb{1}_{[w]}(\mathtt{rep}^*(i)) + \mathcal{O}(G_m)\\ &= \#\{1 \leqslant j \leqslant s-1 \mid k_{j+1} - k_j > \mathtt{val}(w)\} + \mathcal{O}(G_m)\\ &= \sum_{n=1}^{G_m} \#\{1 \leqslant j \leqslant s-1 \mid k_{j+1} - k_j = n\} \sum_{\ell<n} \mathbb{1}_{[w]}(\mathtt{rep}^*(\ell)) + \mathcal{O}(G_m).\end{aligned}$$

Thus we have

$$\frac{1}{N} \sum_{n<N} \mathbb{1}_{[w]}(\tau^n x) = \sum_{n=1}^{G_m} p(n, m, N)\nu_n([w]) + \mathcal{O}(G_m N^{-1}), \tag{4.32}$$

with

$$p(n, m, N) = \frac{n\#\{1 \leqslant j \leqslant s-1 \mid k_{j+1} - k_j = n\}}{N}.$$

Summing this yields

$$\sum_{1 \leqslant n \leqslant G_m} p(n, m, N) = \frac{k_s - k_1}{N} \leqslant 1.$$

Thus there exists an increasing sequence of integers $(N_j)_j$ such that the limits $\lim_j p(n, m, N_j) = \overline{p}(n, m)$ exist for all $n \leqslant G_m$. Define

$$\mu_m = \sum_{n=1}^{G_m} \overline{p}(n, m)\nu_n.$$

Taking $N = N_j$ in (4.32), letting j tend to infinity and using (4.30) we get for all $w \in \mathcal{W}'$ of length m, hence by additivity for all cylinders of length at most m,

$$\mu_m([w]) = \sum_{n=1}^{G_m} \overline{p}(n, m)\nu_n([w]) = \mu([w]) \quad \text{and} \quad \sum_{n=1}^{G_m} \overline{p}(n, m) = 1.$$

Therefore, the sequence $(\mu_m)_m$ converges weakly to μ.

We have proved that $\mathtt{Erg}(\mathcal{K}_G) \subset \overline{\mathrm{Conv}(\mathcal{M})}$, hence $\mathtt{Inv}(\mathcal{K}_G) \subset \overline{\mathrm{Conv}(\mathcal{M})}$, since $\mathtt{Inv}(\mathcal{K}_G)$ is the weakly closed convex hull of $\mathtt{Erg}(\mathcal{K}_G)$.

We now turn to the proof that $\mathtt{Inv}(\mathcal{K}_G)$ is a subset of the cluster points of the sequences in (4.25).

With the notation introduced above, we shall show that the sequence $(\overline{p}(n, m))_m$ tends to 0 when m tends to infinity. This will show that $\mathtt{Erg}(\mathcal{K}_G)$ is a subset of the cluster points of the sequences in (4.25). If we prove that the latter set is convex, this will also show that $\mathtt{Inv}(\mathcal{K}_G)$ is a subset of these cluster points.

We first show convexity. Assume that we have two cluster points μ_i $(i = 1, 2)$ of sequences of the form (4.25)

$$\mu_i = \lim_{j\to\infty} \sum_{k=1}^{N_j^{(i)}} \pi_i(k, N_j^{(i)}) \nu_k.$$

Then take $N_j = \max(N_j^{(1)}, N_j^{(2)})$ and observe that the definitions of μ_i can be rewritten in terms of N_j filling the shorter sum with zeros. Then it is clear that every convex combination of μ_1 and μ_2 can be realised as a limit of a convex combination of sequences of that form.

Let us consider a sequence of probability measures $\xi_m = \sum_{n=1}^{m} \pi(n, m)\nu_n$ on $\mathcal{K}_G$ with $0 \leqslant \pi(n, m)$ and $\sum_{n=1}^{m} \pi(n, m) = 1$ for all m and assume that $(\xi_m)_m$ converges weakly to ξ.

Let $s \geqslant 1$ and $m \geqslant \max(n_0, G_s)$. Then we have

$$\begin{aligned}&\xi_m([\mathtt{rep}^*(n_0 - 1)[s]])\\ &\qquad = \sum_{n=1}^{m} \frac{\pi(n, m)}{n} \#\{k < n \mid \mathtt{rep}^*(n_0 - 1)[s] \overset{\text{pref}}{\preccurlyeq} \mathtt{rep}^*(k)\} \geqslant \frac{\pi(n_0, m)}{n_0}.\end{aligned}$$

Assume that, for some integer n_0, the sequence $(\pi(n_0, m))_m$ does not tend to 0 when m tends to infinity. Then we have

$$\xi([\mathtt{rep}^*(n_0 - 1)[s]]) = \lim_{m\to\infty} \xi_m([\mathtt{rep}^*(n_0 - 1)[s]]) \geqslant \limsup_{m\to\infty} \frac{\pi(n_0, m)}{n_0} > 0,$$

and letting s tend to infinity, we obtain

$$\xi(\{\mathtt{rep}^*(n_0 - 1)\}) = \lim_{s\to\infty} \xi([\mathtt{rep}^*(n_0 - 1)[s]]) \geqslant \limsup_{m\to\infty} \frac{\pi(n_0, m)}{n_0} > 0.$$

Thus ξ is not atom free and cannot be τ-invariant by Proposition 2.

Proof of (iii). Let

$$\mu_n = \sum_{k=0}^{n} \xi(k, n)\nu_{G_k}$$

and assume that, for all k, $\lim \mu_n([\mathtt{rep}(G_k - 1)]) = \alpha_k$. We first show that $(\alpha_m)_m$ satisfies equation (4.11) for all $k \geqslant 0$. For this purpose we use $\nu_{G_k}([\mathtt{rep}(G_\ell - 1)]) = C_k^\ell / G_k$ and the fact that cylinders have empty boundary to write

$$\alpha_\ell = \lim_{n\to\infty} \sum_{k=0}^{n} \xi(k, n) \frac{C_k^\ell}{G_k}.$$

In order to show that $\alpha_k = \sum_{m\geqslant k+1} a_{k,m}\alpha_m$, we write

$$\sum_{m\geqslant k+1} a_{k,m} \lim_{n\to\infty} \sum_{j=0}^{n} \xi(j, n) \frac{C_j^m}{G_j} = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{\xi(j, n)}{G_j} \sum_{m\geqslant k+1} a_{k,m} C_j^m,$$

where the interchange of the limit and the summation is justified, because $C_j^m / G_j \leqslant 1/G_m$ by Lemma 3, which together with (4.27) implies uniformity of the convergence of the series with respect to n. We now use Lemma 2 and the assumption on the limit of the $\xi(k,n)$ to rewrite the inner sum

$$\sum_{m\geqslant k+1} a_{k,m}\alpha_m = \lim_{n\to\infty}\sum_{j=0}^{n}\frac{\xi(j,n)}{G_j}\sum_{m\geqslant k+1} a_{k,m}C_j^m$$
$$= \lim_{n\to\infty}\sum_{j=0}^{n}\frac{\xi(j,n)}{G_j}C_j^k + \lim_{n\to\infty}\frac{\xi(k,n)}{G_k} = \alpha_k.$$

