HARMONIC PROPERTIES OF THE SUM-OF-DIGITS FUNCTION FOR COMPLEX BASES

PETER J. GRABNER† AND PIERRE LIARDET‡

Abstract. We study dynamical properties of digital representations of the Gaussian integers. We are mostly concerned with a specific cocycle defined by the sum-of-digits function s_b with respect to the base b representation of $\mathbb{Z}[i]$. This is used to derive uniform distribution results for the double sequence $z \mapsto \{\alpha s_b(z)\}$ for irrational α and z in large circles and other domains.

1. Introduction

The statistical properties of representations of integers with respect to a given base sequence is a classical area of investigation and there exists a vast literature on this topic. The first works on this subject have dealt with systems of numeration of the integers with respect to a fixed integer basis. The sum-of-digits function related to this representation of the integers plays a special role here, because of its many structural properties and also because of the diversity of the methods used for its investigation. See for example [3, 5, 6, 9, 10, 12, 20, 38]. These investigations have been progressively extended to more general numeration systems given by strictly increasing sequences of integers, especially solutions of linear recurrences [4, 13, 15]. On the other hand the analogous studies for systems of numeration in number fields is more recent. After the arithmetic works [22, 23, 27] which describe the possible canonical radix number systems in the orders of number fields, studies of the statistical properties were started recently [14, 37].

In this paper the sum-of-digits function will also be our principal object of study. We will investigate its properties from the point of view of ergodic theory as it was done in [13, 31]. In order to describe the essential ideas without having to introduce too much terminology, we have restricted this work to canonical number systems in the Gaussian integers. These were characterized in [23] as given by the bases $b = -a \pm i$, with $a \in \mathbb{N} \setminus \{0\}$. These numeration systems lead to a natural $\mathbb{Z}^2$-action on a compact group $\mathcal{K}(b)$. The interest of this study is not the construction of these groups as projective limits of finite groups,

Date: August 11, 2004.

1991 *Mathematics Subject Classification.* Primary: 22D40; Secondary:11A63, 11M41.

Key words and phrases. digital expansions.

† This author is supported by the Austrian Academy of Sciences and the START-project Y96-MAT of the Austrian Science Fund.

‡ This author was supported by MENRT (Ministère de L'Education Nationale de la Recherche et de la Technologie).

but the construction of $\mathbb{Z}^2$-cocycles from the associated sum-of-digits function s_b, whose ergodic properties are studied in detail.

Section 2 provides the ergodic machinery used later. Essentially, this is K. Schmidt's [35] characterization of ergodic extensions of a cocycle. The $\mathbb{Z}^2$-actions T under consideration are uniquely ergodic, but not continuous, which poses the question of identifying the generic points in order to make the results applicable to prove uniform distribution of related sequences. This problem is solved by an extension of a result in [30] obtained for skew products of sufficiently regular transformations. Finally, we have used an immediate extension of theorems of Helson [16] to establish the pure character of the spectrum of the dynamical system associated with the sum-of-digits function.

Section 3 describes the dynamical system associated with a complex basis b for the Gaussian integers $\mathbb{Z}[i]$ and shows the unique ergodicity of the cylindric extension T^Σ given by the cocycle Σ associated to the sum-of-digits function s_b (cf. (3.5) for the definition of Σ). For an arbitrary character $\psi \in \hat{\mathbb{Z}}$ we consider the compact group $\mathcal{K}(b) \times G_\psi$ with $G_\psi = \overline{\psi(\mathbb{Z})}$ equipped with its Haar measure. We introduce the skew product $T_g^{\Sigma_\psi}$: $(x,\xi) \mapsto (x+g, \xi\Sigma_\psi(g,x))$ on $\mathcal{K}(b) \times G_\psi$ with $\Sigma_\psi = \psi \circ \Sigma$ and prove that the spectrum of the dynamical system is purely singular continuous (or purely discrete for a finite number of characters) in the orthocomplement of the $\mathbb{Z}^2$-action T. These spectral properties are quite similar to those observed in the case of the sum-of-digits functions in Cantor scales (see e. g. [34]).

One consequence of the unique ergodicity of T^{Σ_ψ} is the uniform distribution modulo 1 of the double-sequence $z \mapsto \{\alpha s_b(z)\}$ (for irrational α) ($z \in \mathbb{Z}[i]$) in the following sense:

$$\lim_{N\to\infty} \frac{1}{\pi N} \# \left\{ z \in \mathbb{Z}[i] \mid |z|^2 < N, \quad \{\alpha s_b(z)\} \in J \right\} = |J|$$

for all intervals $J \subset [0,1[$ ($\{x\}$ denotes the fractional part of x).

In section 4 we study the distribution of the double sequence $z \mapsto (\arg(z), \{\alpha s_b(z)\})$ in $]-\pi,\pi] \times [0,1[$ for z in large circles by the use of classical methods from analytic number theory. Here we also obtain estimates for the discrepancy of this sequence depending on the approximation type of α.

2. Cocycles and Ergodic Transformation Groups

In this section we recall classical definitions and properties from ergodic theory; we refer the reader to the book of K. Schmidt [35] for the omitted proofs and more details on the subject.

2.1. G**-cocycles.** In the sequel $(X,\mathcal{B})$ will be a standard Borel space (i.e. $\mathcal{B}$ is the Borel σ-algebra of a metrizable and locally compact topology on X). Let μ be a Borel probability measure on $(X,\mathcal{B})$. Let G denote a countable group with identity e. An action T_G (or simply T) of G on $(X,\mathcal{B},\mu)$ is given by a homomorphism

$$T : g \mapsto T_g \in \operatorname{Aut}(X,\mathcal{B},\mu),$$

where $\mathrm{Aut}(X, \mathcal{B}, \mu)$ denotes the group of automorphisms of $(X, \mathcal{B}, \mu)$ (equipped with the weak topology of Halmos).

Let A be an Abelian locally compact metrizable group (the group law will be written additively).

Definition 1. *A T-cocycle (or simply a cocycle, if the underlying action T is fixed) is a Borel map*

$$a : G \times X \to A$$

such that

$$(i) \qquad a(gh, x) = a(g, T_h x) + a(h, x) \quad \mu - a.e.$$

$$(ii) \qquad \mu\left(\bigcup_{g \in G} (\{x \mid T_g x = x\} \cap \{x \mid a(g, x) \neq 0\})\right) = 0.$$

We usually assume that T is aperiodic, i.e. $\mu(\{x \mid \exists g \neq e, T_g x = x\}) = 0$. In this case condition (ii) has to be replaced by

$$(ii') \quad \mu(\{x \mid a(e, x) \neq 0\}) = 0.$$

Definition 2. *A cocycle $a(\cdot, \cdot)$ is called a T-coboundary if there exists a Borel map $c : X \to A$, such that*

$$a(g, x) = c(T_g x) - c(x) \quad \mu - a.e.$$

It is called trivial, if it is the sum of a T-coboundary and a function, which only depends on g.

2.2. **Examples.**

Example 1. *$G = \mathbb{Z}$, $T_n = T^n$ for some aperiodic $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$. For any Borel map $f : X \to A$ we associate the cocycle*

$$a_f(n, x) = \sum_{k=0}^{n-1} f \circ T^k(x) \text{for } n > 0 0 \text{for } n = 0 - \sum_{k=n}^{-1} f \circ T^k(x) \text{for } n < 0.$$

Example 2. *$G = \mathbb{Z}^2$, $X = \mathcal{K}(b)$, the b-compactification of $\mathbb{Z}[i]$ with $b = -a + i$, $a \in \mathbb{N}^*$ (see Section 3):*

$$\begin{aligned} T_{(n,m)} : x &\mapsto x + n + im \\ s_b(n + im) &= \text{b-ary sum of digits of } n + im \text{ (cf. (3.2))}. \end{aligned}$$

$$\Sigma((n, m), x) = \lim_{\substack{r + is \to x \\ \text{in } \mathcal{K}(b) \\ r + is \in \mathbb{Z}[i]}} (s_b(n + r + i(m + s)) - s_b(r + is))$$

(except for x in a set D with $\mu(D) = 0$).

2.3. **Essential Values of a Cocycle.** We assume that the action T is ergodic on $(X,\mathcal{B},\mu)$, i.e.

$$\forall B\in\mathcal{B},\ \forall\, g\in G:\ T_gB=B\Longrightarrow \mu(B)\mu(X\setminus B)=0.$$

We fix a cocycle $a:G\times X\to A$. If A is not compact, let $\bar{A}=A\cup\{\infty\}$, the one point compactification of A and set $a+\infty=\infty$ for all $a\in\bar{A}$.

