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THE DYNAMICAL POINT OF VIEW OF LOW-DISCREPANCY SEQUENCES

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ABSTRACT. In this overview we show by examples, how to associate certain sequences in the higher-dimensional unit cube to suitable dynamical systems. We present methods and notions from ergodic theory that serve as tools for the study of low-discrepancy sequences and discuss an important technique, cutting-and-stacking of intervals.

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1. Introduction

Let X be some nonempty set, let $\omega = (x_n)_{n\geq 0}$ be a sequence in X, and let f be a real or complex valued function on the domain X. In several areas of mathematics, one is interested in the difference between the mean of the values of f taken at the first N points of ω,

$$S_N(f, \omega) = \frac{1}{N} \sum_{n=0}^{N-1} f(x_n),$$

and an integral $I(f) = \int_X f$ of f in some sense (Here, we assume that $I(f)$ is well-defined). For example, in the theory of dynamical systems, ω is the orbit $(T^n x)_{n\geq 0}$ of a point $x \in X$ under some measure-preserving transformation T, and $I(f)$ is the integral of f with respect to the underlying T-invariant measure (see, for example, [76]). In numerical mathematics, in the Monte Carlo method, ω is a sequence of random points in X, and in quasi-Monte Carlo methods, so-called *low discrepancy sequences* ω (see [51]) are employed. In both fields, one is interested in the behaviour of the difference $S_N(f, \omega) - I(f)$:

(i) Under which conditions on ω and f does the limit relation

$$\lim_{N\to\infty} S_N(f, \omega) = I(f)$$

hold?

(ii) For a finite sequence $\omega = (x_n)_{n=0}^{N-1}$, what can be said about the *integration error (quadrature error)* $\mid S_N(f, \omega) - I(f) \mid$?

In the language of stochastics, this error corresponds to the difference between the *sample mean* $S_N(f, \omega)$ and the *expectation* $I(f)$ of the random variable f. It is clear that, in order to provide answers, some measure-theoretic or topological structure on the domain X is required and we restrict our attention to functions f that satisfy some regularity conditions, in addition to integrability. Interestingly enough, the theory of uniform distribution of sequences modulo one, a subfield of metric number theory, provides the most detailed answers to both questions. We refer the reader to the monographs [11, 13, 43] for background information.

In this survey paper, we will deal with a relation between the theory of uniform distribution of sequences and dynamical systems, for particular low-discrepancy sequences.

In Section 2, we will introduce the language and basic notions of ergodic theory. While introducing the theory we will present examples from the theory of uniform distribution in the language of dynamical systems. These will serve as illustrations and motivation. As a motivating problem, bounded remainder sets are investigated for certain constructions of low-discrepancy point sequences. The problem of finding such sets can be directly translated into the language of dynamical systems and can be solved by applying coboundary theorems.

Many constructions of low-discrepancy sequences are based on digital expansions. Possibly, the most prominent example is the classical van der Corput sequence defined by reflecting the digital expansion of an integer at the decimal point to give a real number in $[0, 1)$ (a precise definition will be given later). Several new developments of low-discrepancy sequences are based on this and other classical constructions:

(1) the Halton sequence [29] is produced by taking van der Corput sequences with respect to s pairwise coprime bases to form the components of an s-dimensional vector. In terms of dynamical systems this construction is a product of dynamical systems, which satisfy an independence property in terms of their spectra (cf. Section 2.7).

(2) Several constructions of one- and higher-dimensional sequences are based on “manipulating” the digits before using the van der Corput idea to reflect at the decimal point: one idea studied by Faure and Sobol [16, 73] is to apply matrices to the sequence of digits (viewed as a vector), another idea is to apply a permutation to the digits (cf. [17]). These constructions have proved fruitful for getting better estimates for the discrepancy of such sequences.

(3) The most prominent example of a uniformly distributed sequence is the Kronecker sequence $(n\alpha \mod 1)_{n\geq 0}$ for irrational α. The study of this

sequence in Weyl [77] marks the beginning of the theory of uniform distribution of sequences.

(4) Combinations of different types of sequences like combining the sequences in (3) with the constructions in (1) and (2) are called "hybrid sequences" (cf. [33]). From the dynamical point of view this is again a product of dynamical systems (cf. Section 2.7).

In order to model the different constructions of low-discrepancy sequences by dynamical systems, we need various notions and ideas.

(i) The simplest and most transparent example of the van der Corput sequence provides us with two basic constructions from ergodic theory:
   (a) take the compact abelian group of b-adic integers $\mathbb{Z}_b$ (cf. [37] and Section 2.4) with the addition-by-one map τ (called odometer). Then $(\mathbb{Z}_b, \tau)$ is a topological dynamical system with a unique invariant probability measure, the normalised Haar measure μ on the group $\mathbb{Z}_b$. The map (called "b-adic Monna-map" later)
$$\varphi_b:\ \mathbb{Z}_b \to [0,1), \quad \varphi_b(z) = \sum_{k=0}^{\infty} z_k b^{-k-1} \pmod 1,$$
   where $\sum_{k=0}^{\infty} z_k b^k$ is the Hensel expansion of z with $z_k \in \{0, 1, \dots, b-1\}$ for all indices k, is continuous and transports μ to the Lebesgue measure on $[0,1)$. Furthermore, $(\varphi_b(\tau^n 0))_{n\geq 0}$ is the van der Corput sequence;
   (b) a different approach, which will provide us with a wealth of new constructions in Section 3, is to recognise the van der Corput sequence as the orbit of the real number 0 under application of an interval exchange known as the Kakutani-von Neumann map (see Sections 2.10 and 3.3).

(ii) A natural and attractive generalisation of Kronecker sequences consists in the family of nonperiodic polynomial sequences $(p(n) \bmod 1)_{n\geq 0}$ previously studied by Weyl [77]. In the past, an efficient tool for studying the distribution of these sequences and also many other sequences $\omega = (x_n)_{n\geq 0}$ was to consider the sequences of differences $\Delta_h(\omega)_n = x_{n+h} - x_n \pmod 1$ and the correlation coefficients
$$\gamma_{\omega,k}(h) = \lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} e^{2i\pi k(\Delta_h(\omega)_n)},$$
assuming these limits exist. By the Bochner-Herglotz theorem, a correlation map is the Fourier transform of a positive Borel measure on $[0,1)$. This

leads to a spectral analysis of bounded complex valued sequences which can be interpreted in a dynamical context as it is explained in Section 2.6 with useful incidence in uniform distribution theory. For our example, the first dynamical description of $(p(n) \bmod 1)_{n\geq 0}$ is due to Furstenberg in a general framework [22] and explicitly exhibited in [23] by introducing the uniquely ergodic transformation T_α of the s-dimensional torus $[0,1)^s$ defined by

$$T_\alpha(x_1,\dots,x_s) = (x_1+\alpha, x_2+x_1, x_3+x_2,\dots,x_s+x_{s-1}),$$

with α irrational; see also [59] for a conjugate transformation. The Furstenberg example is built from the translation $x \mapsto x+\alpha \,(\mathrm{mod}\ 1)$ by successive group extensions involving the fundamental notion of a skew product that will be introduced and studied in Section 2.8.

(iii) Some of the matrix constructions such as the Sobol-Faure sequences described above can also be modeled as in (i)a. If the matrices have infinite rows, then the corresponding map φ_b cannot be defined on $\mathbb{Z}_b$. The concept of skew products will play a fundamental role to describe these sequences by dynamical systems and to study their spectral properties. An important combinatorial construction of low discrepancy sequences in dimension s was introduced by Sobol' [73] in binary numeration systems and widely generalized by Niederreiter [50] under the name of (t,m,s)-sets and (t,s)-sequences (see [51] for an extensive study). We show how these sequences are connected with skew products and derive additional properties by employing results from ergodic theory.

As usual in the context of exponential and character sums, we will use the notation

$$e(x) = e^{2\pi i x}, \quad x\in\mathbb{R},$$

for the complex exponential function throughout this paper. In particular, $e(\cdot)$ denotes the generator of the character group of the torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. Frequently, the torus $\mathbb{T}$ will be identified with the half-open interval $[0,1)$.

2. The dynamical point of view

In this section we will show, how certain constructions of low-discrepancy sequences can be seen from the point of view of dynamical systems. This point of view will give a conceptual and unified approach to prove uniform distribution of such sequences. It also allows to answer questions of irregularities of distribution of certain sequences that would be difficult to attack by elementary methods.

2.1. Irregularities of distribution

Let $\omega = (\mathbf{x}_n)_{n\geq 0}$ be a sequence on the torus $[0,1)^s$. For a subset A of $[0,1)^s$, let $\mathbb{1}_A$ denote the indicator function of A, $\mathbb{1}_A(\mathbf{x}) = 1$ if $\mathbf{x} \in A$ and $\mathbb{1}_A(\mathbf{x}) = 0$ otherwise. If A is Lebesgue measurable, then let $\lambda_s(A)$ stand for its Lebesgue (Haar) measure. In the one-dimensional case, we will simply write λ, instead of λ_1. Put $f = \mathbb{1}_A - \lambda_s(A)$. The quantity

$$R_N(A,\omega) = \sum_{n=0}^{N-1} f(\mathbf{x}_n)$$

is called *the local discrepancy* of the first N points of ω for the set A or, in short, *the remainder* of A.

Definition 1. Let $\mathcal{J}$ denote the family of subintervals (or boxes) J of $[0,1)^s$, $J = \prod_{i=1}^{s}[\alpha_i, \beta_i[$, $0 \leq \alpha_i < \beta_i \leq 1$, $1 \leq i \leq s$, and let $\mathcal{J}^*$ be the subfamily of subintervals J anchored at the origin, i.e., $\alpha_i = 0$, $1 \leq i \leq s$. A sequence ω is called *uniformly distributed in* $[0,1)^s$, or *uniformly distributed modulo 1*, if, for all $J \in \mathcal{J}$,

$$\lim_{N\to\infty} N^{-1} R_N(J,\omega) = 0. \tag{1}$$

The elements of $\mathcal{J}^*$ are called *corners* with vertex $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_s)$. Sometimes, one also speaks of *anchored boxes* in $[0,1)^s$. As a simple fact, sequences ω that verify (1) restricted to corners J are also uniformly distributed in $[0,1)^s$. The reader will notice that the expression $N^{-1}R_N(J,\omega)$ is equal to the difference between the relative frequency of hits in J and the volume of J, which is given by the number $N^{-1}\sharp\{n, 0 \leq n < N : \mathbf{x}_n \in J\} - \lambda_s(J)$.

The following numerical quantity is one of the most important figures of merit in the theory of uniform distribution of sequences. In statistics, it is known as the *two-sided Kolmogorov-Smirnov test statistic* (see [42]). We refer to the monographs [43] and [51] for extensive background information and to [9] for the statistical context.

Definition 2. The *star discrepancy* $D_N^*(\omega)$ of the first N points of a sequence ω is defined as the following quantity:

$$D_N^*(\omega) = \sup_{J\in\mathcal{J}^*} \left|N^{-1}R_N(J,\omega)\right|.$$

It is well-known that ω is uniformly distributed in $[0,1)^s$ if and only if $\lim_{N\to\infty} D_N^*(\omega) = 0$ (for a proof see [13] or [43]).

We also consider the *extreme discrepancy* of the first N elements of a sequence $\omega = (\mathbf{x}_n)_{n\geq 0}$ in $[0,1)^s$,

$$D_N(\omega) = \sup_{J\in\mathcal{J}} \left|N^{-1}R_N(J,\omega)\right|.$$

In certain contexts, like with dynamical systems, we will write $D_N(\mathbf{x}_0, \dots, \mathbf{x}_{N-1})$ for the quantity $D_N(\omega)$.

It is elementary to show that, for any sequence ω in $[0,1)^s$, $D_N^*(\omega) \leq D_N(\omega) \leq 2^s D_N^*(\omega)$. Concerning the order of discrepancy, it is one of the most famous conjectures in the theory of irregularities of distribution of sequences that for any sequence ω in $[0,1)^s$, the inequality $D_N^*(\omega) > N^{-1}(\log N)^s$ holds for infinitely many N and $D_N^*(\omega) > N^{-1}(\log N)^{s-1}$ for all N. Sequences ω that satisfy $D_N^*(\omega) = \mathcal{O}(N^{-1}(\log N)^s)$, where the implied constant depends only on the dimension s, are called *low-discrepancy sequences* in the literature.

Roth [64] has shown the bound $D_N^*(\omega) > C_s N^{-1}(\log N)^{(s-1)/2}$, where the constant $C_s > 0$ depends only on s. For $s > 1$ this implies that the sequence $(ND_N^*(\omega))_{N\geq 1}$ is always unbounded.

For a given sequence ω, is there any hope that the remainders $R_N(J,\omega)$ stay bounded in N? In an impressive series of papers, W. M. Schmidt investigated this question (cf. [68]).

Define the family of *corners with bounded remainder* associated with ω as

$$\mathcal{B}(\omega) = \{J \in \mathcal{J}^* : \sup_N |R_N(J,\omega)| < \infty\}.$$

More generally, a (Lebesgue-) measurable subset A of $[0,1)^s$ with

$$\sup_N |R_N(A,\omega)| < \infty$$

is called a set with *bounded remainder* for ω. W. M. Schmidt [67] has shown that, for any sequence ω, the set of so-called *admissible volumes*

$$\Lambda(\omega) = \{\lambda_s(J) : J \in \mathcal{B}(\omega)\}$$

is at most countable. Hence, we will be interested in the following two questions: (i) determining the admissible volumes, and (ii) identifying the family $\mathcal{B}(\omega)$ of corners with bounded remainder.

Question (i) is related to topological dynamics and ergodic theory in a general framework (cf. [24, 70]): the set of admissible volumes comes from eigenvalues of an isometric operator. We will sketch this relation in the following sections.

Answers to the two questions above are only known in several special cases for ω. For example, this is the case for Kronecker sequences $(n\alpha \mod 1)_{n\geq 0}$ in $[0,1)^s$ (see [41] for $s = 1$ and [48] for $s > 1$), for polynomial sequences $(p(n) \mod 1)_{n\geq 0}$ (cf. [48]), and for Halton sequences (cf. [31]). For the latter type of sequences, we will exhibit how the theory of dynamical systems allows us to answer the two questions. The keyword will be 'coboundaries'.

2.2. Halton sequences

A classical family of low-discrepancy sequences are the Halton sequences. They are defined via the b-adic representation of real numbers.

Throughout this section, b denotes a positive integer, $b \geq 2$, and $\mathbf{b}=(b_1, \ldots, b_s)$ a vector of s integers $b_i \geq 2$, $1 \leq i \leq s$, not necessarily distinct. $\mathbb{N}$ stands for the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For a nonnegative integer a, let $a = \sum_{j \geq 0} a_j\, b^j, a_j \in \{0, 1, \ldots, b-1\}$, be the unique b-adic representation of a in base b. With the exception of at most finitely many indices j, the digits a_j are zero.

Every real number $x \in [0, 1)$ has a b-adic representation of the form $x = \sum_{j \geq 0} x_j\, b^{-j-1}$, $x_j \in \{0, 1, \ldots, b-1\}$. If x is a *b-adic rational*, which means that $x = ab^{-g}$, a and g integers, $0 \leq a < b^g$, $g \in \mathbb{N}$, and if $x \neq 0$, then there are two such representations, one of them with the property that $x_j = 0$ for all j sufficiently large, the other one with $x_j = b-1$ for all j sufficiently large. The b-adic representation of x is uniquely determined under the condition that $x_j \neq b-1$ for infinitely many j. In the following, we will call this particular representation *the regular (b-adic) representation of x*. When appropriate, we will write the regular b-adic representation of $x \in [0, 1)$ in the form $x = 0.x_0 x_1 x_2 \ldots$

If $n \in \mathbb{N}_0$ has the b-adic representation $n = n_0 + n_1 b + n_2 b^2 + \cdots$ with digits n_j, then the *radical-inverse function in base b* is defined as the function $\varphi_b(n) = 0.n_0 n_1 n_2 \ldots$. It assigns a number in $[0, 1)$ to the nonnegative integer n. We will encounter this function again in a more general context, under the name 'Monna' map.

Definition 3. Let $\mathbf{b} = (b_1, \ldots, b_s)$ be such that the bases b_i are pairwise coprime. Put $\varphi_{\mathbf{b}}(n) = (\varphi_{b_1}(n), \ldots, \varphi_{b_s}(n))$. The sequence $\omega_{\mathbf{b}} = (\varphi_{\mathbf{b}}(n))_{n \geq 0}$ is called the *Halton sequence* in base $\mathbf{b}$.

It can be shown by elementary counting arguments that every Halton sequence is uniformly distributed in $[0, 1)^s$ (cf. [43]). For a short proof, see [35]. Halton sequences are an important prototype of low discrepancy sequences, see [11] and [51] for comprehensive background information on this topic.

Definition 4. A $\mathbf{b}$-*adic elementary interval* in $[0, 1)^s$ or $\mathbf{b}$-*adic elint* in $[0, 1)^s$ for short, is a subinterval $I_{\mathbf{c},\mathbf{g}}$ of $[0, 1)^s$ of the form

$$I_{\mathbf{c},\mathbf{g}} = \prod_{i=1}^{s} \left[\varphi_{b_i}(c_i), \varphi_{b_i}(c_i) + b_i^{-g_i}\right),$$

where the parameters are subject to the conditions $\mathbf{g} = (g_1, \ldots, g_s) \in \mathbb{N}_0^s$, $\mathbf{c} = (c_1, \ldots, c_s) \in \mathbb{N}_0^s$, and $0 \leq c_i < b_i^{g_i}$, $1 \leq i \leq s$.

In the 'classical' form, the $\mathbf{b}$-adic elementary interval $I_{\mathbf{c},\mathbf{g}}$ is written as

$$I_{\mathbf{c},\mathbf{g}} = \prod_{i=1}^{s} \left[a_i b_i^{-g_i}, (a_i+1) b_i^{-g_i}\right),$$

where $\varphi_{b_i}(c_i) = a_i b_i^{-g_i}$, with $a_i \in \mathbb{N}_0$, $0 \le a_i < b_i^{g_i}$, $1 \le i \le s$.

Definition 5. A $\mathbf{b}$-*adic interval* in $[0,1)^s$ is a nonempty interval I with $\mathbf{b}$-adic rational corners, which means $I = \prod_{i=1}^{s} \left[a_i b_i^{-g_i}, d_i b_i^{-g_i}\right)$, where $a_i, d_i, g_i \in \mathbb{N}_0$, $0 \le a_i < d_i \le b_i^{g_i}$, $1 \le i \le s$. Let $\mathcal{J}_{\mathbf{b}}$ denote the family of $\mathbf{b}$-adic intervals and write $\mathcal{J}_{\mathbf{b}}^* = \mathcal{J}_{\mathbf{b}} \cap \mathcal{J}^*$ for the family of $\mathbf{b}$-adic corners.

Every $\mathbf{b}$-adic corner clearly is a finite disjoint union of $\mathbf{b}$-adic elints. It is elementary to show that a sequence ω is uniformly distributed in $[0,1)^s$ if and only if, for all $I \in \mathcal{J}_{\mathbf{b}}^*$, $\lim_{N\to\infty} N^{-1} R_N(I,\omega) = 0$.

The following result is well-known and easy to verify by counting. For a proof, see for example [31, Lemma 2.4]. A different proof involving dynamical systems will be given later.