Theorem 1 now shows the existence of an invariant measure $\overline{\mu}$ such that

$$\overline{\mu}([\mathtt{rep}(G_k-1)]) = \alpha_k \quad \text{for all } k.$$

It remains to show that $(\mu_n)_n$ converges weakly to $\overline{\mu}$, i.e.

$$\lim_{n\to+\infty}\sum_{k=0}^{n}\xi(k,n)\nu_{G_k}([w]) = \overline{\mu}([w]) \tag{4.33}$$

for all cylinders $[w]$, the existence of the limit in the left-hand side of (4.33) being part of the statement. For this purpose we use (4.6) to write

$$\overline{\mu}([w]) = \sum_{m\geqslant |w|}\alpha_m [\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m}G_j > \mathtt{val}(w)]\!]$$
$$= \sum_{m\geqslant |w|}[\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m}G_j > \mathtt{val}(w)]\!]\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n}\xi(k,n)\frac{C_k^m}{G_k}$$
$$= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{\xi(k,n)}{G_k}\sum_{m\geqslant |w|}C_k^m[\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m}G_j > \mathtt{val}(w)]\!],$$

where the interchange of limit and summation is justified by the same arguments as above. Furthermore, since $C_k^m = 0$ if $m > k$, we have

$$\sum_{k=0}^{n}\frac{\xi(k,n)}{G_k}\sum_{m\geqslant |w|}C_k^m[\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m}G_j > \mathtt{val}(w)]\!]$$
$$= \sum_{k=|w|}^{n}\frac{\xi(k,n)}{G_k}\sum_{m\geqslant |w|}C_k^m[\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m}G_j > \mathtt{val}(w)]\!],$$

hence, by Lemma 1,

$$\overline{\mu}([w]) = \lim_{n\to\infty}\sum_{k=|w|}^{n}\frac{\xi(k,n)}{G_k}\sum_{m\geqslant |w|}C_k^m[\![\textstyle\sum_{j=0}^{|w|-1} a_{j,m}G_j > \mathtt{val}(w)]\!]$$
$$= \lim_{n\to\infty}\sum_{k=|w|}^{n}\frac{\xi(k,n)}{G_k}D_k(w) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{\xi(k,n)}{G_k}D_k(w)$$
$$= \lim_{n\to+\infty}\sum_{k=0}^{n}\xi(k,n)\nu_{G_k}([w])$$

using that $\lim_n \xi(k,n) = 0$, which proves (4.33).

The theorem of Banach–Alaoglu ensures the existence of cluster points of the sequences given by (4.25) or (4.26). By the above arguments condition (4.27) ensures that they give invariant measures.

Milman's theorem [**25**, p. 335] asserts that if a subset M of a locally convex topological vector space has compact closed convex hull, then the extremal points of $\overline{\mathrm{Conv}(M)}$ lie in $\overline{M}$, hence are extremal points of $\overline{M}$ as well. Since ergodic invariant measures are exactly the extremal points of the set of invariant measures, we get the last assertion of (iii). □

COROLLARY 2. *If the series* $\sum G_n^{-1}$ *is convergent, then the odometer* $(\mathcal{K}_G, \tau)$ *admits at least one invariant measure.*

Proof. According to Theorem 2, we just have to prove that the series (4.27) converge for all k. Indeed, we have

$$\sum_{m=k+1}^{\infty} \frac{a_{k,m}}{G_m} \leqslant \frac{G_{k+1}}{G_k} \sum_{m=k+1}^{\infty} \frac{1}{G_m} < \infty.$$

□

4.3. *Conditions for unique ergodicity.* In this section we develop the combinatorial theory of the scale $(G_n)_n$ further to obtain a sufficient condition for unique ergodicity in terms of the greedy recurrence coefficients $a_{k,m}$.

THEOREM 3. *Let* $(G_n)_n$ *be a system of numeration and* $a_{k,m}$ *be given by (2.4). Assume that*

$$\lim_{t\to\infty} \limsup_{n\to\infty} \sum_{i=n+1}^{\infty} \frac{1}{G_i} \sum_{k=0}^{n-t} a_{k,i} G_k = 0. \tag{4.34}$$

Then the odometer $(\mathcal{K}_G, \tau)$ *is uniquely ergodic.*

Proof. According to (iii) in Theorem 2, we have to prove that the sequence $(C_m^k/G_m)_m$ converges for all k.

Fix k. Set $\gamma_k^- = \liminf\,(C_m^k/G_m)$. By Lemma 3, we have $0 \leqslant \gamma_k^- \leqslant 1/G_k$. Then there exists an increasing sequence of integers $(m_j)_j$ such that $\lim_{j\to\infty}(C_{m_j}^k/G_{m_j}) = \gamma_k^-$. Now let $\varepsilon > 0$. There exists an integer j_0 such that

$$\text{for all } m \geqslant m_{j_0}\colon C_m^k \geqslant G_m(\gamma_k^- - \varepsilon) \quad \text{and} \quad \text{for all } j \geqslant j_0\colon C_{m_j}^k \leqslant G_{m_j}(\gamma_k^- + \varepsilon). \tag{4.35}$$

We assume first that $\gamma_k^- > 0$. Using (4.17), we have

$$\begin{aligned}
&a_{m_j-1,m_j} C_{m_j-1}^k \\
&\quad = C_{m_j}^k - \sum_{\ell=0}^{m_{j_0}} a_{\ell,m_j} C_\ell^k - \sum_{\ell=m_{j_0}+1}^{m_j-2} a_{\ell,m_j} C_\ell^k \\
&\quad \leqslant G_{m_j}(\gamma_k^- + \varepsilon) - 0 - (\gamma_k^- - \varepsilon) \sum_{\ell=m_{j_0}+1}^{m_j-2} a_{\ell,m_j} G_\ell \\
&\quad \leqslant G_{m_j}(\gamma_k^- + \varepsilon) - (\gamma_k^- - \varepsilon)\Bigg(G_{m_j} - a_{m_j-1,m_j} G_{m_j-1} - \sum_{\ell=0}^{m_{j_0}} a_{\ell,m_j} G_\ell\Bigg) \\
&\quad \leqslant a_{m_j-1,m_j} G_{m_j-1}(\gamma_k^- - \varepsilon) + 2\varepsilon G_{m_j} + (\gamma_k^- - \varepsilon) G_{m_{j_0}+1}.
\end{aligned}$$

