MINIMA OF DIGITAL FUNCTIONS RELATED TO LARGE DIGITS IN Q-ADIC EXPANSIONS

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Dedicated to Professor Helmut Prodinger on the occasion of his 50th birthday

Abstract. We study the extremal values of fractal continuous functions related to the counting function of the q-ary digits larger than d.

1. Introduction

In the recent paper [1] C. Cooper investigated the number $L_{10}(N)$ of digits ≥ 5 occurring in the decimal expansion of the positive integers $< N$. He gave upper and lower bounds for this number. The purpose of this note is to exhibit a periodic continuous function related to this problem and to study its properties. It turns out that this study gives a sharp lower bound for $L_{10}(N)$, which answers a question posed in [1].

Additive functions related to the q-adic expansion of integers and the behavior of their summatory functions have been studied from various points of view, see for instance [2, 9, 10, 3]. An arithmetic function $f : \mathbb{N} \to \mathbb{R}$ is called completely q-additive, if it satisfies the relation

$$f\left(\sum_{k=0}^{K} \varepsilon_k q^k\right) = \sum_{k=0}^{K} f(\varepsilon_k), \text{ for } \varepsilon_k \in \{0, \ldots, q-1\}. \tag{1.1}$$

The simplest example of such a function is the q-ary sum-of-digits function $s_q(n)$ given by

$$s_q\left(\sum_{k=0}^{K} \varepsilon_k q^k\right) = \sum_{k=0}^{K} \varepsilon_k,$$

which satisfies the following exact formula

$$\sum_{n<N} s(n) = \frac{q-1}{2} N \log_q N + NF(\log_q N), \tag{1.2}$$

where F denotes a continuous, periodic function of period 1. This function is nowhere differentiable and its minima have been computed in [4, 5, 6]. Asymptotic formulæ involving periodically fluctuating terms are frequently encountered in the context of digital functions, see for instance [3]. Formula (1.2) was one of the first occurrences of such behaviour. It

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was discovered by J. Trollope [12] and later reproved by H. Delange [2], who also gave an elementary derivation of the Fourier-coefficients of the periodic function.

The function $L_{10}(N)$ studied by Cooper can now be seen as the summatory function of the 10-additive function

$$\ell_{10}\left(\sum_{k=0}^{K}\varepsilon_k 10^k\right)=\sum_{k=0}^{K}[\varepsilon_k\geq 5],$$

where we use Iverson's notation. Since for any completely q-additive function there holds an exact formula similar to (1.2), one can give an exact expression for

$$L_{10}(N)=\frac{1}{2}N\log_{10}N+NG(\log_{10}N),$$

where G is again a continuous periodic function of period 1. Thus the question asked by Cooper can be translated into finding the minima of G. The periodicity of G reflects the observation made in [1] that $L_{10}(N)/N-\frac{1}{2}\log_{10}N$ attains the smallest value amongst all k-digit integers at $N=4545\ldots45$ or $N=45\ldots455$.

In this paper we will study a more general question. We introduce the q-additive function

$$f_{q,d}\left(\sum_{k=0}^{K}\varepsilon_k q^k\right)=\sum_{k=0}^{K}[\varepsilon_k\geq d] \tag{1.3}$$

which counts the number of occurrences of the digits $\geq d$ in the q-adic expansion of n. We will study the minima of a function on $[0,1]$ related to the summatory function of $f_{q,d}$ and apply these results to find sharp lower bounds for the function $L_q(N)$ counting the number of occurrences of digits $\geq\frac{q}{2}$ in the q-adic expansions of the integers $<N$ for even q. Figure 1 shows a plot of the periodic function $G_{10,5}$ occurring in the investigation of $L_{10}(N)$ below compared to the upper and lower bounds derived in this paper.

Finally, we mention that periodicity phenomena of the type shown above do not only occur in this number-theoretic context, but also in in the field of analytic combinatorics, especially in the average case analysis of recursive algorithms. For a recent survey on such phenomena in the analysis of algorithms we refer to [11].

2. Counting the digits $\geq d$

In order to derive a closed expression for

$$F_{q,d}(N)=\sum_{n<N}f_{q,d}(n)$$

we use a standard way of rewriting the sum of additive functions:

$$F_{q,d}(N)=\sum_{k=0}^{K}\sum_{N_{k+1}\leq n<N_k}f_{q,d}(n),$$

where

$$N_k=\sum_{j=k}^{K}\varepsilon_j q^j,\ \text{if } N=\sum_{j=0}^{K}\varepsilon_j q^j.$$