Theorem 6. *Let I be a $\mathbf{b}$-adic corner. Then, for the Halton sequence $\omega_{\mathbf{b}}$,*

$$\sup_N |R_N(I,\omega_{\mathbf{b}})| < \infty.$$

Corollary 7. *Every $\mathbf{b}$-adic corner is a bounded remainder set for the Halton sequence $\omega_{\mathbf{b}}$. In particular $\mathcal{J}_{\mathbf{b}}^* \subseteq \mathcal{B}(\omega_{\mathbf{b}})$ and, for the admissible volumes, $\{\prod_{i=1}^{s} d_i b_i^{-g_i} : d_i, g_i \in \mathbb{N}_0, 0 \le d_i \le b_i^{g_i}, 1 \le i \le s\} \subseteq \Lambda(\omega_{\mathbf{b}})$.*

Since the indicator function of any $\mathbf{b}$-adic interval is a finite linear combination of indicator functions of $\mathbf{b}$-adic corners, we derive that every $\mathbf{b}$-adic interval is a bounded remainder set for $\omega_{\mathbf{b}}$.

In view of these results, we may conjecture that equality holds in the relations between the sets in Corollary 7. How to prove this conjecture? The following remark provides a clue, it will be our starting point into the theory of dynamical systems.

Remark 8. We observe the following. An interval $J \in \mathcal{J}^*$ is a BRS for the Halton sequence $\omega_{\mathbf{b}} = (\mathbf{x}_n)_{n\ge 0}$ in base $\mathbf{b}$, i.e. $J \in \mathcal{B}(\omega_{\mathbf{b}})$, if and only if, for the function $f = \mathbb{1}_J - \lambda_s(J)$,

$$\sup_N \left| \sum_{n=0}^{N-1} f(\mathbf{x}_n) \right| < \infty. \tag{2}$$

This is the point where dynamical systems come into play. Boundedness conditions like (2) have been studied in the theory of dynamical systems, under the keyword 'coboundary theorems'.

The following measure-theoretic coboundary theorem is due to Petersen [56] and was generalized to the $L^p(\mu)$-case by Liardet [48]. For readers not familiar with the underlying concepts, we provide an introduction to these notions from the theory of dynamical systems in Section 2.3.

Theorem 9 (Measure-theoretic coboundary theorem)**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be measure preserving. Then, for any $f \in L^2(\mu)$,*

$$\sup_N \left\| \sum_{n=0}^{N-1} f \circ T^n \right\|_2 < \infty \Leftrightarrow \exists\, g \in L^2(\mu) : f = g - g \circ T \ \text{in} \ L^2(\mu). \tag{3}$$

This theorem exists also in a topological version, as we shall exhibit in the next theorem, due to Gottschalk and Hedlund [25]. For a newer proof, we refer the reader to [49].

Theorem 10 (Topological coboundary theorem)**.** *Let X be a compact Hausdorff space and let $S : X \to X$ be a homeomorphism. Suppose that the dynamical system (X, S) is minimal (for this notion see Definition 15). Then, for any f belonging to the space $C(X)$ of continuous complex-valued functions on X,*

$$\exists\, x \in X : \ \sup_N \left| \sum_{n=0}^{N-1} f(S^n x) \right| < \infty \Leftrightarrow \exists g \in C(X) : \ f = g - g \circ S. \tag{4}$$

Definition 11. A function f with the property $f = g - g \circ T$ in $L^2(\mu)$ (see Theorem 9) or $f = g - g \circ S$ in $C(X)$ (Theorem 10) is called a *coboundary.* The function g is called a *transfer function* (or *cobounding function*) for f under T or S.

We note that the transfer function g is unique up to an additive constant. Obviously, if f is a coboundary, the sums $\sum_{n=0}^{N-1} f \circ T^n$ are telescopic and collapse to $g - g \circ T^N$. Refinements of the coboundary theorems cited above and a relation to certain spectral measures have been established in [48].

The statements in (3) and (4) closely resemble condition (2). Hence, in order to apply one of the above coboundary theorems, we have to find (i) an appropriate dynamical system (X, T), and (ii) an appropriate function $\varphi : X \to [0, 1)^s$ such that the Halton sequence $\omega_{\mathbf{b}} = (\mathbf{x}_n)_{n \geq 0}$ is the image under φ of the orbit under T of some suitable element x of X:

$$\forall n \in \mathbb{N}_0 : \quad \mathbf{x}_n = \varphi(T^n x).$$

Both versions of the coboundary theorem have their advantages and disadvantages. In the topological version, a single point $x \in X$ with

$$\sup_N \Big| \sum_{n=0}^{N-1} f(T^n x) \Big| < \infty$$

suffices to make f into a coboundary. On the other hand, S has to be a homeomorphism and f has to be continuous.

As a matter of fact, in our case the function f is discontinuous, $f = \mathbb{1}_J - \lambda_s(J)$, $J \in \mathcal{J}^*$. Hence, the measure-theoretic coboundary theorem, Theorem 9, seems more appropriate for our purposes, but, due to its nature as a statement in $L^2(\lambda_s)$, we will only get an 'almost everywhere' result in the relation $f = g - g \circ T$. If $s = 1$, one can overcome the difficulty that f is discontinuous, as has been exhibited by Schoissengeier [69, Theorem 1] for left continuous functions f with finitely many discontinuities.

Remark 12. The following observation will prove itself useful: suppose that we are able to show that $f = \mathbb{1}_J - \lambda_s(J)$, $J \in \mathcal{J}^*$, is an L^2-coboundary, which is to say, $f = g - g \circ T$ in $L^2(\lambda_s)$. If we put $G = e(g)$, we obtain the identity $G \circ T = e(\lambda_s(J))G$ in $L^2(\lambda_s)$. This says that the number $e(\lambda_s(J))$ is an eigenvalue of the isometry U_T on $L^2(\lambda_s)$ induced by the transformation T, $U_T(h) = h \circ T$. Hence, admissible volumes are related to eigenvalues of the operator U_T, in the above sense. For many maps T, the eigenvalues of the induced operator U_T are known (see Section 2.6).

The reader might be interested to know the following variant of Theorem 9 (cf. [48]). We assume that X is a compact topological space and $\mathcal{B}$ is the Borel σ-algebra $\mathcal{B}_X$ on X. Now, we introduce a *regular* measure preserving transformation T on the probability space $(X, \mathcal{B}_X, \mu)$ by requiring that T is *μ-continuous*, that is to say the set of points of discontinuity of T is of μ-measure 0.

Theorem 13 (Pointwise coboundary theorem)**.** *Let T be a regular measure preserving transformation on $(X, \mathcal{B}_X, \mu)$ and let x be a (T, μ)-generic point, i.e., the limit*

$$\lim_{N\to\infty} N^{-1} \sum_{0 \le n < N} \varphi(T^n x) = \int_X \varphi \, d\mu$$

holds for all μ- continuous maps $\varphi : X \to \mathbb{C}$. If $f : X \to \mathbb{C}$ is μ-continuous and bounded with

$$\sup_N \Big| \sum_{0 \le n < N} f(T^n x) \Big| < \infty \,,$$

then there exists $g \in L^\infty(\mu)$ such that $f = g - g \circ T$ μ-almost everywhere.

In the discussion above, we have outlined how to employ certain results from the theory of dynamical systems. In the following section, we will explore these notions in more detail.

2.3. Basic notions of dynamical systems

We start with a collection of notions and definitions that will be used in the sequel. As general reference for an introduction as well as for further reading we refer to [8, 14, 28, 57, 55, 71, 76]. Discrete dynamics studies the behaviour of functions under iteration. Thus the following definition is in the core of the subject.

Definition 14. Let X be a non-empty set and $T : X \to X$ a map (without any further requirements on X and T). For $x \in X$ the sequence $(T^n x)_{n \geq 0}$ is called the *orbit* of x (under T).

Definition 15. A *flow* (or *topological dynamical system*) is a couple $\mathcal{F} = (X, T)$, where X is a compact metric space and $T : X \to X$ is a continuous map. The flow is called *minimal*, if the only closed subsets Y of X, which satisfy $T(Y) = Y$ are $Y = X$ or $Y = \emptyset$.

A simple characterisation of minimality of a $\mathcal{F} = (X, T)$ is that for all $x \in X$ the orbit $(T^n x)_{n \in \mathbb{N}}$ is dense in X.

Definition 16. Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ a measurable transformation, which preserves the measure μ (*i. e.* $\mu \circ T^{-1} = \mu$). Then $\mathcal{T} = (X, T, \mathcal{B}, \mu)$ is called a *measure theoretical dynamical system*.

If the underlying measure space of $\mathcal{T}$ is clear from the context, we write again (X, T) for short or just T.

Definition 17. The system $\mathcal{T}$ is called *ergodic*, if for all $B \in \mathcal{B}$ equality $T^{-1}(B) = B$ implies $\mu(B) = 0$ or $\mu(B) = 1$. The system $\mathcal{T}$ is called *aperiodic*, if

$$\mu\left(\{x \in X\,;\, \exists\, n \in \mathbb{N}, T^n x = x\}\right) = 0.$$

In many situations, different dynamical systems are very similar according to the following definition.

Definition 18. Two measure theoretical dynamical systems $(X_i, \mathcal{B}_i, \mu_i, T_i)$ $(i = 1, 2)$ are called isomorphic, if there exist two sets $A_1 \in \mathcal{B}_1$ and $A_2 \in \mathcal{B}_2$ with $\mu_i(A_i) = 1$ $(i = 1, 2)$ and a bijection $\varphi : A_1 \to A_2$ such that

$$\forall\, x \in A_1,\ T_2 \circ \varphi(x) = \varphi \circ T_1(x).$$

Ergodicity is preserved under isomorphism and has several useful equivalent characterisations:

Theorem 19. *The following statements are equivalent:*

(E1) $\mathcal{T}$ *is ergodic.*

(E2) *All T-invariant measurable functions (* i. e. $f \circ T = f$ *) are constant μ-almost everywhere.*

(E3) *For all $A \in \mathcal{B}$, $\mu(A) > 0$ implies $\mu(\bigcup_{n=1}^{\infty} T^{-n}A) = 1$.*

(E4) *for all $A \in \mathcal{B}$, $\mu(A \triangle T^{-1}A) = 0$ implies $\mu(A) = 0$ or $\mu(A) = 1$.*

The following two lemmas collect basic properties of transformations, which give an intuitive insight into the consequences of measure-preservation and ergodicity.

Lemma 20 (Poincaré recurrence lemma)**.** *Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$ and let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then, for μ-almost all points in A, $T^n(x)$ belongs to A for infinitely many integer $n \geq 1$.*

With notations and assumptions of Lemma 20 the integer $n_A(x) := \min\{k \geq 1\,;\ T^k x \in A\}$, called return time of x to A, exists for μ-almost all $x \in A$.

Definition 21. The map $T_A : A \to A$ defined μ-almost everywhere by $T_A(x) = T^{n_A(x)}(x)$ is called the induced map of T on A.

In dynamic number theory, but not exclusively, this construction plays a fundamental role. Readily, T_A preserves the probability induced by μ on A.

Lemma 22. *Let T be an aperiodic invertible measure-preserving transformation on a Lebesgue space $(X, \mathcal{B}, \mu)$. For all $\varepsilon > 0$ and all integer $N > 0$ there exists a set $F \in \mathcal{B}$ such that the sets F, $TF, \ldots, T^{N-1}F$ are mutually disjoint and*

$$\mu\Big(X \setminus \bigcup_{i=0}^{N-1} T^i(F)\Big) \leq \varepsilon.$$

The hypothesis on $(X, T, \mathcal{B}, \mu)$ in Lemma 22 means that the dynamical system is isomorphic to the one defined by a measure preserving transformation on the torus $[0, 1)$ equipped with the Lebesgue σ-algebra and the Lebesgue measure λ. The family $\Theta(F, N) := \{T^i F\,;\ 0 \leq i < N\}$ is called a Rokhlin tower of base F, height N and measure at least $1 - \varepsilon$. The proof of this lemma in the particular case of ergodic transformations goes back to Kakutani and Rokhlin independently. See also Halmos [28] for a proof in full generality. The lemma can be derived from induced transformations.

Remark 23. Let X be a compact metric space, $\mathcal{B}_X$ the σ-algebra of Borel subsets of X and let $M(X)$ be the set of Borel probability measures on X. It is well known that $M(X)$ is convex and a compact metrisable space in the week topology. Let $T : X \to X$ be a Borel-measurable map. Then the set of T-invariant probability measures on X forms a convex set:

$$\mathrm{Inv}(T) = \left\{\mu \in M(X)\,;\ \mu \circ T^{-1} = \mu\right\}.$$

The set $\mathrm{Erg}(T)$ of measures μ such that $(X, \mathcal{B}, \mu, T)$ is ergodic is precisely the set of extremal points of $\mathrm{Inv}(T)$. If T is continuous, or even only quasi-continuous (see this notion in Section 2.5) the set $\mathrm{Inv}(T)$ is a convex non empty compact subset of $M(X)$ in the weak topology. Hence, in that case, $\mathrm{Inv}(T)$ is the weak convex closure of $\mathrm{Erg}(T)$ according to the Krein-Milman theorem.

Definition 24. Let X be a compact metric space and $T : X \to X$ be Borel-measurable. If $\mathrm{Inv}(T) = \mathrm{Erg}(T) = \{\mu\}$ (there is only one T-invariant measure, which is therefore ergodic), we call $(X, \mathcal{B}, T, \mu)$ *uniquely ergodic*. If the setting is clear, we simply call the transformation T uniquely ergodic.

For measure theoretical dynamical systems we recall the famous individual ergodic theorem:

Theorem 25 (Birkhoff 1931). *Let $\mathcal{T}$ be a measure theoretical dynamical system, then for all $f \in L^1(X, \mu)$ the limit*

$$\tilde{f}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

exists for μ-almost all $x \in X$. If $\mathcal{T}$ is ergodic, this limit equals $\int_X f(x)\,d\mu(x)$ μ-almost everywhere.

Assume that X is a compact topological space, $\mathcal{B}_X$ is the σ-algebra of Borel subsets of X and let $\mathcal{T} = (X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Recall that a point $x \in X$, for which

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f(y)\,d\mu(y)$$

holds for all continuous functions $f : X \to \mathbb{C}$, is called $(T, \mu)-$generic. The following theorem will show how unique ergodicity can be used.

Theorem 26. *Let (X, T) be a flow. Then the following are equivalent:*

(1) *$(X, \mathcal{B}, T, \mu)$ is uniquely ergodic,*

(2) *all points $x \in X$ are (T, μ)-generic*

(3) *for all $f \in C(X)$ the relation*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \int_X f(y)\,d\mu(y)$$

holds uniformly in x.

Notice that a minimal and uniquely ergodic flow is called strictly ergodic.

2.4. An example: the compact group of b-adic integers

For an integer $b \geq 2$, let $\mathbb{Z}_b$ denote the compact group of b-adic integers. We refer the reader to [37] for details.

An element z of $\mathbb{Z}_b$ will be written as $z = \sum_{j\geq 0} z_j\, b^j$, with digits $z_j \in \{0, 1, \ldots, b-1\}$. In certain circumstances, z will also be viewed as a column vector $(z_j)_{j\geq 0}$. If $z \neq 0$ (0 being the neutral element of $\mathbb{Z}_b$ or as well the null vector), then the *b-adic order* $\mathrm{ord}_b(z)$ of z is defined as the first index j such that $z_j \neq 0$. The order of the element 0 is defined as ∞. The *b-adic absolute value* of $z \neq 0$ is given by $|z|_b = b^{-v}$, $v = \mathrm{ord}_b(z)$, and we put $|0|_b = 0$. Finally, we obtain a metric d on $\mathbb{Z}_b$ by defining $d(z, z') = |z - z'|_b$. For an element $z = z_0 + z_1 b + \cdots$ of $\mathbb{Z}_b$, the coset $z + b^g\mathbb{Z}_b$ can be identified with the *cylinder set* $[z_0, z_1, \ldots, z_{g-1}]$. These sets are closed and open in the metric space $(\mathbb{Z}_b, d)$ and they generate the Borel σ-algebra on $\mathbb{Z}_b$. $\mathbb{Z}_b$ is a compact abelian group. We will denote the uniquely determined normalised Haar measure on $\mathbb{Z}_b$ by μ.

The set of integers $\mathbb{Z}$ is embedded in $\mathbb{Z}_b$. If $z \in \mathbb{N}_0$, then at most finitely many digits z_j are different from zero. If $z \in \mathbb{Z}$, $z < 0$, then at most finitely many digits z_j are different from $b-1$. In particular, $-1 = \sum_{j\geq 0}(b-1)\, b^j$.

Definition 27. An element $z \in \mathbb{Z}_b$ will be called *regular* if infinitely many digits z_j are different from $b-1$. Otherwise, z is called *irregular*.

It is easy to see that the set of irregular elements of $\mathbb{Z}_b$ coincides with the set $\{-1, -2, \ldots\}$ of negative integers and that $\mathbb{N}_0$ is contained in the set of regular elements of $\mathbb{Z}_b$.

Definition 28. We define the *b-adic Monna map* φ_b by

$$\varphi_b :\ \mathbb{Z}_b \to [0,1), \quad \varphi_b\left(\sum_{j\geq 0} z_j\, b^j\right) = \sum_{j\geq 0} z_j\, b^{-j-1} \pmod 1.$$

The restriction of φ_b to $\mathbb{N}_0$ is the radical-inverse function in base b that we have considered before. The Monna map is continuous and surjective, but not injective. The function φ_b maps particular cosets of $\mathbb{Z}_b$ to closed b-adic elementary intervals in $[0,1)$, $\varphi_b(z + b^g\mathbb{Z}_b) = [0.z_0 \ldots z_{g-1}, 0.z_0 \ldots z_{g-1} + b^{-g}]$. In particular, $\varphi_b(b - 1 + b\mathbb{Z}_b) = \{0\} \cup [(b-1)b^{-1}, 1) = [1 - b^{-1}, 1] \pmod 1$.

The b-adic elints $I_{c,g}$ generate the Borel σ-algebra on the torus $[0,1)$. We have the relation $\mu(\varphi_b^{-1}(I_{c,g})) = \mu(c + b^g\mathbb{Z}_b) = b^{-g}$, hence the map φ_b is measure preserving from $\mathbb{Z}_b$ onto $[0,1)$. It 'transports' the normalised Haar measure from

$\mathbb{Z}_b$ to normalised Haar measure (Lebesgue measure) on $[0,1)$. Furthermore, φ_b gives a bijection between the subset $\mathbb{N}$ of $\mathbb{Z}_b$ of positive integers and the set $\{ab^{-g} : 0 < a < b^g, g \in \mathbb{N}, b \nmid a\}$ of all reduced b-adic fractions. This will prove useful to enumerate the dual group of $\mathbb{Z}_b$.

The Monna map may be inverted in the following sense.

Definition 29. We define the *pseudo-inverse* φ_b^+ of the b-adic Monna map φ_b by

$$\varphi_b^+ : \ [0,1) \to \mathbb{Z}_b, \quad \varphi_b^+\left(\sum_{j\geq 0} x_j\, b^{-j-1}\right) = \sum_{j\geq 0} x_j\, b^j,$$

where $\sum_{j\geq 0} x_j\, b^{-j-1}$ stands for the regular b-adic representation of the element $x \in [0,1)$.

The image of the torus $[0,1)$ under φ_b^+ is the set of regular elements of $\mathbb{Z}_b$. Furthermore, we have the identities $x = \varphi_b(\varphi_b^+(x))$, for all $x \in [0,1)$, and $z = \varphi_b^+(\varphi_b(z))$, for all regular elements z of $\mathbb{Z}_b$, but, in general, $z \neq \varphi_b^+(\varphi_b(z))$, $z \in \mathbb{Z}_b$. For example, if $z = -1$, then $\varphi_b^+(\varphi_b(-1)) = \varphi_b^+(0) = 0 \neq -1$.

Remark 30. In our notation, the *dual group* $\widehat{\mathbb{Z}}_b$ of $\mathbb{Z}_b$ is given by $\widehat{\mathbb{Z}}_b = \{\chi_k : k \in \mathbb{N}_0\}$, where

$$\chi_k : \ \mathbb{Z}_b \to \{c \in \mathbb{C} : |c| = 1\}, \quad \chi_k\left(\sum_{j\geq 0} z_j b^j\right) = e\left(\varphi_b(k)(z_0 + z_1 b + \cdots)\right), \quad (5)$$

see [35]. If infinitely many digits z_i are different from zero, we will interpret the value of χ_k as an infinite product of complex numbers. All factors of this product with the exception of at most finitely many factors will be equal to one and, hence, the value of χ_k is well-defined.

