L^p-DISCREPANCY AND STATISTICAL INDEPENDENCE OF SEQUENCES

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Dedicated to Prof. Tibor Šalát on the occasion of his 70th birthday

ABSTRACT. We characterize statistical independence of sequences by the L^p-discrepancy and the Wiener L^p-discrepancy. Furthermore, we find asymptotic information on the distribution of the L^2-discrepancy of sequences.

1. Introduction

Let x_n and y_n be two infinite sequences in the unit interval $[0, 1)$. The pair of sequences (x_n, y_n) is called *statistically independent* if

$$\lim_{N \to \infty} \left(\frac{1}{N} \sum_{n=1}^{N} f(x_n)g(y_n) - \frac{1}{N^2} \sum_{n=1}^{N} f(x_n) \sum_{n=1}^{N} g(y_n) \right) = 0$$

for all continuous real functions f, g defined on $[0, 1]$, cf. [11]. In other words, the double sequence (x_n, y_n) is called statistically independent if it has statistically independent coordinate sequences x_n and y_n.

For (x_n, y_n) and any $p > 0$ we define the L^p statistical independence discrepancy $_S D_N^{(p)}$, the Wiener L^p statistical independence discrepancy $_S W_N^{(p)}$, and the statistical independence *star* discrepancy $_S D_N^*$ by the following: denote

$$F_N(x, y) := \frac{1}{N} \sum_{n=1}^{N} \chi_{[0,x)}(x_n)\chi_{[0,y)}(y_n),$$

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where $\chi_{[0,x)}(t)$ is the *characteristic function* of the interval $[0,x)$. Then

$$
{}_sD_N^{(p)} := \int_0^1 \int_0^1 \left| F_N(x,y) - F_N(x,1)F_N(1,y) \right|^p \, \mathrm{d}x \, \mathrm{d}y,
$$

$$
{}_sW_N^{(p)} := \int_{\mathcal{C}_0} \int_{\mathcal{C}_0} \left| \frac{1}{N} \sum_{n=1}^N f(x_n)g(y_n) - \frac{1}{N^2} \sum_{n=1}^N f(x_n) \sum_{n=1}^N g(y_n) \right|^p \, \mathrm{d}f \, \mathrm{d}g, \qquad (1.1)
$$

$$
{}_sD_N^* := \sup_{x,y \in [0,1]} \left| F_N(x,y) - F_N(x,1)F_N(1,y) \right|,
$$

where $\mathrm{d}f$ is the Wiener measure on the set $\mathcal{C}_0$ of all continuous functions defined on $[0,1]$ satisfying $f(0) = 0$. Furthermore, we write ${}_sD_N^{(p)} = {}_sD_N^{(p)}(x_n, y_n)$ and similarly for ${}_sW_N^{(p)}$ and ${}_sD_N^*$.

These definitions of discrepancy originate from the theory of uniform distribution of sequences, where the star discrepancy, the L^p-discrepancy and the Wiener discrepancy are given by

$$
D_N^*(x_n) = \sup_{x \in [0,1]} \left| F_N(x) - x \right|,
$$

$$
D_N^{(p)} = \int_0^1 \left| F_N(x) - x \right|^p \, \mathrm{d}x, \qquad (1.2)
$$

$$
W_N^{(p)} = \int_{\mathcal{C}_0} \left| \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_0^1 f(x) \, \mathrm{d}x \right|^p \, \mathrm{d}f,
$$

where $F_N(x) := \frac{1}{N} \sum_{n=1}^N \chi_{[0,x)}(x_n)$. Again, a sequence x_n is called uniformly distributed, if $D_N^*(x_n)$ tends to 0 for $N \to \infty$. This is equivalent to $\lim_{N \to \infty} D_N^{(p)} = 0$ and $\lim_{N \to \infty} W_N^{(p)} = 0$ (cf. [9]).