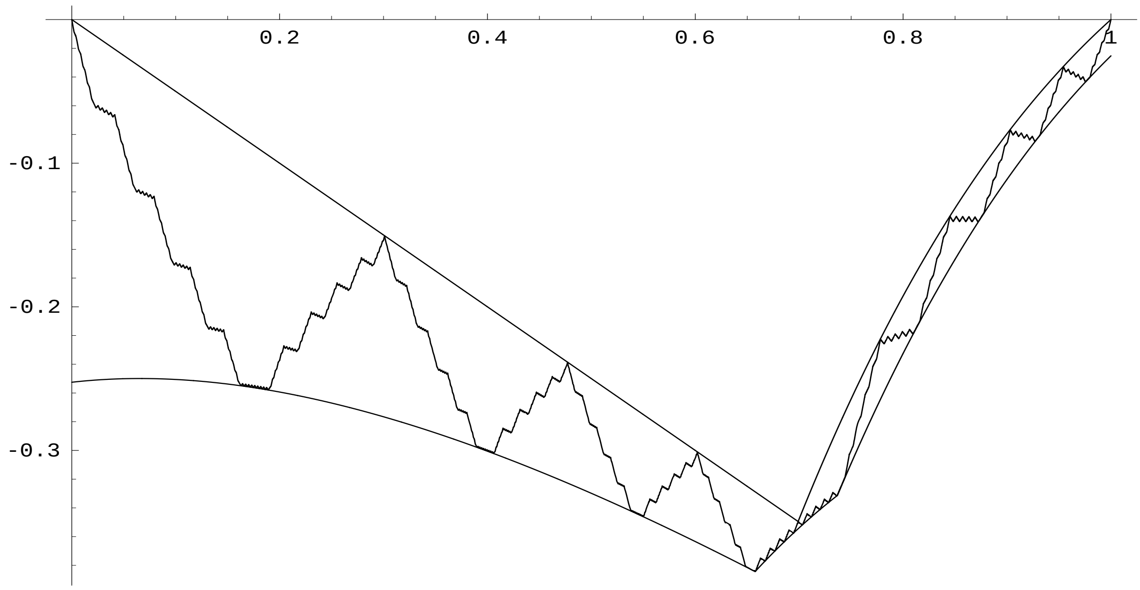

FIGURE 1. Plot of $G_{10,5}$ compared to upper and lower bounds.

Using the additivity of $f_{q,d}$ we obtain

$$F_{q,d}(N) = \sum_{k=0}^{K} \left(\varepsilon_k q^k f_{q,d}(N_{k+1}) + \sum_{m<\varepsilon_k q^k} f_{q,d}(m) \right). \tag{2.1}$$

Thus $F_{q,d}(N)$ can be expressed in terms of the digits of N, if $F_{q,d}(\varepsilon_k q^k)$ can be computed. It is a simple exercise to show that

$$\begin{aligned} F_{q,d}(\varepsilon_k q^k) &= \varepsilon_k F_{q,d}(q^k) + [\varepsilon_k \geq d](\varepsilon_k - d)q^k \\ F_{q,d}(q^k) &= qF_{q,d}(q^{k-1}) + (q-d)q^{k-1}, \end{aligned}$$

which gives

$$F_{q,d}(\varepsilon_k q^k) = \varepsilon_k (q-d)kq^k + [\varepsilon_k \geq d](\varepsilon_k - d)q^k. \tag{2.2}$$

Inserting this into (2.1) we obtain

$$F_{q,d}(N) = \sum_{k=0}^{K} q^k \left(\varepsilon_k \left(\sum_{j=k}^{K} [\varepsilon_j \geq d] + \frac{q-d}{q} k \right) - d[\varepsilon_k \geq d] \right). \tag{2.3}$$

Using the function $r_{q,d} : [0,1] \to \mathbb{R}$ defined by

$$r\left(\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q^k}, q, d \right) = \sum_{k=1}^{\infty} q^{-k} \left(\varepsilon_k \left(\sum_{\ell=1}^{k} [\varepsilon_\ell \geq d] - \frac{q-d}{q} k \right) - d[\varepsilon_k \geq d] \right). \tag{2.4}$$

we can rewrite (2.3) as

$$F_{q,d}(N)=\frac{q-d}{q}(K+1)N+q^{K+1}r\left(Nq^{-K-1},q,d\right).$$

Finally, the expression for $F_{q,d}(N)$ can be given in the form

(2.5)
$$F_{q,d}(N)=\frac{q-d}{q}N\log_q N+N\left(\frac{q-d}{q}\left(1-\left\{\log_q N\right\}\right)+q^{1-\{\log_q N\}}r\left(q^{\{\log_q N\}-1},q,d\right)\right)=\\ \frac{q-d}{q}N\log_q N+G_{q,d}(\{\log_q N\}),$$

where $\{x\}$ denotes the fractional part of x as usual. It is a simple exercise to check that the function

$$G_{q,d}(x)=\frac{q-d}{q}(1-x)+q^{1-x}r\left(q^{x-1},q,d\right)$$

is continuous on $[0,1]$ and satisfies $G_{q,d}(0)=G_{q,d}(1)=0$. Therefore, this function extends to a continuous periodic function on $\mathbb{R}$.