Dividing by $a_{m_j-1,m_j}G_{m_j-1}$, using $G_{m_j} < 2a_{m_j-1,m_j}G_{m_j-1}$ (which comes from $a_{m_j-1,m_j} \neq 0$ and (2.8)) and taking j sufficiently large such that $(G_{m_{j_0}+1}/G_{m_j}) \leqslant \varepsilon$, we get

$$\frac{C^k_{m_j-1}}{G_{m_j-1}} \leqslant (\gamma_k^- - \varepsilon) + \frac{2\varepsilon G_{m_j}}{a_{m_j-1,m_j}G_{m_j-1}} + \frac{\gamma_k^- G_{m_{j_0}+1}}{a_{m_j-1,m_j}G_{m_j-1}} \leqslant \gamma_k^- + 4\varepsilon. \tag{4.36}$$

Using that ε can be chosen arbitrarily and iterating, we have proved

$$\text{for all } t \in \mathbb{N}\colon\ \lim_{j\to\infty} \frac{C^k_{m_j-t}}{G_{m_j-t}} = \gamma_k^-. \tag{4.37}$$

In the case $\gamma_k^- = 0$, we replace the lower bound for C^k_m in (4.35) by 0, and obtain the same conclusion.

We now introduce new coefficients, which are useful for the proof of the theorem. By iteration of (2.8), we set $a^{(1)}_{j,m} = a_{j,m}$ and get

$$\begin{aligned}
G_m &= a^{(1)}_{m-1,m}G_{m-1} + a^{(1)}_{m-2,m}G_{m-2} + \cdots + a^{(1)}_{0,m}G_0\\
&= a^{(1)}_{m-1,m}\sum_{j=0}^{m-2} a^{(1)}_{j,m-1}G_j + \sum_{j=0}^{m-2} a^{(1)}_{j,m}G_j = \sum_{j=0}^{m-2}\left(a^{(1)}_{m-1,m}a^{(1)}_{j,m-1} + a^{(1)}_{j,m}\right)G_j\\
&= \sum_{j=0}^{m-2} a^{(2)}_{j,m}G_j.
\end{aligned}$$

Recursively, we get $G_m = \sum_{j=0}^{m-r} a^{(r)}_{j,m}G_j$, where

$$a^{(r)}_{j,m} = \begin{cases} a^{(r-1)}_{m-r+1,m}a^{(1)}_{j,m-r+1} + a^{(r-1)}_{j,m} & \text{if } j \leqslant m-r,\\ 0 & \text{otherwise.}\end{cases} \tag{4.38}$$

Note in particular that

$$a^{(r)}_{m-r,m}G_{m-r} \leqslant G_m. \tag{4.39}$$

The crucial point is that the iteration of (4.17) gives the same formula with $a^{(r)}_{j,m}$ instead of $a_{j,m}$. Indeed, we have for $m \geqslant k + r$

$$\begin{aligned}
C^k_m &= \sum_{j=0}^{m-1} a^{(1)}_{j,m}C^k_j\\
&= a^{(1)}_{m-1,m}\sum_{j=0}^{m-2} a^{(1)}_{j,m-1}C^k_j + \sum_{j=0}^{m-2} a^{(1)}_{j,m}C^k_j = \sum_{j=0}^{m-2}\left(a^{(1)}_{m-1,m}a^{(1)}_{j,m-1} + a^{(1)}_{j,m}\right)C^k_j\\
&= \sum_{j=0}^{m-2} a^{(2)}_{j,m}C^k_j = \sum_{j=0}^{m-r} a^{(r)}_{j,m}C^k_j.
\end{aligned}$$

We now write

$$\frac{C^k_m}{G_m} = \sum_{j=0}^{m-r} \frac{a^{(r)}_{j,m}G_j}{G_m} \times \frac{C^k_j}{G_j} \tag{4.40}$$

and prove by induction on r that

$$\text{for all } s \leqslant m-r: \sum_{j=0}^{s} \frac{a_{j,m}^{(r)} G_j}{G_m} \leqslant \sum_{i=m-r+1}^{m} \frac{1}{G_i} \sum_{j=0}^{s} a_{j,i} G_j. \tag{4.41}$$

For $r=1$, the inequality (4.41) is an equality. Assume that it is satisfied for $r-1$. For $s \leqslant m-r$, (4.38) yields

$$\begin{aligned}
\sum_{j=0}^{s} \frac{a_{j,m}^{(r)} G_j}{G_m} &= \sum_{j=0}^{s} \left(a_{m-r+1,m}^{(r-1)} a_{j,m-r+1}^{(1)} + a_{j,m}^{(r-1)} \right) \frac{G_j}{G_m} \\
&= \frac{a_{m-r+1,m}^{(r-1)}}{G_m} \sum_{j=0}^{s} a_{j,m-r+1}^{(1)} G_j + \sum_{j=0}^{s} \frac{a_{j,m}^{(r-1)} G_j}{G_m} \\
&\leqslant \frac{1}{G_{m-r+1}} \sum_{j=0}^{s} a_{j,m-r+1}^{(1)} G_j + \sum_{i=m-r+2}^{m} \frac{1}{G_i} \sum_{j=0}^{s} a_{j,i} G_j \\
&\leqslant \sum_{i=m-r+1}^{m} \frac{1}{G_i} \sum_{j=0}^{s} a_{j,i} G_j,
\end{aligned}$$

by (4.39) and the induction hypothesis, which proves (4.41). To complete the proof, we assume that (4.34) is satisfied. Furthermore, we use the sequence $(m_j)_j$ introduced in the first part of the proof. Take a positive integer m. There is a unique ℓ such that $m_\ell < m \leqslant m_{\ell+1}$. Take $r = m - m_\ell$ and $t \geqslant 1$. We use (4.40) and split the sum to get

$$\begin{aligned}
\frac{C_m^k}{G_m} &= \sum_{j=0}^{m_\ell - t} \frac{a_{j,m}^{(m-m_\ell)} G_j}{G_m} \times \frac{C_j^k}{G_j} + \sum_{j=m_\ell-t+1}^{m_\ell} \frac{a_{j,m}^{(m-m_\ell)} G_j}{G_m} \times \frac{C_j^k}{G_j} \\
&= S_1(m) + S_2(m).
\end{aligned} \tag{4.42}$$

To get an upper bound for the $S_1(m)$, we use (4.41) with $s = m_\ell - t$, which yields

$$S_1(m) \leqslant \frac{1}{G_k} \sum_{i=m_\ell+1}^{m} \frac{1}{G_i} \sum_{j=0}^{m_\ell - t} a_{j,i} G_j \leqslant \frac{1}{G_k} \sum_{i=m_\ell+1}^{\infty} \frac{1}{G_i} \sum_{j=0}^{m_\ell - t} a_{j,i} G_j.$$