The dual group $\widehat{\mathbb{Z}}_b$ is an orthonormal basis of $L^2(\mu)$, see [65] and [37] for the background in abstract harmonic analysis. We now 'lift' the characters χ_k to the torus. As in [32], the following function system will be the main tool in our analysis.

Definition 31. For a nonnegative integer k, let

$$\gamma_k : \ [0,1) \to \{c \in \mathbb{C} : |c| = 1\}, \quad \gamma_k(x) = \chi_k(\varphi_b^+(x)).$$

Let $\Gamma_b = \{\gamma_k : k \in \mathbb{N}_0\}$. It is easy to show that

$$\forall\, k \in \mathbb{N} : \quad \int_{[0,1)} \gamma_k \, d\lambda = 0,$$

and that Γ_b is an orthonormal system,

$$\forall\ k \neq \ell : \quad \int_{[0,1)} \gamma_k(x)\overline{\gamma_\ell}(x)\,dx = 0.$$

There is an obvious generalisation of the preceding notions to the higher-dimensional case. In the following, let $\mathbf{b} = (b_1, \ldots, b_s)$ be a vector of s not necessarily distinct integers $b_i \geq 2$, let $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$, and let $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$. We define

$$\begin{aligned}
\varphi_{\mathbf{b}}(\mathbf{k}) &= (\varphi_{b_1}(k_1), \ldots, \varphi_{b_s}(k_s)),\\
\varphi_{\mathbf{b}}^+(\mathbf{x}) &= (\varphi_{b_1}^+(x_1), \ldots, \varphi_{b_s}^+(x_s)),\\
\gamma_{\mathbf{k}}(\mathbf{x}) &= \prod_{i=1}^{s} \gamma_{i,k_i}(x_i), \quad \text{where } \gamma_{i,k_i} \in \Gamma_{b_i}, 1 \leq i \leq s,\\
\Gamma_{\mathbf{b}} &= \left\{\gamma_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^s\right\}.
\end{aligned}$$

Furthermore, for $n \in \mathbb{N}_0$, we will write $\varphi_{\mathbf{b}}(n)$ for the vector $(\varphi_{b_1}(n), \ldots, \varphi_{b_s}(n))$.

It is elementary to show that the family of functions $\Gamma_{\mathbf{b}}$ is an orthonormal system in $L^2(\lambda_s)$. It is even an orthonormal basis, see [35] for a proof. For an integrable function f on $[0,1)^s$, the $\mathbf{k}$-th Fourier coefficient of f with respect to the function system $\Gamma_{\mathbf{b}}$ is defined as

$$\widehat{f}(\mathbf{k}) = \int_{[0,1)^s} f\overline{\gamma_{\mathbf{k}}}\,d\lambda_s.$$

Let $\mathbb{Z}_{\mathbf{b}}$ denote the direct product of the compact groups $\mathbb{Z}_{b_i}$, $1 \leq i \leq s$, and define

$$\tau :\ \mathbb{Z}_{\mathbf{b}} \to \mathbb{Z}_{\mathbf{b}}, \quad \tau(\mathbf{z}) = \mathbf{z} + \mathbf{1} = (z_1 + 1, \ldots, z_s + 1), \tag{6}$$

where $\mathbf{z} = (z_1, \ldots, z_s)$, and $\mathbf{1} = (1, \ldots, 1)$. The transformation τ is a translation of the compact group $\mathbb{Z}_{\mathbf{b}}$. The following result is well-known, see, for example, [76, Ch.6] for background information and [31, Prop. 2.1] for details. The reader might also want to review Theorem 26.

Theorem 32. *Let the integers b_i be pairwise coprime. Then the dynamical system $(\mathbb{Z}_{\mathbf{b}}, \tau)$ has the following properties:*

(1) *τ is a homeomorphism on $\mathbb{Z}_{\mathbf{b}}$,*
(2) *$(\mathbb{Z}_{\mathbf{b}}, \tau)$ is minimal,*
(3) *$(\mathbb{Z}_{\mathbf{b}}, \tau)$ is uniquely ergodic, the uniquely determined τ-invariant probability measure being the normalised Haar measure,*
(4) *the sequence $(\tau^n\mathbf{0})_{n\geq 0}$ is well-distributed in $\mathbb{Z}_{\mathbf{b}}$, $\mathbf{0} = (0, \ldots, 0)$.*

The following corollary is a straightforward consequence.

Corollary 33. *The Halton sequence in base* $\mathbf{b}$ *is given by* $\omega_{\mathbf{b}} = (\varphi_{\mathbf{b}}(\tau^n \mathbf{0}))_{n\geq 0}$. *It is uniformly distributed in* $[0,1)^s$. *In fact,* $\omega_{\mathbf{b}}$ *is even well-distributed (for this notion, see* [43]*).*

In this context, we refer to recent results of Fan, Li, Yao and Zhou [15] where, in the case $s = 1$, the transformation τ is generalised to an affine map on $\mathbb{Z}_b$. Some of these results are already contained in [31] and [34].

The following examples show how the facts and notions we have collected up to now give a rather simple proof for the uniform distribution of the classical van der Corput and more general sequences.

Example 34. Let $b \geq 2$ and $\mathbb{Z}_b$ denote the group of b-adic integers. Then the addition-by-1 map τ on $\mathbb{Z}_b$ is uniquely ergodic with respect to the Haar measure on $\mathbb{Z}_b$, which can be seen by applying Theorem 26 (3) to the continuous characters of $\mathbb{Z}_b$. The continuous map φ_b transports the Haar measure on $\mathbb{Z}_b$ to Lebesgue measure on $[0,1)$. Now, by unique ergodicity of τ, the sequence $(\tau^n z)_{n\geq 0}$ is uniformly distributed in $\mathbb{Z}_b$ for all $z \in \mathbb{Z}_b$, especially for $z = 0$. Thus the van der Corput sequence $(\varphi_b(\tau^n 0))_{n\geq 0} = (\varphi_b(n))_{n\geq 0}$ in base b is uniformly and even well distributed modulo one.

Example 35. The ring $\mathbb{Z}/b\mathbb{Z}$ is identified to $\{0,\dots,b-1\}$. Let $C = (c_{ij})_{i,j\in\mathbb{N}_0}$ be a $\mathbb{N}_0 \times \mathbb{N}_0$-matrix over $\mathbb{Z}/b\mathbb{Z}$ with finite rows, that means for all $i \in \mathbb{N}_0$ there is an $m_i \in \mathbb{N}_0$ be such that $c_{ij} = 0$ for $j > m_i$. For any column vector $x = (x_n)_{n\in\mathbb{N}_0} \in \{0,\dots,b-1\}^{\mathbb{N}_0}$ define the column vector $y = Cx$, with entries

$$y_n := \sum_{j=0}^{\infty} c_{nj} x_j \pmod{b} = \sum_{j=0}^{m_n} c_{nj} x_j \pmod{b}$$

and

$$f(x) = \sum_{n=0}^{\infty} \frac{y_n}{b^{n+1}}.$$

Then $f : \mathbb{Z}_b \to [0,1]$ is continuous by the following observation: fix an integer $I \geq 0$ and set

$$M_I = \max\{m_i \mid i \leq I\}.$$

For two points $x, z \in [\varepsilon_0,\dots,\varepsilon_{M_I}]$ the coordinates y_i coincide for $i \leq I$, which gives $|f(x) - f(z)| \leq b^{-I}$. Keeping in mind the last example, f maps the Haar measure on $\mathbb{Z}_b$ to Lebesgue measure, if the transformation $x \mapsto y$ preserves the Haar measure on $\mathbb{Z}_b$. By linearity, this is the case, if and only if the transformation is surjective. This is the case, if and only if the the rows of C are linearly independent over $\mathbb{Z}/b\mathbb{Z}$. Notice that f corresponds to the Monna map φ_b if C is the unit matrix.

2.5. μ-continuity, quasi-continuity, and uniform quasi-continuity

Let X be a compact metric space, $\mathcal{B}_X$ the σ-algebra of Borel subset of X and $T : X \to X$ a Borel-measurable transformation. For our purposes continuity of the transformation T is too strong a requirement; thus we have to introduce concepts which allow T to have discontinuities, but preserve the statement of Theorem 26 or at least let us keep control on the generic points. Let μ be a Borel-measure on X. We have already encountered the notion of regular dynamical system $(X, T, \mathcal{B}_X, \mu)$ meaning that T is μ-continuous. We remark that regularity requires the presence of a measure *a priori*. This is not the case for the following notions.

Definition 36. A subset A of X is called T-*neglectable*, if for all $\varepsilon > 0$ there exists a continuous function $f : X \to [0, 1]$, such that $\mathbb{1}_A(\cdot) \leq f(\cdot)$ on X and

$$\sup_{x \in X} \Big(\limsup_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \Big) \leq \varepsilon.$$

Definition 37. Let X and Y be compact metric spaces and $T : X \to X$ be a Borel-measurable transformation. A Borel-measurable function $f : X \to Y$ is called T-*quasi-continuous*, if its set of discontinuity points is T-neglectable. If T itself is T-quasi-continuous, we call it *quasi-continuous*.

This definition was introduced in [60]. The advantage of quasi-continuity is that its definition does not require any measure. Moreover, if T is quasi-continuous, then for all T-invariant measures μ, T is also μ-continuous. The following notion of uniform quasi-continuity was introduced in [46].

Definition 38. A map $T : X \to X$ is called *uniformly quasi-continuous*, if for all $\varepsilon > 0$ there exists a $f : X \to [0, 1]$ with $f(x) = 1$ for all discontinuity points x of T and

$$\limsup_{N \to \infty} \Big(\sup_{x \in X} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) \Big) \leq \varepsilon.$$

This notion allows to generalise Theorem 26 (cf. [46]) by relaxing the continuity assumption.

Theorem 39. *Let $T : X \to X$ be uniformly quasi-continuous. Then the following statements are equivalent:*

(1) *T is uniquely ergodic*

(2) *for every real valued $f \in C(X)$ there exists a real number $\lambda(f)$ such that*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) = \lambda(f)$$

holds uniformly in $x \in X$.

Remark 40. Certainly, λ is a positive linear functional on $C(X)$, which can be expressed as $\lambda(f) = \int_X f(x)\,d\mu(x)$ for a Borel-measure μ. Then μ is the unique T-invariant measure on X and T is μ-continuous.

Example 41. A quasi-continuous transformation can be uniquely ergodic without satisfying statement (2) in Theorem 39. The following example is due to I. Abou in her thesis [1]. Let $F : [0,1] \to [0,1]$ be a continuous map strictly increasing with $F(0) = 0$, $F(1) = 1$ and $F(x) < x$ for $0 < x < 1$. Now define $T : [0,1] \to [0,1]$ by $T(0) = 1$ and $T(x) = F(x)$ otherwise. It is clear that T has the only discontinuity point 0 and is quasi-continuous. Moreover, for any continuous map $f : [0,1] \to \mathbb{R}$, the sequence of means $\frac{1}{N}\sum_{0\le n<N} f(T^{n+m}(x))$ converges uniformly in m to $f(1)$. In particular T is uniquely ergodic with invariant measure the Dirac measure $\delta_{\{1\}}$, but T is not uniformly quasi-continuous due to

$$\sup_{0\le x\le 1}\left|\frac{1}{N}\sum_{0\le n<N} f(T^n(x)) - f(1)\right| \ge |f(0) - f(1)|\,.$$

If we are only concerned to relax continuity but conserving genericity for all points (without claiming for uniformity), the following theorem is useful.

Theorem 42. *A quasi-continuous transformation $T : X \to X$ is uniquely ergodic if and only if there is a T-invariant Borel probability μ on X such that all points x in X are (T,μ)-generic.*

2.6. Spectral properties

For a measure theoretic dynamical system $\mathcal{T} = (X, T, \mathcal{B}, \mu)$ the transformation T induces an isometry $U_T : L^2(X,\mu) \to L^2(X,\mu)$ given by $U_T f = f \circ T$. Usually, in the context of dynamical systems which are not necessarily invertible, the spectral measure ρ_f associated to a function f in $L^2(\mu)$, orthogonal to the constants ($\int_X f\,d\mu = 0$) is studied. In fact, the sequence $\gamma_f : n \mapsto \langle U_T^n f, f\rangle$ extended on negative integers by $\gamma_f(-n) := \langle f, U_T^n f\rangle = \overline{\gamma_f(n)}$ is positive definite. By the Bochner-Herglotz theorem, γ_f is the Fourier transform of a Borel measure on the torus $[0,1)$ which is by definition the spectral measure ρ_f above. Therefore

$$\widehat{\rho_f}(k) = \langle U_T^k f, f\rangle = \langle f \circ T^k, f\rangle \tag{7}$$

In case $\mathcal{T}$ is regular and f is μ-continuous and bounded, for any generic point x the spectral measure ρ_f is the one associated to the complex valued sequence $n \mapsto f(T^n x)$. Therefore

$$\widehat{\rho_f}(k) = \lim_N \frac{1}{N}\sum_{0\le n<N} f(T^{n+k}x)\overline{f(T^n x)}$$

and in term of weak convergence

$$\rho_f(dt) = *\text{-}\lim_N \frac{1}{N}\Big| \sum_{0\le n<N} f(T^n x)e(-tn)\Big|^2 dt\,.$$

Properties of spectral measures are usually established for unitary operators U on Hilbert spaces but most of them are also true for isometry. Here we quote properties of ρ_f which are of interest for our purpose.

Remark 43. Let f and g be functions in $L^2(X,\mu)$. Let $\|\cdot\|$ be the total variation for Borel measures on $[0,1)$ and let $\delta_{\{\alpha\}}$ ($\alpha \in [0,1)$) denote the Dirac unit point mass at $\{\alpha\}$. The spectral measures have the following properties.

(i) $\rho_{U_T f} = \rho_f$, $\rho_f(X) = \|f\|_2^2$ and the dependence of ρ_f on f is continuous by the inequality

$$\|\rho_f - \rho_g\| \le \|f+g\|_2\|f-g\|_2.$$

(ii) The dependence on f is homogeneous: $\rho_{cf} = |c|^2\rho_f$ and

$$\rho_{f+g} \le 2(\rho_f + \rho_g)\,.$$

(iii) ρ_f has a point mass at α (*i.e.*, $\rho_f(\{\alpha\}) > 0$) if and only if $e(\alpha)$ is an eigenvalue of U_T whose eigenspace is not orthogonal to f.

(iv) If H_1 and H_2 are U_T-stable subspaces (*i.e.*, $U_T(H_i) \subset H_i$, $i = 1, 2$) and $H_1 \perp H_2$, then for h_i in H_i on has $\rho_{h_1+h_2} = \rho_{h_1} + \rho_{h_2}$.

One eigenfunction h in property (iii) can be built naturally from f. In fact, let P_α be the orthogonal projection to the closed subspace fixed by the isometry $e(-\alpha)U_T$. By a well known von Neumann theorem, the sequence of means $\frac{1}{N}\sum_{0\le n<N} e(-n\alpha)U_T^n f$ converges in L^2 to $P_\alpha f$ and $h = P_\alpha f$ verifies (iii). Then, one has the orthogonal decomposition $f = g + h$ (with $\langle U_T^n g, h\rangle = \langle g, U_T^n h\rangle = 0$ for all $n \ge 0$) giving $\rho_f = \rho_g + \|h\|^2\delta_{\{\alpha\}}$ with $\rho_g(\{\alpha\}) = 0$. For more details and complements about this approach of the spectral measure of U_T we refer to [58]. Furthermore, continuity of ρ_f can be characterised by the Wiener-Schoenberg theorem:

Theorem 44 (Wiener-Schoenberg)**.** *Let λ be a measure on $\mathbb{T}$ and $(\widehat{\lambda}(n))_{n\in\mathbb{Z}}$ denote its sequence of Fourier-coefficients. Then one has*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \left|\widehat{\lambda}(n)\right|^2 = \sum_{z\in\mathbb{T}} \lambda(\{z\})^2.$$

Thus the measure λ is continuous, if and only if this limit is 0.

Whereas the Fourier coefficients of the spectral measure ρ_f are rather easy to compute, the finer properties, such as continuity, absolute continuity, or mutual singularity with some other spectral measure are not easy to obtain. By Theorem 19, $(X, T, \mathcal{B}, \mu)$ is ergodic, if and only if the eigenspace of 1 of the operator U_T is one-dimensional (and therefore contains only the constants). Equivalently:

Theorem 45. *A measure theoretical dynamical system $(X, T, \mathcal{B}, \mu)$ is ergodic if and only if for all functions f in $L^2(X, \mu)$ equality $\langle f, 1\rangle = 0$ implies equality $\rho_f(\{0\}) = 0$.*

Definition 46. A dynamical system $(X, T, \mathcal{B}, \mu)$ has a discrete spectrum (or pure point spectrum) if all spectral measures are purely atomic. This is equivalent to the fact that there exists an orthonormal basis for $L^2(X, \mu)$ which consists of eigenfunctions of U_T.

A fundamental result of Halmos [28] says that a dynamical system has a discrete spectrum if and only it is isomorphic to a translation $x \mapsto x + g$ on a compact metrisable abelian group G which has in turn the same set of eigenvalues $\{\chi(g) \mid \chi \in \widehat{G}\}$ of T.

Example 47. The b-adic integers $\mathbb{Z}_b$ form a group under addition. Therefore the spectrum of the addition-by-one map τ (see Equation (6)) is discrete. The continuous characters of $\mathbb{Z}_b$ (cf. [37]) are given by (5):

$$\chi_k\left(\sum_{j\geq 0} z_j b^j\right) = e\left(\varphi_b(k)\sum_{j=0}^{K-1} z_j b^j\right), \quad \text{if } k < b^K.$$

The character χ_k is then an eigenfunction of U_τ with eigenvalue $\chi_k(1)$. Therefore, the eigenvalues of the operator U_τ are given by the set

$$\left\{z \in \mathbb{C} \mid \exists k \in \mathbb{N} : z^{b^k} = 1\right\}.$$

Example 48. Let α be irrational. Then the rotation $R_\alpha : x \to x + \alpha \mod 1$ is an ergodic transformation on the torus $\mathbb{T}$ and the spectrum of $(\mathbb{T}, R_\alpha)$ is discrete. The eigenfunctions are simply the characters $e(\ell\cdot)$ and the eigenvalues are the numbers $e(\ell\alpha)$, $\ell \in \mathbb{Z}$ (cf. [76]).

For a detailed study of the spectral properties of digital functions we refer to [58].

Bounded remainder sets are related to spectral measures through Theorem 9 and the following general result is easy to prove:

Theorem 49. *Let U be an isometry of a Hilbert space H (with scalar product $\langle\cdot|\cdot\rangle$) and let x be a vector in H. Then $x = y - U(y)$ for some $y \in H$, if and only if the map $t \mapsto 1/\sin^2 \pi t$ from $[0, 1)$ to $\overline{\mathbb{R}}$, defined as $+\infty$ for $t = 0$, is* *integrable with respect to the spectral measure* ρ_x *associated to* x *and* U *defined by its Fourier coefficients:* $\widehat{\rho_x}(k) = \langle U^k x | U^n x \rangle$ *and* $\widehat{\rho_x}(-k) = \overline{\widehat{\rho_x}(k)}$ *for* $k \geq 0$.

2.7. Product construction

Assume that we have two ergodic dynamical systems $\mathcal{T}_i = (X_i, T_i, \mathcal{B}_i, \mu_i)$ $(i = 1, 2)$. It is natural to consider the product $\mathcal{T}_1 \times \mathcal{T}_2 = (X_1 \times X_2, T_1 \times T_2, \mathcal{B}_1 \otimes \mathcal{B}_2, \mu_1 \otimes \mu_2)$, where $T_1 \times T_2(x, y) = (T_1 x, T_2 y)$. The following theorem gives a complete answer to the question, when the product of two ergodic dynamical systems is again ergodic. The proof derives from Theorem 43 and Theorem 45.