3. Study of the function $r(x,q,d)$

In this section we will first collect some simple properties of $r(x,q,d)$ and use these properties to find all minima of $r(x,q,d)$ for given q and d.

Lemma 1. *The function $r(x,q,d)$ is continuous on $[0,1]$ and satisfies the following relations:*

(3.1) $$r(x,q,d)=r(1-x,q,q-d)$$

(3.2) $$r(0,q,d)=r(1,q,d)=0$$

(3.3) $$\forall x\in[0,1]: r(x,q,d)\le\max\left(-\frac{q-d}{q}x,-\frac{d}{q}(1-x)\right)$$

(3.4) $$r\left(\frac{\varepsilon}{q},q,d\right)=\begin{cases}-\frac{\varepsilon(q-d)}{q^2} & \text{for } \varepsilon<d\\ -\frac{d(q-\varepsilon)}{q^2} & \text{for } \varepsilon\ge d\end{cases}.$$

Furthermore, r satisfies the following functional equation

(3.5)
$$r\left(\sum_{k=1}^{L}\frac{\varepsilon_k}{q^k}+q^{-L}y,q,d\right)=r\left(\sum_{k=1}^{L}\frac{\varepsilon_k}{q^k},q,d\right)+q^{-L}r(y,q,d)+q^{-L}y\left(\sum_{k=1}^{L}[\varepsilon_k\ge d]-\frac{q-d}{q}L\right).$$

Proof. The equations (3.1), (3.2), and (3.4) are immediate. The functional equation (3.5) can be proved by inserting the definition of $r(x,q,d)$. The upper bound (3.3) is proved by induction using (3.5) as follows: assume that (3.3) holds for all $x=\sum_{k=1}^{L}\frac{\varepsilon_k}{q^k}$ (for $L=1$

this is simply (3.4)). Write $y = \sum_{k=1}^{L+1} \frac{\delta_k}{q^k} = \frac{\delta_1}{q} + \frac{x}{q}$. Then we have for $\delta_1 < d$

$$r\left(\frac{\delta_1}{q} + \frac{y}{q}, q, d\right) = -\frac{q-d}{q}\frac{\delta_1}{q} + \frac{1}{q} r(x,q,d) - \frac{q-d}{q}\frac{x}{q} \le -\frac{q-d}{q}\left(\frac{\delta_1}{q} + \frac{x}{q}\right) = -\frac{q-d}{q} y,$$

where we have used $r(x,q,d) \le 0$. Similarly, for $\delta_1 \ge d$ we have

$$r\left(\frac{\delta_1}{q} + \frac{y}{q}, q, d\right) = -\frac{d}{q}\left(1 - \frac{\delta_1}{q}\right) + \frac{1}{q} r(x,q,d) + \frac{d}{q}\frac{x}{q} \le -\frac{d}{q}(1-y).$$

By continuity of the function $r(x,q,d)$ the inequality holds for all $x \in [0,1]$. □

From (3.5) we derive immediately by letting $y \to 1$ and using continuity

$$r\left(\sum_{\ell=1}^{L} \frac{\varepsilon_\ell}{q^\ell} + q^{-L}, q, d\right) = r\left(\sum_{\ell=1}^{L} \frac{\varepsilon_\ell}{q^\ell}, q, d\right) + q^{-L}\left(\sum_{\ell=1}^{L} [\varepsilon_\ell \ge d] - \frac{q-d}{q} L\right). \tag{3.6}$$

Similarly, we get for $\varepsilon_L \neq 0$:

$$r\left(\sum_{\ell=1}^{L} \frac{\varepsilon_\ell}{q^\ell} - q^{-L}, q, d\right) = r\left(\sum_{\ell=1}^{L} \frac{\varepsilon_\ell}{q^\ell}, q, d\right) - q^{-L}\left(\sum_{\ell=1}^{L} [\varepsilon_\ell \ge d] - \frac{q-d}{q} L - [\varepsilon_L = d]\right) \tag{3.7}$$

and for $\varepsilon_K \neq 0$, $\varepsilon_{K+1} = \cdots = \varepsilon_L = 0$

$$\begin{aligned} r\left(\sum_{\ell=1}^{L} \frac{\varepsilon_\ell}{q^\ell} - q^{-L}, q, d\right) &= \\ r\left(\sum_{\ell=1}^{L} \frac{\varepsilon_\ell}{q^\ell}, q, d\right) &- q^{-L}\left(\sum_{\ell=1}^{L} [\varepsilon_\ell \ge d] - \frac{q-d}{q} L + L - K - [\varepsilon_K = d]\right) \end{aligned} \tag{3.8}$$