According to (4.34), we have

$$\lim_{t\to\infty} \limsup_{m\to\infty} S_1(m) = 0. \tag{4.43}$$

We treat the second sum as follows,

$$S_2(m) \leqslant \max_{m_\ell - t+1 \leqslant j \leqslant m_\ell} \frac{C_j^k}{G_j} \sum_{j=m_\ell-t+1}^{m_\ell} \frac{a_{j,m}^{(m-m_\ell)} G_j}{G_m} \leqslant \max_{m_\ell - t+1 \leqslant j \leqslant m_\ell} \frac{C_j^k}{G_j}. \tag{4.44}$$

Finally, (4.43), (4.44) and (4.37) yield

$$\limsup_{m\to\infty} \frac{C_m^k}{G_m} = \liminf_{m\to\infty} \frac{C_m^k}{G_m} = \gamma_k^-,$$

and the theorem is proved. And this also shows that $\gamma_k^- > 0$ by Theorem 2. □

Remark 6. Assume that the conditions of Proposition 1 are satisfied. Writing

$$\sum_{i=n+1}^{\infty}\frac{1}{G_i}\sum_{k=0}^{n-t}a_{k,i}G_k \leqslant \sum_{i=n+1}^{\infty}\frac{G_{n-t+1}}{G_i} = \frac{G_{n-t+1}}{G_n}\left(G_n\sum_{i=n+1}^{\infty}\frac{1}{G_i}\right),$$

it follows from (i) and (iv) in Proposition 1 that (4.34) holds. Therefore, the equivalent conditions in Proposition 1 imply (4.34). Moreover, the condition that $\lim_n(G_{n+1}-G_n) = \infty$ is not needed any more.

5. *Examples*

In this section we collect a number of examples that show how the machinery developed in this paper can be used to understand invariant measures. In §4.1 we have already given simple and classical examples illustrating how Theorem 1 can be used to show the unique ergodicity of the odometer $(\mathcal{K}_G, \tau)$ or the lack of invariant measures.

The investigation of Example 4 has been the subject of [**10**]. The equation (4.11) have a unique solution, which shows unique ergodicity.

Example 5 is closely related to β-numeration on the real numbers. For every $\beta > 1$ a sequence of integers $(G_n)_n$ has been constructed in [**22**] using the β-numeration introduced in [**29**]. This sequence grows like $C\beta^n$, ensuring unique ergodicity by Remark 3. In this example it is not difficult to construct an explicit solution of (4.11).

The next examples are more involved and are given to show what Corollary 2 and Theorem 3 can and cannot achieve. These examples do not correspond to classically studied numeration systems. They are constructed using the same principle: alternating expressions of G_{n+1} where $G_{n+1} - G_n$ is small with respect to G_n or not, with the very lacunary relations (2.6). It allows us to control both local and global growth of the scale, constructing scales satisfying (or not) the assumptions of Proposition 1 or (4.34). The lacunarity of the $a_{k,m}$ yields systems of equations (4.10) that are explicitly solvable, so that Theorem 1 can be applied, which allows us to test the efficiency of Corollary 2 and Theorem 3. In general, Theorem 1 is of rather theoretical interest (except for very specific examples), whereas Corollary 2 and Theorem 3 can be checked for many scales given in any form. It turns out that the conditions given in Corollary 2 and Theorem 3 are not necessary, and that Theorem 3 is strictly stronger than [**3**, Théorème 8]. Notice by the way that Example 9 uses Theorem 1 in both directions.

Finally, Example 10 describes an odometer with exactly two ergodic invariant probability measures.

Example 4. The method we develop here has been used in [**10**]. Let $q \geqslant 2$ and, for all n, $G_{n+1} = qG_n + 1$. Then, we have

$$a_{0,1} = q+1,\ a_{n,n+1} = q \text{ if } n \geqslant 1 \quad \text{and} \quad a_{0,n} = 1 \text{ if } n \geqslant 1;\ G_n = \frac{q^{n+1}-1}{q-1}.$$

The equations give

$$\begin{aligned} 1 = \alpha_0 &= (q+1)\alpha_1 + \alpha_2 + \alpha_3 + \cdots \\ \alpha_n &= q\alpha_{n+1} \quad \text{for } n \geqslant 1 \end{aligned}$$

with the unique solution

$$\alpha_n = \frac{q-1}{q^{n+1}} \quad \text{for } n \geqslant 1.$$

The odometer is thus uniquely ergodic by Theorem 1. Furthermore, the point $x = q(q-1)^{(\infty)} \in \mathcal{K}_G$ satisfies

$$\tau^n(x) = 0^{(n)} q(q-1)^{(\infty)}$$

and is therefore not dense in $\mathcal{K}_G$.

Example 5. Let $\beta > 1$. Parry [**29**] has introduced and studied the transformation $T_\beta : [0, 1) \to [0, 1)$, $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$. Writing $\varepsilon(x) = \lfloor \beta x \rfloor$, we get a fibred number system in the sense of [**32**] as described in [**1**]. One obtains an expansion of any real number $x \in [0, 1]$ as a convergent series $x = \sum_{k \geqslant 0} \varepsilon(T_\beta^k x)\beta^{-k-1}$. We write $c_k = \varepsilon(T_\beta^k(1))$. Then a series $\sum_{k \geqslant 0} \xi_k \beta^{-k-1}$ is the expansion of a real number x if, and only if, for any integer k, one has $\xi_k \xi_{k+1} \cdots <_{\text{lex}} c_0 c_1 \ldots$.

One associates a system of numeration to the sequence $(c_n)_n$ by setting, for every n, $G_{n+1} = c_0 G_n + c_1 G_{n-1} + \cdots + (c_n + 1)G_0$. It turns out that this is also the greedy recurrence relation (2.6), and that $G_n \sim C\beta^n$ for some $C > 0$ (see [**22**] and also [**18**]). Therefore, we have

$$\begin{cases} a_{0,m} = c_{m-1} + 1 & \text{for } m \geqslant 1, \\ a_{k,m} = c_{m-k-1} & \text{for } 1 \leqslant k \leqslant m-1, \end{cases}$$
$$\begin{cases} 1 = \alpha_0 = (c_0 + 1)\alpha_1 + (c_1 + 1)\alpha_2 + \cdots, \\ \alpha_k = c_0 \alpha_{k+1} + c_1 \alpha_{k+2} + \cdots & \text{for } k \geqslant 1. \end{cases} \tag{5.1}$$

By construction of the sequence $(c_j)_j$, $\alpha_j = c\beta^{-j}$ $(j \geqslant 0)$ gives a solution of the equations $\alpha_k = c_0\alpha_{k+1} + c_1\alpha_{k+2} + \cdots$. Inserting into the first equation yields $c = 1 - 1/\beta$, which gives an invariant measure. Furthermore, the function $e(n)$ defined in (3.2) is stationary; thus the scale $(G_n)_n$ satisfies (iii) in Proposition 1, hence the underlying odometer is uniquely ergodic by Theorem 3 or even by [**2**, Théorème 8].

