On the asymptotic behaviour of the zeros of the solutions of a functional-differential equation with rescaling

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Dedicated to Heinz Langer on the occasion of his 80^{th} birthday.

Abstract. We study the asymptotic behaviour of the solutions of a functional-differential equation with rescaling, the so-called pantograph equation. From this we derive asymptotic information about the zeros of these solutions.

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1. Introduction

1.1. Historical remarks

The importance of functional and functional-differential equations with rescaling is being increasingly recognised in the last decades, as is their relevance to a wide range of application areas. Such equations – functional and functional-differential, linear and nonlinear – appear as adequate tools in a number of phenomena that display some kind of self-similarity. It is next to impossible to describe all recent activities in this area. For general references and bibliography we refer the reader to survey papers by Derfel and co-authors [1, 6, 7].

One of the best known examples of equations with rescaling is the celebrated *pantograph equation*:

$$y'(z) = ay(\lambda z) + by(z). \tag{1.1}$$

This equation was introduced by Ockendon & Tayler [20] as a mathematical model of the overhead current collection system on an electric locomotive. (The term 'pantograph equation' was coined by Iserles [13].) This equation and its ramifications have emerged in a striking range of applications, including number theory [18], astrophysics [3], queues & risk theory [10], stochastic games [9], quantum theory [23], population dynamics [12], and graph theory [21]. The common feature of all such examples is some self-similarity of the system under study.

In 1972, Morris, Feldstein and Bowen [19] studied functional-differential equations (FDE) of the form:

$$y'(z) = \sum_{k=1}^{\ell} a_k y(\lambda_k z) \tag{1.2}$$

with $1 > \lambda_\ell > \lambda_{\ell-1} > \cdots > \lambda_1 > 0$, $a_1, \ldots, a_\ell \in \mathbb{C}$. They were able to obtain deep results about the existence, uniqueness, and asymptotic behaviour of solutions of (1.2) in the complex plane $\mathbb{C}$. (For more about FDE in the complex plane see also [8].)

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Among other things, Morris *et al.* [19] gave a detailed analysis of (1.1) in the special case $a = -1, b = 0$, i.e.,

$$y'(z) = -y(\lambda z) \tag{1.3}$$

In particular, they proved that (1.1), supplemented by the initial condition

$$y(0) = 1 \tag{1.4}$$

has the unique solution

$$y(z) = \sum_{n=0}^{\infty} (-1)^n \lambda^{\frac{n(n-1)}{2}} \frac{z^n}{n!}. \tag{1.5}$$

Moreover, $y(z)$ is an entire function of order zero, and it has infinitely many positive zeros, but no other zeros in the complex plane. The entire function (1.5) is sometimes called the *deformed exponential function* (see Sokal [22]).

A number of conjectures on the zeros $0 < t_0 < t_1 < t_2 \ldots$ of (1.5) have been made by Morris, Feldstein and Bowen [19] and by Iserles [13]. In particular, Morris, Feldstein and Bowen conjectured that

$$\lim_{n\to\infty} t_{n+1}/t_n = 1/\lambda := q \tag{1.6}$$

Also, in what follows we shall use the notation $q := 1/\lambda > 1$.

It is notable that in 1973, independently of [19], Robinson, in his paper on counting of acyclic digraphs [21], derived the same FDE as (1.3) and conjectured

$$t_n = (n+1)q^n + o(q^n), \tag{1.7}$$

Clearly, (1.7) is stronger than (1.6). In 2000, Langley [15] resolved Morris's *et al.* conjecture by proving that

$$t_n = nq^{n-1}(\gamma + o(1)), \tag{1.8}$$

where γ is a positive constant. In 2005, Grabner and Steinsky [11], independently of [15], proved a weaker form of Robinson's conjecture: namely, there exists k_0 such that

$$t_{k_0+k} = (k+1)q^k + o(q^k/k^{(1-\varepsilon)}), \tag{1.9}$$

for all $\varepsilon > 0$.

Recently, Zhang [24] proved that $\gamma = 1$ in (1.8) and, moreover,

$$t_n = nq^{n-1}(1 + \psi(\lambda)n^{-2} + o(n^{-2})), \tag{1.10}$$

where $\psi(q)$ is the generating function of the sum-of-divisors function $\sigma(k)$. Also, he derived an asymptotic formula for the oscillation amplitude A_n of $y(x)$, i.e., $A_n = |y(qt_n)|$.