Now want to study the minima of the function $r(x,q,d)$ for fixed q and d. Thus we make the following two definitions

$$m_L(q,d) = \min\left\{ r(x,q,d) \mid x = \sum_{k=1}^{L} \frac{\varepsilon_k}{q^k} \right\}$$

$$\mathcal{M}_L(q,d) = \left\{ x \mid r(x,q,d) = m_L(q,d), \quad x = \sum_{k=1}^{L} \frac{\varepsilon_k}{q^k} \right\}.$$

In the following we will frequently make use of the notation

$$Q = \frac{q}{\gcd(q,d)}.$$

Lemma 2. *Let $x = \sum_{\ell=1}^{L} \frac{\varepsilon_\ell}{q^\ell} \in \mathcal{M}_L(q,d)$. Then*

- *for $Q \nmid L$*

$$\sum_{\ell=1}^{L} [\varepsilon_\ell \ge d] = \left\lceil \frac{q-d}{q} L \right\rceil \quad \textit{and } \varepsilon_L = d \tag{3.9}$$

- *for $Q \mid L$*

$$\sum_{\ell=1}^{L}[\varepsilon_\ell\geq d]=\frac{q-d}{q}L \text{ and } \begin{cases}\varepsilon_{L-1}=d \text{ and } 0\leq\varepsilon_L<d \text{ or}\\ \varepsilon_{L-1}=d-1 \text{ and } d\leq\varepsilon_L\leq q-1\end{cases} \tag{3.10}$$

or

$$\sum_{\ell=1}^{L}[\varepsilon_\ell\geq d]=\frac{q-d}{q}L+1 \text{ and } \varepsilon_{L-1}=\varepsilon_L=d.$$

Furthermore, for $Q \mid L$ we have $m_{L-1}(q,d)=m_L(q,d)$.

Proof. Assume $x\in\mathcal{M}_L(q,d)$. Then by minimality of $r(x)$ we obtain

$$\sum_{\ell=1}^{L}[\varepsilon_\ell\geq d]\geq\frac{q-d}{q}L \tag{3.11}$$

from (3.6) and

$$\sum_{\ell=1}^{L}[\varepsilon_\ell\geq d]\leq\frac{q-d}{q}L-L+K+[\varepsilon_K=d] \tag{3.12}$$

from (3.8).

Assume now that $Q\nmid L$. Then $\frac{q-d}{q}L\notin\mathbb{Z}$ and the two inequalities above are strict. Thus we have $L<K+[\varepsilon_K=d]$, which is only possible, if $K=L$ and $\varepsilon_L=d$. Observing that

$$\frac{q-d}{q}L<\sum_{\ell=1}^{L}[\varepsilon_\ell\geq d]<\frac{q-d}{q}L+1$$

finishes the first case.

In the second case we have $Q\mid L$. Then $\frac{q-d}{q}L\in\mathbb{Z}$ and we can have equality in the above inequalities. In this case we have $K\leq L\leq K+[\varepsilon_K=d]$, which allows $K=L$ or $K=L-1$, which means that $\varepsilon_{L-1}=d$ and $\varepsilon_L=0$. It remains to discuss the case $K=L$. In this case we can have (3.11), which allows $\sum_{\ell=1}^{L}[\varepsilon_\ell\geq d]=\frac{q-d}{q}L$ or $\frac{q-d}{q}L+1$. The second alternative implies $\varepsilon_L=d$ by (3.12). Furthermore, we have

$$r\left(\sum_{\ell=1}^{L-1}\frac{\varepsilon_\ell}{q^\ell}+\frac{\delta}{q^L}\right)=r\left(\sum_{\ell=1}^{L-1}\frac{\varepsilon_\ell}{q^\ell}\right)+\frac{1}{q^{L-1}}r\left(\frac{\delta}{q}\right)+\frac{\delta}{q^L}\left(\sum_{\ell=1}^{L-1}[\varepsilon_\ell\geq d]-\frac{q-d}{q}(L-1)\right)=$$
$$r\left(\sum_{\ell=1}^{L-1}\frac{\varepsilon_\ell}{q^\ell}\right)+\begin{cases}0 & \text{for } 0\leq\delta\leq d\\ \frac{1}{q^L}(\delta-d) & \text{for } \delta>d,\end{cases}$$

where we used that $\sum_{\ell=1}^{L-1}[\varepsilon_\ell\geq d]=\frac{q-d}{q}L-1$. This implies that $\sum_{\ell=1}^{L-1}\frac{\varepsilon_\ell}{q^\ell}\in\mathcal{M}_{L-1}(q,d)$, since $m_{L-1}(q,d)\geq m_L(q,d)$ and the value for $\delta=0$ is an $(L-1)$-digit number. Since $Q\nmid(L-1)$ this implies $\varepsilon_{L-1}=d$ by (3.9).