Definition 3. *$\alpha\in\bar{A}$ is said to be an essential value of a if for every neighbourhood $N(\alpha)$ of α in $\bar{A}$ and for every $B\in\mathcal{B}$ with $\mu(B)>0$,*

$$\mu\left(\bigcup_{g\in G}\left(B\cap T_g^{-1}(B)\cap\{x\mid a(g,x)\in N(\alpha)\}\right)\right)>0. \tag{2.1}$$

Let

$$\begin{aligned}\bar{E}(a) &= \{\alpha\in\bar{A}\mid\alpha\quad \textit{is an essential value of } a\}\\ E(a) &= \bar{E}(a)\cap A.\end{aligned}$$

We have the following properties:

1. *If $b:G\times X\to A$ is a coboundary then $\bar{E}(a+b)=\bar{E}(a)$ and $E(a+b)=E(a)$.*
2. *$E(a)$ is a closed subgroup of A.*
3. *a is a coboundary $\Leftrightarrow\bar{E}(a)=\{0\}$.*

2.4. **The Skew Product T^a.** We assume that $(X,\mathcal{B},\mu)$ is non-atomic. Let h_A be the Haar measure on A. If A is compact, h_A will be normalized by $h_A(A)=1$. If A is discrete, h_A will be normalized by $h_A(\{0\})=1$ or by $h_A(\{0\})=\frac{1}{\#A}$, if A is finite. Let

$$\tilde{X}=(X\times A,\mathcal{B}\otimes\mathcal{B}_A,\mu\otimes h_A),$$

where $\mathcal{B}_A$ is the Borel σ-algebra of A. We define a G-action T^a on $\tilde{X}$ by

$$T_g^a(x,\alpha)=(T_gx,\alpha+a(g,x)).$$

Clearly, $T_g^a\in\mathrm{Aut}(\tilde{X})$. The action T^a is called the *skew product* of T with respect to a.

If $a'=a+b$, where b is a coboundary, then $T^a\simeq T^{a'}$. More explicitly, if $b(g,x)=c(T_gx)-c(x)$, the conjugate automorphism is given by

$$\begin{aligned}\Phi_c:\qquad & X\times A\to X\times A\\ & (x,\alpha)\mapsto(x,\alpha+c(x)).\end{aligned}$$

Let $\mathcal{I}$ be the set of T^a-invariant elements in $\mathcal{B}\otimes\mathcal{B}_A$ and put

$$I(a)=\{\beta\in A\mid\mu\otimes h_A(\tau_\beta B\bigtriangleup B)=0\quad\text{for every } B\in\mathcal{I}\}$$

where $\tau_\beta:X\times A\to X\times A$ is given by

$$\tau_\beta(x,\alpha)=(x,\alpha+\beta).$$

Theorem 1. *(K. Schmidt [35], Theorem 5.2) Let T be an ergodic action on $(X,\mathcal{B},\mu)$ which is assumed to be non-atomic. Then for any cocycle $a:G\times X\to A$:*

$$I(a)=E(a).$$

Remark 1. *We may define $I(a)$ as follows*

$$I(a) = \{\beta \in A \mid \forall f \in L^\infty(\mu \otimes h_A) : (\forall g \in G : f \circ T^a_g = f \Rightarrow f \circ \tau_\beta = f)\}.$$

Corollary 1. *If T is ergodic:*

$$T^a \quad \text{is ergodic } \Leftrightarrow E(a) = A.$$

2.5. **Functional equations and ergodicity.** In this section we assume that A is compact.

Theorem 2. *Assume that T is an ergodic action of G on on the probability space $(X, \mathcal{B}, \mu)$. Then the following are equivalent:*

(i) *T^a is ergodic for $\mu \otimes h_A$*
(ii) *$\forall \chi \in \hat{A}$, $\chi \not\equiv 1$ the functional equation*

$$\forall g \in G : F(x) = \chi(a(g,x))F(T_g(x)) \quad \mu - a.e. \tag{2.2}$$

has no measurable solution F, except the trivial one $F \equiv 0$.

Remark 2. *The proof of Theorem 2 is classical for an ergodic $\mathbb{Z}$-action, (see* [33] *and also* [11] *and* [39]*). The extension to a more general action is straight forward and left to the reader, we only point out the Hilbert decomposition*

$$L^2(X \times A, \mu \otimes h_A) = \bigoplus_{\chi \in \hat{A}} L^2(X,\mu) \otimes \chi \quad \textit{as a Hilbert orthogonal sum.} \tag{2.3}$$

Remark 3. *For $\psi \in \hat{A}$ the action T^a induces a unitary group representation*

$$\begin{aligned} g \mapsto \quad & U^{a_\psi}_g \\ & U^{a_\psi}_g(f) = (\psi \circ a)(g, \cdot) f \circ T_g(\cdot) \end{aligned} \tag{2.4}$$

on the summands in the orthogonal decomposition (2.3)*. These were introduced in the proof of Theorem 2 and will be used later in the discussion of the spectrum of the action T^a.*

2.6. **Generic points.** In this section we assume that $G = \mathbb{Z}^d$ and X a compact metric space. The action T is said to be continuous if T_g is a homeomorphism of X for every $g \in G$.

For an integer $N \geq 1$ we set

$$\Delta_N = \{(g_1, \ldots, g_d) \in \mathbb{Z}^d \mid 0 \leq g_i < N, i = 1, \ldots, d\}. \tag{2.5}$$

Definition 4. *Let T be a continuous $\mathbb{Z}^d$-action on (X, μ). A point $x \in X$ is said to be (T, μ)-generic if*

$$\forall f \in C(X) : \quad \lim_{N \to \infty} \frac{1}{N^d} \sum_{g \in \Delta_N} f \circ T_g(x) = \int_X f \, d\mu. \tag{2.6}$$

We may extend this definition and instead of the family $\{\Delta_N, N = 1, 2, \ldots\}$ consider any family $\mathcal{Q} = \{Q_N, N = 1, 2, \ldots\}$ of subsets $Q_N \subset \mathbb{Z}^d$ which satisfies the following assumptions:

$$\begin{aligned} &(i) && Q_N \subset Q_{N+1} \quad \text{for } N \geq 1 \\ &(ii) && \#Q_N > 0 \\ &(iii) && \frac{\#(Q_N \vartriangle (g + Q_N))}{\#Q_N} \to 0, \quad \forall g \in \mathbb{Z}^d \end{aligned} \tag{2.7}$$

For the family $\{\Delta_N\}$ we have a multidimensional individual ergodic theorem due to Tempel'man (cf. [28], p.205). This theorem implies that (T, μ)-generic points exist. If in addition T is uniquely ergodic, all points are (T, μ)-generic. The theorem remains valid for any family $\mathcal{Q}$ satisfying the above mentioned assumptions.

We need to extend the notion of generic points to the case where T is μ-continuous, that is to say, for any $g \in G$ there exists an $E_g \subset X$ such that $\mu(E_g) = 0$ and T_g is continuous at any point of $X \setminus E_g$.

For a μ-continuous function $f : X \to \mathbb{C}$ we denote by $D(f)$ its set of discontinuity points, which has by assumption $\mu(D(f)) = 0$. If T is μ-continuous then $f \circ T_g$ is also μ-continuous. We denote by $\mathcal{R}_\mu(X)$ the space of bounded μ-continuous functions $f : X \to \mathbb{C}$.

Lemma 1. *Assume that T is μ-continuous and let $x \in X$ such that there exists $J \subset \mathbb{N}$, J infinite, and a Borel measure λ on X, absolutely continuous with respect to μ, such that*

$$\forall f \in C(X), \qquad \lim_{\substack{N \to \infty \\ N \in J}} \frac{1}{N^d} \sum_{g \in \Delta_N} f \circ T_g(x) = \int_X f \, d\lambda \tag{2.8}$$

then $\mathcal{R}_\mu(X) \subset \mathcal{R}_\lambda(X)$ and

$$\forall f \in \mathcal{R}_\lambda(X), \qquad \lim_{\substack{N \to \infty \\ N \in J}} \frac{1}{N^d} \sum_{g \in \Delta_N} f \circ T_g(x) = \int_X f \, d\lambda. \tag{2.9}$$

Moreover, λ is T-invariant.