1.2. Main results

All aforementioned results were concerned with the analytic function (1.5), which is the unique solution of the Cauchy problem (1.3)–(1.4). In contrast to this, in the present paper we deal with *all solutions* of (1.3), not necessarily also satisfying (1.4), but rather *defined on an arbitrary half-line.*

The following natural definition is commonly accepted in the theory of functional-differential equations:

Definition 1.1. If x_0 is a real number, then a real or complex function $y(x)$, defined and continuous for $x \geq \lambda x_0$, is said to be a solution of (1.3) for $x \geq x_0$, if it satisfies (1.3) for all $x \geq x_0$.

Thus, instead of the Cauchy problem for ODE, we have an *initial value* problem for FDE of *retarded type* (i.e., $0 < \lambda < 1$): for an arbitrary *initial function* φ defined on $[\lambda x_0, x_0]$, a solution of the initial value problem is a function $y(x)$ defined and continuous for $x \geq \lambda x_0$, which satisfies (1.3) and the initial condition:

$$y(x) = \varphi(x), \qquad x \in [\lambda x_0, x_0]. \tag{1.11}$$

Thus, the general solution of FDEs normally consists of an *infinite family of solutions*, depending on an *arbitrary function.*

Evidently, any function that is continuous in $[\lambda x_0, x_0]$ can be continued uniquely to a solution for $x \geq x_0$, and it makes sense to discuss the asymptotic behaviour of all solutions as $x \to \infty$.

The aim of this paper is to show that the above stated asymptotic behaviour of zeros is true not only for the analytic solution (1.5), but for *all* solutions of (1.3).

Namely, in Section 3, below, we prove that for every solution $y(x)$ of (1.3), the following asymptotic formula for zeros t_n is valid:

$$t_n = nq^{n-1}\left(\gamma + \mathcal{O}\left(\frac{\log n}{n}\right)\right) \tag{1.12}$$

It is worth noting that all solutions of (1.3), except for (1.5), are *non-analytic* ones at $z = 0$, but (1.12) remains true for all of them.

In summary, we can say that the asymptotic behaviour (1.12) is an *intrinsic property of the equation* (1.3) itself, and only the constant γ in (1.12) depends on the specific solution.

Our second objective (see Section 2) is an asymptotic analysis of the solutions of (1.3).

In Section 2.1 we derive an asymptotic formula of the de Bruijn, Kato and McLeod type by a method different from [2] and [14].

Furthermore, in Section 2.2 we give a brief summary of related results from [4] and [5] for *general FDE of higher order with several scaling factors*

$$y^{(m)}(x) = \sum_{j=0}^{\ell}\sum_{k=0}^{m-1} a_{jk}y^{(k)}(\alpha_j t + \beta_j), \tag{1.13}$$

where all α_j are less than 1 in modulus i.e. $|\alpha_j| < 1$.

Throughout the paper we denote the by $\log(z)$ the **natural** logarithm. as it is accepted in complex analysis. Also, for complex argument z by $\log(z)$ we mean the principal branch of the complex logarithm.

2. Asymptotic behaviour of analytic solutions

2.1. Asymptotic behaviour of the solution of FDE $y'(z) = -y(\lambda z)$

We first observe that the equation

$$y'(z) = ay(\lambda z) \tag{2.1}$$

can be simplified to

$$g'(z) = -g(\lambda z); \tag{2.2}$$

i.e. every solution of (2.1) can be written as $y(z) = g(-az)$ for a solution g of (2.2).

We start with an ansatz as a power series

$$g(z) = \sum_{n=0}^{\infty} g_n z^n,$$

which gives the recursion

$$(n+1)g_{n+1} = -\lambda^n g_n.$$

From this we obtain

$$g(z) = \sum_{n=0}^{\infty}(-1)^n \lambda^{\binom{n}{2}} \frac{z^n}{n!}, \tag{2.3}$$

if we assume $g_0 = 1$.