In [**24**, §2] the system of equation (5.1) occurs in the context of additive functions on the above mentioned β-expansion. There the question concerning uniqueness of the solutions has been posed as an open problem.

Example 6. We construct a scale G as follows. We choose an increasing sequence of integers $0 = k_0 < k_1 < k_2 < \cdots$ with $k_{j+1} - k_j \geqslant 2$ and set, for all j,

$$\begin{cases} G_{k_j+1} = 2G_{k_j}, \\ G_{k_j+\ell+1} = G_{k_j+\ell} + G_{k_j} & \text{for } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1. \end{cases}$$

The corresponding greedy recurrence coefficients given by (2.2) are

$$\begin{cases} a_{k_j,k_j+1} = 2 & \text{for } j \geqslant 0, \\ a_{k_j,k_j+\ell+1} = a_{k_j+\ell,k_j+\ell+1} = 1 & \text{for } j \geqslant 0 \text{ and } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1, \\ a_{k,m} = 0 & \text{otherwise.} \end{cases}$$

According to (4.11), we obtain the following equations for $\alpha_k = \mu_G([\mathtt{rep}(G_k - 1)])$:

$$\alpha_{k_j} = 2\alpha_{k_j+1} + \alpha_{k_j+2} + \cdots + \alpha_{k_{j+1}},$$
$$\alpha_{k_j+\ell} = \alpha_{k_j+\ell+1} \text{ for } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1.$$

This gives

$$\alpha_{k_j} = (k_{j+1} - k_j + 1)\alpha_{k_{j+1}},$$
$$\alpha_{k_j+1} = \cdots = \alpha_{k_{j+1}}.$$

Hence the solution of (4.11) is unique and the odometer $(\mathcal{K}_G, \tau)$ is uniquely ergodic. On the other hand we get

$$G_{k_j+\ell} = (\ell + 1)G_{k_j} \quad \text{for } 0 \leqslant \ell \leqslant k_{j+1} - k_j,$$

hence

$$G_{k_n+\ell} = (\ell + 1) \prod_{j=0}^{n-1} (k_{j+1} - k_j + 1) \quad \text{for } 0 \leqslant \ell \leqslant k_{n+1} - k_n.$$

It follows from the calculations above that

$$G_{k_j+\ell}\alpha_{k_j+\ell} = \frac{\ell + 1}{k_{j+1} - k_j + 1} \quad \text{for } 0 \leqslant \ell \leqslant k_{j+1} - k_j.$$

Condition (iii) in Proposition 1 corresponds to the boundedness of the sequence $(k_{j+1} - k_j)_j$. Therefore, if $\limsup_{j\to\infty}(k_{j+1} - k_j) = \infty$, then we have an example of a continuous (by Remark 1) uniquely ergodic odometer with locally slow growth.

However, Theorem 3 does not provide a better result than [**3**, Théorème 8] here. Indeed, for given t and j, and $t \leqslant \ell < k_{j+1} - k_j$, we have

$$\sum_{i=k_j+\ell}^{\infty} \frac{1}{G_i} \sum_{k=0}^{k_j+\ell-t} a_{k,i} G_k = G_{k_j} \sum_{i=k_j+\ell}^{k_{j+1}} \frac{1}{G_i}, \text{ therefore} \tag{5.2}$$

$$\limsup_{n\to\infty} \sum_{i=n+1}^{\infty} \frac{1}{G_i} \sum_{k=0}^{n-t} a_{k,i} G_k = \limsup_{j\to\infty} \left(G_{k_j} \sum_{i=k_j+t}^{k_{j+1}} \frac{1}{G_i} \right),$$

noting that the left-hand side in (5.2) is 0 if $\ell < t$. The above computation of $G_{k_j+\ell}$ yields

$$G_{k_j} \sum_{i=k_j+t}^{k_{j+1}} \frac{1}{G_i} = \sum_{m=t+1}^{k_{j+1}-k_j+1} \frac{1}{m} = \log \frac{k_{j+1} - k_j + 1 - t}{t} + \mathcal{O}\left(\frac{1}{t}\right).$$

Hence (4.34) is not fulfilled as soon as $(k_{j+1} - k_j)_j$ is not bounded. This gives an example of a uniquely ergodic odometer, which does not satisfy (4.34).

Example 7. We construct a scale G as follows. We choose an increasing sequence of integers $0 = k_0 < k_1 < k_2 < \cdots$ with $k_{j+1} - k_j \geqslant 2$ and set, for all j,

$$\begin{cases} G_{k_j+1} = 2G_{k_j}, \\ G_{k_j+\ell+1} = G_{k_j+\ell} + 1 \quad \text{for } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1. \end{cases} \tag{5.3}$$

The corresponding greedy recurrence coefficients given by (2.2) are

$$\begin{cases} a_{k_j,k_j+1} = 2, & \\ a_{k_j+\ell,k_j+\ell+1} = a_{0,k_j+\ell+1} = 1 & \text{for } j \geqslant 0 \text{ and } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1, \\ a_{k,m} = 0 & \text{otherwise.} \end{cases}$$

According to (4.11), we obtain the following equations for $\alpha_k = \mu_G([\mathtt{rep}(G_k - 1)])$

$$\begin{aligned} 1 = \alpha_0 = 2\alpha_1 + \sum_{j=0}^{\infty} \sum_{\ell=2}^{k_{j+1}-k_j} \alpha_{k_j+\ell}, \\ \alpha_{k_j} = 2\alpha_{k_j+1} \quad \text{for } j \geqslant 1, \\ \alpha_{k_j+\ell} = \alpha_{k_j+\ell+1} \quad \text{for } j \geqslant 0 \text{ and } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1. \end{aligned}$$

From the preceding equations we derive the equality

$$\begin{aligned} 1 &= (k_1 + 1)\alpha_1 + \sum_{j=2}^{\infty} (k_j - k_{j-1} - 1)\alpha_{k_j} \\ &= \alpha_{k_1} \left((k_1 + 1) + \sum_{j=2}^{\infty} \frac{k_j - k_{j-1} - 1}{2^{j-1}} \right) = \alpha_{k_1} \sum_{j=1}^{\infty} \frac{k_j}{2^j}. \end{aligned} \tag{5.4}$$

It turns out that either the series $\sum k_j 2^{-j}$ converges, hence the solution of (4.11) is unique and the odometer $(\mathcal{K}_G, \tau)$ is uniquely ergodic, or it diverges, hence (4.11) has no solution and the odometer does not have any invariant probability measure.