Proof. Since $f \mapsto f \circ T_g$ are positive operators, standard arguments easily show (2.9). Since $\{\Delta_N\}$ satisfies property (iii) and T is λ-continuous, we have for any $f \in \mathcal{R}_\lambda$

$$\int_X f \, d\lambda = \int_X f \circ T_g \, d\lambda$$

for any $g \in G$. □

Definition 5. *If T is μ-continuous, a point $x \in X$ is said to be (T, μ)-generic if* (2.9) *holds with $J = \mathbb{N}$ and $\lambda = \mu$.*

The above lemma shows that generic points for μ-continuous actions exist.

Theorem 3. *Assume that $T_{\mathbb{Z}^d}$ acts μ-continuously on $(X, \mathcal{B}, \mu)$, X a compact metric space, and let $a : \mathbb{Z}^d \times X \to A$ be a μ-continuous A-valued cocycle with A compact. Assume further that T^a is ergodic for $\mu \otimes h_A$. If x is (T, μ)-generic, then for all $\alpha \in A$, the point (x, α) is $(T^a, \mu \otimes h_A)$-generic.*

Remark 4. (1) *Since T^a is ergodic, $\mu \otimes h_A$-almost all points (x, α) are $(T^a, \mu \otimes h_A)$-generic and then it is easy to see that if (x, α) is $(T^a, \mu \otimes h_A)$-generic, then (x, β) is $(T^a, \mu \otimes h_A)$-generic for all β.*

(2) *The theorem was proved in the one-dimensional case in* [30] *(see also* [2]*).*

Proof. We assume that x is a (T, μ)-generic point but there exists an $\alpha \in A$ such that (x, α) is not $(T^a, \mu \otimes h_A)$-generic. Then there exists $J \subset \mathbb{N}$, J infinite, and a Borel measure λ on $X \times A$ such that

$$\forall F \in \mathcal{R}_\lambda(X \times A), \quad \lim_{N \in J} \frac{1}{N^d} \sum_{g \in \Delta_N} F \circ T^a_g(x, \alpha) = \int_{X \times A} F \, d\lambda$$

and

$$\exists \chi \in \hat{A}, \quad \chi \not\equiv 1, \quad \exists f \in C_{\mathbb{R}}(X), \int_{X \times A} f \otimes \chi \, d\lambda \neq 0 = \int_{X \times A} f \otimes \chi \, d(\mu \otimes h_A).$$

In particular, for all $\phi \in \mathcal{R}_\lambda(X) = \{\phi \mid \phi \otimes 1_A \in \mathcal{R}_\lambda(X \times A)\}$ we have

$$\lim_{N \in J} \frac{1}{N^d} \sum_{g \in \Delta_N} \phi \otimes \chi(T^a_g(x, \alpha)) = \int_{X \times A} \phi \otimes \chi \, d\lambda. \tag{2.10}$$

Clearly λ projects on μ and $\mathcal{R}_\mu(X) = \mathcal{R}_\lambda(X)$. Moreover, for any $g \in \mathbb{Z}^d$, the set of discontinuous points of T^a_g is contained in a set of the form $E'_g \times A$ where $\mu(E'_g) = 0$. Therefore T^a is λ-continuous. This implies from (2.10) that λ is T^a-invariant.

Letting $L : L^2(X, \mu) \to \mathbf{C}$ be the continuous linear form defined by $L(f) = \int_{X \times A} f \otimes \chi \, d\lambda$, there exists an $F \in L^2(X, \mu)$, $F \neq 0$, such that $L(\cdot) = \langle \cdot | F \rangle$. Since λ is T^a-invariant, one has for all $f \in L^2(X, \mu)$:

$$\forall g \in G : \quad L(U^{a_\chi}_g f) = L(f),$$

where $U^{a_\chi}_g : L^2(X, \mu) \to L^2(X, \mu)$ is defined by (2.4). Moreover the space $\mathcal{R}_\mu(X)$ is everywhere dense in $L^2(X, \mu)$. Hence we get also $\langle U^{a_\chi}_g f | F \rangle = \langle f | F \rangle$ for any $f \in L^2(X, \mu)$, so that

$$\forall g \in G : \quad (U^{a_\chi}_g)^* F = F$$

which implies that F is invariant under the unitary operator $U^{a_{\bar{\chi}}}_g$ *i.e.,*

$$F \circ T_g = \bar{\chi}(a(g, x)) F(x) \quad \mu - \text{a.e.}$$

This contradicts the ergodicity of T^a by Theorem 2. □

Corollary 2. *Under the hypotheses of Theorem 3, if the action $T_{\mathbb{Z}^d}$ is uniquely ergodic, then T^a is uniquely ergodic.*

2.7. **Purity of the spectrum.** We now consider the special case where X is a compact abelian metrizable group and T acts by translation. In other words, T is given by a group homomorphism $\theta : G \to X$ such that $T_g x = x + \theta(g)$.

As above we have the restriction of the T^a-action to each component of the Hilbert sum (2.3) given by

$$U_g^{a_\chi} f = \chi(a(g,\cdot)) f \circ T_g(\cdot).$$

In particular, we have a Weyl commutation relation

$$U_g^{a_\chi} M_\gamma = \psi(\theta(g)) M_\gamma U_g^{a_\chi}, \tag{2.11}$$

where γ is any character on X and M_γ is the operation of multiplication by γ:

$$M_\gamma f(\cdot) = \gamma(\cdot) f(\cdot) \quad \text{on } L^2(X,\mu).$$

The following two theorems are straightforward generalizations of Theorems 3, 4 and 5 in [16]:

Theorem 4. *Let X be a compact Abelian group and the actions T and T^a defined as above. Assume that the orbit of 0 (the neutral element of X) under T is dense (T is therefore ergodic). Then for any $\chi \in \hat{A}$ the spectral measure of the G-action $g \mapsto U_g^{a_\chi}$ on $L^2(X,\mu)$ is either purely Lebesgue, or purely singular-continuous, or purely discrete with respect to the Haar measure $h_{\hat{G}}$ on $\hat{G}$. Moreover, the spectral multiplicity is uniform.*

Theorem 5. *A cocycle a is trivial, if and only if for any $\chi \in \hat{A}$ the spectral measure of the G-action $U_g^{a_\chi}$ is (purely) discrete.*

3. Dynamics related to complex bases

In this section we will study properties of digital expansions of integers in number fields. For any order $\mathcal{O}$ in a number field K a base $b \in \mathcal{O}$ and a set of digits $\mathcal{D} \subset \mathcal{O}$ define a system of numeration, if every element $z \in \mathcal{O}$ can be written uniquely as a sum

$$z = \sum_{\ell=0}^{L} \varepsilon_\ell b^\ell, \quad \text{with } \varepsilon_\ell \in \mathcal{D}. \tag{3.1}$$

If $\mathcal{D}$ is restricted to be the set $\{0, 1, \ldots, |N(b)|-1\}$, the number system is called a canonical number system; here N denotes the norm of the field extension $K \supset \mathbb{Q}$. For the question of existence of canonical number systems we refer to [23, 26, 27]. In the case of quadratic number fields the bases of canonical number systems could be characterized completely in [21, 22]. For the Gaussian integers the bases of canonical number systems are given by $b = -a \pm i$ for $a \in \mathbb{N} \setminus \{0\}$ (cf. also [25]). Here we will restrict ourselves to this case.

If z is given by (3.1) we define the sum-of-digits function of z by

$$s_b(z) = \sum_{\ell=0}^{L} \varepsilon_\ell. \tag{3.2}$$

3.1. **Generalities about the b-compactification of $\mathbb{Z}[i]$.** For $b = -a+i$ we imitate the construction of **a**-adic integers (cf. [17])

$$\mathcal{K}(b) = \varprojlim_n \mathbb{Z}[i]/(b^n). \tag{3.3}$$

There is a natural identification of this projective limit to the compact product space $\{0,1,\ldots,a^2\}^{\mathbb{N}}$ endowed with the following group law: for two sequences $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ in $\mathcal{K}(b)$ we define a sequence $z = (z_0, z_1, \ldots)$ as follows. Write $x_0 + y_0 = t_1 b + z_0$ with $z_0 \in \{0, \ldots, a^2\}$ and $t_1 \in \mathbb{Z}[i]$. Suppose now that $z_0, z_1, \ldots, z_{k-1}$ and $t_1, \ldots, t_k$ have been defined inductively; then $z_k \in \{0, \ldots, a^2\}$ is given by $x_k + y_k + t_k = t_{k+1} b + z_k$. We write $x + y$ for z. The proof of the group axioms and continuity for this operation follows the same lines as the proof for **a**-adic integers (cf. [17]).