In order to study the asymptotic behaviour of $g(z)$ for large z, we transform (2.3) into an integral representation inspired by the inversion formula for the Mellin transform

$$g(z) = \frac{1}{2\pi i}\int_{\mathcal{H}} \Gamma(s)\lambda^{\binom{-s}{2}} z^{-s}\, ds = \frac{1}{2\pi i}\int_{\mathcal{H}} \frac{\pi}{\sin(\pi s)} \frac{\lambda^{\binom{-s}{2}} z^{-s}}{\Gamma(1-s)}\, ds, \tag{2.4}$$

$$g(-z) = \frac{1}{2\pi i}\int_{\mathcal{H}} \pi\cot(\pi s)\frac{\lambda^{\binom{-s}{2}} z^{-s}}{\Gamma(1-s)}\, ds, \tag{2.5}$$

where $\mathcal{H}$ is a contour encircling the negative real axis counterclockwise, like the Hankel-contour used in the theory of the Γ-function (see Figure 1). We will use the representation (2.4) for deriving an asymptotic formula for $g(z)$ for $|\arg(z)| \leq \pi - \varepsilon$ (for $\varepsilon > 0$), whereas (2.5) will be used for the asymptotic of $g(z)$ for $|\arg(-z)| \leq \pi - \varepsilon$. These representations can be proved by residue calculus and taking care of the growth order of the integrand along the contour. In the case $\lambda = \frac{1}{2}$ a similar representation was used in [11]. A similar integral representation is given in [2].

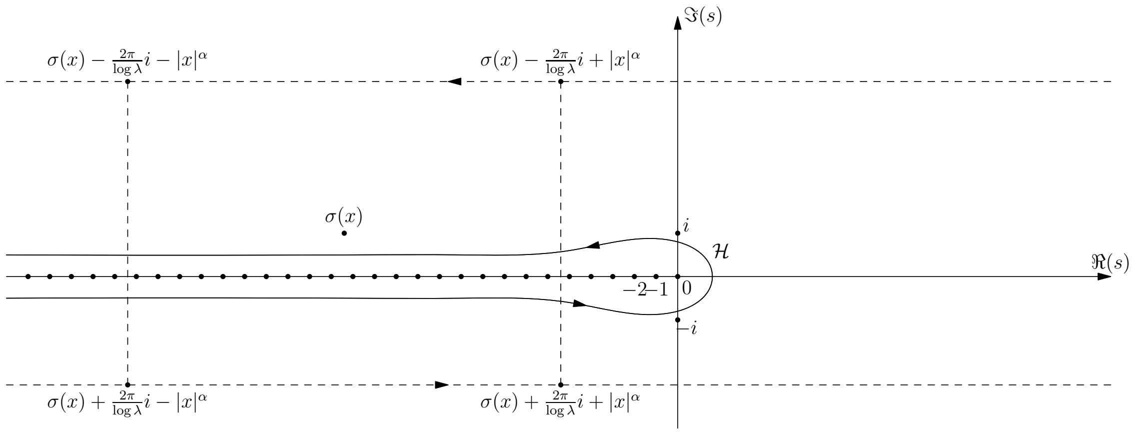

Figure 1. The contour of integration $\mathcal{H}$

Similarly, the representation

$$g(z) = \frac{1}{2\pi i}\left(\int_{-\infty-i}^{\infty-i} - \int_{-\infty+i}^{\infty+i}\right)\Gamma(s)\lambda^{\binom{-s}{2}}z^{-s}\,ds$$

can be shown. Here the contour of integration is deformed into two horizontal lines above and below the real axis.

Theorem 2.1. *Let g be the entire solution of* (2.2) *with $g(0) = 1$. Then for $\varepsilon > 0$ and $|\arg(z)| \leq \pi - \varepsilon$ and $z \to \infty$ the asymptotic expansion*

$$g(z) \sim Cz^A \log(z)^B \exp\left(-\frac{1}{2\log\lambda}\left(\log(z) - \log(\log(z))\right)^2\right) \times H\left(\frac{1}{\log\lambda}\left(\log(z) - \log(\log(z))\right) - \frac{1}{2} + \frac{\log(-\log\lambda)}{\log\lambda}\right) \tag{2.6}$$

holds, where the periodic function H of period 2 is given by the Fourier series

$$H(x) = \sqrt{\frac{2\pi}{-\log\lambda}}\sum_{k\in\mathbb{Z}} e^{\frac{(2k+1)^2\pi^2}{2\log\lambda}} e^{\pi i(2k+1)x}, \tag{2.7}$$

and

$$\begin{aligned} A &= \frac{1}{2} - \frac{1}{\log\lambda} - \frac{\log(-\log\lambda)}{\log\lambda} \\ B &= \frac{\log(-\log\lambda)}{\log\lambda} - 1 \\ C &= \exp\left(\frac{1}{2} - \frac{\log\lambda}{8} + \log(-\log\lambda) - \frac{\log(-\log\lambda)}{\log\lambda} - \frac{(\log(-\log\lambda))^2}{2\log\lambda} - \frac{1}{2}\log(2\pi)\right). \end{aligned} \tag{2.8}$$

Remark 2.1. This theorem extends the real asymptotic of **all** solutions on $\mathbb{R}^+$ given in [14] to the complex asymptotic of the **entire** solution in an angular region avoiding the negative real axis.