It remains to discuss the case $\sum_{\ell=1}^{L}[\varepsilon_\ell \geq d] = \frac{q-d}{q}L$. In this case we have

$$r\left(\sum_{\ell=1}^{L}\frac{\varepsilon_\ell}{q^\ell}+\frac{1}{q^L}\right)=r\left(\sum_{\ell=1}^{L}\frac{\varepsilon_\ell}{q^\ell}\right)$$

by (3.6). Assume first that $\varepsilon_L < d$. Then we repeat this procedure until we reach $\varepsilon_L = d$. Then $\sum_{\ell=1}^{L}[\varepsilon_\ell \geq d]$ increases by 1 and we are in the case $\sum_{\ell=1}^{L}[\varepsilon_\ell \geq d] = \frac{q-d}{q}L+1$, which has been treated above, and which shows that $\varepsilon_{L-1} = d$ and $0 \leq \varepsilon_L \leq d$. Finally, assume that $\varepsilon_L \geq d$. In this case we can again iterate addition by q^{-L} until $\varepsilon_L = q-1$. If we add one more q^{-L} we arrive at the case $\varepsilon_L = 0$, which by the same arguments as above implies that the $(L-1)$-st digit was d. Since there was a carry in the last step, this means that $\varepsilon_{L-1} = d-1$ for $d \leq \varepsilon_L \leq q-1$. This finishes the proof. □

Lemma 3. *Let $1 \leq k < Q$. Then*

- $\left\{\frac{q-d}{q}k\right\} \geq \frac{q-d}{q} \Rightarrow m_k(q,d) = m_{k-1}(q,d) - \frac{q-d}{q^k}\left(1-\left\{\frac{q-d}{q}k\right\}\right).$
  In this case $\mathcal{M}_k(q,d) = \{x - \frac{q-d}{q^k} \mid x \in \mathcal{M}_{k-1}(q,d)\}$.
- $\left\{\frac{q-d}{q}k\right\} < \frac{q-d}{q} \Rightarrow m_k(q,d) = m_{k-1}(q,d) - \frac{d}{q^k}\left\{\frac{q-d}{q}k\right\}.$
  In this case $\mathcal{M}_k(q,d) = \{x + \frac{d}{q^k} \mid x \in \mathcal{M}_{k-1}(q,d)\}$.

Proof. Let $x \in \mathcal{M}_{k-1}(q,d)$ and $y = \sum_{\ell=1}^{k}\varepsilon_\ell q^{-\ell} \in \mathcal{M}_k(q,d)$. Then we compute

$$\begin{aligned} r\left(x+\frac{d}{q^k},q,d\right) &= m_{k-1}(q,d)+\frac{d}{q^k}\left(\frac{d}{q}-\left\{\frac{q-d}{q}(k-1)\right\}\right) \geq m_k(q,d)\\ r\left(x-\frac{q-d}{q^k},q,d\right) &= m_{k-1}(q,d)+\frac{q-d}{q^k}\left(\left\{\frac{q-d}{q}(k-1)\right\}-\frac{d}{q}\right) \geq m_k(q,d)\end{aligned} \tag{3.13}$$

and (using $\varepsilon_k = d$, which follows from $Q \nmid k$ by Lemma 2)

$$\begin{aligned} r\left(y-\frac{d}{q^k},q,d\right) &= m_k(q,d)+\frac{d}{q^k}\left\{\frac{q-d}{q}k\right\} \geq m_{k-1}(q,d)\\ r\left(y+\frac{q-d}{q^k},q,d\right) &= m_k+\frac{q-d}{q^k}\left(1-\left\{\frac{q-d}{q}k\right\}\right) \geq m_{k-1}(q,d).\end{aligned} \tag{3.14}$$

We now consider two cases:

- $\left\{\frac{q-d}{q}k\right\} \geq \frac{q-d}{q}$: in this case we have $\left\{\frac{q-d}{q}(k-1)\right\} = \left\{\frac{q-d}{q}k\right\} - \frac{q-d}{q}$. Inserting this into (3.13) it turns out that the left-hand-side of the second inequality is smaller and gives

  $$m_{k-1}(q,d)-\frac{q-d}{q^k}\left(1-\left\{\frac{q-d}{q}k\right\}\right) \geq m_k(q,d).$$

  The second inequality in (3.14) gives the opposite inequality, which implies that the asserted equality holds in the first case.

- $\left\{\dfrac{q-d}{q}k\right\} < \dfrac{q-d}{q}$: the proof in this case works in the same way as in the first case by taking the first inequalities in (3.13) and (3.14).