The following explicit formulæ for statistical independence discrepancies are known. In [5] the following formula is given:

$$
{}_sD_N^{(2)} = \frac{1}{16\pi^4} \sum_{\substack{k,l=-\infty \\ k,l \neq 0}}^{\infty} \frac{1}{k^2 l^2} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i(kx_n + ly_n)} - \frac{1}{N^2} \sum_{n=1}^N \sum_{m=1}^N e^{2\pi i(kx_n + ly_m)} \right|^2.
$$

$$(1.3)$$

Furthermore, in [13] an alternative expression is presented:

$$
\begin{aligned}
{}_sD_N^{(2)} =& \frac{1}{N^2} \sum_{m,n}^N \left(1 - \max(x_m, x_n)\right)\left(1 - \max(y_m, y_n)\right) \\
&+ \frac{1}{N^4} \sum_{m,n,k,l=1}^N \left(1 - \max(x_m, x_k)\right)\left(1 - \max(y_n, y_l)\right) \\
&- \frac{2}{N^3} \sum_{m,k,l=1}^N \left(1 - \max(x_m, x_k)\right)\left(1 - \max(y_m, y_l)\right). \qquad (1.4)
\end{aligned}
$$

For the Wiener L^2 statistical independence discrepancy in [13] we have

$$
\begin{aligned}
_SW_N^{(2)} =& \frac{1}{N^2} \sum_{m,n}^N \frac{\min(x_m, x_n)}{2} \frac{\min(y_m, y_n)}{2} + \frac{1}{N^4} \sum_{m,n,k,l=1}^N \frac{\min(x_m, x_n)}{2} \frac{\min(y_k, y_l)}{2} \\
& - \frac{2}{N^3} \sum_{m,k,l=1}^N \frac{\min(x_m, x_k)}{2} \frac{\min(y_m, y_l)}{2}.
\end{aligned}
\tag{1.5}
$$

These are extensions of classical formulæ, which can be found in [9]. The notion of Wiener discrepancy was introduced in [13].

In [5] it is proved that $\lim_{N\to\infty} {}_SD_N^* = 0$ does not characterize the statistical independence of (x_n, y_n). On the other hand, $\lim_{N\to\infty} {}_SD_N^{(p)} = 0$ for $p = 2$ is a characterization and it has been conjectured that the same is true also for any $p > 0$. In Section 2 we will prove this conjecture and we will also prove the same for the Wiener discrepancy ${}_SW_N^{(p)}$. Moreover, we will see that the statistical independence is fully described by the set of distribution functions of a given sequence (x_n, y_n).

In [13] it is proved that ${}_SW_N^{(2)} = \frac{1}{4} {}_SD_N^{(2)}$, but a similar relation for ${}_SW_N^{(p)}$, $p > 0$ is not valid, which we will demonstrate in Section 4.

In Section 3 of this paper we will discuss the asymptotical distribution of L^2-discrepancy. This continues investigations of the star discrepancy due to Kolmogorov [8]. It is now well-known that

$$
\lim_{N\to\infty} \mathbb{P}\left(\sqrt{N} D_N^*(x_n) < t\right) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 t^2}.
\tag{1.6}
$$

We will make use of a heuristic approach to this result due to Doob [4], which has been justified by Donsker [3]. The heuristic states that the discrepancy function $F_N(x) - x$ behaves like a trajectory of the Wiener process. Especially this behaviour holds for continuous functionals of the discrepancy function, as the supremum or the L^p-norm.

2. Statistical independence

As we have mentioned in the introduction, the equivalence

$$
(x_n, y_n) \text{ is statistically independent} \iff \lim_{N\to\infty} {}_SD_N^{(2)} = 0
$$

was proved in [5]. We shall extend this characterization of statistical independence to any $p > 0$. To do this we need the following notation:

For a given infinite sequence (x_n, y_n) in $[0, 1)^2$, let $G(x_n, y_n)$ be the set of all distribution functions of (x_n, y_n).

Here $g : [0, 1]^2 \to [0, 1]$ is a distribution function of (x_n, y_n) if there exists an increasing sequence of indices $N_1 < N_2 < \ldots$ such that $\lim_{k\to\infty} F_{N_k}(x, y) = g(x, y)$ for every point $(x, y) \in [0, 1]^2$. Following [9, p. 54] two distribution functions g_1 and g_2 are considered to be equivalent, if $g_1(x, y) = g_2(x, y)$ a.e. on $[0, 1]^2$ or equivalently, $g_1(x, y) = g_2(x, y)$ for every $(x, y) \in [0, 1]^2$ where both g_1 and g_2 are continuous.