Theorem 2.2. *Let g be the entire solution of* (2.2) *with $g(0) = 1$. Then for $\varepsilon > 0$ and $|\arg(-z)| \leq \pi - \varepsilon$ and $z \to \infty$ the asymptotic expansion*

$$g(-z) \sim Cz^A \log(-z)^B \exp\left(-\frac{1}{2\log\lambda}\left(\log(-z) - \log(\log(-z))\right)^2\right) \times K\left(\frac{1}{\log\lambda}\left(\log(-z) - \log(\log(-z))\right) - \frac{1}{2} + \frac{\log(-\log\lambda)}{\log\lambda}\right) \quad (2.9)$$

holds, where the periodic function K of period 1 is given by the Fourier series

$$K(x) = \sqrt{\frac{2\pi}{-\log\lambda}} \sum_{k\in\mathbb{Z}} e^{\frac{2k^2\pi^2}{\log\lambda}} e^{2\pi ikx}, \quad (2.10)$$

and A, B, and C are given by (2.8).

Proof of Theorem 2.1. We apply a saddle point approximation combined with residue calculus to the second integral representation given in (2.4). For this purpose, we consider

$$\frac{\lambda^{\binom{-s}{2}} z^{-s}}{\Gamma(1-s)} = \exp\left(\frac{s(s+1)}{2}\log\lambda - s\log(z) - \log(-s) - \log(\Gamma(-s))\right). \quad (2.11)$$

We find the saddle point as the stationary point of the argument of the exponential:

$$\left(s + \frac{1}{2}\right)\log\lambda - x - \frac{1}{s} + \psi(-s) = 0, \quad (2.12)$$

where $\psi = \frac{\Gamma'}{\Gamma}$; for simplicity, we set $x = \log(z)$. From [17] we infer

$$\psi(-s) = \log(-s) + \frac{1}{2s} + \mathcal{O}\left(\frac{1}{s^2}\right),$$

valid for $|\arg(-s)| \leq \pi - \varepsilon$ for $\varepsilon > 0$. This shows that (2.12) has a unique solution $\sigma(x)$ satisfying

$$\sigma(x) = \frac{1}{\log\lambda}\left(x - \log(x)\right) - \frac{1}{2} + \frac{\log(-\log\lambda)}{\log\lambda} + \mathcal{O}(\log(x)/x), \quad (2.13)$$

which lies close to the negative real axis; notice that the imaginary part of $\sigma(x)$ is bounded by $\frac{\pi}{|\log\lambda|}$. Around $s = \sigma(x)$ we have the following approximation

$$\begin{aligned} &\frac{s(s+1)}{2}\log\lambda - sx - \log(s) - \log(\Gamma(-s)) \\ &\qquad = \frac{\sigma(x)(\sigma(x)+1)}{2}\log\lambda - x\sigma(x) - \log(-\sigma(x)) - \log(\Gamma(-\sigma(x))) \\ &\qquad\quad + \frac{1}{2}\left(\log\lambda + \frac{1}{\sigma(x)^2} - \psi'(-\sigma(x))\right)(s-\sigma(x))^2 + \mathcal{O}\left(x^{-2}(s-\sigma(x))^3\right) \end{aligned} \quad (2.14)$$

for $s - \sigma(x) = \mathcal{O}(x^\alpha)$ for $\alpha < \frac{2}{3}$.