By induction on j, the construction given in (5.3) yields

$$G_{k_j} = k_j + 2^{j-1} \sum_{i=1}^{j-1} \frac{k_i}{2^i} + 1. \tag{5.5}$$

Assume now that the series $c = \sum_{i=1}^{\infty} k_i 2^{-i-1}$ converges. Then (5.5) shows that $G_{k_j} \sim c2^j$. For $t \geqslant 1$, the sums arising from condition (4.34) become

$$\sum_{i=n+1}^{\infty} \frac{1}{G_i} \sum_{k=0}^{n-t} a_{k,i} G_k = \sum_{i=n+1}^{\infty} \frac{1}{G_i} [\![\forall j : i \neq k_j + 1]\!]. \tag{5.6}$$

Clearly, the series $\sum_i (1/G_i)[\![\text{for all } j : i \neq k_j + 1]\!]$ converges if, and only if, the series $\sum_i (1/G_i)$ converges. Furthermore, using $G_{k_j} \sim c2^j$, we get

$$\sum_{i=k_{j-1}+1}^{k_j} \frac{1}{G_i} = \mathcal{O}\left(\frac{k_j - k_{j-1}}{G_{k_j}} \right) = \mathcal{O}\left(\frac{k_j}{G_{k_j}} \right), \tag{5.7}$$

hence the convergence of $\sum_i 1/G_i$.

In this example, Theorem 3 gives the unique ergodicity in all cases where it holds. Notice that (i) in Proposition 1 is equivalent to the fact that the sequence $(k_j - k_{j-1})_j$ is bounded, which is much stronger than the convergence of $\sum_{i=1}^{\infty} k_i 2^{-i}$.

Example 8. The scale G is constructed as follows. Let $2 = k_1 < k_2 < \cdots$ be an increasing sequence of integers. We define

$$\begin{cases} G_1 = 2 = 2G_0, \\ G_2 = 5 = 2G_1 + G_0, \\ G_{n+1} = G_n + G_j + G_0 \quad \text{for } k_j \leqslant n < k_{j+1} \text{ and } j \geqslant 1. \end{cases} \tag{5.8}$$

The corresponding greedy recurrence coefficients given by (2.2) are

$$\begin{cases} a_{0,1} = 2, & \\ a_{0,n} = 1 & \text{for } n \geqslant 2, \\ a_{1,2} = 2, & \\ a_{1,2+\ell} = 1 & \text{for } 1 \leqslant \ell \leqslant k_2 - 2, \\ a_{j,j+1} = a_{j,k_j+\ell} = 1 & \text{for } j \geqslant 2, \text{ and } 1 \leqslant \ell \leqslant k_{j+1} - k_j, \\ a_{i,m} = 0 & \text{otherwise.} \end{cases}$$

We obtain the following equations for the α_k:

$$1 = \alpha_0 = 2\alpha_1 + \sum_{i \geqslant 2} \alpha_i, \tag{5.9}$$

$$\alpha_1 = 2\alpha_2 + \sum_{i=3}^{k_2} \alpha_i, \tag{5.10}$$

$$\alpha_j = \alpha_{j+1} + \sum_{i=k_j+1}^{k_{j+1}} \alpha_i, \quad (j \geqslant 2). \tag{5.11}$$

Note that for $i \geqslant 1$, $a_{i,j} = 0$ for all but finitely many j; hence condition (4.27) is satisfied for $k \geqslant 1$.

We now consider the system of equations given by (5.10) and (5.11) and replace α_k by $\widetilde{\alpha}_{k-1}$ for $k \geqslant 2$ and set $\widetilde{\alpha}_0 = 1$,

$$1 = \widetilde{\alpha}_0 = 2\widetilde{\alpha}_1 + \sum_{i=2}^{k_2-1} \widetilde{\alpha}_i, \tag{5.12}$$

$$\widetilde{\alpha}_j = \widetilde{\alpha}_{j+1} + \sum_{i=k_{j+1}}^{k_{j+2}-1} \widetilde{\alpha}_i, \ (j \geqslant 1). \tag{5.13}$$

This gives the system of equations for the invariant measures of the odometer defined by the scale $(\widetilde{G}_n)$ given by

$$\begin{cases} \widetilde{G}_1 = 2 = 2\widetilde{G}_0, \\ \widetilde{G}_{n+1} = \widetilde{G}_n + \widetilde{G}_{j-1} \quad \text{for } k_j - 1 \leqslant n < k_{j+1} - 1 \text{ and } j \geqslant 1. \end{cases}$$

Then we can apply Theorem 2 to the scale $(\widetilde{G}_n)_n$ to obtain a non-zero solution $(\widetilde{\alpha}_k)_k$ of the equations (5.12) and (5.13), since all series (4.27) are finite, and hence convergent.

We now observe that the equations (5.12) and (5.13) show that the sequence $(\widetilde{\alpha}_k)_k$ is strictly decreasing, thus convergent. Summing equation (5.13) for $1 \leqslant j < m$ gives

$$\widetilde{\alpha}_1 - \widetilde{\alpha}_m = \sum_{i=k_2}^{k_{m+2}-1} \widetilde{\alpha}_i .$$

Letting m tend to ∞ shows the convergence of the series

$$\sum_{i=k_2}^{\infty} \widetilde{\alpha}_i$$

and $\lim_m \widetilde{\alpha}_m = 0$. We now set

$$\widetilde{\alpha}_{-1} = 2\widetilde{\alpha}_0 + \sum_{i=1}^{\infty} \widetilde{\alpha}_i .$$

Then

$$\alpha_i = \frac{\widetilde{\alpha}_{i-1}}{\widetilde{\alpha}_{-1}} \quad \text{for } i \geqslant 0$$

gives a solution of the equations (5.9), (5.10), and (5.11). Therefore we can claim that any choice of the sequence $(k_j)_j$ in (5.8) yields an odometer with at least one invariant probability measure.

However, the divergence of the harmonic series ensures that we can choose the sequence $(k_j)_j$ such that $\sum_{i=k_j+1}^{k_{j+1}} 1/G_i \geqslant 1$ for all j (notice that the sequence $(G_n)_n$ is piecewise linear). Then the series $\sum_j (a_{0,j}/G_j)$ diverges, which gives an example of an odometer with invariant measure for which the assumptions of Theorem 2 are not fulfilled.