The Gaussian integers can be isomorphically embedded into $\mathcal{K}(b)$ via the b-ary digital expansion. $\mathcal{K}(b)$ can also be viewed as the closure of the set of all b-ary digital expansions of Gaussian integers in the product space $\{0,1,\ldots,a^2\}^{\mathbb{N}}$ equipped with the product topology of the discrete spaces. The action T of $\mathbb{Z}[i]$ on $\mathcal{K}(b)$ is given as the continuation of addition by $g \in \mathbb{Z}[i]$. We write $T_{m+in}x = T_{(m,n)}x = x + m + in$. We use the notation $e(x) = e^{2\pi i x}$.

Lemma 2. *The characters of $\mathcal{K}(b)$ are given by*

$$\gamma(x) = e\left(\Re\left(\frac{z}{b^k}\sum_{j=0}^{k-1} x_j b^j\right)\right)$$

for $k \geq 1$ and $z \in \mathbb{Z}[i]$ (defined up to congruence modulo (b^k)).

Proof. Every character γ of $\mathcal{K}(b)$ is determined by its restriction γ' to $\mathbb{Z}[i]$. By the definition of $\mathcal{K}(b)$ (3.3) this restriction has to be trivial on a subgroup of $\mathbb{Z}[i]$ of the form $b^k\mathbb{Z}[i]$ for some integer k. Thus we have

$$\gamma'(n+im) = e\left(\Re\left(\frac{z(m+in)}{b^k}\right)\right)$$

for a Gaussian integer z and this proves the lemma. □

We identify the dual group of $\mathcal{K}(b)$ with the discrete subgroup

$$\Gamma(b) = \left\{u = \frac{z}{b^k} \mid k \in \mathbb{N}, \quad 0 \leq \Re u, \Im u < 1, \quad z \in \mathbb{Z}[i]\right\}$$

of the two-dimensional torus; we write $u(\gamma)$ for the point in $\mathbb{T}^2$ identified with γ. For a given point $u = z/b^k \in \Gamma(b)$ we set $\gamma_u(x) = e(\Re(u \sum_{j=0}^{k-1} x_j b^j))$.

Using the shift σ on $\mathcal{K}(b)$ (viewed as the product space $\prod_{k=0}^{\infty}\{0,\ldots,a^2\}$) we can write

$$T_{b^k}(x_0x_1\ldots) = x_0x_1\ldots x_{k-1}T_1(\sigma^k(x_0x_1\ldots)), \tag{3.4}$$

where the element on the right hand side is formed by concatenation of $x_0 \ldots, x_{k-1}$ and $T_1(\sigma^k(x_0x_1\ldots))$. Furthermore, σ preserves the Haar measure on $\mathcal{K}(b)$.

We set if the limit exists

$$\Sigma(m+in,x)=\Sigma((m,n),x)=\lim_{\substack{z\in\mathbb{Z}[i]\\ z\to x}}(s_b(z+m+ni)-s_b(z)), \tag{3.5}$$

where s_b denotes, as in the introduction, the b-ary sum-of-digits function (see (3.2)). For $\psi\in\widehat{\mathbb{Z}}$ we set $\Sigma_\psi=\psi\circ\Sigma$.

Proposition 1. *Σ is μ-continuous, where μ denotes the Haar measure on $\mathcal{K}(b)$.*

Proof. At first we investigate where the limit in the definition of Σ exists. As in [14] this can be studied using the properties of the "addition automaton" (a transducer automaton, cf. [8]). Figure 1 shows the automaton which produces the digital expansion of $z+1$ from the digital expansion of z, if the initial state is chosen as P. A sequence of digits $(\varepsilon_0,\varepsilon_1,\dots)$ is mapped to another sequence of digits $\eta_0,\eta_1,\dots)$ according to the following rules:

(1) start in the indicated initial state and let $k=0$.
(2) read ε_k and move along the vertex of the graph marked with $\varepsilon_k\mid\delta$.
(3) let $\eta_k=\delta$ and increase k by 1. Then go to 2.

The digital expansion of $z-a-i$ is produced, if the initial state is chosen R; the initial states $-P$ and $-R$ correspond to the inverse operations (cf. [14]). We note here, that the states of the automaton correspond to the possible "carries" t_k (in the discussion at the beginning of this paragraph) for $y=(1,0,0,\dots)$.

Thus addition of any complex integer $m+ni$ can be performed by applying a suitable finite sequence of additions of ±1 and $\pm(a+i)$ thus defining a new addition automaton, which contains Figure 1 as a subgraph and all paths followed in the computation of $z+m+ni$ end up in this subgraph.

The only possibility for the limit (3.5) not to exist is that the path corresponding to the digits of x does not end in one of the two terminal states $\bullet$. Since all the digits are reproduced after a terminal state is reached, the difference of the sum-of-digits functions is produced before hitting $\bullet$. Thus the preimages of the limit (where it exists) can be written as a union of cylinder sets, and therefore the function defined by the limit is continuous where the limit exists.

We now study the set of points x, where the limit does not exist for the addition of ±1 and $\pm(a+i)$. These points correspond to paths in the graph, which do not reach a terminal state. In order to bound the measure of the set of those points we count P_n, the number of paths of length n in the graph with the two terminal states removed. This can be described by the entries of the n-th power of the matrix

$$\left(010000002a00(a-1)^2(a-1)^2+1002a-100002a-100(a-1)^2+1(a-1)^2002a00000010\right)$$

whose characteristic polynomial is given by

$$(\lambda^2+2a\lambda+a^2+1)(\lambda-1)(\lambda^3-(2a-1)\lambda^2-(a-1)^2\lambda-a^2-1).$$

It can be easily seen that the root of largest modulus comes from the last factor and it satisfies $|\lambda|<(1+\sqrt{2})a$ for $a\geq2$ and $|\lambda|<\sqrt{3}$ for $a=1$. Thus the number of paths P_n

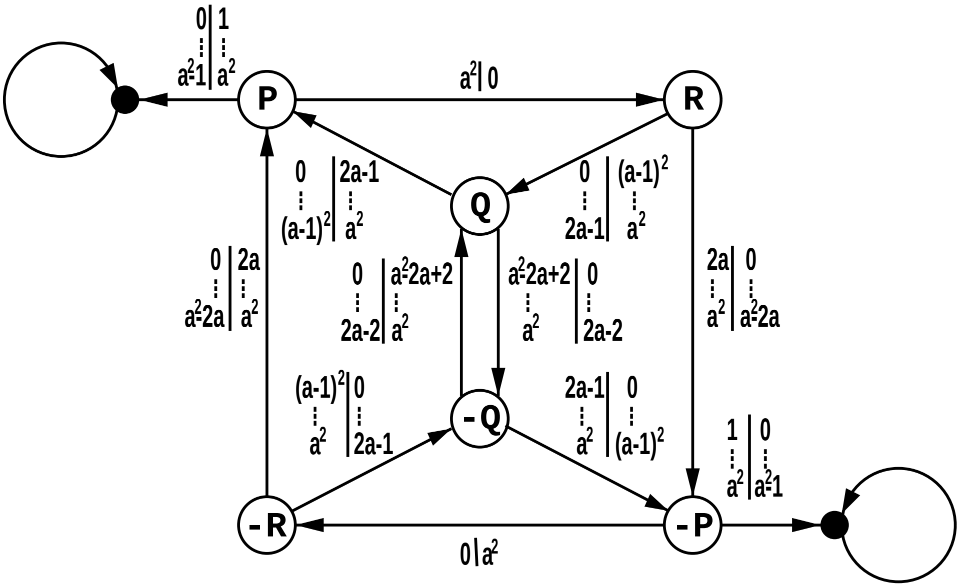

FIGURE 1. The addition automaton

satisfies

$$P_n = \mathcal{O}\left((1+\sqrt{2})^n a^n\right) \text{ for } a \geq 2 \mathcal{O}\left(\sqrt{3}^n\right) \text{ for } a = 1.$$

Since the total number of digital sequences of length n is $(a^2+1)^n$ this implies that the set of points x, where the limit does not exist, has measure 0 (this set can be written as a countable intersection of cylinder sets whose measure tends to 0). $\square$

The next lemma is an immediate consequence of (3.4).