Inserting the approximation (2.14) into the integral representation (2.4) (and splitting the range of integration into $|s - \sigma(x)| \leq x^\alpha$ and $|s - \sigma(x)| \geq x^\alpha$) yields

$$\begin{aligned} g(z) = \frac{\lambda^{\binom{-\sigma(x)}{2}}}{\Gamma(1-\sigma(x))} e^{-x\sigma(x)} \frac{1}{2\pi i}\oint_{R_x} \exp\left(\frac{1}{2}(s-\sigma(x))^2\log\lambda\right)\frac{\pi}{\sin(\pi s)}\,ds\left(1 + \mathcal{O}(x^{2\alpha-1})\right) \\ + \frac{1}{2\pi i}\int_{R'_x} \Gamma(s)\lambda^{\binom{-s}{2}} z^{-s}\,ds, \end{aligned} \quad (2.15)$$

if we choose $\alpha < \frac{1}{2}$. Here R_x denotes the positively oriented rectangle with corners $\sigma(x) \pm |x|^\alpha \pm \frac{2\pi i}{\log\lambda}$ and R'_x denotes the remaining part of the dashed contour in Figure 1.

The first integral in (2.15) can be evaluated by residue calculus

$$\frac{1}{2\pi i}\oint_{R_x}\exp\left(\frac{1}{2}(s-\sigma(x))^2\log\lambda\right)\frac{\pi}{\sin(\pi s)}\,ds$$
$$=\sum_{\substack{n\in\mathbb{Z}\\ |n-\sigma(x)|<|x|^\alpha}}(-1)^n e^{\frac{1}{2}(n-\sigma(x))^2\log\lambda}=\sum_{n\in\mathbb{Z}}(-1)^n e^{\frac{1}{2}(n-\sigma(x))^2\log\lambda}+\mathcal{O}\left(e^{\frac{1}{2}|x|^{2\alpha}\log\lambda}\right). \quad (2.16)$$

This series represents a continuous periodic function of period 2, whose Fourier coefficients can be computed by a variant of Poisson's summation formula

$$H(y)=\sum_{n\in\mathbb{Z}}(-1)^n e^{\frac{1}{2}(n-y)^2\log\lambda}=\frac{\sqrt{2\pi}}{\sqrt{-\log\lambda}}\sum_{k\in\mathbb{Z}}e^{\frac{(2k+1)^2\pi^2}{2\log\lambda}}e^{i\pi(2k+1)y}. \quad (2.17)$$

For estimating the remaining integral over R'_x in (2.15), we use the estimates (cf. [17])

$$|\Gamma(t\pm iC)|\le\sqrt{2\pi}(t^2+C^2)^{\frac{1}{2}(t-\frac{1}{2})}e^{-\frac{\pi C}{2}+\frac{1}{6C}}$$
$$|\Gamma(-t\pm iC)|\le\frac{\pi}{|C|\cosh(\pi C)^{\frac{1}{2}}\Gamma(t)}$$

valid for $t\ge 1$ and $C>0$.

Inserting the asymptotic information about $\sigma(x)$ from (2.13) into (2.11) and using Stirling's formula for the Γ-function yields

$$\frac{\lambda^{\binom{-\sigma(x)}{2}}e^{-x\sigma(x)}}{\Gamma(1-\sigma(x))}=Ce^{Ax}x^B\exp\left(-\frac{1}{2\log\lambda}(x-\log(x))^2\right),$$

with A, B, and C as given in (2.8). □

Proof of Theorem 2.2. For the proof of (2.9) we use the integral representation (2.5) and argue along the same lines as in the proof of Theorem 2.1. The only technical difference is that the residues of $\pi\cot(\pi s)$ are all equal to 1, which avoids the sign change occurring in (2.16). □

2.2. FDE of higher order with compressed arguments

In this section we consider the general FDE with rescaling

$$y^{(m)}(x)=\sum_{j=0}^{\ell}\sum_{k=0}^{m-1}a_{jk}y^{(k)}(\alpha_j t+\beta_j), \quad (2.18)$$

where $a_{jk}\in\mathbb{C}$ and $\alpha_j,\beta_j\in\mathbb{R}$. From here on we consider, solutions of (2.18) defined on the whole real line $\mathbb{R}$, when there exists at least one $\beta_j\ne 0$, and possibly defined on a half-line $\mathbb{R}_+$, or $\mathbb{R}_-$, when all $\beta_j=0$.

For such equations, it is hard to expect the existence of an asymptotic formula similar to (2.6). However, we can derive sharp estimates from above and below for solutions of (2.18).

Below, we give a brief summary of the related results from [4] and [5].

Denote:

$$\alpha=\min_{0\le j\le\ell}|\alpha_j|,\qquad A=\max_{0\le j\le\ell}|\alpha_j|. \quad (2.19)$$

and assume that $A<1$. Then:

(i) Every solution $y(x)$ of (2.18) is an analytic function, that can be extended as an entire function $y(z)$ of order zero in $\mathbb{C}$.