□

Lemma 4.

$$r\left(\sum_{\ell=1}^{L}\frac{\varepsilon_\ell}{q^\ell}\right) \geq m_L(q,d) - q^{-L}\left(\sum_{\ell=1}^{L}[\varepsilon_\ell \geq d] - \frac{q-d}{q}L\right) \tag{3.15}$$

Proof. From (3.5) we have

$$r\left(\sum_{\ell=1}^{L}\frac{\varepsilon_\ell}{q^\ell} + \frac{x}{q^L}\right) = r\left(\sum_{\ell=1}^{L}\frac{\varepsilon_\ell}{q^\ell}\right) + q^{-L}r(x) + \frac{x}{q^L}\left(\sum_{\ell=1}^{L}[\varepsilon_\ell \geq d] - \frac{q-d}{q}L\right).$$

Letting x tend to 1 and using continuity of r we obtain

$$r\left(\sum_{\ell=1}^{L}\frac{\varepsilon_\ell}{q^\ell}\right) = r\left(\sum_{\ell=1}^{L}\frac{\varepsilon_\ell}{q^\ell} + \frac{1}{q^L}\right) - \frac{1}{q^L}\left(\sum_{\ell=1}^{L}[\varepsilon_\ell \geq d] - \frac{q-d}{q}L\right) \geq$$
$$m_L(q,d) - q^{-L}\left(\sum_{\ell=1}^{L}[\varepsilon_\ell \geq d] - \frac{q-d}{q}L\right).$$

□

Lemma 5. *The minima of* $m_k(q,d)$ *satisfy*

$$m_{kQ+K}(q,d) = m_{kQ}(q,d) + q^{-kQ}m_K(q,d). \tag{3.16}$$

Proof. For any sequence of digits we have by (3.5)

$$r\left(\sum_{\ell=1}^{kQ+K}\frac{\varepsilon_\ell}{q^\ell}\right) = \tag{3.17}$$
$$r\left(\sum_{\ell=1}^{kQ}\frac{\varepsilon_\ell}{q^\ell}\right) + q^{-kQ}r\left(\sum_{\ell=1}^{K}\frac{\varepsilon_{kQ+\ell}}{q^\ell}\right) + q^{-kQ}\left(\sum_{\ell=1}^{K}\frac{\varepsilon_{kQ+\ell}}{q^\ell}\right)\left(\sum_{\ell=1}^{kQ}[\varepsilon_\ell \geq d] - \frac{q-d}{q}kQ\right).$$

If $\sum_{\ell=1}^{kQ}[\varepsilon_\ell \geq d] - \frac{q-d}{q}kQ > 0$, then the right-hand-side can be estimated from below by $m_{kQ}(q,d) + q^{-kQ}m_K(q,d)$.

Assume now that $\sum_{\ell=1}^{kQ}[\varepsilon_\ell \geq d] - \frac{q-d}{q}kQ \leq 0$. Then the first sum on the right-hand-side of (3.17) can be estimated from below by $m_{kQ}(q,d) - q^{-kQ}(\sum_{\ell=1}^{kQ}[\varepsilon_\ell \geq d] - \frac{q-d}{q}kQ)$ by

Lemma 4. Thus we have the lower bound

$$r\left(\sum_{\ell=1}^{kQ+K}\frac{\varepsilon_\ell}{q^\ell}\right)\geq m_{kQ}(q,d)-q^{-kQ}\left(\sum_{\ell=1}^{kQ}[\varepsilon_\ell\geq d]-\frac{q-d}{q}kQ\right)+q^{-kQ}m_K(q,d)$$
$$+q^{-kQ}\left(\sum_{\ell=1}^{K}\frac{\varepsilon_{kQ+\ell}}{q^\ell}\right)\left(\sum_{\ell=1}^{kQ}[\varepsilon_\ell\geq d]-\frac{q-d}{q}kQ\right)\geq m_{kQ}(q,d)+q^{-kQ}m_K(q,d).$$

Since by (3.11) there exists an element of $\mathcal{M}_{kQ}(q,d)$ with $\sum_{\ell=1}^{kQ}[\varepsilon_\ell\geq d]=\frac{q-d}{q}kQ$ we can obtain equality in the estimate

$$r\left(\sum_{\ell=1}^{kQ+K}\frac{\varepsilon_\ell}{q^\ell}\right)\geq m_{kQ}(q,d)+q^{-kQ}m_K(q,d).$$

□

The next lemma follows immediately from Lemma 5.