Lemma 3. *Where Σ is defined, the following equation holds*

$$\Sigma(b^k, x) = \Sigma(1, \sigma^k x).$$

3.2. **Ergodicity of T^Σ.** We compute the set $E(\Sigma)$ of essential values of Σ.

Lemma 4. *Let k be a non-negative integer and $x = (\varepsilon_0, \varepsilon_1, \ldots) \in \mathcal{K}(b)$.*

(1) *if $\varepsilon_k < a^2$ then $\Sigma(b^k, x) = 1$*
(2) *if $(\varepsilon_k, \varepsilon_{k+1}, \varepsilon_{k+2}) = (a^2, a^2, a^2)$ then $\Sigma(b^k, x) = -(a+1)^2$.*

Proof. For the first part we just observe that the digit by digit addition of b^k and $x_N = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N, 0, 0, \ldots)$ gives a valid element of $\mathcal{K}(b)$ and $s_b(b^k + x_N) - s_b(x_N) = 1$ for $N > k$.

For the second statement we use the automaton in Figure 1 to observe that digits $(\varepsilon_k, \varepsilon_{k+1}, \varepsilon_{k+2}) = (a^2, a^2, a^2)$ produce digits $(0, a^2-2a, a^2-1)$ after addition of x_N and b^k (again for $N > k$), which yields $s_b(b^k + x_N) - s_b(x_N) = -(a+1)^2$. $\square$

Theorem 6. $E(\Sigma) = \mathbb{Z}$.

Proof. Since $E(\Sigma)$ is a group it is enough to show that $1 \in E(\Sigma)$.

Let $B = C_{\varepsilon_0\varepsilon_1\ldots\varepsilon_k}$ be a cylinder set. Choose $L \geq k$ and define

$$B_L = \{x \in B \mid \varepsilon_{L+1} > 0\}.$$

Then

$$x \in B \cap T_{-b^{L+1}} B_L \Leftrightarrow x = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, x_{k+1}, x_{k+2}, \ldots) \quad \text{with } x_{L+1} < a^2.$$

For such an x we have $\Sigma(b^{L+1}, x) = 1$ by Lemma 4, and

$$\mu\left(B \cap T_{-b^{L+1}} B_L\right) \geq \frac{a^2}{a^2+1}\mu(B), \tag{3.6}$$

from which we deduce that

$$\mu\left(\bigcup_{g\in\mathbb{Z}[i]} B \cap T_{-g}B \cap \{x \mid \Sigma(g,x) = \ell\}\right) \geq \frac{a^2}{a^2+1}\mu(B). \tag{3.7}$$

Inequality (3.7) can be easily extended to countable unions of cylinder sets, hence to open sets and then to Borel sets. This proves that $1 \in E(\Sigma)$. $\square$

Corollary 3. *The action T^{Σ} is ergodic.*

Corollary 4. *For any character $\psi \in \widehat{\mathbb{Z}}$ the action T^{Σ_ψ} on $\mathcal{K}(b) \times G_\psi$ is ergodic ($G_\psi = \overline{\psi(\mathbb{Z})}$).*

Proof. By the definition of the set of essential values, $\psi(E(\Sigma)) \subset E(\psi \circ \Sigma)$. Since $E(\psi\circ\Sigma)$ is closed, it is equal to $G_\psi = \overline{\psi(E(\Sigma))} = \overline{\psi(\mathbb{Z})}$. $\square$

We are now going to study the spectral properties of the action T^{Σ_ψ} which we will denote shortly by T^ψ; in the following we will always identify (m,n) and $(m+ni)$ as well as $\mathbb{Z}^2$ and $\mathbb{Z}[i]$. Furthermore, we will write U_g^ψ for the operators $U_g^{\Sigma_\psi}$ defined by (2.4).

The following Lemma has several consequences.

Lemma 5.

$$\mu\left(\{x \in \mathcal{K}(b) \mid s_b(x+1) - s_b(x) = 1\}\right) = \frac{(a+1)^2}{(a+1)^2+1} \tag{3.8}$$

$$\mu\left(\left\{x \in \mathcal{K}(b) \mid s_b(x+1) - s_b(x) = -(a+1)^2\right\}\right) = \frac{1}{(a+1)^2+1}.$$

Proof. As in the proof of Proposition 1 we make use of the automaton in Figure 1; it can be easily seen by tracing paths from the initial state to the two terminal states $\bullet$, that the upper left terminal state corresponds to $s_b(x+1) - s_b(x) = 1$ and the lower right one corresponds to $s_b(x+1) - s_b(x) = -(a+1)^2$. The lemma follows by a simple application of finite Markov chains. $\square$

We introduce $I(\psi) = \int_{\mathcal{K}(b)} \psi(\Sigma(1,x))\,\mu(dx)$. Then Lemma 5 yields

$$I(\psi) = \frac{(a+1)^2\zeta + \zeta^{-(a+1)^2}}{(a+1)^2+1} \neq 0 \tag{3.9}$$

with $\zeta = \psi(1)$.

Lemma 6. *For different characters ψ_1 and ψ_2 in$\hat{}$
Zthevalues$I(\psi_1)$ and $I(\psi_2)$ are different.*

Proof. We set $\zeta_1 = \psi_1(1)$ and $\zeta_2 = \psi_2(1)$ and suppose that $I(\psi_1) = I(\psi_2)$. Then rewriting this last equation yields

$$(a+1)^2 = \zeta_1^{1-(a+1)^2} + \zeta_1^{2-(a+1)^2}\zeta_2^{-1} + \cdots + \zeta_2^{1-(a+1)^2},$$

from which we conclude by a simple convexity argument that $\zeta_1^k\zeta_2^\ell = 1$ for $k+\ell = (a+1)^2-1$. This implies that $\zeta_1 = \zeta_2$ and consequently equality of the characters. □

Theorem 7. *For all f and g in $L^2(\mathcal{K}(b),\mu)$ we have*

$$\lim_{k\to\infty} \left\langle U^{\psi}_{b^k} f, g \right\rangle = I(\psi)\langle f, g\rangle. \tag{3.10}$$

Proof. Clearly, it suffices to proof the result only for characters γ_1 and γ_2. By Lemma 2 these depend only on a finite number of coordinates, say the first n. We take $k > n$ and compute using Lemma 3

$$\begin{aligned}
\left\langle U^{\psi}_{b^k} f, g \right\rangle &= \int_{\mathcal{K}(b)} \psi(\Sigma(b^k,x))\gamma_1(T_{b^k}x)\bar{\gamma}_2(x)\,\mu(dx) \\
&= \int_{\mathcal{K}(b)} \psi(\Sigma(1,\sigma^k x))\gamma_1(b^k)\gamma_1(x)\bar{\gamma}_2(x)\,\mu(dx) \\
&= \gamma_1(b^k)\int_{\mathcal{K}(b)} \psi(\Sigma(1,\sigma^k x))\,\mu(dx)\int_{\mathcal{K}(b)} \gamma_1(x)\bar{\gamma}_2(x)\,\mu(dx).
\end{aligned}$$

The last equality holds because the arguments of the γ's depend only on the first n coordinates and the argument of ψ only on the rest. Finally, $\gamma_1(b^k) = 1$. □

In the following let $\chi_g(x+iy) = e(\Re(x\bar{g}))$ $(g \in \mathbb{Z}[i])$ be a character of $\mathbb{T}^2$.