Theorem 50. *Let* $\mathcal{T}_i = (X_i, \mathcal{B}_i, \mu_i, T_i)$ $(i = 1, 2)$ *be two ergodic dynamical systems. Then the dynamical system* $\mathcal{T}_1 \times \mathcal{T}_2$ *is ergodic, if and only if the discrete parts of the two spectra intersect only at* 1.

Example 51. Let b_1 and b_2 be positive integers and let τ_{b_i} denote the addition-by-one map on $\mathbb{Z}_{b_i}$, $i = 1, 2$. We want to study, when the two dynamical systems $(\mathbb{Z}_{b_1}, \tau_{b_1})$ and $(\mathbb{Z}_{b_2}, \tau_{b_2})$ are spectrally disjoint. By Example 47 the corresponding spectra intersect only at 1, if and only if b_1 and b_2 are coprime.

A different way to view this result is a projective version of the Chinese remainder theorem, which reads as

$$\mathbb{Z}_{b_1 b_2} \simeq \mathbb{Z}_{b_1} \oplus \mathbb{Z}_{b_2},$$

if b_1 and b_2 are coprime. This provides the idea of the proof of Theorem 32.

Example 52. Let $b \geq 2$ be an integer and α an irrational. Let $f(n)$ be a digital function as in Example 35. Then the sequence $(f(n), n\alpha \mod 1)_{n \geq 0}$ is well distributed in $\mathbb{T}^2$. This is an immediate consequence of the disjointness of the spectra computed in Examples 47 and 48 that implies the ergodicity of the translation $(x, t) \to (x + 1, t + \alpha)$ of the group $\mathbb{Z}_b \times [0, 1)$ and consequently its unique ergodicity. The result now follows from Theorem 26. In the same manner, the sequence $(f(n), p(n) \bmod 1)_{n \geq 0}$ with any non periodic polynomial sequence $n \mapsto p(n) \mod 1$ is well distributed modulo 1.

2.8. Skew products

Examples 34 and 35 represent the simplest way of applying ideas from ergodic theory to low-discrepancy sequences. For more complicated sequences, such as the various modifications of the van der Corput sequence, the construction used in these two examples breaks down. In most cases where this happens, the map f (see Example 35) cannot be defined on a large enough set. In order to cope with this problem, we introduce the concept of cocycles and associated skew products, which allows to replace the map f by its first differences, which can be defined almost everywhere on X.

In this section, let $(X, \mathcal{B})$ be a standard measure space *i.e.*, X is a compact metric space and $\mathcal{B}$ is the Borel σ-algebra. Let μ be a Borel probability measure on $(X, \mathcal{B})$. Let $T : X \to X$ be a Borel automorphism of $(X, \mathcal{B}, \mu)$ and let A be a compact metrisable abelian group with normalised Haar measure h_A.

Definition 53. A *T-cocycle* (or simply a cocycle, if the underlying action T is fixed) is a Borel map

$$a : \mathbb{Z} \times X \to A$$

such that

$$(i) \quad \forall m, n \in \mathbb{Z} : a(m+n, x) = a(m, T^n x) + a(n, x) \quad \mu - \text{a.e.}$$

$$(ii) \quad \mu\left(\bigcup_{n \in \mathbb{Z}} (\{x \mid T^n x = x\} \cap \{x \mid a(n,x) \neq 0\})\right) = 0\,.$$

Notice that condition (i) implies $a(0,x) = 0$ μ-a.e. We usually assume that T is aperiodic so that condition (ii) is satisfied automatically.

Remark 54. Relation (i) in Definition 53 is the source of the general definition of T^G-cocycles where G is a group (usually discrete) and T^G a G-action on X (the action of $g \in G$ being an automorphism denoted by T^g). For $G = \mathbb{Z}$, the corresponding action is given by an automorphism T of $(X, \mathcal{B}, \mu)$ and the T cocycle a is constructed from the Borel map $\gamma(\cdot) := a(1, \cdot)$:

$$a(n,x) = \begin{cases} \sum_{k=0}^{n-1} \gamma \circ T^k(x) & \text{for } n > 0, \\ 0 & \text{for } n = 0, \\ -\sum_{k=n}^{-1} \gamma \circ T^k(x) & \text{for } n < 0. \end{cases} \tag{8}$$

Notice that in [34] page 113, Formula (10) must be corrected according to (8).

Definition 55. A cocycle $a(\cdot, \cdot)$ is called a *T-coboundary* if there exists a Borel map $c : X \to A$, such that

$$\forall n \in \mathbb{Z} : a(n,x) = c(x) - c(T^n x) \quad \mu - \text{a.e.} \tag{9}$$

It is called *trivial*, if it is the sum of a T-coboundary and a cocycle which only depends on n.

From now on we assume that the action T is ergodic on $(X, \mathcal{B}, \mu)$ and fix a T-cocycle $a : \mathbb{Z} \times X \to A$.

Definition 56 (K. Schmidt [66])**.** An element $\alpha \in A$ is said to be an *essential value* of the cocycle a if for every neighbourhood $N(\alpha)$ of α in A and for every

$B \in \mathcal{B}$ with $\mu(B) > 0$,

$$\mu\left(\bigcup_{n\in\mathbb{Z}} \left(B \cap T^{-n}(B) \cap \{x \mid a(n,x) \in N(\alpha)\}\right)\right) > 0. \tag{10}$$

Let $E(a)$ denote the set of essential values of the cocycle a (the reader should note that these definitions do not require ergodicity of T).

Theorem 57 (K. Schmidt [66]). *The set of essential values $E(a)$ has the following properties:*

(1) *If $b : \mathbb{Z} \times X \to A$ is a coboundary then $E(a+b) = E(a)$.*
(2) *$E(a)$ is a closed subgroup of A.*
(3) *$E(a) = G$ if and only if there exists a G-valued cocycle a' such that $a - a'$ is a coboundary. If $E(a) = \{0\}$ this means that a is itself a coboundary.*

We assume that $(X, \mathcal{B}, \mu)$ is non-atomic. Recall that h_A denotes the Haar measure on A. Let

$$\tilde{X} = (X \times A, \mathcal{B} \otimes \mathcal{B}_A, \mu \otimes h_A),$$

where $\mathcal{B}_A$ is the Borel σ-algebra of A. We define the $\mathbb{Z}$*-action* T_a on $\tilde{X}$ by $T_a(x,\alpha) = (Tx, \alpha + a(1,x))$. Therefore, by definition

$$T_a^n(x,\alpha) = (T^n x, \alpha + a(n,x)). \tag{11}$$

Clearly, T_a is an automorphism of $\tilde{X}$. The action T_a is called the *skew product* of T with respect to a.

Theorem 58 (K. Schmidt [66]). *If T is ergodic, then*

$$T_a \textit{ is ergodic } \Leftrightarrow E(a) = A.$$

Remark 59. Let a be a T-cocycle and $c : X \to A$ be a Borel-map. Then a and

$$a'(n,x) = a(n,x) + c(x) - c(T^n x)$$

define isomorphic skew product actions: let $\varphi_c : X \times A \to X \times A$ be given by $\varphi_c(x,\alpha) = (x, \alpha - c(x))$. Then, by a straightforward computation,

$$\varphi_c \circ T_a(x,\alpha) = T_{a'} \circ \varphi_c(x,\alpha),$$

which defines an isomorphism between the dynamical systems $(X \times A, T_a)$ and $(X \times A, T_{a'})$.

The following theorem was first proved in special cases by Conze (unpublished) and in a series of papers by Veech[74, 75]. A complete proof with A not necessary abelian is given in [47].

Theorem 60. *Let $(X, T, \mathcal{B}, \mu)$ be a dynamical system, and let $a : \mathbb{Z} \times X \to A$ be a T-cocycle. Then the dynamical system $(X \times A, \mathcal{B} \otimes \mathcal{B}_A, \mu \otimes h_A, T_a)$ is ergodic, if and only if T is ergodic and the functional equation*

$$f(x) = \chi(a(1, x)) f \circ T(x) \quad \mu\text{-almost everywhere} \tag{12}$$

has no measurable non-zero solution f for any non-trivial character $\chi \in \widehat{A}$.

Remark 61. Notice that equation (12) is equivalent to the coboundary relation (9) with cocycle $\chi \circ a$, if there exists a non-zero solution f. By ergodicity of T, we can and do assume that the measurable transfer function f is unimodular.

Let

$$\Psi = \{\chi \in \widehat{A} \mid \exists f_\chi \neq 0 : f_\chi(x) = \chi(a(1, x)) f_\chi \circ T(x)\}. \tag{13}$$

Then we have

$$E(a) = \{g \in A \mid \forall \chi \in \Psi : \chi(g) = 1\} = \operatorname{Ann} \Psi. \tag{14}$$

The following theorem was proved in a more general setting in [26]. We present it in a version appropriate for our present purposes.

Theorem 62. *Let X be a compact metric space and assume that T acts μ-continuously on $(X, \mathcal{B}, \mu)$. Further, let $a : \mathbb{Z} \times X \to A$ be a μ-continuous A-valued cocycle for a compact group A. Assume further that T_a is ergodic for $\mu \otimes h_A$ (h_A the Haar measure on A). If x is (T, μ)-generic, then the point (x, α) is $(T_a, \mu \otimes h_A)$-generic for all $\alpha \in A$.*

Example 63. In the context of digital functions $f : \mathbb{N} \to \mathbb{R}$, such as modified versions of the van der Corput sequence, the cocycle is usually defined as

$$a_f(n, x) = \lim_{\substack{m \to x \\ m \in \mathbb{N}}} (f(m + n) - f(m)).$$

The reason behind this definition is the following. For given $n \in \mathbb{N}$, the sum $n + x$ in $\mathbb{Z}_b$ involves only a finite number of carries except in the case where $x = -n$. Thus, for any $n \in \mathbb{N}$, the above limit may exist except eventually at $x = -1, -2, \ldots$. In that latter case, a_f should be extended from $a_f(1, x)$, $x \neq -1$, by assigning an arbitrary value to $a_f(1, -1)$. In our case, we put $a(1, -1) = 0$. Actually, there is a dichotomy in this context: either f admits an extension to $\mathbb{Z}_b$ (as in Example 34), which implies that the cocycle is trivial, or f cannot be extended continuously to $\mathbb{Z}_b$, which makes the cocycle the object to be studied. The prototype of such a function f is the classical sum-of-digits function $s_b(\cdot)$ in base b, which certainly has no meaningful extension to $\mathbb{Z}_b$. Nevertheless, the associated cocycle is defined almost everywhere and has been studied in [40, 39, 38].

Example 64. As in Example 35 we take an $\mathbb{N}_0 \times \mathbb{N}_0$-matrix C over $\mathbb{Z}/b\mathbb{Z}$ with linearly independent rows. Assume that at least one of the rows contains infinitely many non-zero entries. Taking the b-ary expansion of $n \in \mathbb{N}$, $n = \sum_{k\geq 0} n_k b^k$, we compute

$$y_i = \sum_{k=0}^{\infty} c_{ik} n_k \pmod{b}$$

and put

$$f(n) = \sum_{i=0}^{\infty} y_i b^{-i-1}.$$

There exists no continuous extension of f from $\mathbb{N}$ to $\mathbb{Z}_b$ for the following reason: clearly, y_i can only be computed for a sequence $(x_k)_{k\geq 0} \in (\mathbb{Z}/b\mathbb{Z})^{\mathbb{N}_0}$, if there are only finitely many non-zero x_k. Nevertheless, the limit

$$a_f(n,x) = \lim_{\substack{m\to x\\ m\in\mathbb{N}}} (f(m+n) - f(m)) \tag{15}$$

exists for all $x \in \mathbb{Z}_b$ if $n = 0$, and for all $x \in \mathbb{Z}_b \setminus \{-n\}$ if $n \in \mathbb{N}$. We now define the skew product transformation on $(\mathbb{Z}_b \times \mathbb{T}, h_{\mathbb{Z}_b} \otimes h_{\mathbb{T}})$ by

$$T_a(x,z) = (x+1, z + a_f(1,x) \pmod{1}).$$

This transformation is $h_{\mathbb{Z}_b} \otimes h_{\mathbb{T}}$-continuous, since the set of its points of discontinuity is $\{-1\} \times \mathbb{T}$, which is a $h_{\mathbb{Z}_b} \otimes h_{\mathbb{T}}$-negligible set. In fact, T_a is uniformly quasi-continuous.

The following example will show that T_a need not be ergodic.

Example 65. Let $C = (c_{ij})_{i,j\in\mathbb{N}_0}$ be the matrix given by $c_{0j} = 1$ for all $j \in \mathbb{N}_0$ and $c_{ij} = \delta_{ij}$ for $i \geq 1$. For $x \neq -1$ set $m(x) = \min\{k \in \mathbb{N}_0 \mid x_k < b-1\}$. Then the cocycle a_f satisfies

$$a_f(1,x) = \frac{m(x)+1 \pmod{b}}{b} + \sum_{i=1}^{m(x)} \frac{1}{b^{i+1}}.$$

We have for $n \in \mathbb{N}_0$

$$f(n) = \varphi_b(n) - \frac{n_0}{b} + \frac{s_b(n)}{b},$$

where $\varphi_b(n)$ denotes the n-th term of the van der Corput sequence and $s_b(n)$ the b-adic sum of digits function. Thus

$$a_f(1,x) = \frac{a_{s_b}(1,x) - 1}{b} + \varphi_b(x+1) - \varphi_b(x),$$

where

$$a_{s_b}(1,x) = \lim_{\substack{m\to x\\ m\in\mathbb{N}}} (s_b(m+1) - s_b(m)),$$

which exists for $x \neq -1$. Thus a_f is the sum of a $\frac{1}{b}\mathbb{Z}/\mathbb{Z}$-valued cocycle and a continuous coboundary. The essential values of $\frac{1}{b}a_{s_b}$ are $\{0, \frac{1}{b}, \dots, \frac{b-1}{b}\}$. This shows that the skew product is not ergodic in this case.

Now $((T^n x, z + a'_f(n,x)))_{n\geq 1}$ is uniformly distributed in $\mathbb{Z}_b \times (\frac{1}{b}\mathbb{Z}/\mathbb{Z})$ by Theorem 62. Furthermore, $c : \mathbb{Z}_b \to [0, \frac{1}{b}]$, $c(x) = \sum_{k=1}^{\infty} x_k b^{-k-1}$, transports the Haar-measure on $\mathbb{Z}_b$ to the Lebesgue measure on $[0, \frac{1}{b}]$. Taking the orbit of $(0,0)$ under the action of $T_{a'_f}$ we get that $((c(n), \frac{1}{b}s_b(n)))_{n\geq 1}$ is uniformly distributed in $[0, \frac{1}{b}) \times (\frac{1}{b}\mathbb{Z}/\mathbb{Z})$. Finally, this implies that $(f(n))_{n\geq 1} = (c(n) + \frac{1}{b}s_b(n))_{n\geq 1}$ is uniformly distributed in $[0,1)$.

The computations can be generalised to the case that only finitely many rows of the matrix are infinite.

Example 66. Consider now a matrix C as in Example 64 with infinitely many infinite rows. Assume that all rows of C are linearly independent. Assume furthermore that C has infinitely many different columns. Then the cocycle a_f is $h_{\mathbb{Z}_b} \otimes h_{\mathbb{T}}$-continuous. In order to compute its essential values we first consider a cylinder set $B = [a_0, a_1, \dots, a_K]$ and take $L > K$. Then for $x \in B$ with $0 < x_L < b-1$ (let $b > 2$ for simplicity) we have $x \in B \cap \tau^{-b^L}(B)$. On the other hand we have

$$a_f(b^L, x) = \sum_{i=0}^{\infty} \frac{c_{iL}}{b^{i+1}} = f(b^L).$$

Every measurable set B can be approximated in proportion arbitrarily closely to 1 by finite unions of cylinder sets. Thus every value attained by $f(b^L)$ is indeed essential for the cocycle a_f. Since there are infinitely many different columns of the matrix C and every real number in $[0,1)$ has at most two different expansions to base b, $a_f(b^L, x)$ attains infinitely many different values in $\mathbb{T}$. Since the essential values form a closed subgroup of $\mathbb{T}$, this shows that $E(a_f) = \mathbb{T}$. Thus the transformation τ_{a_f} is ergodic.

Since $(\tau^n 0)_{n\geq 1}$ is uniformly distributed in $\mathbb{Z}_b$, Theorem 62 implies that the sequence $(\tau^n 0, a_f(n,0))_{n\geq 1}$ is uniformly distributed in $\mathbb{Z}_b \times \mathbb{T}$. Hence $(f(n))_{n\in\mathbb{N}} = (a_f(n,0))_{n\geq 1}$ is uniformly distributed in $[0,1)$.

For $b = 2$ the argument has to be adapted slightly.

2.9. Spectral properties of cocycles

We will now investigate the spectrum of the operator U_{T_a} given by

$$U_{T_a}\Phi(x,z) = \Phi(Tx, z + a_f(1,x))$$

for functions f as in Example 64 and we assume throughout this section that the matrix C has infinitely many infinite rows. Since the Hilbert space $L^2(\mathbb{Z}_b \times \mathbb{T}, \mu_b \otimes h_{\mathbb{T}})$ decomposes as a Hilbert orthogonal sum

$$L^2(\mathbb{Z}_b \times \mathbb{T}, \mu_b \otimes h_{\mathbb{T}}) = \bigoplus_{\ell \in \mathbb{Z}} L^2(\mathbb{Z}_b, \mu_b) \otimes e(\ell \cdot),$$

we can study the spectrum separately on each summand. The operator U_{T_a} acts on $\Phi = \varphi \otimes e(\ell \cdot)$ for $\varphi \in L^2(\mathbb{Z}_b, \mu_b)$ as

$$U_{T_a}\varphi(x)e(\ell z) = \varphi(Tx)e(\ell z)e(\ell a_f(1,x)) = e(\ell z)U_{T_a,\ell}\varphi(x).$$

The Fourier coefficients of the spectral measure $\rho_{\varphi,\ell}$ associated with $U_{T_a,\ell}$ are given by

$$\widehat{\rho_{\varphi,\ell}}(n) = \int_{\mathbb{Z}_b} \varphi(T^n x)e(\ell a_f(n,x))\overline{\varphi(x)}\, d\mu_b(x).$$

By [26, Theorems 4 and 5], which are consequences of results in [36], the spectral measure $\rho_{\varphi,\ell}$ is either purely absolutely continuous, purely singular continuous with respect to Lebesgue measure, or purely discrete.

Remark 67. If the spectrum of $U_{T_a,\ell}$ is discrete for $\ell \in m\mathbb{Z}$, then there exist a $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$-valued cocycle a', a measurable function g and a real number ξ such that

$$a(n,x) = a'(n,x) + g(x) - g(T^n x) + n\xi.$$

This case (with $\xi = 0$) occurs, if the matrix C has only finitely many infinite rows.