(ii) Every solution of (2.18) satisfies the estimate

$$|y(z)|\le C\exp\{\gamma\log^2(1+|z|)\},\qquad z\in\mathbb{C}, \quad (2.20)$$

for some $C>0$, and

$$\gamma>m/(2|\log A|) \quad (2.21)$$

(iii) Every solution of (2.18) is unbounded *on any ray* emanating from the origin $z = 0$ (as it is an entire function of order zero). In particular, every solution is unbounded on $\mathbb{R}_+$ and $\mathbb{R}_-$.

This result cannot be strengthened in general due to the existence of polynomial solutions. It was proved in [4] and [5] that:

(iv) A necessary and sufficient condition for the existence of polynomial solutions of (2.18) is that:

$$\sum_{j=0}^{\ell} a_{j0}\alpha_j^n = 0 \tag{2.22}$$

for some $n \in \mathbb{N}$.

Under the assumption that (2.18) has no polynomial solutions, and $\beta_j = 0$ for all j, one can prove a result stronger than (iii). Roughly speaking, every nontrivial solution of (2.18) grows as $|z| \to \infty$ faster than $\exp\{\gamma \log^2 |z|\}$ for some $\gamma > 0$. More precisely:

(v) Every solution of (2.18), which *at least on one ray* emanating from the origin satisfies (2.20) for some $C > 0$, and

$$\gamma < 1/(2|\log \alpha|) \tag{2.23}$$

vanishes identically.

Remark 2.2. The above stated results, formulated for the equation (2.18), with *constant coefficients*, remain true with minor changes also for FDE of higher order, with several compressed arguments and *polynomial coefficients*:

$$y^{(m)}(x) = \sum_{k=0}^{m-1}\sum_{j=0}^{\ell}\sum_{\nu=0}^{r} a_{jk\nu}x^{\nu}y^{(k)}(\alpha_j t + \beta_j), \tag{2.24}$$

3. Asymptotic behaviour of the zeros

The central result of this section is the following theorem.

Theorem 3.1. *Let $y(x)$ be a solution of (1.3), and $0 < x_0 < x_1 < x_2 < \dots$ be the zeros of $y(x)$. Then there exists a positive constant γ, such that:*

$$x_n = nq^{n-1}\left(\gamma + \mathcal{O}\left(\frac{\log n}{n}\right)\right). \tag{3.1}$$

The proof is based on a result of Kato & McLeod [14] and de Bruijn [2] on the asymptotic behaviour of *all solutions* of the equation

$$y'(z) = ay(\lambda z) \tag{3.2}$$

and Lemma 3.1 below.

According to Kato and McLeod [14, Theorem 7(iii)] and de Bruijn [2, Sections 1.3-1.4] every solution of (3.2) has the following asymptotic behaviour

$$\begin{aligned} y(x) = x^{A_1}\log(x)^{B_1}\exp\left(-\frac{1}{2\log\lambda}\left(\log(x) - \log(\log(x))\right)^2\right) \\ \times\left(h\left(\frac{1}{\log\lambda}\left(\log(x) - \log(\log(x))\right)\right) + o(1)\right), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} A_1 &= \frac{1}{2} - \frac{1}{\log\lambda} - \frac{\log(-a\log\lambda)}{\log\lambda}, \\ B_1 &= \frac{\log(-a\log\lambda)}{\log\lambda} - 1, \end{aligned} \tag{3.4}$$

and $h(x)$ is a periodic function of period $\log q = |\log\lambda|$, with some additional assumptions on its Fourier coefficients (see [14, (6.2)–(6.3)]).

Remark 3.1. Notice that this asymptotic behaviour is in accordance with the behaviour stated in Theorems 2.6 and 2.2. If the value $-a\log\lambda$ is negative, the complex values of the power $(x/\log(x))^{-\log(-a\log\lambda)/\log\lambda}$ are compensated by complex values of the periodic function $h(x)$ and a doubling of the period to accommodate the sign change. This is the explanation of the fact that the periodic function in Theorem 2.6 has period 2.