Corollary 5. *For every $f \in L^2(\mathcal{K}(b),\mu)$ and every $\varphi \in L^2(\mathbb{T}^2,\nu_f)$ we have*

$$\lim_{k\to\infty} \int_{\mathbb{T}^2} \varphi(x)\chi_{b^k}(x)\,\nu_f(dx) = I(\psi)\int_{\mathbb{T}^2} \varphi(x)\,\nu_f(dx), \tag{3.11}$$

where ν_f denotes the spectral measure associated with f under the unitary representation $g \mapsto U^{\psi}_g$ (cf. [1]) defined by

$$\langle U^{\psi}_g f, f\rangle = \int_{\mathbb{T}^2} \chi_g(x)\,\nu_f(dx). \tag{3.12}$$

Proof. It suffices to prove (3.11) for the characters of $\mathbb{T}^2$. Then we have by (3.12)

$$\int_{\mathbb{T}^2} \chi_g(x)\chi_{b^k}(x)\,\nu_f(dx) = \langle U^\psi_{g+b^k} f, f\rangle = \langle U^\psi_{b^k} U^\psi_g f, f\rangle,$$

from which we conclude (3.11) using (3.10) and (3.12). □

Remark 5. *Corollary 5 implies Theorem 7; just take $\phi \equiv 1$.*

Remark 6. *Property* (3.10) *means that the unitary representation U^ψ is weakly ξ-mixing with $\xi = I(\psi)$. This notation was introduced for ergodic transformations in* [24], *where skew products of rotations on the circle are presented as examples with $\xi = -1$.*

Theorem 8. *The spectral measure of the unitary representation U^ψ is singular with respect to the Haar-Lebesgue measure λ. Moreover, if R_a denotes the group of the (a^2+2a+2)-nd roots of unity, we have:*

(i) *if $\psi(1) \notin R_a$, then the spectral measure of U^ψ is purely singular continuous,*
(ii) *if $\psi(1) \in R_a$, then $\chi_b : g \mapsto \psi(s_b(g))$ is a character of $\mathbb{Z}[i]$ and the spectral measure of U^ψ is discrete; in this case $T^\psi : \mathcal{K}(b) \times R_a \mapsto \mathcal{K}(b) \times R_a$ is given by*

$$T^\psi_g(x,\zeta) = (x+g, \zeta\chi_b(g)).$$

Proof. By Theorems 4, 7 and (3.9) the spectral measures are singular with respect to the Haar-Lebesgue measure on the two-torus $\mathbb{T}^2$. It remains to characterize the cases where the spectrum is discrete. In this case the representation U^ψ has an eigenfunction (and by Theorem 4 this property is characteristic for this case). Suppose now that there is a measurable function $f : \mathcal{K}(b) \to \mathbb{C}$ and a character $\eta = e(\Re(z\cdot))$ of $\mathbb{Z}[i]$ such that

$$U^\psi_g f = \eta(g) f, \quad \mu - a.e. \tag{3.13}$$

for all $g \in \mathbb{Z}[i]$. In particular $|f| \circ T_g = |f|$ and thus ergodicity allows to choose $|f| = 1$. From (3.13) we derive

$$\left\langle U^\psi_{b^k} f, f \right\rangle = \eta(b^k)\langle f, f\rangle = \eta(b^k). \tag{3.14}$$

Theorem 7 yields

$$\lim_{k\to\infty} \left\langle U^\psi_{b^k} f, f \right\rangle = I(\psi).$$

This and (3.14) imply

$$\lim_{k\to\infty} \eta(b^k) = I(\psi) \tag{3.15}$$

and consequently $|I(\psi)| = 1$. This together with (3.9) yields $\psi(1) = \psi(1)^{-(a+1)^2}$, which is equivalent to $\psi(1) \in R_a$.

For the following we set $\psi(1) = e(\frac{\ell}{a^2+2a+2})$. In order to prove the remaining assertions of (ii) we need the following lemma, which can be read off immediately from the automaton in Figure 1.

Lemma 7. *For two Gaussian integers g and h we have*

$$s_b(g+h) \equiv s_b(g) + s_b(h) \mod (a+1)^2+1 \tag{3.16}$$

As an immediate consequence of this lemma $\chi_b = \psi \circ s_b$ is a character of $\mathbb{Z}[i]$ and more precisely

$$\chi_b(z) = e\left(\Re\left(\frac{\ell(1-(a+1)i)}{a^2+2a+2}z\right)\right),$$

which can be easily seen by inserting $z = 1$ and $z = b$. This implies also that the cocycle Σ_ψ is trivial and given by

$$\Sigma_\psi(g,x) = \chi_b(g),$$

which finishes the proof of (ii) and (i). □

Theorem 9. *If ψ_1 and ψ_2 are different characters of $\mathbb{Z}$, then the two representations U^{ψ_1} and U^{ψ_2} have mutually singular spectral measures.*

Proof. This result is classical for $\mathbb{Z}$-actions. We give a short proof of this more general result for the reader's convenience. For $f_1, f_2 \in L^2(\mathcal{K}(b),\mu)$ denote ν_{f_1} and ν'_{f_2} the spectral measures associated to the two representations U^{ψ_1} and U^{ψ_2}, respectively. Assume now, that ν_{f_1} and ν'_{f_2} are not mutually singular. Then there exists a Borel probability measure τ on $\mathbb{T}^2$ absolutely continuous with respect to both ν_{f_1} and ν'_{f_2}, and let ρ and ρ' denote the respective densities. Then (3.11) implies that $\hat\tau(b^k) = \int_{\mathcal{K}(b)} \chi_{b^k}(x)\rho(x)\,\nu_{f_1}(dx)$ tends to $I(\psi_1)$. Similarly, we have $\hat\tau(b^k) = \int_{\mathcal{K}(b)} \chi_{b^k}(x)\rho'(x)\,\nu'_{f_2}(dx)$, which implies $\lim_k \hat\tau(b^k) = I(\psi_2)$ and therefore $I(\psi_1) = I(\psi_2)$. Now Lemma 6 implies that $\psi_1 = \psi_2$. □

We now consider the orthogonal decomposition (2.3) (with $A = G_\psi$) and let $\sigma^\chi = \nu_{f_\chi}$ (for $\chi \in \widehat{G_\psi}$ and $f_\chi \in L^2(\mathcal{K}(b),\mu)$) a probability measure of maximal spectral type for the representation $U^{\chi\circ\psi}$ determined by the restriction of T^ψ to $L^2(\mathcal{K}(b),\mu)\otimes\chi$. By Theorem 9 the measures σ^χ are mutually singular so that the spectral type of T^ψ is given by a probability measure $\nu_M = \sum_{\chi\in\widehat{G_\psi}} a_\chi \nu_{f_\chi}$ obtained by choosing $a_\chi > 0$ with $\sum_{\chi\in\widehat{G_\psi}} a_\chi = 1$.

The Weyl commutation relation (2.11) shows that for every $f \in L^2(\mathcal{K}(b),\mu)$ and every character γ of $\mathcal{K}(b)$ the spectral measure of $M_\gamma(f)$ is obtained by translating the one of f by the translation $x \mapsto x + u(\gamma)$ on $\mathbb{T}^2$. This shows that the spectral type of $U^{\chi\circ\psi}$ is invariant under the action of $\Gamma(b)$ (the dual of $\mathcal{K}(b)$) on $\mathbb{T}^2$ by translation. We show that this action is ergodic on $(\mathbb{T}^2,\sigma^\chi)$. Suppose that B is a Borel set invariant (σ^χ-a.e.) under the action of $\Gamma(b)$ and let H_B be the set of functions $f \in L^2(\mathcal{K}(b),\mu)$ of spectral measure ν_f (associated to the representation $U^{\chi\circ\psi}$) such that $\nu_f(B) = 0$. Clearly, H_B is a closed subspace of $L^2(\mathcal{K}(b),\mu)$, invariant under $U^{\chi\circ\psi}$. Moreover, the invariance property of B under $\Gamma(b)$ implies the invariance of H_B under M_γ, for any character γ of $\mathcal{K}(b)$; it is therefore either equal to $L^2(\mathcal{K}(b),\mu)$, which implies $\sigma^\chi(B) = 0$ or to $\{0\}$, which implies $\sigma^\chi(B) = 1$. Finally, we note that (3.11) implies that the spectral measures σ^χ for $\chi \neq 1$ are mutually singular to every probability measure τ on $\mathbb{T}^2$ with $\lim_k \hat\tau(b^k) = 0$. This property is preserved in the case $\chi = 1$.

We collect these facts in a theorem.

Theorem 10. *The spectral type of T^ψ is singular and mutually singular with respect to all measures τ on $\mathbb{T}^2$ such that $\lim_k \hat\tau(b^k) = 0$. For every measure σ^χ of the spectral type of $U^{\chi\circ\psi}$ the action of the dual $\Gamma(b)$ of $\mathcal{K}(b)$ on $\mathbb{T}^2$ is ergodic and non-singular.*

4. Distribution with Respect to the Argument

In this section we study uniform distribution of the sequence $(\arg z, \{\alpha s_b(z)\})$ in $]-\pi,\pi]\times[0,1[$. We will prove the following theorem.