In the following we will show that $\rho_{\varphi,\ell}$ can never be absolutely continuous, if $b > 2$. For this purpose we investigate the behaviour of the sequence $(\widehat{\rho_{\chi,\ell}}(b^k))_{k \geq 1}$ for a character χ of $\mathbb{Z}_b$. We compute

$$\widehat{\rho_{\chi,\ell}}(b^k) = \int_{\mathbb{Z}_b} \chi(x+b^k)\overline{\chi(x)}e(\ell a_f(b^k,x))\, d\mu_b(x) = \chi(b^k)\int_{\mathbb{Z}_b} e(\ell a_f(b^k,x))\, d\mu_b(x).$$

We notice that $\chi(b^k) = 1$ for k large enough. Thus it remains to compute the integral. For this purpose we define the sets

$$B_K = \{x \in \mathbb{Z}_b \mid x_k = \cdots = x_{K-1} = b-1, x_K < b-1\}.$$

Then (by slight abuse of notation)

$$y_m(x+b^k) - y_m(x) = -\sum_{r=k}^{K} c_{mr} \pmod{b}, \text{ for } x \in B_K.$$

Therefore $a_f(b^k, \cdot)$ is constant on every B_K. For $b > 2$ we estimate

$$\left|\widehat{\rho_{\chi,\ell}}(b^k)\right| \geq \left|\int_{B_k} e(\ell a_f(b^k,x))\,d\mu_b(x)\right| - \sum_{r=k+1}^{\infty}\left|\int_{B_r} e(\ell a_f(b^k,x))\,d\mu_b(x)\right|$$
$$= \mu_b(B_k) - \sum_{r=k+1}^{\infty}\mu_b(B_r) = 1 - \frac{2}{b}.$$

Thus for $b > 2$ the Fourier coefficients $(\widehat{\rho_{\chi,\ell}}(b^k))_k$ do not tend to zero. Therefore the measure $\rho_{\chi,\ell}$ cannot be absolutely continuous by the Riemann-Lebesgue lemma.

2.10. An application: bounded remainder sets

We are now in a position to solve the problem posed in Section 2.2 concerning the bounded remainder sets for the Halton sequence $\omega_{\mathbf{b}}$.

Remark 68. Due to the simple fact that $\chi_{\mathbf{k}}(\tau(\mathbf{z})) = \chi_{\mathbf{k}}(\mathbf{1})\chi_{\mathbf{k}}(\mathbf{z})$, for all $\mathbf{z} \in \mathbb{Z}_{\mathbf{b}}$, $\mathbf{1} = (1,\ldots,1)$, all characters $\chi_{\mathbf{k}}$ are eigenfunctions of the operator U_τ on $L^2(\mu)$. For this reason, the eigenvalues of U_τ are given by the set $\{\chi_{\mathbf{k}}(\mathbf{1}) : \mathbf{k} \in \mathbb{N}_0^s\}$, which is equal to the set of complex numbers $\{\prod_{i=1}^s e(a_i b_i^{-g_i}) : a_i, g_i \in \mathbb{N}_0, 0 \leq a_i < b_i^{g_i}, 1 \leq i \leq s\}$.

The following transformation T on the torus $[0,1)^s$ will allow us to apply the measure-theoretic coboundary theorem to solve the problem of bounded remainder sets in the class $\mathcal{J}^*$ for Halton sequences. For readers interested in the mixing properties of T and in its measure-theoretic background, we recommend Silva [72, Ch.3.8].

Definition 69. Let $b \geq 2$ be an integer. We define the *b-adic Kakutani-von Neumann odometer* (or *b-adic adding machine transformation*) on $[0,1)$ as the map

$$T : [0,1) \to [0,1), \quad T(x) = \varphi_b \circ \tau \circ \varphi_b^+(x).$$

We observe that the Kakutani-von Neumann map T is a piecewise translation map given by

$$T(x) = x - 1 + b^{-k} + b^{-k-1} \text{ for } x \in \left[1 - b^{-k}, 1 - b^{-k-1}\right),$$

with $k = 0,1,2,\ldots$. Let us write λ instead of λ_1 in dimension $s = 1$. Clearly, T is continuous at all points distinct of the points $1 - b^k$, $k = 0,1,2,\ldots$. In fact, T is uniformly quasi-continuous. Furthermore T preserves the Haar measure λ and the dynamical system $([0,1),T)$ is ergodic. It is in fact uniquely ergodic and isomorphic to the system $(\mathbb{Z}_b,\tau)$. As a consequence, the associated operator U_T has the same eigenvalues as U_τ, which means that the eigenvalues of T are given by the set $\{e(ab^{-g}) : a,\ g \in \mathbb{N}_0, 0 \leq a < b^g\}$, see Remark 68.

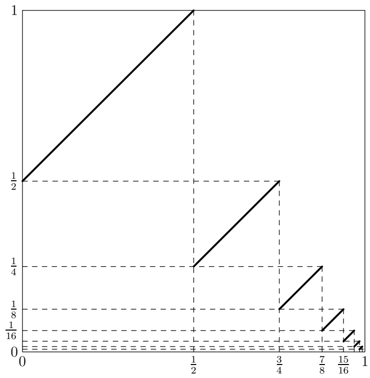

FIGURE 1. The graph of the Kakutani-von Neumann map for $b = 2$

The generalisation to the s-dimensional case is simple. We define $T : [0,1)^s \to [0,1)^s$, $T(\mathbf{x}) = \varphi_{\mathbf{b}} \circ \tau \circ \varphi_{\mathbf{b}}^{+}(\mathbf{x})$. The orbit of $\mathbf{0}$ i.e., the sequence $(T^n\mathbf{0})_{n\geq 0}$, is the Halton sequence $\omega_{\mathbf{b}}$ in base $\mathbf{b}$. We are now in a position to apply the above theory to the problem of bounded remainder sets.

THEOREM 70. *Let the corner $J = \prod_{i=1}^{s}[0,\beta_i)$ be a bounded remainder set for the Halton sequence $\omega_{\mathbf{b}}$ in base $\mathbf{b}$, $\mathbf{b} = (b_1,\dots,b_s)$, with pairwise coprime integers b_i. Then every side-length β_i of J is a b_i-adic rational, $\beta_i = a_i b_i^{-g_i}$, where $0 \leq a_i \leq b_i^{g_i}$, $a_i, g_i \in \mathbb{N}_0$, $1 \leq i \leq s$.*

Proof. Put $f = \mathbb{1}_J - \lambda_s(J)$ and write $f_N(\mathbf{x}) = \sum_{n=0}^{N-1} f\circ T^n(\mathbf{x})$, $\mathbf{x} \in [0,1)^s$. The fact that J belongs to the set $\mathcal{B}(\omega_{\mathbf{b}})$ may be written as $C = \sup_N |f_N(\mathbf{0})| < \infty$.

The function f is Riemann-integrable on $[0,1)^s$. Due to the fact that the transformation T is continuous almost everywhere and measure preserving, the function $f\circ T$ is also Riemann-integrable. Hence, the uniform distribution of the Halton sequence $\omega_{\mathbf{b}} = (T^k\mathbf{0})_{k\geq 0}$ implies that

$$||f_N||_2^2 = \lim_{K\to\infty} K^{-1}\sum_{k=0}^{K-1} |f_N(T^k\mathbf{0})|^2 = K^{-1}\sum_{k=0}^{K-1} |f_{N+k}(\mathbf{0}) - f_k(\mathbf{0})|^2 \leq (2C)^2.$$

An application of Theorem 9, the measure-theoretic coboundary theorem, proves the existence of $g \in L^2(\lambda_s)$ with $f = g - g\circ T$ almost everywhere.

Without loss of generality, we may assume that g has integral zero. Furthermore, for any index $\mathbf{k}$, we have the identity $\widehat{(g\circ T)}(\mathbf{k}) = \chi_{\mathbf{k}}(\mathbf{1})\widehat{g}(\mathbf{k})$ between the Fourier coefficients of $g\circ T$ and g. Hence, the equality $f = g - g\circ T$ in $L^2(\lambda_s)$

yields

$$\|g\|_2^2 = \sum_{\mathbf{k}\neq\mathbf{0}} |\widehat{f}(\mathbf{k})|^2/|1-\chi_{\mathbf{k}}(\mathbf{1})|^2 < \infty. \tag{16}$$

We notice that $\widehat{f}(\mathbf{k}) = \widehat{\mathbb{1}}_J(\mathbf{k})$ and that $\chi_{\mathbf{k}}(\mathbf{1}) \neq 1$, for all $\mathbf{k} \neq \mathbf{0}$.

The s-dimensional corner J is the Cartesian product of one-dimensional corners, $J = \prod_{i=1}^{s} J_i$, $J_i = [0, \beta_i[$, $0 < \beta_i \leq 1$. As a consequence, the $\mathbf{k}$-th Fourier coefficient of the function $\mathbb{1}_J$ is given by

$$\widehat{\mathbb{1}}_J(\mathbf{k}) = \prod_{i=1}^{s} \widehat{\mathbb{1}}_{J_i}(k_i).$$

We now consider one-dimensional projections. Suppose that the index $\mathbf{k}$ is of the special form $\mathbf{k} = (0,\dots,0,k_i,0,\dots,0)$, where $k_i \neq 0$. In this case,

$$\widehat{f}(\mathbf{k}) = \widehat{\mathbb{1}}_J(\mathbf{k}) = \widehat{\mathbb{1}}_{J_i}(k_i) \prod_{j\neq i} \lambda(J_j).$$

Put $f_i = \mathbb{1}_{J_i} - \lambda(J_i)$. Identity (16) implies that

$$\sum_{k\neq 0} |\widehat{f}_i(k)|^2/|1-\chi_{i,k}(1)|^2 < \infty, \tag{17}$$

where the s-dimensional character $\chi_{\mathbf{k}}$, $\mathbf{k} = (k_1,\dots,k_s)$, is the product of the one-dimensional characters χ_{i,k_i}, $\chi_{\mathbf{k}} = \prod_{i=1}^{s} \chi_{i,k_i}$. This proves that, for every coordinate i, there is a function $g_i \in L^2(\lambda)$ such that $f_i = g_i - g_i \circ T_i$ in $L^2(\lambda)$, where T_i stands for the Kakutani-von Neumann transformation in base b_i. From Remark 12 we derive that $e(\lambda(J_i))$ is an eigenvalue of T_i, which implies that $\lambda(J_i)$ is of the form $a_i b_i^{-g_i}$, $0 \leq a_i \leq b_i^{g_i}$, with $a_i, g_i \in \mathbb{N}_0$. This proves the theorem. $\square$

Remark 71. As noted in Section 2.2, we will give a proof of Theorem 6 in a dynamical context. To this aim we notice that by additivity it is enough to show that a functions of the form $f = \mathbb{1}_I - a$ where I is a $\mathbf{b}$-adic elint, i.e., $I = \prod_{i=1}^{s} [a_i b_i^{-g_i}, (a_i+1) b_i^{-g_i})$ with $0 \leq a_i < b_i^{g_i}$, $1 \leq i \leq s$ and $a = \lambda_s(I)$, is coboundary. The family of such elints I with fixed g_i forms a partition of $[0,1)^s$ into $g = b_1^{g_1} \cdots b_s^{g_s}$ elements and T acts cyclically on this family. This means that

$$1 = \sum_{j=0}^{g-1} \mathbb{1}_I \circ T^j \quad \lambda_s - a.e.,$$

and a straightforward computation shows that

$$f = \mathbb{1}_I - a = \varphi \circ T - \varphi, \quad \lambda_s - a.e., \quad \text{with} \quad \varphi = \sum_{j=1}^{g-1} ja\mathbb{1}_I \circ T^j\,. \tag{18}$$

Theorem 70 allows us to characterize the family of sets with bounded remainder as well as the set of admissible volumes for the Halton sequence in base $\mathbf{b}$.

Corollary 72. *For the Halton sequence $\omega_{\mathbf{b}}$ in base $\mathbf{b}$, the set $\mathcal{B}(\omega_{\mathbf{b}})$ is equal to the family of $\mathbf{b}$-adic corners and the set $\Lambda_{\mathbf{b}}(\omega_{\mathbf{b}})$ of admissible volumes is equal to the set $\{\prod_{i=1}^{s} a_i b_i^{-g_i}\ ;\ a_i, g_i \in \mathbb{N}_0, 0 \le a_i \le b_i^{g_i}, 1 \le i \le s\}$.*

Remark 73. For a $\mathbf{b}$-adic corner I in $[0,1)^s$, the series expansion S_f of the function $f = \mathbb{1}_I - \lambda_s(I)$ in the orthonormal basis $\Gamma_{\mathbf{b}}$ (see Section 2.4) is finite (see [35]). Since any non-constant function χ in $\Gamma_{\mathbf{b}}$ is an eigenfunction for T, it is also a coboundary with transfer function $\gamma = \frac{1}{\chi(1)-1}\chi$. This leads easily to the series expansion S_φ of the transfer function φ given in (18); see [32] for additional results.

An extension of these results to certain Cantor series representations of real numbers is known, see Hellekalek [31], where it is proved in particular that intervals I in $[0,1)$ of bounded remainder for the analogous Halton sequence are exactly those of length ℓ such that $e(\ell)$ is an eigenvalue of the underlying add-one transformation.

Historically, the question of bounded remainder sets has been studied first for sequences $(n\alpha \pmod 1)_{n\ge 0}$. For any irrational translation $R_\alpha : x \mapsto x + \alpha \pmod 1$ on $[0,1)$, bounded remainder intervals $I = [a,b)$ $(0 \le a < b < 0)$ are those of length in $\alpha\mathbb{Z} + \mathbb{Z}$. This condition is clear since the eigenvalues of R_α are $e(n\alpha)$, $n \in \mathbb{Z}$. The fact that if $b - a$ belongs to $\alpha\mathbb{Z} + \mathbb{Z}$ then the interval I is a bounded remainder set was first observed by Hecke [30]. A simple proof derives directly from the relation $\mathbb{1}_{[a,b)}(x) - (b-a) = \langle x - b\rangle - \langle x - a\rangle$, $x \in [0,1)$, with $\langle t\rangle = t - \lfloor t\rfloor$, that leads easily to a transfer function. The converse was proved by Kesten [41] with a complicated proof. The case of an ergodic translation $R_{(\alpha_1,\dots,\alpha_s)}$ on the s-dimensional torus $[0,1)$, $s \ge 2$, is more difficult. In fact, in dimension 2, some particular examples of bounded remainder sets are due to Rauzy [63]. Non-empty intervals $I_1 \times \cdots \times I_s$ with bounded remainder have been characterised in [48]. They are of the form $I_j = [0,1)$ for all j except for at most one index j_0 for which I_{j_0} is of bounded remainder for the translation $x \mapsto x + \alpha_{j_0} \pmod 1$. The proof use the spectral criterion (see Theorems 9 and 49) and simultaneously diophantine properties of the α_i. The same spectral criterion also can be used to prove that non-empty bounded remainder intervals $[a,b[$ are just $[0,1[$ in cases of the non-periodic polynomial sequences $n \mapsto p(n) \pmod 1$ and sequences $n \mapsto \alpha s_b(n)$ where $s_b(\cdot)$ denotes the b-adic sum of digits function and α is any irrational real number (see [48]).

2.11. Dynamics behind (m, s)-digital sequences

The functions $f : n \mapsto f(n)$ studied in Examples 35 and 64 in base b verify the remarkable equality

$$f(m + b^k n) = f(m) + f(b^k n), \quad 0 \le m < b^k,\ k \in \mathbb{N}_0,$$

that characterises b-additive real-valued functions (or sequences). The most famous example is the sequence $n \mapsto s_b(n)\alpha$, which is known to be well distributed modulo 1 if α is irrational. For a given abelian group A, the definition of b-additivity (or b-multiplicativity if the law of A is noted multiplicatively) is easily extended to A-valued sequences. In this section A is assumed to be the infinite metrisable compact product group $\mathbb{A}_b = (\mathbb{Z}/b\mathbb{Z})^{\mathbb{N}_0}$. Elements z of $\mathbb{A}_b$ are columns with entries z_j in $\mathbb{Z}/b\mathbb{Z}$, $j \in \mathbb{N}_0$. Notice that the b-adic Monna map φ_b can be also defined on $\mathbb{A}_b$, analogously to $\mathbb{Z}_b$, by setting $\varphi_b(z) = \sum_{j=0}^{\infty} z_j b^{-j-1}$. Characters of $\mathbb{A}_b$ are in one-to-one correspondence with t-tuples $(k_0, \dots, k_{t-1}) \in (\mathbb{Z}/b\mathbb{Z})^t$, $t \in \mathbb{N}$, by

$$\chi(z) = \prod_{j=0}^{t-1} e(k_j z_j / b)\,.$$

Let π_j be the projection map $\pi_j : \mathbb{A}_b \to \mathbb{Z}/b\mathbb{Z}$ at rank j, defined by $\pi_j(z) = z_j$. An $\mathbb{A}_b$-valued b-additive function F is given by the coordinate functions $f_j = \pi_j \circ F$ and clearly the maps $f_j : \mathbb{N}_0 \to \mathbb{Z}/b\mathbb{Z}$ are b-additive. We use F to define a cocycle $a_F : \mathbb{Z}_b \to \mathbb{A}_b$ in the same way as above:

$$a_F(n, x) = \lim_{\substack{m \to x \\ m \in \mathbb{N}}} \big(F(m + n) - F(m)\big)$$

for $x \ne -n$ if $n \ne 0$ (and set the value $a_F(1, -1) = 0_{\mathbb{A}_b}$ if the corresponding limit does not exist). Now we introduce the skew product $T_{a_F} : \mathbb{Z}_b \times \mathbb{A}_b \to \mathbb{Z}_b \times \mathbb{A}_b$ by

$$T_{a_F}(x, z) = \big(x + 1, z + a_F(1, x)\big)\,.$$

Naturally, T_{a_F} is an automorphism of the probability space $(\mathbb{Z}_b \times \mathbb{A}_b, \mu)$ where μ denotes the normalised Haar measure on the compact group $\mathbb{Z}_b \times \mathbb{A}_b$. In addition, T_{a_F} is uniformly quasi-continuous. In order to give a characterisation of the ergodicity of T_a (that implies its unique ergodicity) we consider the $\mathbb{Z}/b\mathbb{Z}$-module Φ of all $\mathbb{Z}/b\mathbb{Z}$-valued b-additive functions and the sub-module Φ_0 of those functions g such that there exists a positive integer ℓ with

$$g(n + b^\ell) = g(n)$$

for all integers $n \ge 0$. A family $\mathcal{E} := \{e_j\,;\ j \in \mathbb{N}_0\}$ of elements in Φ is said to be independent modulo Φ_0 if for all integers $m \ge 0$ and all $(m{+}1)$-tuple $(\lambda_0, \dots, \lambda_m)$ in $(\mathbb{Z}/b\mathbb{Z})^{m+1}$, the relation $\sum_{0 \le k \le m} \lambda_k e_k \in \Phi_0$ implies $\lambda_0 = \lambda_1 = \dots = \lambda_m = 0$. We are ready to state the important theorem proved in [34]:

Theorem 74. *With the above notations, the skew product T_{a_F} is ergodic if and only if the family of function $\mathcal{F} := \{f_j\,;\ j \in \mathbb{N}_0\}$ is independent modulo Φ_0.*

Example 75. We return to Examples 35 and 64 but with an arbitrary $\mathbb{N}_0 \times \mathbb{N}_0$-matrix C over $\mathbb{Z}/b\mathbb{Z}$. Let $F : \mathbb{N}_0 \to \mathbb{A}_b$ be the b-additive function with coordinate functions $f_j : \mathbb{N}_0 \to \mathbb{Z}/b\mathbb{Z}$ given by

$$f_j(n) = \sum_{k\geq 0} \varepsilon_k(n) c_{jk},$$

where $n = \sum_{k\geq 0} \varepsilon_k(n) b^k$ is the b-adic expansion of n. By construction $f = \varphi_b \circ F = \sum_{j=0}^{\infty} f_j b^{-j-1}$ where f is the function defined in the examples. But the cocycle a_f is generally not equal to $\varphi(a_F)$ since $\varphi_a : \mathbb{A}_b \to \mathbb{T}$ is not a group homomorphism. Moreover, Example 66 shows that T_{a_F} can be non-ergodic while T_{a_f} is, so that the sequence $n \mapsto F(n)$ is not uniformly distributed in $\mathbb{A}_b$ while its image $n \mapsto f(n)$ by φ_a is uniformly distributed modulo 1.