Lemma 3.1. *Let G be a periodic function of period 2, x_0 the minimal positive zero of G, and all zeros of G located at the points: $x_0 + k;\quad k = 0, 1, 2, \ldots$. Then the zeros of the function $F(x) := G\left(\frac{1}{\log\lambda}(\log(x) - \log(\log(x))\right)$ have the following asymptotic behaviour*

$$x_n = nq^{n-1}\left(\gamma + \mathcal{O}\left(\frac{\log n}{n}\right)\right) \tag{3.5}$$

Proof of Lemma 3.1. Let us observe first that the zeros of $F(x)$ are located at the points x, such that

$$\log x - \log x \log x = (x_0 + k)\log q \tag{3.6}$$

We shall seek solutions of (3.6) of the form

$$x_k = x(k) = C(k)kq^k, \tag{3.7}$$

where $C(k)$ is an unknown function of no more than power growth, i.e., there exists $\alpha > 0$ such that $C(k) = \mathcal{O}(k^\alpha)$. To prove Lemma 3.1 it is enough to show that

$$C(k) = \gamma + \mathcal{O}\left(\frac{\log k}{k}\right), \tag{3.8}$$

where γ is positive constant. For that, substitute (3.7) in (3.6). It follows from (3.7) that

$$\log x = k\log q\left(1 + \frac{\log k}{k\log q} + \frac{\log C(k)}{k\log q}\right), \tag{3.9}$$

and

$$\log\log x = \log k + \frac{\log k}{k\log q} + \frac{\log C(k)}{k\log q} + \log\log q + o\left(\frac{\log k}{k\log q}\right), \tag{3.10}$$

Combining (3.6), (3.9) and (3.10), after some elementary calculations we obtain that

$$\left(1 - \frac{1}{k\log q}\right)\log C(k) = x_0\log q + \log\log q + \frac{\log k}{k\log q} + o\left(\frac{\log k}{k\log q}\right) \tag{3.11}$$

Next, let $k \to \infty$, then from (3.11) we obtain

$$\log C(k) = (x_0\log q + \log\log q) + \mathcal{O}\left(\frac{\log k}{k}\right), \tag{3.12}$$

or

$$C(k) = q^{x_0}\log q\left(1 + \mathcal{O}(\frac{\log k}{k})\right). \tag{3.13}$$

Finally, (3.13) implies (3.8) and (3.5). □

Proof of Theorem 3.1. First, we apply the asymptotic formula (3.3) to the solutions of (1.3). Observe that in this case $a = -1$ and $\log\lambda = -\log q < 0$, and therefore the expression $\log(-a\log\lambda) = \log(\log\lambda)$ in (3.4) is a complex number.

Having this in mind, we can rewrite (3.3) and (3.4) in the form:

$$\begin{aligned} y(x) = x^{A_2}\log(x)^{B_2}\exp&\left(\frac{1}{2\log\ q}(\log(x) - \log(\log(x)))^2\right) \\ \times&\left(\cos\frac{\pi}{\log q}(\log(x) - \log(\log(x)) + i(\sin\frac{\pi}{\log q}(\log(x) - \log(\log(x))\right) \\ &\times\left(h\left(\frac{1}{\log q}(\log(x) - \log(\log(x)))\right) + o(1)\right), \end{aligned} \tag{3.14}$$

where

$$A_2 = \frac{1}{2} + \frac{1}{\log q} + \frac{\log(\log\ q)}{\log q}$$
$$B_2 = -1 - \frac{\log(\log q)}{\log q}, \tag{3.15}$$

Both the real part and the imaginary parts of (3.14) provide asymptotic formulas for solutions of (1.3), and the assumptions of Lemma 3.1 are fulfilled.

Then, in accordance with Lemma 3.1, there exists k_0 such that for all $k = 0, 1, 2, \ldots$

$$x_{k_0+k} = kq^{k-1}\left(\gamma_1 + \mathcal{O}(\frac{\log k}{k})\right), \tag{3.16}$$