Theorem 11. *Let α be an irrational number, then the sequence $(\arg z, \{\alpha s_b(z)\})$ is uniformly distributed, in the following sense:*

$$\lim_{N\to\infty}\frac{1}{\pi N}\#\{|z|^2<N \mid \arg z\in I,\quad \{\alpha s_b(z)\}\in J\}=\frac{1}{2\pi}|I|\cdot|J|$$

for all intervals $I\subset]-\pi,\pi]$ and $J\subset[0,1[$.

Proof. We first study the corresponding Weyl sum

$$S_N(h,k)=\sum_{|z|^2<N}e(h\alpha s_b(z))\chi_k(z), \tag{4.1}$$

where $\chi_k(z)$ denotes the Hecke character $\exp(ik\arg z)$. Asymptotic information on $S_N(h,k)$ is encoded in the analytic behaviour of the Dirichlet series

$$\zeta(s)=\sum_{z\in\mathbb{Z}[i]\setminus\{0\}}\frac{e(h\alpha s_b(z))\chi_k(z)}{|z|^{2s}}, \tag{4.2}$$

which is absolutely convergent for $\Re s>1$.

We split the range of summation according to the last digit and use $s_b(\ell+bz)=s_b(z)+\ell$ to obtain

$$\zeta(s)=\sum_{\ell=1}^{a^2}\frac{e(h\alpha\ell)}{\ell^{2s}}+\frac{\chi_k(b)}{|b|^{2s}}\sum_{\ell=0}^{a^2}e(h\alpha\ell)\sum_{z\in\mathbb{Z}[i]\setminus\{0\}}\frac{e(h\alpha s_b(z))\chi_k(z+\frac{\ell}{b})}{\left|z+\frac{\ell}{b}\right|^{2s}}. \tag{4.3}$$

We observe that for $s=\sigma+it$

$$\left|\chi_k\left(z+\tfrac{\ell}{b}\right)-\chi_k(z)\right| \le \min\left(2,\frac{2|k||b|}{|z|}\right) \tag{4.4}$$

$$\left|\frac{1}{\left|z+\frac{\ell}{b}\right|^{2s}}-\frac{1}{|z|^{2s}}\right| \le \frac{1}{|z|^{2\sigma}}\min\left(2,\frac{2|t||b|}{|z|}\right). \tag{4.5}$$

In order to find an analytic continuation and a growth estimate for $\zeta(s)$ we compare the last sum in (4.3) with $\zeta(s)$:

$$\begin{aligned}F_\ell(s) &= \sum_{z\in\mathbb{Z}[i]\setminus\{0\}}e(h\alpha s_b(z))\left(\frac{\chi_k\left(z+\frac{\ell}{b}\right)}{\left|z+\frac{\ell}{b}\right|^{2s}}-\frac{\chi_k(z)}{|z|^{2s}}\right)\\ &= \sum_{z\in\mathbb{Z}[i]\setminus\{0\}}e(h\alpha s_b(z))\left[\frac{\chi_k\left(z+\frac{\ell}{b}\right)-\chi_k(z)}{\left|z+\frac{\ell}{b}\right|^{2s}}+\chi_k(z)\left(\frac{1}{\left|z+\frac{\ell}{b}\right|^{2s}}-\frac{1}{|z|^{2s}}\right)\right].\end{aligned}$$

Inserting (4.4) into the last sum yields that $F_\ell(s)$ is holomorphic in $\Re s>\frac12$. For $\frac12<\sigma<1$ we have by splitting the range of summation at $|k|$ for the first summand and at $|t|$

for the second summand

$$|F_\ell(\sigma+it)| \le \sum_z \left(\frac{|b|}{|z|^{2\sigma}}\min\left(2,\frac{2|k||b|}{|z|}\right)+\frac{1}{|z|^{2\sigma}}\min\left(2,\frac{2|t||b|}{|z|}\right)\right) \tag{4.6}$$

$$= \mathcal{O}_{b,\sigma}\left(|k|^{1-\sigma}+|t|^{1-\sigma}\right). \tag{4.7}$$

Inserting $\zeta(s)+F_\ell(s)$ for the last summand in (4.3) we obtain

$$\zeta(s)=\left(1-\frac{\chi_k(b)}{|b|^{2s}}\sum_{\ell=0}^{a^2}e(h\alpha\ell)\right)^{-1}\left(\sum_{\ell=1}^{a^2}\frac{e(h\alpha\ell)}{\ell^{2s}}+\frac{\chi_k(b)}{|b|^{2s}}\sum_{\ell=1}^{a^2}e(h\alpha\ell)F_\ell(s)\right),$$

where the second factor is holomorphic in $\Re s>\frac{1}{2}$. Thus the only possible poles of $\zeta(s)$ are in the points where the denominator vanishes. These points lie on a vertical line.

In order to study the behaviour of $S_N(h,k)$ we introduce the sums

$$T_N=\sum_{|z|^2<N}e(h\alpha s_b(z))\chi_k(z)\left(1-\frac{|z|^2}{N}\right)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\zeta(s)\frac{ds}{s(s+1)},$$

where the last equality holds by the Mellin-Perron summation formula (cf. [36], Chapter II.2). Shifting the line of integration to $\Re s=\frac{3}{4}$ and taking the residues at the possible poles into account we obtain (notice that the remaining integral converges by the growth estimate (4.6)

$$T_N=\mathcal{O}_b\left(|k|^{\frac{1}{4}}N^{\frac{3}{4}}\right)\text{ if there were no poles}\mathcal{O}_b\left(|k|^{1-\beta}N^{\beta}\right)\text{ if there were poles,} \tag{4.8}$$

where $\beta=\log|\sum_{\ell=0}^{a^2}e(h\alpha\ell)|/\log(a^2+1)$. Notice that by a trivial estimate

$$\beta\le\frac{\log(a^2+1-2\pi\|h\alpha\|^2)}{\log(a^2+1)}\le 1-\frac{2\pi\|h\alpha\|^2}{(a^2+1)\log(a^2+1)} \tag{4.9}$$

where $\|x\|$ denotes the distance to the nearest integer. We define $\beta(h)=\max(\beta,\frac{3}{4})$.

Finally, we have to retrieve the behaviour of $S_N(h,k)$ from the behaviour of T_N. For this purpose we let $t=1+k^{\frac{1-\beta(h)}{2}}N^{-\frac{1-\beta(h)}{2}}$ for $k=o(N)$ and compute the difference of

$$\begin{aligned}T_N &= \sum_{|z|^2<N}\left(1-\frac{|z|^2}{N}\right)\chi_k(z)e(h\alpha s_b(z))=\mathcal{O}\left(k^{1-\beta(h)}N^{\beta(h)}\right)\\ tT_{tN} &= \sum_{|z|^2<tN}\left(t-\frac{|z|^2}{N}\right)\chi_k(z)e(h\alpha s_b(z))=\mathcal{O}\left(k^{1-\beta(h)}N^{\beta(h)}\right)\end{aligned}$$

to obtain

$$\begin{aligned}S_N(h,k) &= \sum_{|z|^2<N}\chi_k(z)e(h\alpha s_b(z)) = \\ \frac{tT_{tN}-T_N}{t-1} &- \frac{1}{t-1}\sum_{N\le|z|^2<tN}\left(t-\frac{|z|^2}{N}\right)\chi_k(z)e(h\alpha s_b(z))=\mathcal{O}\left(k^{\frac{1-\beta(h)}{2}}N^{\frac{\beta(h)+1}{2}}\right),\end{aligned} \tag{4.10}$$

where we have used that the summand in the second sum can be estimated by $(t-1)$ and the number of points in the area $N \leq |z|^2 < tN$ is $\mathcal{O}(N(t-1))$. From (4.10) we conclude that $\frac{S_N(h,k)}{N} \to 0$ for any fixed h and k, which by Weyl's criterion (cf. [29]) proves uniform distribution. □

In fact it is easy to derive an estimate for the discrepancy of the sequence from (4.10). The discrepancy is defined as the deviation from uniformity:

$$\begin{aligned} D_N &= D_N(\arg z, \{\alpha s_b(z)\}) = \\ &\sup_{I,J} \left| \frac{1}{\pi N} \# \left\{ |z|^2 < N \mid \arg z \in I, \quad \{\alpha s_b(z)\} \in J \right\} - \frac{1}{2\pi} |I| \cdot |J| \right|, \end{aligned}$$

where I and J run over all subintervals of $]-\pi, \pi]$ and $[0,1[$, respectively.