Lemma 6.

$$m_{kQ}(q,d)=m_Q(q,d)\frac{1-q^{-kQ}}{1-q^{-Q}}.$$

Lemma 7. *For $q,d\in\mathbb{N}\setminus\{0\}$ and $k\in\mathbb{N}$ the following relation holds*

$$\sum_{\ell=1}^{k}\left[\left\{\frac{d\ell}{q}\right\}<\frac{q-d}{q}\right]=\frac{q-d}{q}k+\left\{\frac{dk}{q}\right\}-\left[\left\{\frac{dk}{q}\right\}\geq\frac{q-d}{q}\right].\tag{3.18}$$

Proof. First we notice that it is enough to prove the Lemma for $\gcd(q,d)=1$. Clearly, (3.18) is equivalent to

$$\sum_{\ell=0}^{k-1}\left[\left\{\frac{d\ell}{q}\right\}<\frac{q-d}{q}\right]=\frac{q-d}{q}k+\left\{\frac{dk}{q}\right\}.$$

We use finite Fourier transforms to write the sum as a character sum

$$\left[\left\{\frac{x}{q}\right\}<\frac{q-d}{q}\right]=\sum_{m=0}^{q-1}\hat{\chi}(m)e\left(\frac{mx}{q}\right),$$

where we write $e(t)=\exp(2\pi it)$ as usual and denote

$$\hat{\chi}(m)=\frac{1}{q}\sum_{r=0}^{q-d-1}e\left(-\frac{mr}{q}\right)=\begin{cases}\frac{q-d}{q} & \text{if } m=0\\ -\frac{e(\frac{m}{q})}{q}\frac{e(\frac{md}{q})-1}{e(\frac{m}{q})-1} & \text{otherwise.}\end{cases}$$

Then we have

$$\sum_{\ell=0}^{k-1}\left[\left\{\frac{d\ell}{q}<\frac{q-d}{q}\right\}\right]=\frac{q-d}{q}k+\sum_{m=1}^{q-1}\hat{\chi}(m)\sum_{\ell=0}^{k-1}e\left(\frac{md\ell}{q}\right)=$$
$$\frac{q-d}{q}k-\sum_{m=1}^{q-1}\frac{e(\frac{m}{q})}{q}\cdot\frac{e(\frac{mdk}{q})-1}{e(\frac{m}{q})-1}=\frac{q-d}{q}k-\frac{1}{q}\sum_{r=1}^{dk \bmod q}\sum_{m=1}^{q-1}e\left(\frac{mr}{q}\right).$$

The inner sum is now either $q-1$, if $q\mid r$, or -1, if $q\nmid r$. Thus we can rewrite the last expression as

$$\frac{q-d}{q}k+\frac{dk \bmod q}{q}$$

and the Lemma is proved. □

Putting Lemma 5 and Lemma 6 together yields

Theorem 1. *For any infinite sequence of digits* $(\varepsilon_1,\varepsilon_2,\ldots)$ *there holds*

$$r\left(\sum_{\ell=1}^{\infty}\frac{\varepsilon_\ell}{q^\ell}\right)\geq\frac{m_Q(q,d)}{1-q^{-Q}}=m(q,d).$$

Equality holds if and only if $(\varepsilon_1,\varepsilon_2,\ldots)\in(\mathcal{B}(q,d))^\omega$, *where* $\mathcal{B}(q,d)$ *is given as follows: Let* $\varepsilon_\ell=d-[\{\frac{d}{q}\ell\}\geq\frac{q-d}{q}]$ *for* $1\leq\ell\leq Q$. *Then*

$$\mathcal{B}(q,d)=\Big\{(\varepsilon_1,\ldots,\varepsilon_Q),(\varepsilon_1,\ldots,\varepsilon_Q+1),\ldots(\varepsilon_1,\ldots,q-1),$$
$$(\varepsilon_1,\ldots,\varepsilon_{Q-1}+1,0),\ldots,(\varepsilon_1,\ldots,\varepsilon_{Q-1}+1,d-1)\Big\}.$$

Furthermore, we have the following expression for $m_Q(q,d)$*:*

$$m_Q(q,d)=-\sum_{\ell=1}^{Q}q^{-\ell}\left(d-\left[\left\{\frac{d}{q}\ell\right\}\geq\frac{q-d}{q}\right]\right)\left(1-\left\{\frac{d\ell}{q}\right\}\right).$$

Remark. From Theorem 1 it follows that $r(x,q,d)$ attains its minimum on a set of Hausdorff-dimension $\frac{1}{Q}$.

Lemma 8. *The following lower bound holds for all* $x\in[0,1]$

$$r(x,q,d)\geq\max\left(m(q,d),\frac{m(q,d)}{q}-\frac{q-d}{q}x,\frac{m(q,d)}{q}-\frac{d}{q}(1-x)\right).\tag{3.19}$$

Proof. Applying (3.5) to $x=\frac{\varepsilon+y}{q}$ for $\varepsilon\in\{0,\ldots,q-1\}$ and $y\in[0,1]$ yields the desired estimate, if we use the lower bound $r(y)\geq m$. □

4. Application to large digits in even bases

In this section we will study the minima of the function $G_{2t,t}$ for $t \geq 1$. For $t = 5$ this is the function whose minimum is estimated in [1]. We will prove the following theorem.