Theorem 1. *For any sequence (x_n, y_n) in $[0, 1)^2$ and any $p > 0$ we have*

$$(x_n, y_n) \text{ is statistically independent} \iff \lim_{N \to \infty} {}_S D_N^{(p)} = 0.$$

Proof. By the well known first Helly lemma and the Lebesgue theorem of dominated convergence we have

$$\lim_{N \to \infty} \int_0^1 \int_0^1 \left| F_N(x, y) - F_N(x, 1) F_N(1, y) \right|^p \mathrm{d}x\,\mathrm{d}y = 0 \iff$$

$$\forall \big(g \in G(x_n, y_n)\big) \int_0^1 \int_0^1 |g(x, y) - g(x, 1)g(1, y)|^p \mathrm{d}x\,\mathrm{d}y = 0.$$

The right hand side is true for all $p > 0$, and for $p = 2$, the left hand side characterizes the statistical independence. Thus the proof is complete. $\square$

The following is an immediate consequence of the above proof:

Theorem 2. *For every $(x_n, y_n) \in [0, 1)^2$,*

$$(x_n, y_n) \text{ is statistically independent} \iff$$
$$\forall \big(g \in G(x_n, y_n)\big) g(x, y) = g(x, 1)g(1, y) \text{ a.e. on } [0, 1]^2.$$

Using the proof of Theorem 1 with Remark 1 in [13] and observing that any neighbourhood in the supremum topology in $\mathcal{C}_0$ has a positive Wiener's measure, we have a condition for statistical independence in terms of the Wiener statistical independence discrepancy.

Theorem 3. *For any $p > 0$ the sequence (x_n, y_n) is statistically independent, if and only if*

$$\lim_{N \to \infty} {}_S W_N^{(p)} = 0.$$

Using Theorem 2 we can describe the case when the star discrepancy ${}_S D_N^*$ tends to 0.

Theorem 4. *If $G(x_n, y_n)$ contain only continuous distribution functions, then*

$$(x_n, y_n) \text{ is statistically independent} \iff \lim_{N \to \infty} {}_S D_N^* = 0.$$

Proof. The case $\Longleftarrow$ follows immediately. The implication $\Longrightarrow$ follows from Theorem 2 and the fact that, for continuous $g \in G(x_n, y_n)$, the convergence

$$\lim_{k \to \infty} F_{N_k}(x, y) = g(x, y)$$

is uniform in $[0, 1]^2$. Hence we have $\lim_{k \to \infty} {}_S D_{N_k}^* = 0$ and this leads to $\lim_{N \to \infty} {}_S D_N^* = 0$. $\square$

In [14] it is shown that one can use the Wiener-Schoenberg theorem for the proof of continuity of $g \in G(x_n)$ (cf. the monograph of L. Kuipers and H. Niederreiter [9, Th. 7.5, p. 55]). The same method can be used for $G(x_n, y_n)$.

3 Uniform distribution

In order to describe the asymptotic distribution function of the L^2-discrepancy, we use a theorem due to Donsker [3] and the well-known Feynman-Kac formula (cf. [7]). Donsker's theorem states that for a functional F, which is continuous in the uniform topology on the space of sample paths of the Wiener process, the following limit relation holds:

$$\lim_{N\to\infty} \mathbb{P}\left(F\left(\sqrt{N}\left(F_N(x)-x\right)\right)\le\alpha\right)=\mathbb{P}\left(F\left(x(.)\right)\le\alpha\right), \qquad (3.1)$$

where $x(t)$ is a trajectory of the Wiener process with $x(0)=x(1)=0$.

The Feynman-Kac formula relates the Laplace transform of the distribution function of the integral $\int_0^t V(x(\tau))\,d\tau$ (V is a positive function) to the solutions of the eigenvalue problem

$$\frac{1}{2}\psi''(x)-V(x)\psi(x)=-\lambda\psi(x), \quad \psi\in L^2(-\infty,\infty). \qquad (3.2)$$

The relation is given by the formula

$$\mathbb{E}\left(\exp\left(-\int_0^t V(x(\tau))\,d\tau\right)\,\bigg|\,x(t)=0\right)=\sqrt{2\pi t}\sum_n e^{-\lambda_n t}\psi_n(0)^2, \qquad (3.3)$$

where λ_n are the eigenvalues and ψ_n are the corresponding normalized eigenfunctions of (3.2).

In order to get information on the distribution function of L^2-discrepancy we have to study equation (3.2) for $V(x)=x^2$. Clearly, this procedure could also be applied for $V(x)=|x|^p$ to study the distribution of L^p-discrepancy, but it is not enough known to get as precise information as in the L^2-case. We will write

$$\Phi(T)=\lim_{N\to\infty}\mathbb{P}\left(\sqrt{N}D_N^{(2)}<T\right) \qquad (3.4)$$

for the limit distribution of the L^2-discrepancy.