Remark 76. If T_{a_F} is ergodic, we readily deduce that the sequence $n \mapsto (\varphi_b(n), \varphi_b(F(n)))$ is well distributed in $[0,1)^2$. Moreover, from $n \mapsto \varphi_b(F(n))$ we can construct, for any positive integer s fixed, sequences in $[0,1)^s$ which are well distributed. For example, with the family of functions $\mathcal{F}^{(i)} := \{f_{i+js}\,;\ j \in \mathbb{N}_0\}$ $(1 \leq i \leq s-1)$ one constructs the b-additive $(\mathbb{A}_b)^{s-1}$-valued sequence $G : n \mapsto (F^{(1)}(n), \ldots, F^{(s-1)}(n))$. Then the cocycle $a_G = (a_{F^{(1)}}, \ldots, a_{F^{(s-1)}}) : \mathbb{Z}_b \to (\mathbb{A}_b)^{s-1}$ leads to an ergodic skew product transformation T_{a_G} on $\mathbb{Z}_b \times (\mathbb{A}_b)^{s-1}$. As before, by a standard argument, the sequence

$$n \mapsto (\varphi_b(n), \varphi_b(F^{(1)}(n)), \ldots, \varphi_b(F^{(s-1)}(n)))$$

is well distributed in $[0,1)^s$.

An interesting and non-trivial problem is to find families of b-additive functions $\{f_i : \mathbb{N}_0 \to \mathbb{Z}/b\mathbb{Z}\,;\ i \in \mathbb{N}_0\}$ that satisfy the independence condition of Theorem 74. Once such a family has been found, we may apply the construction in Remark 76 to get the uniquely ergodic skew product T_{a_G} on $\mathbb{Z}_b \times (A_b)^{s-1}$. This problem is related to the construction of (t,s)-sequences that we are going to define.

Definition 77. A set Y of $N = b^m$ points y in $[0,1)^s$ is said to be a (t,m,s)-net in base b if

$$\sum_{y\in Y} (\mathbb{1}_J(y) - \lambda_s(J)) = 0$$

for every interval J of the form

$$J = \prod_{i=1}^{s} \Big[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}\Big)$$

with integers d_i and a_i such that $d_i \geq 0$, $0 \leq a_i < b^{d_i}$ $(1 \leq i \leq s)$, and $\lambda_s(J) = 1/b^{m-t}$ (i.e., $\sum_i d_i = m - t$).

This definition expresses the fact that Y has many local discrepancies equal to 0.

Definition 78. An infinite sequence $\omega = (\mathbf{x}_n)_{n\geq 0}$ of points in $[0,1)^s$ is called a (t,s)-sequence in base b if for all integers $k \geq 0$ and $m > t$, the set

$$\{\mathbf{x}_n \mid kb^m \leq n < (k+1)b^m\}$$

forms a (t,m,s)-net in base b.

A necessary condition for existence is that $t \geq \frac{s}{b} - \log_b \frac{(b-1)s+b+1}{2}$. For an extensive study of these notions we refer to [51, 50].

Remark 79. A $(0,s)$-sequence exists if and only if $s \leq b$.

Theorem 80 (see [34])**.** *With notations of Remark 76, if*

$$n \mapsto (\varphi_b(n), \varphi_b(F^{(1)}(n)), \ldots, \varphi_b(F^{(s-1)}(n))), \quad n \in \mathbb{N}_0,$$

is a (t,s)-sequence, then the skew product $T_{(a_{F^{(1)}},\ldots,a_{F^{(s-1)}})} : \mathbb{Z}_b \times (A_b)^{s-1} \to \mathbb{Z}_b \times (A_b)^{s-1}$ is uniquely ergodic.

The usual approach to construct (t,s)-sequences is to chose $s-1$ $\mathbb{N}_0 \times \mathbb{N}_0$-matrices $C^{(i)}$ over $\mathbb{Z}/b\mathbb{Z}$ and $s-1$ families of bijections $\{\psi_j^{(i)} : \mathbb{Z}/b\mathbb{Z} \to \mathbb{Z}/b\mathbb{Z}\,;\ j \in \mathbb{N}_0\}$ $(1 \leq i \leq s-1)$. For simplicity we assume that $\psi_j^{(i)}(0) = 0$ for all indices. Let $F^{(i)} : \mathbb{N}_0 \to \mathbb{A}_b$ be defined by the coordinate maps $f_j^{(i)}(n) = \sum_{k\geq 0} \psi_k^{(i)}(\varepsilon_k(n))c_{jk}$ which are b-additive and take $x_n = (\varphi_b(n), \varphi_b(F^{(1)}(n)), \ldots, \varphi_b(F^{(s-1)}(n)))$. For conditions on matrices $C^{(i)}$ such that $n \mapsto x_n$ is a (t,s)-sequence we refer to the monograph [51].

3. New constructions

In this section we study constructions of one-to-one maps $T : [0,1) \to [0,1)$ which are given by a partition of $[0,1)$ into a family of non-empty intervals $[a,b)$ and such that the restrictions of T on these intervals are translations. We always identify the interval $[0,1)$ with the one dimensional torus $\mathbb{T}$, hence $[0,1)$ is equipped with the compact metrisable topology given by the metric $d(x,y) := \|x-y\|$ where $\|t\| := \min\{|t-n|;\ n \in \mathbb{Z}\}$, $t \in \mathbb{R}$.

We study in particular the cutting-stacking construction applied to rank-one transformations and maps deduced from cutting-stacking processes associated to appropriate substitutions. Then connections with the previous sections are investigated. This will yield classical as well as new sequences arising from these dynamical systems, with good discrepancy.

3.1. Interval exchanges

3.1.1. Main definitions and basic constructions

Definition 81. A map $T : [0,1) \to [0,1)$ is said to be an *interval exchange* on $[0,1)$ (or a *piecewise translation map*) if there is a family $\mathcal{F}$ of non empty subintervals $I_n := [a_n, b_n)$ of $[0,1)$ and a family of corresponding real numbers t_n $(0 \leq n < N = \mathrm{card}(\mathcal{F}))$ such that:

(i) $I_m \cap I_n = \emptyset$ and $(I_m + t_m) \cap (I_n + t_n) = \emptyset$ if $m \neq n$,

(ii) $T(x) = x + t_n$ if $x \in I_n$,

(iii) $\lambda\big(\bigcup_{0 \leq n < N} I_n\big) = \lambda\big(\bigcup_{0 \leq n < N} (I_n + t_n)\big) = 1$.

By construction, any interval exchange T preserves the Lebesgue measure λ, is one-to-one and invertible up to a set of λ-measure 0.

One interest of this construction is that every aperiodic invertible measure-preserving transformation T on a Lebesgue space $(X, \mathcal{B}, \mu)$ can be identified with a possibly infinite interval exchange [2]. Such constructions play a fundamental role for our purpose and are deeply related to Rokhlin towers (cf. Lemma 22).

3.1.2. Irrational translation modulo one

We illustrate the Rokhlin tower constructions in the case of a translation $\tau_\alpha : [0,1) \to [0,1)$ by an irrational number α modulo one. As usual, we identify $[0,1)$ with the torus $\mathbb{R}/\mathbb{Z}$ and assume $0 < \alpha < 1$ so that, by definition, $\tau_\alpha(t) := t + \alpha - \lfloor t + \alpha \rfloor$. It appears readily that τ_α is an interval exchange between $[0, 1-\alpha)$ and $[1-\alpha, 1)$. To go deeply inside the combinatorial and diophantine properties of τ_α it is instructive to construct τ_α from a sequence of pairs of Rokhlin towers derived from the regular continued fraction expansion $\alpha = [0; a_1, a_2, a_3, \ldots]$, where the a_n are integers ≥ 1. A straightforward computation shows that the return time map $n_{[0,\alpha)}(\cdot)$ takes the values a_1 and a_1+1, leading to the induced transformation on the interval $[0,\alpha)$, which exchanges intervals $[0, 1 - a_1\alpha)$ and $[1 - a_1\alpha, \alpha)$. Therefore, $[1-a_1\alpha, \alpha)$ is translated to $[0, (a_1+1)\alpha-1)$ so that $\tau_\alpha|_{[0,\alpha)}$ corresponds to the translation by $(a_1+1)\alpha-1$ modulo $\alpha\mathbb{Z}$. Let $T : (0,1) \to (0,1)$ be the Gauss transformation given by $T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ and $S : x \mapsto 1-x$ the reflection map on $(0,1)$. Then, after normalisation, the induced transformation $\tau_\alpha|_{[0,\alpha)}$ corresponds on $[0,1)$ to the translation $\tau_{S \circ T(\alpha)}$. Notice that the transformation $T_1 := S \circ T$ corresponds to the so-called semi-regular continued fraction expansion of an irrational number x in $(0,1)$, given by

$$x = \frac{1}{b(x) - T_1(x)} = \frac{1}{a(x) + 1 - (1 - T(x))}$$

where $a(x) := \lfloor \frac{1}{x} \rfloor$. In other words,

$$\alpha = \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots}}}}$$

with $b_1 = a_1 + 1$ and $b_n = b(T_1^{n-1}(\alpha))$. Also, τ_α induces a translation on $[\alpha, 1)$ which corresponds, after normalisation, to the translation $\tau_{T\circ S(\alpha)}$. These facts was investigated for interval exchanges by Rauzy in two seminal papers [61, 62]. The induced map on $[0,\alpha)$ (but also on $[\alpha,1)$ as well as on $[0,1-\alpha)$ and $[1-\alpha,1)$) allows to identify τ_α to the discrete special flow S_α built above the translation $\tau_{\alpha|_{[0,\alpha)}}$ and under the map $f : [0,\alpha) \to \mathbb{N}$ defined by

$$f(t) := \begin{cases} a_1 & \text{if } 0 \le t < 1 - a_1\alpha, \\ a_1 - 1 & \text{if } 1 - a_1\alpha \le t < \alpha. \end{cases}$$

More precisely, S_α is defined on

$$\Omega_\alpha := \Big(\bigcup_{0 \le j \le a_1} \tau_\alpha^j [0, 1 - a_1\alpha) \times \{j\} \Big) \cup \Big(\bigcup_{0 \le j < a_1} \tau_\alpha^j [1 - a_1\alpha, \alpha) \times \{j\} \Big)$$

by $S_\alpha(x,j) = (x, j+1)$ if $0 \le j < f(x)$ and $S_\alpha(x, f(x)) = (\tau_{\alpha|_{[0,\alpha)}}(x), 0)$. The special flow S_α for α with $a_1 = 2$ is depicted by the left graphics in Figure 2 and the same construction is continued but now with the induced translation $\tau_{\alpha|_{[0,\alpha)}}$ in place of τ_α, leading to a new special flow representing τ_α and depicted by the right graphics in Figure 2.

3.2. Cutting-stacking

The cutting-stacking method given below is quite general and useful to construct step by step interval exchanges. It was used initially in 1940 by von Neumann and Kakutani, then generalised and popularised by Friedman in his monograph [20] and used by many authors to produce various examples and counterexamples of dynamical systems. We intend to use this construction to build sequences in the unit interval with good discrepancy.

The basic objects of this method are columns $C = (I_1, \ldots, I_h)$ (also called towers) of disjoint subintervals $I_j := [c_j, d_j)$ of $[0,1)$ of same length called the

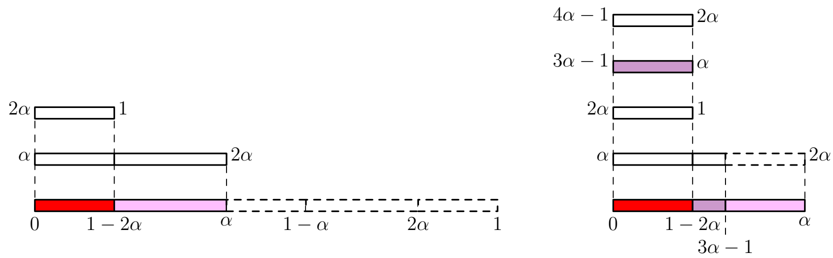

FIGURE 2. Geometric representations of τ_α as special flows in the case $\alpha = [0; 2, \dots]$. On the left, the special flow is above the induced map on $[0, \alpha)$ which is a translation of $3\alpha - 1$ modulo $\alpha\mathbb{Z}$. On the right, the special flow is above the induced transformation of τ_α on $[0, 3\alpha - 1)$.

width of C and denoted by $\ell(C)$. The interval I_1 is called the bottom of C, the interval I_h is called the top of C, the union $\operatorname{supp}(C) := \bigcup_{1\le j\le h} I_j$ is the support of C and the integer h its height. With the column C is associated a translation map $T_C : \operatorname{supp}(C) \setminus I_h \to \operatorname{supp}(C) \setminus I_1$ defined by $T_C(x) = x + (c_{j+1} - c_j)$ if $x \in I_j$, $1 \le j < h$. It is convenient to represent a column $C = (I_1, \dots, I_h)$ by drawing each interval I_{j+1} $(1 \le j < d)$ above the interval I_j completed if necessary by vertical arrows to figure the map T_C, the arrow issuing from the top interval I_h being labelled with an interrogation mark.

For a given finite set of columns $S := \{C_1, \dots, C_s\}$ with disjoint supports we associate the map T_S which coincides with T_{C_i} for $1 \le i \le s$. Also, by definition, $\operatorname{supp}(S) := \bigcup_{1\le i\le s} \operatorname{supp}(C_i)$ is the support of S and $\ell(S) := \sum_{1\le i\le s} \ell(C_i)$ is the width of S. In the sequel, we usually assume that the columns C_i of S are indexed according to the order of their bottoms, the one induced by the natural order of $[0, 1)$. A cutting of a column $C := (I_1, \dots, I_h)$ in t columns is by definition a set of columns $C_i := \{I_{i,1}, \dots, I_{i,h}\}$ such that $\bigcup_{1\le i\le t} I_{i,1} = I_1$ and each map T_{C_i} is the restriction of T_C on C_i. More generally, a cutting of a set S of columns is obtained by collecting all columns resulting by cutting part or all columns from S and then producing a new set of columns $S' := \{C'_1, \dots, C'_{s'}\}$. Now, a stacking of a column $C' := (I'_1, \dots, I'_{h'})$ above a column $C := (I_1, \dots, I_h)$ having same width and disjoint support is by definition the column $C * C' = (I_1, \dots, I_h, I'_1, \dots, I'_{h'})$. The map $T_{C*C'}$ extends both T_C and $T_{C'}$ and $T_{C*C'}$ translates I_h onto I'_1.

Notice that the stacking law $*$ is associative but not commutative. It is convenient to introduce the empty column $(\ \ )$ of height 0 and to set by definition $C * (\ \ ) = (\ \ ) * C = C$ for any column C. A sequence $\Sigma := (S_m)_{m\ge 0}$ of sets S_m of columns is said to be complete if $\operatorname{supp}(S_0) = [0, 1)$, $\lim_m \ell(S_m) = 0$ and for each $m \ge 1$, S_{m+1} is built from S_m by performing cutting and stacking

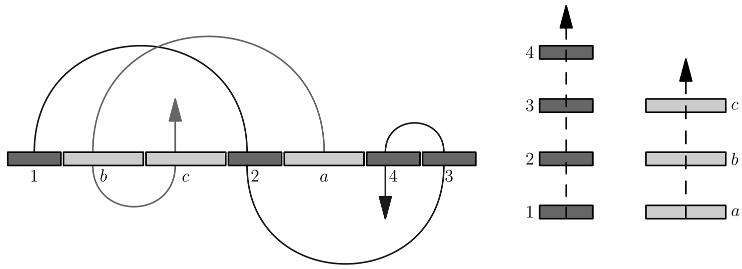

Figure 3. Cutting-stacking: an example starting from the unit interval. On the left, the interval is cut into a set of 7 subintervals noted from left to right $I_1, I_b, I_c, I_2, I_a, I_4, I_3$ and stacked following the arrows to give two columns $\{I_1, I_2, I_3, I_4\}$ and $\{I_a, I_b, I_c\}$.

but a finite number of times. By construction $T_{S_{m+1}}$ extends T_{S_m}. Let $\mathrm{top}(S_m)$ (resp. $\mathrm{bot}(S_m)$) be the union of top (resp. bottom) intervals of columns in S_m. Clearly $\mathrm{top}(S_{m+1}) \subset \mathrm{top}(S_m)$, $\mathrm{bot}(S_{m+1}) \subset \mathrm{bot}(S_m)$ and the intersections $\mathrm{top}(\Sigma) := \cap_{m\geq 0} \mathrm{top}(S_m)$, $\mathrm{bot}(\Sigma) := \cap_{m\geq 0} \mathrm{bot}(S_m)$ are at most countable, finite if the numbers of columns in infinitely many S_n are bounded. Clearly, no map T_{S_m} is defined on $\mathrm{top}(\Sigma)$ but it is easy to prove that for a complete sequence Σ there is a unique transformation $T : [0,1) \setminus \mathrm{top}(\Sigma) \to [0,1)$ which extends all T_{S_m}. Moreover, T is a measure-preserving map of $([0,1), \lambda)$ (well defined on $[0,1) \setminus \mathrm{top}(\Sigma)$) and invertible on $[0,1) \setminus \mathrm{bot}(\Sigma)$. We illustrate this construction with basic examples.

3.3. Kakutani-von Neumann transformation in base 2 revisited

Let us start with the interval $[0,1)$ viewed as a column of height 1, cut it into intervals $[0,1/2)$, $[1/2,1)$ and form the column $S_2 := ([0,1/2),[1/2,1))$. In the next step, cut the column S_2 into two columns of equal width

$$\{([0,1/4),[1/2,3/4)),([1/4,1/2),[3/4,1))\}$$

and stack the second column over the first one to get the column

$$S_3 = ([0,1/4),[1/2,3/4),[1/4,1/2),[3/4,1)).$$

Continuing in this way, we get the sequence of columns

$$S_n = ([v_0, v_0 + 2^{-n}),[v_1, v_1 + 2^{-n}),\ldots,[v_{2^n-1}, 1))$$

, where $(v_n)_n$ is the well known van der Corput sequence. In fact, cutting S_n into two columns of equal width and stacking the second column above the first one leads to S_{n+1} as a consequence of the classical formula $v_{j+2^n} = v_j + 1/2^{n+1}$ for $0 \leq j < 2^n$. Notice that the width of S_n is 2^{-n} and its height is 2^n.

The map $T_2 : [0, 1) \to [0, 1)$ obtained by this construction is well defined on $[0, 1)$, invertible on $(0, 1)$ and corresponds to the dyadic Kakutani-von Neumann map (see Definition 69). The above construction can be generalised in two directions. The first one consisting in cutting the current column S_n into $b \geq 2$ columns of equal width to get the column S_{n+1} by stacking these b columns from right to left, leading to the construction of the so-called b-adic Kakutani-von Neumann transformation T_b (see Figure 4 for $b = 3$, cf. [54, 45, 44]). The second generalisation consists in permuting the columns in every step of the stacking procedure. In fact this is the idea behind the modification of the van der Corput sequence introduced in [18]. It was later put into the dynamical context in [52].

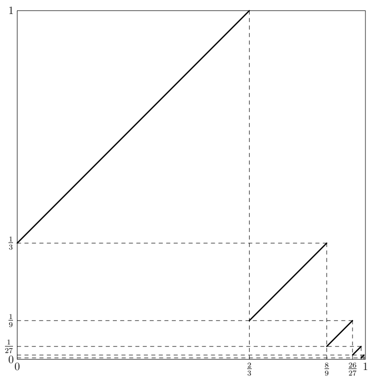

FIGURE 4. Partial graph of the Kakutani-von Neumann map in base 3 built by cutting-stacking

EXAMPLE 82. The orbit of 0 under T_b is the classical Halton sequence in dimension 1. Notice that for $b = b_1 \cdots b_s$ with the b_j pairwise primes, then $\mathbb{Z}_b$ is isomorphic to $\mathbb{Z}_{b_1} \times \cdots \times \mathbb{Z}_{b_s}$ so that Halton sequences in dimension 1 are also Halton sequences in dimension s by considering the product transformation $T_{b_1} \times \cdots \times T_{b_s}$.