Theorem 2. *Let $t \geq 1$, then the function $G_{2t,t}$ satisfies the following inequality*

$$-\frac{t}{2t-1} - \frac{1}{2}\log_{2t}\frac{t}{2t+1} \leq G_{2t,t}(x) \leq 0. \tag{4.1}$$

Equality on the left hand side holds if and only if $x = \frac{t}{2t+1}$; equality on the right hand side holds if and only if $x \in \{0, 1\}$.

Proof. From Theorem 1 we infer that $m(2t,t) = -\frac{t^2}{4t^2-1}$. From Lemma 8 we get that

$$G_{2t,t}(1 + \log_{2t} x) \geq \max\left(-\frac{t^2}{4t^2-1}\frac{1}{x} - \frac{1}{2}\log_{2t} x, -\frac{t}{4t^2-1}\frac{1}{2x} - \frac{1}{2} - \frac{1}{2}\log_{2t} x\right). \tag{4.2}$$

Notice that the first and second entry in the maximum in (3.19) are equal, if $x = \frac{t}{2t+1}$. For this value of x we also have equality in (3.19) (this occurs only in the case $q = 2d$). Discussing the two functions under the maximum in (4.2) on the interval $[\frac{1}{2t}, 1]$ we see that the lower bound of the right hand side is attained exactly at $x = \frac{t}{2t+1}$. Since there is equality in (4.2) for this value of x, this proves the lower bound in (4.1).

For the upper bound we use the inequality (3.3). Again a discussion of the function $-\frac{1}{2}\min(x, 1-x)/x - \frac{1}{2}\log_{2t} x$ yields the desired bound. ☐

Remark. Experiments with different values of q and d show that the minima of $G_{q,d}$ are not always attained at the values of x corresponding to the minima of $r(\cdot, q, d)$.

Remark. The example of the digital function $2^{s(n)}$, where $s(n)$ denotes the binary sum-of-digits function, shows that the minimum of the corresponding fractal function need not always be attained in rational values of the argument (cf. [7, 8]).

Remark. For $q = 10$ and $d = 5$ (the case studied in [1]) we obtain the following sharp bound for $L_{10}(N)$

$$\frac{1}{2}N\log_{10} N - \left(\frac{5}{9} + \frac{1}{2}\log_{10}\frac{5}{11}\right)N \leq L_{10}(N);$$

the coefficient of N equals $0.38434421514445\ldots$.

In [1] it was observed numerically for small values of k that $L_{10}(N) - \frac{1}{2}N\log_{10} N$ attains its minima for $10^k \leq N < 10^{k+1}$ at the point $N = 4545\ldots45$ (if k is even) or $N = 4545\ldots455$ (if k is odd). The following theorem shows that this conjecture is indeed true even in the general case.

Theorem 3. *Let q be an even positive integer and denote by $L_q(N)$ the number of digits $\geq \frac{q}{2}$ occurring in the q-adic representation of the integers less than N. Then for $q^k \leq N < q^{k+1}$ the quantity*

$$\frac{L_q(N)}{N} - \frac{1}{2}\log_q N$$

attains its minimum value for $N = \frac{q}{2}\frac{q^k-1}{q+1}$ *for* k *even and* $N = \frac{q}{2}\frac{q^k+1}{q+1}$ *for* k *odd.*

Proof. Since $f_q(x) = r(x,q,q/2)/x - \frac{1}{2}\log_q x$ attains its minimum in $x = \frac{q}{2(2q+1)}$ by Theorem 2 and the three functions that have been used to estimate $f_q(x)$ in the proof of Theorem 2 are monotone, it suffices to compare the values

$$f_q\left(\frac{q}{2(q+1)}\left(1-(-q)^{-k}\right)\right) \text{ and } f_q\left(\frac{q}{2(q+1)}\left(1-(-q)^{-k}\right)+(-q)^{-k}\right) \tag{4.3}$$

(the arguments of f_q above are those two rational numbers with exactly k digits after the decimal point, which come closest to the point $\frac{q}{2(2q+1)}$, where f_q attains its minimum). Using the fact that

$$r\left(\frac{q}{2(q+1)}\left(1-(-q)^{-k}\right)+(-q)^{-k},q,\frac{q}{2}\right) = r\left(\frac{q}{2(q+1)}\left(1-(-q)^{-k}\right)\right) + \left\{\frac{k}{2}\right\}q^{-k}$$

it is a simple exercise to check that the first value in (4.3) is always smaller. □

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