First, we notice that by the rescaling property of the Wiener process we have

$$\mathbb{E}\left(\exp\left(-\int_0^t x(\tau)^2\,d\tau\right)\,\bigg|\,x(t)=0\right)=\mathbb{E}\left(\exp\left(-t^2\int_0^1 x(\tau)^2\,d\tau\right)\,\bigg|\,x(1)=0\right). \qquad (3.5)$$

For the case studied here equation (3.2) has the form

$$\frac{1}{2}\psi''(x)-x^2\psi(x)=-\lambda\psi(x),$$

which is the differential equation for the Hermite functions (cf. [10,p.253]). Thus we have $\lambda_n=\frac{2n+1}{\sqrt{2}}$ and

$$\psi_n(x)=\frac{\sqrt[8]{2}}{\sqrt[4]{\pi}}\frac{1}{2^n\sqrt{(2n)!}}e^{-\frac{x^2}{\sqrt{2}}}H_n\left(\sqrt[4]{2}x\right),$$

where H_n are the Hermite polynomials as defined in [10,p.249]. Hence we derive

$$\mathbb{E}\left(\exp\left(-\int_0^t x(\tau)^2\,\mathrm{d}\tau\right)\,\bigg|\,x(t)=0\right) = \sqrt{2\sqrt{2}t}\sum_{n=0}^{\infty}\exp\left(-\frac{4n+1}{\sqrt{2}}t\right)\frac{1}{4^n}\binom{2n}{n} =$$

$$= \sqrt{\frac{\sqrt{2}t}{\sinh\sqrt{2}t}}.$$

Using (3.5) we obtain

$$\mathbb{E}\left(\exp\left(-s\int_0^1 x(\tau)^2\,\mathrm{d}\tau\right)\,\bigg|\,x(1)=0\right) = \sqrt{\frac{\sqrt{2s}}{\sinh\sqrt{2s}}}$$

for the Laplace transform of the distribution function of the limit distribution of $N(D_N^{(2)})^2$. Notice that this function is holomorphic in the region $\Re s > -\frac{\pi^2}{2}$. Furthermore, it has a branch cut of the square-root type at the point $s = -\frac{\pi^2}{2}$. Thus using the Laplace inversion theorem and asymptotic techniques for the Laplace transform (cf. [2]) we obtain

$$\Phi(T) = 1 - \frac{1}{\sqrt{\pi T}}e^{-\frac{\pi^2}{2}T} + O\left(\frac{1}{T^{\frac{3}{2}}}e^{-\frac{\pi^2}{2}T}\right). \tag{3.6}$$

We remark here that for the case of L^p-discrepancy the whole procedure also works. Again the Laplace transform of the distribution function is holomorphic in a region $\Re s > -\varepsilon$ for some $\varepsilon > 0$, but this is a consequence of (1.6). We could not derive this analytic information from the knowledge of the asymptotics of the eigenvalues and eigenfunctions (cf. [15], [12]), nor could we find the location of the singularity of the largest real part, whose type would yield asymptotic information on the limiting distribution of the L^p-discrepancy.

4. RELATION BETWEEN WIENER'S AND CLASSICAL L^2 DISCREPANCY

We start with the Paley-Wiener formula (cf. [1]):

$$\int_{\mathcal{C}_0} F\left[\int_0^1 f(x)\,\mathrm{d}m(x)\right]\mathrm{d}f = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}F(bu)\,\mathrm{d}u, \quad b^2 = \int_0^1 m^2(t)dt,$$

where $F(u)$ is a (real or complex-valued) measurable function defined on $(-\infty,\infty)$ such that $e^{-u^2}F(bu)$ is of class L_1 and $m(1) = 0$. Thus, putting $F(u) = |u|^p$ and $m(x) = F_N(x) - x$, in the classical case we have

$$W_N^{(p)} = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)\left(D_N^{(2)}\right)^{\frac{p}{2}}.$$

Assuming $m(x,y) = m_1(x)m_2(y)$ on $[0,1]^2$ and $m_1(1) = m_2(1) = 0$, the Paley-Wiener formula can also be used for computing the two dimensional integral

$$\int_{\mathcal{C}_0}\int_{\mathcal{C}_0} F\left[\int_0^1\int_0^1 f(x)g(y)\,\mathrm{d}m(x,y)\right]\mathrm{d}f\,\mathrm{d}g.$$

For any x_1, x_2 and y_1, y_2 in $[0, 1)$, there exist $m_1(x)$ and $m_2(y)$, $m_1(1) = m_2(1) = 0$, such that $F_2(x, y) - F_2(x, 1)F_2(1, y) = m_1(x)m_2(y)$ $(x, y \in [0, 1])$. Hence

$$_sW_2^{(p)} = \frac{1}{\pi}\Gamma^2\left(\frac{p+1}{2}\right)\left(_sD_2^{(2)}\right)^{\frac{p}{2}}$$

for every $p > 0$.