More generally, for a given sequence $\mathbf{q} := (q_1, q_2, q_3, \dots)$ of integers q_i greater than 1, by cutting S_n into q_{n+1} columns and then build S_{n+1} by stacking these columns from left to right above the first to left column, we define in this way the standard transformation $T_{\mathbf{q}}$. Another simple generalisation is to start from an initial column of height $a \geq 1$ that plays the role of the initial interval. The second direction is to stack the q_{n+1} columns $C_{n,1}, \dots, C_{n,q_{n+1}}$ according to the multi-stacking $S_{n+1} : C_{n,\sigma_n(1)} * \cdots * C_{n,\sigma_n(q_{n+1})}$ where $\Sigma := (\sigma_n)_{n \geq 0}$ is a sequence of permutations of $\{1, \dots, q_{n+1}\}$. Hence we get a uniquely ergodic transformation that we denote by $T_{\mathbf{q},\Sigma}$. These latter constructions include all the previous ones and define uniquely ergodic transformations which are all metrically isomorphic to the translation $x \mapsto x + 1$ on the corresponding projective adic group $\lim_{\leftarrow} \mathbb{Z}/q_1 \dots q_n\mathbb{Z}$ (see [37] for details on these groups). It is of course not so evident that, for $\mathbf{q}$ fixed, all transformations $T_{\mathbf{q},\Sigma}$ are metrically isomorphic. One way to see this without exhibiting any isomorphism consists in proving that $T_{\mathbf{q},\Sigma}$ has discrete spectrum, in other words that the family of normalised eigenfunctions of $T_{\mathbf{q},\Sigma}$ form an orthonormal basis of the Hilbert space $L^2([0,1],\lambda)$. This fact follows easily from the observation that, for any integer $n \geq 0$, $T_{\mathbf{q},\Sigma}$ acts as a cyclic permutation on the set of intervals $\{[0, 1/p_n), [1/p_n, 2/p_n), \dots, [(p_n - 1)/p_n, 1)\}$ with $p_n = q_1 \dots q_n$.

Example 83. The above construction with constant base $\mathbf{b} = (b, b, b \dots)$ and any sequence $\Sigma = (\sigma_n)_{n\geq 0}$ of permutations of $\{0, 1, \dots, b-1\}$ is connected with Faure sequences S_b^Σ introduced in [16]. More precisely, one has

$$S_b^\Sigma(n) = \sum_{j=0}^{\infty} \sigma_j(e_j(n))b^{-j-1} = T^n_{\mathbf{b},\Sigma}(S_b^\Sigma(0)),$$

where $n = \sum_{j=0}^{\infty} e_j(n)b^j$ is the usual b-adic expansion of n.

3.4. Chacon transformation

Another celebrated example of a transformation built by cutting-stacking is the Chacon transformation introduced in [5]. Here S_1 is formed of two columns of one interval, precisely, $S_2 := \{([0, 2/3)), ([2/3, 1))\}$. The next step leads to (see Figure 5):

$$S_2 := \{([0, 2/9), [2/9, 4/9), [2/3, 8/9), [4/9, 2/3)), ([8/9, 1))\}\,.$$

At step n, one has two columns $S_n = \{\Theta_n, C_n\}$ where Θ_n is of height $h_n = \frac{3^n-1}{2}$,

Figure 5. Chacon construction: first step

with bottom $B_n = [0, \frac{2}{3^n})$ and support $\operatorname{supp}(\Theta_n) = [0, 1 - \frac{1}{3^n})$, and $C_n = ([1 - \frac{1}{3^n}, 1))$. Now S_{n+1} is built by cutting Θ_n in three columns $\{\Theta_n^0, \Theta_n^1, \Theta_n^2\}$ of width

$\frac{2}{3^{n+1}}$, C_n is cut in two intervals $C_n^0 := [1-\frac{1}{3^n}, 1-\frac{1}{3^{n+1}})$ and $C_n^1 = [1-\frac{1}{3^{n+1}}, 1)$. These columns are stacked to produce $S_{n+1} = \{\Theta_{n+1}, C_{n+1}\}$ with $\Theta_{n+1} = \Theta_n^0 * \Theta_n^1 * C_n^0 * \Theta_n^2$ and $C_{n+1} = C_n^1$. The Chacon transformation $C : [0,1) \to [0,1)$ defined in this way is one-to-one from $[0,1)$ onto $(0,1)$, is continuous at all points except those which are extremities of intervals involved in the cutting-stacking construction but is left continuous at these points. Friedman [21] proved that the dynamical system $(C, [0,1), \lambda)$ is ergodic, weakly mixing but not strongly mixing with singular spectrum. In addition, all points x in $[0,1)$ are λ-generic for C. Moreover, the set of discontinuities of C has two accumulation points, namely $2/3$ and 1 (identified to 0). Consequently, C is quasi-continuous and in fact, as we shall exhibit in the next section more generally, C is uniformly quasi-continuous.

3.5. Transformations of rank one

The Chacon transformation is a particular example of a rank-one transformation introduced by Ornstein [53]. We put apart the general definition to give an equivalent one within in the framework of interval exchanges.

Definition 84. A measure-preserving transformation is said to be *of rank one* if it is metrically isomorphic to an interval exchange $T : [0,1) \to [0,1)$ built by cutting-stacking in the following way: there exists $b_0 \in (0,1)$, a sequence $(q_n)_{n\geq 0}$ of integers and a family of integers $\{a_{n,i}\,;\ n \geq 0\ \&\ 0 \leq i < q_n\}$ such that $q_n \geq 1$, $a_{n,i} \geq 0$, and the sequence $(h_n)_n$ defined by the recurrence

$$h_0 = 1, \quad h_{n+1} = q_n h_n + \sum_{0 \leq i < q_n} a_{n,i},$$

verifies

$$\lim_n \frac{h_{n+1} b_0}{q_0 \cdots q_n} = 1.$$

Furthermore, T is constructed from a sequence of two columns $S_n := \{\Theta_n, C_n\}$ where Θ_n is a column of height h_n of bottom B_n, C_n is a tower of height 1 (the flat column) and the cutting-stacking transformation of S_n to S_{n+1} runs as follow:

(i)$\forall n \in \mathbb{N}_0 : B_n = [0, b_n)$, $C_n = [c_n, 1)$ and $h_n b_n = c_n$; (ii) Θ_n is cut in q_n sub-columns $\Theta_{n,i}$ of bottoms $[\frac{ib_n}{q_n}, \frac{(i+1)b_n}{q_n})$, $0 \leq i < q_n$ and C_n is cut in $a_n + 1$ intervals with $a_n = \sum_{i=0}^{q_n-1} a_{n,i}$ intervals of length $\frac{b_n}{q_n}$ namely $I(n,j) := [c_n + \frac{jb_n}{q_n}, c_n + \frac{(j+1)b_n}{q_n})$ for $0 \leq j < a_n$.

(iii) Let

$$C_{n,i} = I(n, d_{n,i}) * I(n, d_{n,i}+1) * \cdots * I(n, d_{n,i+1}-1),$$

$0 \leq i \leq q_n$, where $d_{n,i} := \sum_{0\leq k<i} a_{n,k}$ ($d_{n,0} = 0$, $d_{n,q_n} = a_n$), eventually $C_{n,i}$ is empty if $a_{n,i} = 0$. Each column $C_{n,i}$ is stacked above the column $\Theta_{n,i}$ to form the columns $\Theta'_{n,i} = \Theta_{n,i} * C_{n,i}$;

(iv) All columns $\Theta'_{n,i}$ are stacked to form the column $\Theta_{n+1} = \Theta'_{n,0} * \cdots * \Theta'_{n,q_n-1}$ of height h_{n+1} and width $b_{n+1} = \frac{b_n}{q_n}$. In addition $c_{n+1} = c_n + \frac{a_n b_n}{q_n}$.

The map T built by this construction is well-defined on $[0, 1)$, one-to-one and is referred as a standard rank-one transformation. In Figure 6 we have represented the n-th cutting-stacking construction that built T step by step. The dynamical

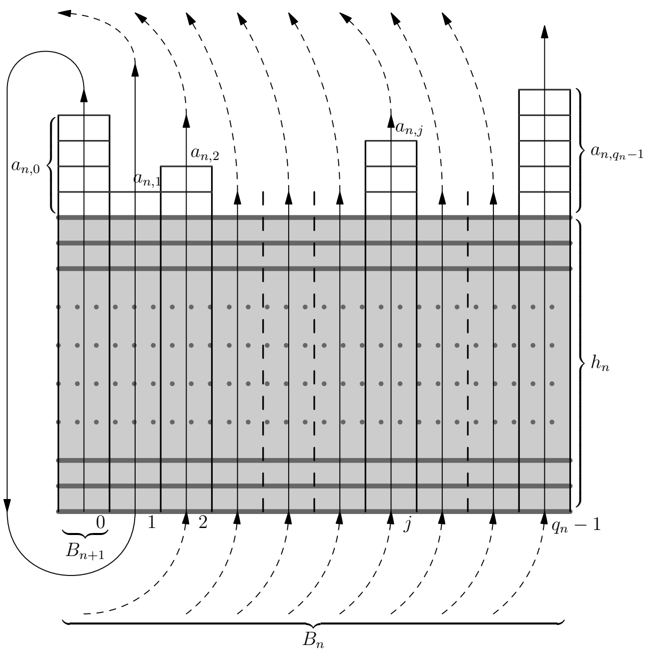

FIGURE 6. Rank one construction: the n-th step

system (T, I, λ) is ergodic. This result follows from the observation that the induced maps on each column Θ_n is nothing but a Kakutani-von Neumann-like transformation which is ergodic. According to Chacon [6], the spectral type of a rank one transformation is simple and Choksi and Nadkarni [7] have shown that, with the notations of Definition 84, the spectral type T is given by the

Riesz product

$$\prod_{n\geq 0}\frac{1}{q_n}\Big|\sum_{j=0}^{q_n-1} e^{2i\pi x(jh_n+d_{n,j})}\Big|^2\lambda(dx)\,.$$

3.5.1. Discrepancy of orbit sequences of rank-one transformations

Let T be a rank-one transformation defined with notations as above. In order to estimate the discrepancy of any orbit sequence $n\mapsto T^n x$ ($x\in[0,1)$), we introduce the scale $H:=(h_n)_n$. It is well known that any nonnegative integer n can be expanded in a sum $n=e_0(n)h_0+e_1(n)h_1+\cdots+e_k(n)h_k$ with $e_k\neq 0$. The uniqueness of the integers $e_j(n)$ is guaranteed if all inequalities

$$e_0(n)h_0+\cdots+e_j(n)h_j<h_{j+1}\quad (0\leq j\leq k(n))$$

hold. The integer $k(n)$ is the height of n and the digits $e_j(n)$ are equal to 0 for $j>k$. The sum of digits function s_H in scale H is defined by $s_H(n)=\sum_{j\geq 0}e_j(n)$. For numeration from scales we refer to the seminal paper of Fraenkel [19] and for more dynamical investigations, we refer to [3, 4, 27].

Theorem 85 (see [12]). *Let $(T,[0,1),\lambda)$ be a rank one transformation defined by Definition 84. Assume $q_n\geq 2$ for all indices n, the set of all $a_{n,i}$ is bounded, say by M, and $a_{n,q_n-1}=0$ for n large enough. Then, for any continuous map $f:[0,1)\to\mathbb{C}$, one has*

$$\lim_{N\to\infty}\sup_{0\leq x<1}\Big|\frac{1}{N}\sum_{0\leq n<N}f(T^n(x))-\int_0^1 f(t)\lambda(dt)\Big|=0$$

This theorem has two interesting consequences: first it shows that $(T,[0,1),\lambda)$ is uniquely ergodic and consequently the sequence $(T^n(x))_{n\in\mathbb{N}}$ is well distributed modulo 1 for each $x\in[0,1)$. Moreover, using the fact that the set of discontinuity points of T has at most two accumulation points, one gets that T is uniformly quasi-continuous. The discrepancy is estimated by the next result.

Theorem 86 (see [12]). *Under the assumption of Theorem 85,*

$$ND_N(x,Tx,\ldots,T^{N-1}x)\leq\sum_{j\geq 0}e_j(N)h_j\Big(\frac{q_0\cdots q_{j-1}}{b_0h_j}-1\Big)+s_H(N)M.$$

As a consequence,

Corollary 87.

$$\lim_{N\to\infty}\sup_{0\leq x<1}D_N(x,Tx,\ldots,T^{N-1}x)=0.$$

Moreover, we have

Corollary 88. *If the sequence* $(q_n)_n$ *and the set of integers* $a_{n,j}$ *are bounded with* $a_{n,q_n-1} = 0$ *for* n *large enough, then*

$$D_N(x, Tx, \ldots, T^{N-1}x) = \mathcal{O}\left(\frac{\log N}{N}\right).$$

This result is a consequence of the assumptions which imply that the sequence $(h_n(1-c_n))_n$ is bounded and $s_H(N) = \mathcal{O}(\log N)$.

Remark 89. If we omit the assumption that the set of integers $a_{n,j}$ is bounded or that if $a_{n,q_{n-1}} \neq 0$ for infinitely many n then one can find a sequence of points x_N such that such that $\lim_{N\to\infty} D_N(x_N, Tx_N, \ldots, T^{N-1}x_N) = 1$.

3.6. Substitution maps on the unit interval

Let $\mathcal{A}$ be a non empty set, also called alphabet, of s elements also called letters. Usually we take $\mathcal{A} := \{1, \ldots, s\}$. A word w of length $|w| := n$ on $\mathcal{A}$ is a string $w_1 \cdots w_n$ of n letters w_j in $\mathcal{A}$. Formally the word of length 0 is introduced, called empty word and denoted by $\wedge$. For any letter a, the number of occurrences of a in w is denoted $|w|_a$. Hence $|w| = \sum_{a\in\mathcal{A}} |w|_a$. The set $\mathcal{A}^*$ of words on $\mathcal{A}$, equipped with the concatenation law $(v_1 \cdots v_m, w_1 \cdots w_n) \mapsto v_1 \cdots v_m w_1 \cdots w_n$ is the free monoid generated by $\mathcal{A}$, the empty word being the neutral element.

Definition 90. A monoid endomorphism $\sigma : \mathcal{A}^* \to \mathcal{A}^*$ is called a *substitution on* $\mathcal{A}$ (or simply a *substitution* if the reference to alphabet $\mathcal{A}$ is unambiguous) if $|\sigma(a)| \geq 1$ for all letters a in $\mathcal{A}$. If $\sigma(a) = \wedge$ for at least one letter, we say that σ is a *pseudo-substitution*.

In case $\mathcal{A} = \{1, \ldots, s\}$, the following matrix $M(\sigma)$ associated with the substitution σ

$$M(\sigma) := \begin{pmatrix} |\sigma(1)|_1 & \ldots & |\sigma(s)|_1 \\ \vdots & & \vdots \\ |\sigma(1)|_s & \ldots & |\sigma(s)|_s \end{pmatrix}$$

is called companion matrix of σ. It will play a fundamental role. Let $\mathcal{P}_s$ be the cone of positive column vectors ℓ in $\mathbb{R}^s$, that is to say all entries ℓ_i of ℓ are positive. For any couple (ℓ, ℓ') of vectors in $\mathcal{P}_s$, we say that ℓ' derives from ℓ by σ, and we write $\ell \xrightarrow{\sigma} \ell'$, if the relation

$$\ell = M(\sigma)\ell'$$

holds.

3.6.1. A basic construction

Now, we apply the cutting-stacking construction to a set of column $S := \{C_1, \ldots, C_s\}$ according to σ to build a set of columns $S' := \{C'_1, \ldots, C'_s\}$. To make this construction possible, let ℓ be the column vector in $\mathcal{P}_s$ with entries $\ell_i = \ell(C_i)$ and assume that there exists $\ell' \in \mathcal{P}_s$ such that $\ell \overset{\sigma}{\to} \ell'$. Now cut each column C_j in order to create a set of $|\sigma|_j := \sum_{1\le k\le s} |\sigma(k)|_j$ (sum of entries of the j-th line of $M(\sigma)$) sub-columns $S_j := \{C_{j,1}, \ldots, C_{j,|\sigma|_j}\}$ such that $|\sigma(k)|_j$ of them have width ℓ'_k. This cutting is of course not unique but will be selected later. Then, for each k, build the column C'_k by stacking $\sum_{1\le j\le s} |\sigma(k)|_j$ sub-columns such that:

(i) $|\sigma(k)|_j$ sub-columns come from the sub-columns of width ℓ'_k in S_j;

(ii) from bottom to the top the column C'_k is built according to the word $\sigma(k) := \sigma_{k,1} \cdots \sigma_{k,|\sigma(k)|}$:

$$C'_k = T^{(1)}_{\sigma_{k,1}} * T^{(2)}_{\sigma_{k,2}} * \cdots * T^{(|\sigma(k)|)}_{\sigma_{k,|\sigma(k)|}},$$

where $T^{(j)}_{\sigma_{k,j}}$ is a column from $S_{\sigma_{k,j}}$ not yet used. The word $\sigma(k)$ is called the label of C'_k. In the standard construction, we select the successive $T^{(j)}_{\sigma_{k,j}}$ in C'_k from left to right.

This construction is not unique but at least a standard one exists due to $\ell = M(\sigma)\ell'$. When we use a standard construction we shall say that the couple (S', ℓ') derives from (S, ℓ) by taking into account the derivation $\ell \overset{\sigma}{\to} \ell'$ and write $(S, \ell) \overset{\sigma}{\to} (S', \ell')$.

Example 91. Figure 7 illustrates the first two steps of this construction (the standard one) with the substitution

$$\sigma(1) = 122 \quad \sigma(2) = 13 \quad \sigma(3) = 4 \quad \sigma(4) = 3. \tag{19}$$

In that case, the companion matrix is

$$M(\sigma) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{20}$$

Step 0 consist in cutting $[0, 1)$ into the 4 intervals $I_k := [\frac{k-1}{4}, \frac{k}{4})$ producing the initial set of columns $S^{(0)}$ (but we can start as well with any set $S^{(0)}$ of four columns of width $1/4$). Now, cut and stack twice according to σ in a standard manner to get the sets of columns $S^{(1)} := \{C^{(1)}_1, \ldots, C^{(1)}_4\}$ and

$S^{(2)} := \{C_1^{(2)}, \ldots, C_4^{(2)}\}$ with successive vectors of widths

$$\ell^{(0)} = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}, \quad \ell^{(1)} = \begin{pmatrix} 1/8 \\ 1/8 \\ 1/4 \\ 1/8 \end{pmatrix}, \quad \ell^{(2)} = \begin{pmatrix} 1/16 \\ 1/16 \\ 1/8 \\ 3/16 \end{pmatrix}, \tag{21}$$

chosen such that $\ell^{(0)} = M(\sigma)\ell^{(1)}$ and $\ell^{(1)} = M(\sigma)\ell^{(2)}$. Notice that there are many ways to stack intervals at each steps but the standard choice is tacitly assumed in Figure 7. Now if we replace each interval J that occurs in the construction of $S^{(1)}$ and $S^{(2)}$ by the letter j if $J \subset I_j$ then the columns $C_i^{(1)}$ and $C_i^{(2)}$ $(1 \le i \le 4)$ are replaced by words written vertically such that from bottom to top we can read respectively the word $\sigma(i)$ for $C_i^{(1)}$ and the word $\sigma^2(i)$ for $C_i^{(2)}$ (see Figure 7).

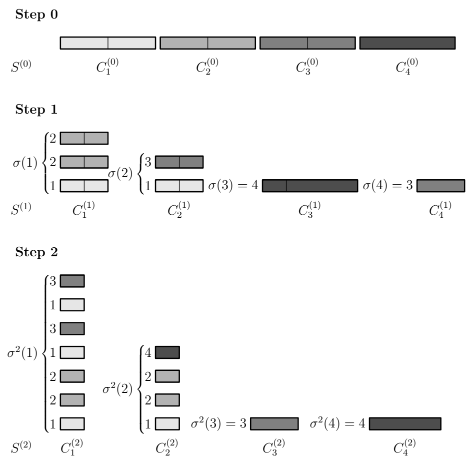

FIGURE 7. The two initial steps with the adapted substitution σ defined in (19) and the vectors of widths given in (21). Columns $C_i^{(k)}$ are built with intervals labelled from bottom to top according to $\sigma^k(i)$.