Example 9. The scale G is constructed as follows. We construct an increasing sequence of integers $0 = k_0 < k_1 < k_2 < \cdots$. We take k_1 arbitrary and define G_m for $1 \leqslant m \leqslant k_1$ arbitrary as well. For a while, we consider the values of k_j to be indeterminate if $j \geqslant 2$. For $m > G_{k_1}$, that is for $j \geqslant 1$ below, the scale is defined piecewise by

$$G_{k_j+\ell+1} = G_{k_j+\ell} + G_{k_{j-1}} \quad \text{for } 0 \leqslant \ell \leqslant k_{j+1} - k_j - 1.$$

The corresponding greedy recurrence coefficients given by (2.2) are

$$\begin{cases} a_{k_j+\ell,k_j+\ell+1} = 1 & \text{for } 0 \leqslant \ell \leqslant k_{j+1} - k_j - 1 \text{ and } j \geqslant 1, \\ a_{k_{j-1},k_j+\ell+1} = 1 & \text{for } 0 \leqslant \ell \leqslant k_{j+1} - k_j - 1 \text{ and } j \geqslant 1, \\ a_{i,m} = 0 & \text{otherwise and if } m > k_1. \end{cases}$$

According to (4.11), we obtain the following recursions for α_k:

$$1 = \alpha_0 = \sum_{m=1}^{k_1} a_{0,m}\alpha_m + \sum_{m=k_1+1}^{k_2} \alpha_m,$$

$$\alpha_j = \sum_{m=j+1}^{k_1} a_{j,m}\alpha_m \quad \text{for } 1 \leqslant j \leqslant k_1 - 1,$$

$$\alpha_{k_j} = \alpha_{k_j+1} + \sum_{\ell=0}^{k_{j+2}-k_{j+1}-1} \alpha_{k_{j+1}+\ell+1} \quad \text{for } j \geqslant 1,$$

$$\alpha_{k_j+\ell} = \alpha_{k_j+\ell+1} \quad \text{for } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1 \text{ and } j \geqslant 1.$$

These equations transform into

$$1 = \sum_{m=1}^{k_1} a_{0,m}\alpha_m + (k_2 - k_1)\,\alpha_{k_2}, \tag{5.14}$$

$$\alpha_j = \sum_{m=j+1}^{k_1} a_{j,m}\alpha_m \quad \text{for } 1 \leqslant j \leqslant k_1 - 1, \tag{5.15}$$

$$\alpha_{k_j+\ell} = \alpha_{k_{j+1}} \quad \text{for } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1 \text{ and } j \geqslant 1, \tag{5.16}$$

$$\alpha_{k_j} = \alpha_{k_j+1} + (k_{j+2} - k_{j+1})\,\alpha_{k_{j+2}} \quad \text{for } j \geqslant 1. \tag{5.17}$$

The equations (5.15) form a triangular system. Therefore, we can express α_j $(1 \leqslant j < k_1)$ linearly in α_{k_1}. Hence, we obtain

$$\begin{cases} \alpha_j = \lambda_j \alpha_{k_1} & \text{for } 1 \leqslant j \leqslant k_1 - 1,\\ 1 = c\alpha_{k_1} + (k_2 - k_1)\,\alpha_{k_2}, & \\ \alpha_{k_j+\ell} = \alpha_{k_j+\ell+1} & \text{for } 1 \leqslant \ell \leqslant k_{j+1} - k_j - 1 \text{ and } j \geqslant 1,\\ \alpha_{k_j} = \alpha_{k_j+1} + (k_{j+2} - k_{j+1})\,\alpha_{k_{j+2}} & \text{for } j \geqslant 1. \end{cases}$$

where the λ_j and c are explicit positive constants depending on the greedy recurrence coefficients $a_{k,m}$ for $0 \leqslant m < k \leqslant k_1$. Therefore, setting $u_j = \alpha_{k_j}$ and $M_j = k_j - k_{j-1}$, the previous system reduces to

$$\begin{cases} 1 = cu_1 + M_2 u_2,\\ u_j = u_{j+1} + M_{j+2} u_{j+2} \quad \text{for } j \geqslant 1. \end{cases} \tag{5.18}$$

Clearly, if one fixes the value of u_1, then all the u_j are determined. The solution is acceptable if and only if all values u_i are positive. An easy induction shows that

$$u_n = \frac{(-1)^n}{M_2 \cdots M_n}\left(P_n(M_3, \ldots, M_{n-1}) - Q_n(M_2, \ldots, M_{n-1})u_1\right),$$

where P_n and Q_n are polynomials with non-negative coefficients in the indicated variables satisfying

$$P_{n+1} = P_n + M_n P_{n-1} \quad \text{and} \quad Q_{n+1} = Q_n + M_n Q_{n-1}.$$

The conditions that all u_n have to be positive translates to the inequalities

$$\text{for all } n \geqslant 1\colon \quad \frac{P_{2n+1}}{Q_{2n+1}} < u_1 < \frac{P_{2n}}{Q_{2n}}.$$

We choose $M_2 \geqslant 1$. Then we choose two sequences $(a_n)_n$ and $(b_n)_n$ with the properties

$$\frac{1}{c + M_2} = a_2 < a_3 = a_4 < a_5 \cdots < a_{2n-1} = a_{2n} < a_{2n+1} < \lim a_m = \alpha < \beta$$

$$\alpha < \beta = \lim b_m < \cdots < b_{2n+1} = b_{2n} < b_{2n-1} \cdots < b_3 = b_2 < b_1 = \frac{1}{c}.$$

Note that

$$\frac{1}{c + M_2} = \frac{P_3}{Q_3} < a_3 < b_2 < \frac{P_2}{Q_2} = \frac{1}{c}.$$

Assume now that $M_2, \dots, M_{n-1}$ have been chosen such that the inequalities

$$\frac{P_r}{Q_r} < a_r < b_s < \frac{P_s}{Q_s}$$

are satisfied for all odd r and even s with $r \leqslant n$ and $s \leqslant n$. Notice that on $\mathbb{R}_+$, the function $t \mapsto (P_n + tP_{n-1})/(Q_n + tQ_{n-1})$ is increasing if n is odd and decreasing if n is even.

Then for even n we choose M_n so that

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{P_n + M_n P_{n-1}}{Q_n + M_n Q_{n-1}} < a_{n+1},$$

which is possible, since the limit for $M_n \to \infty$ of the quotient equals $P_{n-1}/Q_{n-1} < a_{n-1} < a_{n+1}$; for odd n we choose M_n so that

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{P_n + M_n P_{n-1}}{Q_n + M_n Q_{n-1}} > b_{n+1},$$

which is possible, since the limit for $M_n \to \infty$ of the quotient equals $P_{n-1}/Q_{n-1} > b_{n-1} > b_{n+1}$. Given a sequence of positive integers $(M_n)_n$ as constructed above (and therefore the sequence $(k_j)_j$), all values $(u_n)_n$ are positive, if

$$u_1 \in \bigcap_n \left(\frac{P_{2n+1}}{Q_{2n+1}}, \frac{P_{2n}}{Q_{2n}} \right) \supset [\alpha, \beta].$$

Since every such value of u_1 yields an invariant measure, we have infinitely many invariant measures. There are exactly two ergodic invariant measures, corresponding to the extremal values of u_1.

Acknowledgements. The authors are indebted to an anonymous referee for many valuable remarks and suggestions that helped to improve the presentation of the paper. The first author is supported by the Austrian Science Fund (FWF): projects F5503 (part of the Special Research Program (SFB) ‘Quasi-Monte Carlo Methods: Theory and Applications’), W1230 (Doctoral Program ‘Discrete Mathematics’), and I1136 (French-Austrian international cooperation ‘Fractals and Numeration’).