The proof of $_sW_N^{(2)} = \frac{1}{4}{_sD_N^{(2)}}$ in [13] is also extremely simple: Using (1.3) we have

$$_sD_N^{(2)}(x_n, y_n) = {_sD_N^{(2)}}(1 - x_n, 1 - y_n)$$

and using $1 - \max(x_m, x_n) = \min(1 - x_m, 1 - x_n)$ and (1.5) we have the result.

These results give rise to the question whether there is a relation of the type

$$_sW_N^{(p)} = c_p\left(_sD_N^{(2)}\right)^{\frac{p}{2}} \tag{4.1}$$

between the different notions of statistical independence discrepancy. In the following we give explicit formulae for these discrepancies which lead to the negative answer.

The Paley-Wiener formula is equivalent to

$$\int_{\mathcal{C}_0}\left(\int_0^1 f(x)\,dm(x)\right)^{2k}df = \frac{(2k-1)!!}{2^k}\left(\int_0^1 dt\left(\int_0^1 \chi_{[t,1]}(x)\,dm(x)\right)^2\right)^k,$$

where $k = 1, 2, \ldots$, $(2k - 1)!! = (2k - 1)(2k - 3)\cdots 3 \cdot 1$ and for the exponent $2k + 1$ the left hand integral is zero. (For this formula the assumption $m(1) = 0$ is superfluous.) The formal two-dimensional analogue is the relation $A = cB$, where

$$A := \int_{\mathcal{C}_0}\int_{\mathcal{C}_0}\left(\int_0^1\int_0^1 f(x)g(y)\,dm(x, y)\right)^{2k}df\,dg,$$

$$B := \left(\int_0^1\int_0^1\left(\int_0^1\int_0^1 \chi_{[t_1,1]}(x)\chi_{[t_2,1]}(x)\,dm(x, y)\right)^2 dt_1\,dt_2\right)^k$$

and c is independent of $m(x, y)$. These integrals can be expressed as

$$A = \int_0^1\cdots\int_0^1\left(\int_{\mathcal{C}_0} f(u_1)\ldots f(u_{2k})\,df\right)\left(\int_{\mathcal{C}_0} g(v_1)\ldots g(v_{2k})\,dg\right)$$
$$dm(u_1, v_1)\ldots dm(u_{2k}, v_{2k}),$$

$$B = \int_0^1\cdots\int_0^1 \left(\min(u_1, u_2)\ldots\min(u_{2k-1}, u_{2k})\right)\left(\min(v_1, v_2)\ldots\min(v_{2k-1}, v_{2k})\right)$$
$$dm(u_1, v_1)\ldots dm(u_{2k}, v_{2k}).$$

Furthermore, by the well known formula (which can also be proved by applying the above Paley-Wiener formula)

$$\int_{\mathcal{C}_0} f(u_1)\ldots f(u_{2k})\,df = \frac{(2k-1)!!}{2^k(2k)!}\sum_\pi \min(u_{\pi(1)}, u_{\pi(2)})\ldots\min(u_{\pi(2k-1)}, u_{\pi(2k)}),$$

where the summation $\sum_\pi$ ranges over all permutations π of $(1, \ldots, 2k)$. For the odd case $2k+1$ the integral vanishes. Next we choose $m(x, y)$ such that $\mathrm{d}m(a_i, b_i) = z_i$ for $i = 1, \ldots 2k$, and $\mathrm{d}m(x, y) = 0$ otherwise. Here we shall view z_i as independent variables. Assuming $A = cB$ and comparing the coefficients at $z_1 \ldots z_{2k}$, we have $C = c'D$, where

$$C := \sum_\pi \left(\min(a_{\pi(1)}, a_{\pi(2)}) \ldots \min(a_{\pi(2k-1)}, a_{\pi(2k)}) \right) \times$$

$$\times \sum_\pi \left(\min(b_{\pi(1)}, b_{\pi(2)}) \ldots \min(b_{\pi(2k-1)}, b_{\pi(2k)}) \right),$$

$$D := \sum_\pi \left(\min(a_{\pi(1)}, a_{\pi(2)}) \ldots \min(a_{\pi(2k-1)}, a_{\pi(2k)}) \right) \times$$

$$\times \left(\min(b_{\pi(1)}, b_{\pi(2)}) \ldots \min(b_{\pi(2k-1)}, b_{\pi(2k)}) \right).$$