3.6.2. Adapted substitutions

In order to identify a substitution σ for which the above derivation process can be iterated to produce a suitable transformation T_σ, we introduce some definitions attached to any substitution on $\mathcal{A}$. For any part $\mathcal{B}$ of $\mathcal{A}$ we associate the pseudo-substitution $\tau_{\mathcal{B}}$ defined on $\mathcal{A}$ by $\tau_{\mathcal{B}}(x) = x$ if $x \in \mathcal{B}$ and $\tau_{\mathcal{B}}(x) = \wedge$ otherwise.

The map $\tau_{\mathcal{B}}$ is obviously extended to a morphism of monoids from $\mathcal{A}^*$ to $\mathcal{B}^*$. Now we define the pseudo-substitution $\sigma_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^*$ induced by σ on $\mathcal{B}$ by $\sigma_{\mathcal{B}}(b) = \tau_{\mathcal{B}}(\sigma(b))$. For any substitution $\sigma : \mathcal{A} \to \mathcal{A}^*$, the following definition is classical:

Definition 92. A letter a is said to be *expansive for* σ if the increasing sequence $n \mapsto |\sigma^n(a)|$ is unbounded.

Now we associate to σ the sets:

$$\begin{aligned} &E(\sigma) := \{a \in \mathcal{A}\,;\ a \text{ is expansive}\},\\ &L(\sigma) := \{a \in \mathcal{A}\,;\ \forall\, n,\ |\sigma^n(a)| = 1\},\\ &B(\sigma) := \mathcal{A} \setminus (E(\sigma) \cup L(\sigma)). \end{aligned}$$

Notice that if $E := E(\sigma)$ is non-empty, the pseudo-substitution σ_E induced by σ on E is a substitution but $E(\sigma_E)$ could be empty (for example, the substitution σ given by $\sigma(a) = ab$ and $\sigma(b) = b$ verifies $E(\sigma) = \{a\}$, $L(\sigma) = \{b\}$ and σ_E has no expansive letter). The substitution σ is said to be expansive if $E(\sigma) = \mathcal{A}$.

According to Dekking [10] the substitution σ is called semi-primitive if $E(\sigma) \neq \emptyset$ and the substitution induced by σ on $E(\sigma)$ is primitive, that is to say, there exists an integer $k \geq 1$ such that $|\sigma^k(a)|_b \geq 1$ for all couples (a, b) of expansive letters. We shall use a somewhat weaker form of this definition, namely:

Definition 93. A substitution σ is called *adapted* if $E(\sigma) \neq \emptyset$ and for all expansive letters a and all letters x from the alphabet, there exists an integer $k \geq 1$ such that $|\sigma^k(a)|_x \geq 1$.

Notice that from the definition, if σ is adapted then $\sigma_{E(\sigma)}$ is irreducible. The next definition is related to the period of irreducible matrices:

Definition 94. Let σ be an adapted substitution. The *period h of σ* is the period of the companion matrix of $\sigma_{E(\sigma)}$.

Therefore, h is given from any expansive letter a by

$$h := \gcd\{k \geq 1\,;\ |\sigma_E^k(a)|_a \geq 1\}\,.$$

Primitive substitutions are adapted and characterised by all letters expansive and $h = 1$. Semi-primitive substitutions are not adapted if there is an expansive letter a and a letter b such that $|\sigma^k(a)|_b = 0$ for all integers $k \geq 1$.

Examples 95.

95.1 If $\mathcal{A} = \{1\}$, the substitution $1 \to 1^b$ for any integer $b \geq 1$ is primitive and is expansive if $b \geq 2$.

95.2 The Thue-Morse substitution μ on $\{0, 1\}$ given by

$$\mu(0) := 01 \quad \mu(1) := 10$$

is primitive and expansive (and so expansive since the alphabet has more than one letter). In particular μ is adapted.

95.3 The Rudin-Shapiro substitution ρ on $\{a, b, c, d\}$ defined by

$$\rho(a) := ab \quad \rho(b) := ac \quad \rho(c) := db \quad \rho(d) := dc$$

is also primitive.

95.4 The Fibonacci substitution

$$f_1(0) = 01 \quad f_1(1) = 0$$

is primitive. More generally, all substitutions of the form

$$f_a(0) = 01^a \quad f_a(1) = 01^{a-1}$$

are primitive.

95.5 The so-called period-doubling substitution

$$\sigma(0) = 11 \quad \sigma(1) = 10$$

is primitive.

95.6 The Chacon substitution $c(0) := 0010 \quad c(1) := 1$ is not primitive, but semi-primitive and adapted with $E(c) = \{0\}$ and $L(c) = \{1\}$.

95.7 The substitution $0 \to 002$, $1 \to 2$, $2 \to 2$ is semi-primitive but not adapted.

95.8 The substitution $0 \to 11$, $1 \to 00$ is adapted but not semi-primitive. Its period is 2.

95.9 All letters of the substitution $0 \to 00$, $1 \to 01$ are expansive but the substitution is neither adapted nor semi-primitive.

The following theorem generalises the classical Perron-Frobenius relative to primitive matrices with non-negative entries:

Theorem 96 (Frobenius theorem revisited)**.** *Let σ be an adapted substitution with companion matrix $M := M(\sigma)$. Then*

(i) M has an eigenvalue $\theta > 1$ and $\theta \geq |\lambda|$ for all eigenvalues λ of M,

(ii) θ has an eigenvector with positive entries,

(iii) θ is simple.

The eigenvalue θ of the above theorem is called the dominant eigenvalue of $M(\sigma)$ or of σ and the unique eigenvector associated with θ such that the sum of its entries is equal to 1 is called the unitary dominant eigenvector of $M(\sigma)$, or of σ. If σ has p expansive letters we may assume that they form the set $E = \{1, \ldots, p\}$ so that the matrix $M(\sigma)$ takes the form

$$M = \begin{pmatrix} M(\sigma_E) & \mathbf{0} \\ A & B \end{pmatrix}.$$

The dominant eigenvalue θ of M is also the dominant eigenvalue of $M(\sigma_E)$ and the first p entries of the dominant eigenvector ℓ of M, after normalisation is the dominant eigenvector of $M(\sigma_E)$.

3.6.3. The cutting-stacking process

We are ready to construct an interval exchange $T_\sigma : [0,1) \to [0,1)$ by a cutting-stacking process associated to an adapted substitution σ with dominating eigenvalue θ. The construction is generally not unique but all of them lead to metrically conjugate transformations. We use the above notation and denote by ℓ the unitary dominant eigenvector of $M(\sigma)$. We start by cutting $[0,1)$ in s intervals $I_r = [\,\sum_{1\le j<r} \ell_j, \sum_{1\le j\le r} \ell_j\,)$ $(1 \le r \le s)$ to form the set of column $S_0 := \{I_1, \ldots, I_s\}$ and then apply the basic standard construction of Section 3.6.1 to build a set of columns $S_1 := \{C_1^{(1)}, \ldots, C_s^{(1)}\}$ such that $(S_0, \ell) \xrightarrow{\sigma} (S_1, \theta^{-1}\ell)$. We iterate this derivation to built successive sets of columns $S_n := \{C_1^{(n)}, \ldots, C_s^{(n)}\}$ given by successive derivations $(S_n, \theta^{-n}\ell) \xrightarrow{\sigma} (S_{n+1}, \theta^{-n-1}\ell)$. The resulting sequence $\Sigma := (S_n)_{n\ge 0}$ is complete and defines a map $T_\sigma : [0,1) \setminus \mathrm{top}(\Sigma) \to [0,1)$.

Remark 97. In general, even if we choose each derivation

$$(S_n, \theta^{-n}\ell) \xrightarrow{\sigma} (S_{n+1}, \theta^{-n-1}\ell)$$

standard it may happen that $\mathrm{top}(\Sigma)$ is not empty but in any case if at each step of the derivation we form columns of label $\sigma^{n+1}(k)$ by selecting first all expansive letters k and end with a non expansive letter, then T will be well-defined on each interval I_j with non expansive letter j. Concerning the construction of columns issuing from expansive letters, we can select each derivative and order which lead to $\mathrm{top}(\Sigma) = \emptyset$. This is clear in the case one has only one expansive letter and there are just one possible standard construction. In the other cases, we use the following fact true for every expansive letter k: there are infinitely many integers n such that the derivation $(S_n, \theta^{-n}\ell) \xrightarrow{\sigma} (S_{n+1}, \theta^{-n-1}\ell)$ puts at least one sub-column, say $C_{k,1}^{(n)}$, below a sub-column coming form the other sub-columns produced by the cutting of S_n. Since we are free to manage the cutting of $C_k^{(n)}$ we may assume that $C_{k,1}^{(n)}$ is the left most sub-interval of $\mathrm{top}(C_k^{(n)})$. This construction repeated infinitely many times for each expansive letter k determines a sequence $\Sigma := (S_n)_{n\ge 0}$ such that the resulting map T_σ is defined on $[0,1)$ (hence $\mathrm{top}(\Sigma) = \emptyset$).

From the above remark, for any adapted substitution σ there exists a map T_σ which is defined on $[0,1)$. The following theorems classify the ergodic structure of T_σ.

Theorem 98 (see [12]). *Let σ be an adapted substitution, let E be the set of expansive letters, and let T_σ be a map built by the cutting-stacking process associated to σ. Let h be the period of the substitution σ_E. Then T_σ has h ergodic components and is uniquely ergodic on each such component.*

The substitution τ on $\{1, 2\}$ given by $1 \to 112$ and $2 \to 2$ generates a transformation T_τ which also corresponds to the construction of a rank one transformation viewed before. Then T_τ is uniquely ergodic but notice that, with the notations of the above construction, for any integer $N \geq 1$ there exists x_N in the interval I_2 $(= [0, 1/2))$ such that $T_\tau^n(x_N)$ belongs to I_2 for all integers n with $0 \leq n < N$. Hence T_σ is not uniformly quasi-continuous. Nevertheless, T_τ is quasi-continuous.

3.7. Applications

We return to the previous examples to illustrate the cutting-stacking process.

3.7.1. Kakutani-von Neumann transformation once again

The standard construction of T_σ in case of the substitution $0 \to 0^b$ with $b \geq 2$ is unique and corresponds to the construction of the Kakutani-von Neumann transformation described in Section 3.3. The variation presented in Example 83 according to a sequence of permutations σ_n of $\{0, \ldots, b-1\}$ is obtained by a cutting-stacking process where the n-th derivation is determined by the permutation σ_n.

3.7.2. Thue-Morse transformation T_μ.

Recall that $\mu(0) = 01$ and $\mu(1) = 10$. The current n-th derivation in the cutting-stacking process used to define T_μ is given in Figure 8. The map T_μ is defined on $[0, 1)$, is uniquely ergodic and uniformly quasi-continuous. The interest of our choice of derivation is to get rather easily that T_μ is metrically conjugate by the map $F : [0, 1) \to [0, 1) \times \{+1, -1\}$ with

$$F(x) = (2x - \lfloor 2x \rfloor, (-1)^{\lfloor 2x \rfloor}),$$

to the skew product $T_\varphi : [0, 1) \times \{+1, -1\}$ defined by

$$T_\varphi(t, \varepsilon) = (Tx, \varepsilon\varphi(t)),$$

where T is the Kakutani-von Newmann transformation in base two and φ : $[0, 1) \to \{+1, -1\}$ is the cocycle defined by

$$\varphi(t) = \sum_{k=1}^{\infty} (-1)^k \mathbb{1}_{I_k}(t),$$

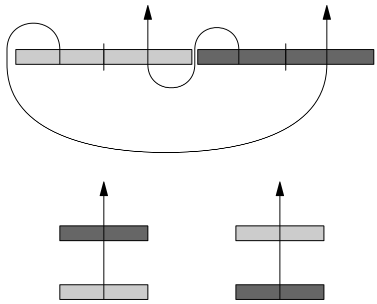

FIGURE 8. The current derivative in the cutting-stacking process associated to the Thue-Morse substitution μ

with $I_k := [1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k})$ (see [12]). The points $T_\mu^n(0)$ are related to the van der Corput sequence v_n by the formula

$$T_\mu^n(0) = \frac{v_n}{2} + \frac{1-(-1)^{s_2(n)}}{4}\,,$$

where $s_2(n)$ denote the sum of digits of n in base 2. It is easy to see that there is an absolute constant c such that for all $x \in [0,1)$,

$$D_N((T_\mu^n(x))_n) \leq c\frac{\log N}{N}\,.$$

3.7.3. The Rudin-Shapiro transformation T_ρ

We consider the substitution ρ of the Example 95.3. and use the derivation depicted in Figure 9 to define T_ρ by cutting-stacking. Notice that the selected derivation is regular and T_ρ is defined on $[0,1)$. The map is also uniformly quasi-continuous.

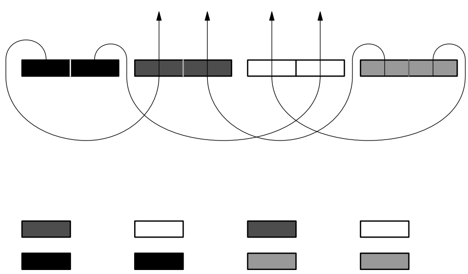

FIGURE 9. The current derivative in the cutting-stacking process associated to the Rudin-Shapiro substitution ρ

Let $R := R_0R_1R_2\dots$ be the fixed infinite word under the substitution ρ with $R_0 = a$, that is to say $R = \rho(R_0)\rho(R_1)\rho(R_2)\dots$. The binary Rudin-Shapiro sequence $r := (r_n)_{n\geq 0}$ is derived from R by substituting 0 to a and b, and 1 to c and d. Hence $r = 00010010\dots$. It is well known that $r_n = \frac{1-(-1)^{s_{11}(n)}}{2}$ where $s_{11}(n)$ count the number of occurrence of the block-digit 11 in the usual binary expansion of n. According to the construction of T_ρ one has also

$$r_n = \mathbb{1}_{[1/4,1)}(T_\rho^n(0)) .$$

3.7.4. The Fibonacci transformation

With the Fibonacci substitution $f = f_1$ (see example 95.4) the companion matrix is $\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ with dominant eigenvalue the golden ratio $\theta = \frac{1+\sqrt{5}}{2}$ and normalised dominant eigenvector the transpose of the line $[\alpha, 1-\alpha]$ with $\alpha = \theta^{-1} = \theta - 1$. Consequently, one has the relation $1 = F_{n+1}\alpha^{n+1} + F_n\alpha^{n+2}$ where $(F_n)_{n\geq 0}$ is the usual Fibonacci sequence $F_0 = 1$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. We also have by induction $\alpha^{n+1} = (-1)^n(F_n\alpha - F_{n-1}) = \|F_n\alpha\|$. Since we want to build a Fibonacci transformation T_f that is related to the translation $x \mapsto x+\alpha$ modulo 1 we use the cutting-stacking derivations $(S_n, \ell^{(n)}) \xrightarrow{f} (S_{n+1}, \ell^{(n+1)})$ where the first two steps are

$$S^{(0)} = \{[1-\alpha, 1), [0, 1-\alpha)\}, \quad \ell^{(0)} = \begin{pmatrix}\alpha\\ \alpha^2\end{pmatrix}$$

$$S^{(1)} = \left\{\begin{array}{ll} [0, 1-\alpha) & \\ [1-\alpha, 2-2\alpha)\ , & [2-2\alpha, 1) \end{array}\right\}, \quad \ell^{(1)} = \begin{pmatrix}\alpha^2\\ \alpha^3\end{pmatrix} .$$

$$S^{(2)} = \left\{\begin{array}{ll} [2-2\alpha, 1) & \\ [2-3\alpha, 1-\alpha) & [0, 2-3\alpha) \\ [3-4\alpha, 2-2\alpha)\ , & [1-\alpha, 3-4\alpha) \end{array}\right\}, \quad \ell^{(1)} = \begin{pmatrix}\alpha^3\\ \alpha^4\end{pmatrix} .$$

The next derivation depends on the parity of n. S^n is constituted by two columns $C_0^{(n)}$ and $C_1^{(n)}$ of height respectively F_{n+1} and F_n with $\ell(C_0^{(n)}) = \alpha^{n+1}$, $\ell(C_1^{(n)}) = \alpha^{n+2}$. Moreover, defining $\langle x\rangle := x - \lfloor x\rfloor$ one as

$$\mathrm{top}(C_0^{(2m)}) = [\langle -F_{2m}\alpha\rangle, 1), \quad \mathrm{top}(C_1^{(2m)}) = [0, \langle -F_{2m+1}\alpha\rangle)$$

and

$$\mathrm{top}(C_0^{(2m+1)}) = [0, \langle -F_{2m+1}\alpha\rangle), \quad \mathrm{top}(C_1^{(2m+1)}) = [\langle -F_{2m+2}\alpha\rangle, 1).$$

Figure 10 depicts $S^{(n)}$ for $n = 2m$ and $n = 2m+1$. In the figure only the tops and bottoms of the columns are depicted.

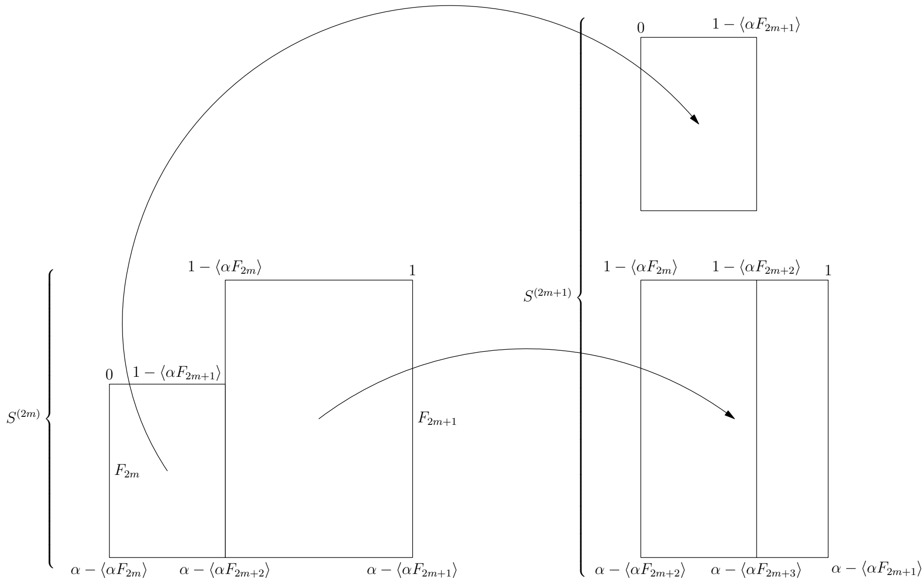

FIGURE 10. The derivation $S_{2m} \xrightarrow{f} S_{2m+1}$: the small tower is put left above the big one. The derivation $S_{2m+1} \xrightarrow{f} S_{2m+2}$ is similar: the small tower is put right above the big one

Notice that in this construction T_f is not defined at the point 0. If we translate the construction of S_n by F_{n+2} we get two Rokhlin towers with bottoms $[1-\|F_{2m+1}\alpha\|,1)$ and $[0,\|F_{2m}\alpha\|)$ if $n=2m$ and $[1-\|F_{2m+1}\alpha\|,1)$ and $[0,\|F_{2m+2}\alpha\|)$ if $n=2m+1$. We let the interested reader to extend this construction with the substitutions f_a given in Example 95.4 and then the general case of any irrational translation modulo one. Also the classical theorem of three lengths clearly is obtained through such a construction.

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