Now (3.8) follows from (3.16), and from (3.8) finally we obtain (3.1) . □

4. The case $b \neq 0$

The behaviour of the entire solution of (1.1) changes considerably, if $b \neq 0$. This is quite obvious from the observation that the equation can be viewed as a perturbed differential equation $y' = by$. Thus we expect an asymptotic behaviour of the form $f(x) \sim Ce^{bx}$ for $\Re(bx) \to +\infty$. We again start with an ansatz as a power series

$$f(z) = \sum_{n=0}^{\infty} f_n z^n,$$

from which we derive the recurrence formula

$$(n+1)f_{n+1} = (a\lambda^n + b)f_n \text{ for } n \geq 0.$$

This gives

$$f_n = \frac{b^n}{n!}\prod_{k=0}^{n-1}\left(1 + \frac{a}{b}\lambda^k\right), \tag{4.1}$$

if we set $f_0 = 1$. In order to simplify notation, we introduce

$$Q_\lambda(\alpha) = \prod_{k=0}^{\infty}\left(1 + \alpha\lambda^k\right).$$

This gives

$$f(z) = Q_\lambda\left(\frac{a}{b}\right)\sum_{n=0}^{\infty}\frac{(bz)^n}{n!Q_\lambda\left(\frac{a}{b}\lambda^n\right)}, \tag{4.2}$$

if $\frac{a}{b}\lambda^n \neq -1$ for all $n \in \mathbb{N}$. In the case that $\frac{a}{b}\lambda^N = -1$ for some $N \in \mathbb{N}$, the solution degenerates to a polynomial

$$f(z) = Q_\lambda\left(\frac{a}{b}\right)\sum_{n=0}^{N-1}\frac{(bz)^n}{n!Q_\lambda\left(\frac{a}{b}\lambda^n\right)},$$

In order to derive an expression for $f(z)$, which allows for determining its asymptotic behaviour, we recall the well known power series expansion

$$\frac{1}{Q_\lambda(\alpha)} = \sum_{n=0}^{\infty}(-1)^n\prod_{k=1}^{n}\frac{1}{1-\lambda^k}\alpha^n \tag{4.3}$$

valid for $|\alpha| < 1$.

We now choose N as the smallest non-negative integer such that $|\frac{a}{b}\lambda^N| < 1$. Then we rewrite (4.2) as

$$f(z) = Q_\lambda\left(\frac{a}{b}\right)\sum_{n=0}^{N-1}\frac{(bz)^n}{n!Q_\lambda\left(\frac{a}{b}\lambda^n\right)} + Q_\lambda\left(\frac{a}{b}\right)\sum_{n=N}^{\infty}\frac{(bz)^n}{n!Q_\lambda\left(\frac{a}{b}\lambda^n\right)}. \tag{4.4}$$

We replace $\frac{1}{Q_\lambda}$ in the second sum by (4.3) to obtain

$$f(z)=Q_\lambda\left(\frac{a}{b}\right)\sum_{n=0}^{N-1}\frac{(bz)^n}{n!Q_\lambda\left(\frac{a}{b}\lambda^n\right)}+Q_\lambda\left(\frac{a}{b}\right)\sum_{m=0}^{\infty}(-1)^m\prod_{k=1}^{m}\frac{1}{1-\lambda^k}\left(\frac{a}{b}\right)^m\sum_{n=N}^{\infty}\frac{(b\lambda^m z)^n}{n!},\tag{4.5}$$

which simplifies to

$$f(z)=Q_\lambda\left(\frac{a}{b}\right)\sum_{n=0}^{N-1}\frac{(bz)^n}{n!Q_\lambda\left(\frac{a}{b}\lambda^n\right)}\\+Q_\lambda\left(\frac{a}{b}\right)\sum_{m=0}^{\infty}(-1)^m\prod_{k=1}^{m}\frac{1}{1-\lambda^k}\left(\frac{a}{b}\right)^m\left(e^{b\lambda^m z}-\sum_{n=0}^{N-1}\frac{(b\lambda^m z)^n}{n!}\right).\tag{4.6}$$

Remark 4.1. Notice that the expansion (4.6) converges for all values of a and b (after a suitable choice of N). This is in contrast to a similar series expansion given in [16], which converges only for $|a|<|b|$.

Remark 4.2. In [16] a generalisation of the classical pantograph equation, the multi-pantograph equation

$$f'(z)=\sum_{k=1}^{\ell}a_k f(\lambda_k z)+bf(z)\tag{4.7}$$

with $1>\lambda_\ell>\lambda_{\ell-1}>\cdots>\lambda_1>0$, $a_1,\dots,a_\ell,b\in\mathbb{R}$ is studied. In [16, Lemma 3.1] a series expansion for the
solution of (4.7) is given, which is shown to converge for

$$\sum_{k=1}^{\ell}|a_k|<|b|.$$

The truncation method presented above can easily be adapted to provide a series representation similar to (4.6) for f, which does not require this condition.

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