Putting $a_i = b_i$, $i = 1, \ldots, 2k$, we have

$$\left(\sum_\pi \left(\min(a_{\pi(1)}, a_{\pi(2)}) \ldots \min(a_{\pi(2k-1)}, a_{\pi(2k)}) \right) \right)^2$$

$$= c' \sum_\pi \left(\min(a_{\pi(1)}, a_{\pi(2)}) \ldots \min(a_{\pi(2k-1)}, a_{\pi(2k)}) \right)^2,$$

which is impossible, for $k > 1$ and general a_i.

The proof of impossibility of (4.1) is more difficult. First, we have mentioned that for

$$m(x, y) = F_N(x, y) - F_N(x, 1)F_N(1, y)$$

we have $A = {}_sW_N^{(2k)}$ and $B = ({}_sD_N^{(2)})^k$. Moreover, $\mathrm{d}m(x, y) \neq 0$ only for $x = x_m$ and $y = y_n$, where $1 \leq m, n \leq N$. Precisely, assuming that $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ are one-to-one we have

$$\mathrm{d}m(x_m, y_n) = \begin{cases} \frac{1}{N} - \frac{1}{N^2} & \text{if } m = n, \\ -\frac{1}{N^2} & \text{in other cases.} \end{cases}$$

For brevity, we shall use the following notation:

$$\mathbf{m} := (m_1, \ldots, m_{2k}),$$
$$\pi(\mathbf{m}) := (m_{\pi(1)}, \ldots, m_{\pi(2k)}),$$
$$\mathbf{x_m} := (x_{m_1}, \ldots, x_{m_{2k}}),$$
$$\mathbf{1} \leq \mathbf{m} \leq \mathbf{N} \iff 1 \leq m_1 \leq N \wedge \cdots \wedge 1 \leq m_{2k} \leq N,$$
$$l(\mathbf{m}, \mathbf{n}) := \#\{1 \leq i \leq 2k; m_i = n_i\},$$
$$\mu(\mathbf{x_m}) := \prod_{i=1}^k \min(x_{m_{2i-1}}, x_{m_{2i}}).$$

Computing the integrals A and B for such $m(x, y)$ we can find

$$_SW_N^{(2k)} = \frac{1}{N^{4k}} \left(\frac{1}{2^{2k}k!}\right)^2 \sum_{\substack{1 \leq \mathbf{m} \leq \mathbf{N} \\ 1 \leq \mathbf{n} \leq \mathbf{N}}} \mu(\mathbf{x_m})\mu(\mathbf{y_n}) \times$$

$$\times \sum_{\pi_1, \pi_2} (N-1)^{l(\pi_1(\mathbf{m}), \pi_2(\mathbf{n}))} \cdot (-1)^{2k - l(\pi_1(\mathbf{m}), \pi_2(\mathbf{n}))},$$

$$\left(_SD_N^{(2)}\right)^k = \frac{1}{N^{4k}} \sum_{\substack{1 \leq \mathbf{m} \leq \mathbf{N} \\ 1 \leq \mathbf{n} \leq \mathbf{N}}} \mu(\mathbf{x_m})\mu(\mathbf{y_n}) \times$$

$$\times (N-1)^{l(\mathbf{m}, \mathbf{n})} \cdot (-1)^{2k - l(\mathbf{m}, \mathbf{n})}.$$

We can regard $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ as independent variables. Then we see that $_SW_N^{(2k)}$ and $\left(_SD_N^{(2)}\right)^k$ are homogeneous polynomials of the degree k in $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$, respectively.

In the following denote

$$x_a = \max_{1 \leq i \leq N} x_i, \quad x_b = \max_{1 \leq i \leq N, i \neq a} x_i, \quad y_c = \max_{1 \leq i \leq N} y_i, \quad y_d = \max_{1 \leq i \leq N, i \neq c} y_i,$$

and let $a \neq c$ and $b = d$. Next we shall find coefficients of $x_a^{k-1}x_b y_c^{k-1} y_d$ in $_SW_N^{(2k)}$ and $\left(_SD_N^{(2)}\right)^k$, respectively.