REFERENCES

[**1**] G. Barat, V. Berthé, P. Liardet and J. Thuswaldner. Dynamical directions in numeration. *Ann. Inst. Fourier (Grenoble)* **56**(7) (2006), 1987–2092.

[**2**] G. Barat, T. Downarowicz, A. Iwanik and P. Liardet. Propriétés topologiques et combinatoires des échelles de numération. *Colloq. Math.* **84/85** (part 2) (2000), 285–306, Dedicated to the memory of Anzelm Iwanik.

[**3**] G. Barat, T. Downarowicz and P. Liardet. Dynamiques associées à une échelle de numération. *Acta Arith.* **103**(1) (2002), 41–78.

[**4**] G. Barat and P. Liardet. Dynamical systems originated in the Ostrowski alpha-expansion. *Ann. Univ. Sci. Budapest. Sect. Comput.* **24** (2004), 133–184.

[**5**] V. Berthé and M. Rigo. Odometers on regular languages. *Theory Comput. Syst.* **40**(1) (2007), 1–31.

[**6**] H. Bruin, G. Keller and M. St. Pierre. Adding machines and wild attractors. *Ergod. Th. & Dynam. Sys.* **17**(6) (1997), 1267–1287.

[**7**] L. Carlitz, R. Scoville and V. E. Hoggatt Jr. Fibonacci representations of higher order I. *Fibonacci Quart.* **10**(1) (1972), 43–69.

[**8**] L. Carlitz, R. Scoville and V. E. Hoggatt Jr. Fibonacci representations of higher order II. *Fibonacci Quart.* **10**(1) (1972), 71–80.

[**9**] A. H. Dooley. Markov odometers. *Topics in Dynamics and Ergodic Theory (London Mathematical Society Lecture Note Series, 310)*. Cambridge University Press, Cambridge, 2003, pp. 60–80.

[**10**] M. Doudékova-Puydebois. On dynamics related to a class of numeration systems. *Monatsh. Math.* **135**(1) (2002), 11–24.

[**11**] J.-M. Dumont. Formules sommatoires et systèmes de numération lies aux substitutions. *Séminaires de Théor. des Nombres, Bordeaux* **16** (1987/88), Exp. Nr. 39, 12 pp.

[**12**] J.-M. Dumont and A. Thomas. Systèmes de numération et fonctions fractales relatifs aux substitutions. *Theoret. Comput. Sci.* **65** (1989), 153–169.

[**13**] A. S. Fraenkel. Systems of numeration. *Amer. Math. Monthly* **92** (1985), 105–114.

[**14**] C. Frougny. Fibonacci representations and finite automata. *IEEE Trans. Inform. Theory* **37** (1991), 393–399.

[**15**] C. Frougny. On the successor function in non-classical numeration systems. *Proc. S.T.A.C.S. 96 (Lecture Notes in Computer Science, 1046)*. Springer, Berlin, 1996, pp. 543–553.

[**16**] C. Frougny. On the sequentiality of the successor function. *Inform. and Comput.* **139** (1997), 17–38.

[**17**] C. Frougny. Number representation and finite automata. *Topics in Symbolic Dynamics and Applications (Temuco, 1997) (London Mathematical Society Lecture Notes Series, 279)*. Cambridge University Press, Cambridge, 2000, pp. 207–228.

[**18**] C. Frougny. Numeration systems. *Algebraic Combinatorics on Words (Encyclopedia of Mathematics and its Applications, 90)*. Ed. M. Lothaire. Cambridge University Press, Cambridge, 2002, pp. 230–268.

[**19**] C. Frougny. Non-standard number representation: computer arithmetic, beta-numeration and quasicrystals. *Physics and Theoretical Computer Science (NATO Security through Science Series D: Information and Communication Security, 7)*. IOS, Amsterdam, 2007, pp. 155–169.

[**20**] C. Frougny and C. Solomyak. On representation of integers in linear numerations systems. *Ergodic Theory of $\mathbb{Z}^d$-Actions (London Mathematical Society Lecture Note Series, 228)*. Eds. M. Pollicott and K. Schmidt. Cambridge University Press, Cambridge, 1996, pp. 345–368.

[**21**] P. J. Grabner, P. Liardet and R. F. Tichy. Odometers and systems of numeration. *Acta Arith.* **70** (1995), 103–123.

[**22**] P. J. Grabner and R. F. Tichy. α-expansions, linear recurrences and the sum-of-digits function. *Manuscripta Math.* **70** (1991), 311–324.

[**23**] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis*. Springer, Berlin, 1970.

[**24**] I. Kátai. Open problems originated in our research work with Zoltán Daróczy. *Publ. Math. Debrecen* **75** (2009), 149–165.

[**25**] G. Köthe. *Topological Vector Spaces. I (Die Grundlehren der Mathematischen Wissenschaften, Band 159)*. Springer, New York, 1969 (translated from the German by D. J. H. Garling).

[**26**] P. B. A. Lecomte and M. Rigo. Numeration systems on a regular language. *Theory Comput. Syst.* **34**(1) (2001), 27–44.

[**27**] M. Lothaire. *Algebraic Combinatorics on Words (Encyclopedia of Mathematics and its Applications, 90)*. Cambridge University Press, Cambridge, 2002.

[**28**] H. L. Montgomery and U. M. A. Vorhauer. Greedy sums of distinct squares. *Math. Comp.* **73**(245) (2004), 493–513 (electronic).

[**29**] W. Parry. On the β-expansions of real numbers. *Acta Math. Acad. Sci. Hung.* **11** (1960), 401–416.

[**30**] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France* **110**(2) (1982), 147–178.

[**31**] M. Rigo. Automates et systèmes de numération. *Bull. Soc. Roy. Sci. Liège* **73**(5–6) (2005), 257–270.

[**32**] F. Schweiger. *Ergodic Theory of Fibred Systems and Metric Number Theory*. Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1995.

[**33**] N. Sidorov. Arithmetic dynamics. *Topics in Dynamics and Ergodic Theory (London Mathematical Society Lecture Note Series, 310)*. Eds. S. Bezuglyi and S. Kolyada. Cambridge University Press, Cambridge, 2003, pp. 145–189.

[**34**] A. M. Vershik. The adic realizations of the ergodic actions with the homeomorphisms of the Markov compact and the ordered Bratteli diagrams. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **223** (1995), 120–126.

[**35**] A. M. Yaglom and I. M. Yaglom. *Challenging Mathematical Problems with Elementary Solutions. Vol. II (Problems from Various Branches of Mathematics)*. Dover Publications Inc, New York, 1987 (translated from the Russian by James McCawley, Jr., Reprint of the 1967 edition).