Corollary 6. *Let α be of approximation type η, i.e. $\|h\alpha\| h^{\eta+\varepsilon} \geq C(\alpha, \varepsilon)$ for every $\varepsilon > 0$. Then we have for the discrepancy of the sequence* $(\arg z, \{\alpha s_b(z)\})$

$$D_N \leq \frac{C(a, \alpha, \varepsilon)}{(\log N)^{\frac{1}{2\eta} - \varepsilon}}.$$

Proof. Since the proof runs along the same lines as the proof of Theorem 2 in [15], we only give a sketch here. Inserting (4.10) into the classical Erdős-Turán-Koksma inequality (cf. [7], [29]) yields

$$D_N \ll \frac{1}{H} + \sum_{h,k=1}^{H} \frac{1}{hk} k^{\frac{1-\beta(h)}{2}} N^{-\frac{1-\beta(h)}{2}} + \sum_{h=1}^{H} \frac{1}{h} N^{-\frac{1-\beta(h)}{2}} + \sum_{k=1}^{H} \frac{1}{k} \frac{k}{\sqrt{N}}$$

(the last estimate is the classical estimate for the sum of the Hecke character in a large circle, cf; [18, 19, 32]). We first perform the sum over k in the first sum and use that $\beta(h) \geq \frac{3}{4}$ and (4.9) to obtain

$$D_N \ll \frac{1}{H} + H^{\frac{1}{8}} \sum_{h=1}^{H} \frac{1}{h} \exp\left(-\frac{\pi \|h\alpha\|^2}{(a^2+1)\log(a^2+1)} \log N \right)$$

(the remaining sums are easily seen to be of smaller order of magnitude). Using the lower estimate for $\|h\alpha\|$ and summing up yields

$$D_N \ll \frac{1}{H} + H^{\frac{1}{8}} \log H \exp(-D(a, \alpha, \varepsilon) H^{-2\eta - 2\varepsilon} \log N),$$

which gives the desired estimate for $H = [(\log N)^{\frac{1}{2\eta+2\varepsilon} - \varepsilon}]$. □

Remark 7. *Corollary 6 could also be formulated as a quantitative version of the multidimensional ergodic theorem for the family of sets $\mathcal{Q}_I$ given by*

$$Q_N = \{z \in \mathbb{Z}[i] \mid |z|^2 < N, \quad \arg z \in I\}$$

(cf. (2.7)*).*

References

[1] W. Ambrose. Spectral resolution of groups of unitary operators. *Duke Math. J.*, 11:589–595, 1944.

[2] J.-P. Conze. Equirépartition et ergodicité de transformations cylindriques. *Séminaire de Probabilité, Université de Rennes*, 1976.

[3] J. Coquet. Power sums of digital sums. *J. Number Th.*, 22:161–176, 1986.

[4] J. Coquet and P. van den Bosch. A summation formula involving fibonacci digits. *J. Number Th.*, 22:139–146, 1986.

[5] H. Delange. Sur la fonction sommatoire de la fonction "somme des chiffres". *l'Enseignement Math.(2)*, 21:31–47, 1975.

[6] M. P. Drazin and J. S. Griffith. On the decimal representation of integers. *Proc. Cambridge Philos. Soc.*, 48:555–565, 1952.

[7] M. Drmota and R.F. Tichy. *Sequences, Discrepancies and Applications*. Number 1651 in Lecture Notes in Mathematics. Springer, 1997.

[8] S. Eilenberg. *Automata, Languages and Machines*, volume A. Academic Press, 1974.

[9] N. J. Fine. The distribution of the sum of digits (mod p). *Bull. Amer. Math. Soc.*, 71:651–652, 1965.

[10] P. Flajolet, P. J. Grabner, P. Kirschenhofer, H. Prodinger, and R. F. Tichy. Mellin transforms and asymptotics: digital sums. *Theor. Comput. Sci.*, 123:291–314, 1994.

[11] H. Furstenberg. Ergodic behaviour of the diagonal measures and the theorem of Szemeredi on arithmetic progressions. *J. Analyse Math.*, 34:275–291, 1978.

[12] A. O. Gelfond. Sur les nombres qui ont des propriétés additives et multiplicatives données. *Acta Arith.*, 13:259–266, 1968.

[13] P. J. Grabner, P. Liardet, and R. F. Tichy. Odometers and systems of numeration. *Acta Arith.*, 70:103–123, 1995.

[14] P.J. Grabner, P. Kirschenhofer, and H. Prodinger. The sum-of-digits function for complex bases. *J. London Math. Soc.*, 57:20–40, 1998.

[15] P.J. Grabner and R.F. Tichy. α-expansions, linear recurrences, and the sum-of-digits function. *manuscripta math.*, 70:311–324, 1991.

[16] H. Helson. Cocycles on the circle. *J. Operator Theory*, 16:189–199, 1986.

[17] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis*, volume 1. Springer, Berlin, 1963.

[18] H. Hida. *Elementary Theory of L-Functions and Eisenstein Series*. Cambridge University Press, 1993.

[19] E. Hlawka, J. Schoissengeier, and R. Taschner. *Geometric and Analytic Theory of Numbers*. Springer, 1991.

[20] T. Kamae. Mutual singularity of spectra of dynamical systems given by the "sum of digits" to different bases. *Soc. Math. France, Asterisque*, 49:109–114, 1977.

[21] I. Kátai and B. Kovács. Kanonische Zahlsysteme in der Theorie der quadratischen algebraischen Zahlen. *Acta Sci. Math. Szeged*, 42:99–107, 1980.

[22] I. Kátai and B. Kovács. Canonical number systems in imaginary quadratic fields. *Acta Math. Hung.*, 37:159–164, 1981.

[23] I. Kátai and J. Szabó. Canonical number systems for complex integers. *Acta Sci. Math. Hung.*, 37:255–260, 1975.

[24] A. B. Katok and A. M. Stepin. Approximations in ergodic theory. *Uspehi Mat. Nauk*, 22(5):81–106, 1967.

[25] D.E. Knuth. *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms*. Addison Wesley, London, 1981.

[26] B. Kovács. Canonical number systems in algebraic number fields. *Acta Math. Acad. Sci. Hung.*, 37:405–407, 1981.

[27] B. Kovács and A. Pethő. Number systems in integral domains, especially in orders of algebraic number fields. *Acta Sci. Math. Szeged*, 55:287–299, 1991.

[28] U. Krengel. *Ergodic Theorems*. W. de Gruyter, 1985.

[29] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences*. J. Wiley, 1974.
[30] P. Liardet. Répartition et ergodicité. Séminaire de Théorie des Nombres D.P.P., 19e année Paris, 1977/78. 12 p.
[31] P. Liardet. Propriétés harmoniques de la numération suivant Jean Coquet. In *Colloque de théorie analytique des nombres "Jean Coquet"*, volume 88-02 of *Publ. Math. Orsay*, pages 1–35. CIRM, 1985.
[32] W. Narkiewicz. *Elementary and Analytic Theory of Algebraic Numbers*. Springer, Berlin, 1990.
[33] W. Parry. Compact abelian group extensions of discrete dynamical systems. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, 13:95–113, 1969.
[34] M. Queffelec. Mesures spectrales associées à certaines suites arithmétiques. *Bull. Soc. Math. France*, 107:385–421, 1979.
[35] K. Schmidt. *Lectures on cocycles and ergodic transformation groups*. McMillan, Delhi, 1977.
[36] G. Tenenbaum. *Introduction to analytic and probabilistic number theory*. Number 46 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.
[37] J. M. Thuswaldner. The sum of digits function in number fields. *Bull. London Math. Soc.*, 30:37–45, 1998.
[38] J. R. Trollope. An explicit expression for binary digital sums. *Math. Mag.*, 41:21–27, 1968.
[39] R. J. Zimmer. Extension of ergodic group actions. *Ill. J. Math.*, 20:373–409, 1976.

(P. Grabner) Institut für Mathematik A, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
E-mail address: grabner@weyl.math.tu-graz.ac.at

(P. Liardet) Université de Provence, CMI, Château Gombert, 39, rue Joliot-Curie, 13453 Marseille, Cedex 13, France
E-mail address: liardet@gyptis.univ-mrs.fr