First, $\mu(\mathbf{x_m}) = x_a^{k-1}x_b$ only for

$$\mathbf{m} = \begin{cases} (a, \ldots, a, b, a, \ldots, a) & \text{(type I)}, \\ (a, \ldots, a, b, b, a, \ldots, a) & \text{(type II)}, \end{cases}$$

where the couple (b, b) lies at the place with indices $(2i-1, 2i)$. We have $2k$ vectors of type I and $k(2k-1)$ vectors of type II. If $\mathbf{m}$ is of type I and π ranges over all permutations of $(1, \ldots, 2k)$, then all vectors of type I occur in $\pi(\mathbf{m})$ $(2k-1)!$ times. If $\mathbf{m}$ is of type II, then all vectors of the form

$$(a, \ldots, a, b, a, \ldots, a, b, a, \ldots, a) \text{ (type II')}$$

occur in $\pi(\mathbf{m})$ with multiplicity $2.(2k-2)!$. For $(\mathbf{m}, \mathbf{n})$ of type (I,I) we have $l(\mathbf{m}, \mathbf{n}) = 1$ in $2k$ cases and $l(\mathbf{m}, \mathbf{n}) = 0$ in $(2k)^2 - k$ cases. For $(\mathbf{m}, \mathbf{n})$ of type (I,II) we have $l(\mathbf{m}, \mathbf{n}) = 1$ in $2k$ cases and $l(\mathbf{m}, \mathbf{n}) = 0$ in $2k^2 - 2k$ cases. For type (II,II) we have only $l(\mathbf{m}, \mathbf{n}) = 2$ in k cases and $l(\mathbf{m}, \mathbf{n}) = 0$ in $k^2 - k$ cases. Similarly, for type (I,II') we have

$$l(\mathbf{m}, \mathbf{n}) = \begin{cases} 1 \text{ in } 2k(2k-1) \text{ cases}, \\ 0 \text{ in } k(2k-1)(2k-2) \text{ cases}, \end{cases}$$

and for (II',II') we have

$$l(\mathbf{m}, \mathbf{n}) = \begin{cases} 2 \text{ in } k(2k-1) \text{ cases}, \\ 1 \text{ in } 2k(2k-1)(2k-2) \text{ cases}, \\ 0 \text{ in } k(2k-1)(k-1)(2k-3) \text{ cases}. \end{cases}$$

Summing up all of the above we have

$$\sum_{\substack{1 \le \mathbf{m} \le \mathbf{N} \\ 1 \le \mathbf{n} \le \mathbf{N} \\ \mu(\mathbf{x_m}) = x_a^{k-1} x_b \\ \mu(\mathbf{y_n}) = y_c^{k-1} y_d}} (N-1)^{l(\mathbf{m},\mathbf{n})} \cdot (-1)^{2k-l(\mathbf{m},\mathbf{n})}$$

$$= k(N-1)^2 - 6k(N-1) + 9k^2 - 7k,$$

$$\sum_{\substack{1 \le \mathbf{m} \le \mathbf{N} \\ 1 \le \mathbf{n} \le \mathbf{N} \\ \mu(\mathbf{x_m}) = x_a^{k-1} x_b \\ \mu(\mathbf{y_n}) = y_c^{k-1} y_d}} \sum_{\pi_1, \pi_2} (N-1)^{l(\pi_1(\mathbf{m}), \pi_2(\mathbf{n}))} \cdot (-1)^{2k-l(\pi_1(\mathbf{m}), \pi_2(\mathbf{n}))} =$$

$$((2k)!)^2 \left((2k^2 - k)(N-1)^2 - (8k^3 - 4k^2 + 2k)(N-1) + (4k^4 - 4k^3 + 3k^2 - k) \right)$$

which is a contradiction to

$$_S W_N^{(2k)} = c_{2k} \left(_S D_N^{(2)} \right)^k.$$

5. EXAMPLES AND FURTHER RESULTS ON STATISTICAL INDEPENDENCE

Using the expressions (1.3), (1.4) and (1.5) we immediately have:

Theorem 5.
  (i)   *The sequences* (x_n, y_n), (y_n, x_n), $(1-x_n, y_n)$, $(1-x_n, 1-y_n)$ *and* $(t_1 x_n, t_2 x_n)$ *are simultaneously statistically independent. Here* $t_1, t_2 \in (0, 1]$ *and in the case* $x_n = 0$ *we reduce* $1 - x_n \bmod 1$.
  (ii) (c, y_n) *is statistically independent with any* $y_n, c \in [0, 1)$, *where* c *is a constant.*

Using an example given in [5] we will generalize (ii) in the following way. Define, for $\alpha \in [0, 1]$, the *one-jump* distribution function $c_\alpha(x)$ as

$$c_\alpha(x) = \begin{cases} 0, & \text{for } 0 \le x < \alpha, \\ 1, & \text{for } \alpha < x \le 1. \end{cases}$$

Theorem 6. *Assume that the sequence* x_n *in* $[0, 1)$ *has the limit law* c_α, *i.e.* $\lim_{N \to \infty} F_N(x) = c_\alpha(x)$ *a.e. Then for any sequence* y_n *in* $[0, 1)$ (x_n, y_n) *is statistically independent.*

Proof. For a continuous $g : [0, 1] \to \mathbb{R}$ we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) g(y_n) - \frac{1}{N^2} \sum_{n=1}^{N} f(x_n) \sum_{n=1}^{N} g(y_n) \right| \le 2 \sup_{x \in [0,1]} |g(x)| \frac{1}{N} \sum_{n=1}^{N} |f(x_n) - f(\alpha)|,$$

and for a continuous $f : [0, 1] \to \mathbb{R}$ we have

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |f(x_n) - f(\alpha)| = \int_0^1 |f(x) - f(\alpha)| \, \mathrm{d}c_\alpha(x) = 0. \quad \square$$

Theorem 7. *For sequences x_n, y_n, x'_n and y'_n in $[0, 1)$ we assume that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (|x_n - x'_n| + |y_n - y'_n|) = 0.$$

Then the sequences (x_n, y_n) and (x'_n, y'_n) are simultaneously statistically independent.

Proof. This follows from the expression (1.5) and from the fact that

$$||x - y||u - v| - |x' - y'||u' - v'|| \le |x - x'| + |y - y'| + |u - u'| + |v - v'|$$

for $x, y, u, v, x', y', u', v' \in [0, 1]$. $\square$

Motivated by Theorem 2, a trivial example of statistical independence is given by a sequence (x_n, y_n) which is uniformly distributed in the square. Another example is any sequence (x_n, y_n) which has only one-jump distribution functions. A more general example:

Let G_1 and G_2 be any nonempty closed and connected sets of one-dimensional distribution functions. Denote

$$G_1 \cdot G_2 := \{g_1(x)g_2(y); g_1 \in G_1, g_2 \in G_2\}.$$

Again $G_1 \cdot G_2$ is nonempty closed and connected and thus by R. Winkler [16] there exists a sequence (x_n, y_n) in $[0, 1)^2$ such that $G(x_n, y_n) = G_1 \cdot G_2$. By Theorem 2, this sequence is statistically independent.

Furthermore, Theorem 2 may be used for a generalization of the notion of statistical independence to the multidimensional sequence $(x_n, y_n, z_n, \dots)$ in $[0, 1)^s$ (precisely, the statistical independence of its coordinate sequences $x_n, y_n, z_n, \dots$) as follows:

$(x_n, y_n, z_n, \dots)$ is statistically independent if, for every distribution function $g \in G(x_n, y_n, z_n, \dots)$ we have

$$g(x, y, z, \dots) = g(x, 1, 1, \dots)g(1, y, 1, \dots)g(1, 1, z, \dots)\dots$$

a.e. on $[0, 1]^s$. As an example we give the following sequences described in [6]:

Let $\mathbf{x}_n$ be defined by

$$\mathbf{x}_n = \left((-1)^{[[\log^{(j)} n]^{1/p_1}]}[\log^{(j)} n]^{1/p_1}, \dots, (-1)^{[[\log^{(j)} n]^{1/p_s}]}[\log^{(j)} n]^{1/p_s}\right) \quad \mod 1,$$

where $\log^{(j)} n$ denotes the jth iterated logarithm $\log\dots\log n$, and $p_1, \dots, p_s$ are coprime positive integers. Then, for $j > 1$, the set of all distribution functions of $\mathbf{x}_n$ coincides (under equivalence) with the set of all one-jump distribution functions on $[0, 1]^s$ and thus the sequence $\mathbf{x}_n$ is statistically independent.

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