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TOPICAL REVIEW

Laplace operators on fractals and related functional equations

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Abstract
We give an overview over the application of functional equations, namely the classical Poincaré and renewal equations, to the study of the spectrum of Laplace operators on self-similar fractals. We compare the techniques used to those used in the Euclidean situation. Furthermore, we use the obtained information on the spectral zeta function to compute the Casimir energy of fractals. We give numerical values for this energy for the Sierpiński gasket.

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1. Introduction

The initial interest in stochastic processes and analysis on fractals came from physicists working in the theory of *disordered media*. It turns out that heat and wave transfer in disordered media (such as polymers, fractured and porous rocks, amorphous semiconductors, etc) can be adequately modelled by means of fractals and random walks on them. (See the initial papers by Alexander and Orbach [1] and Rammal and Toulouse [2]. See also the survey by Havlin and Ben-Avraham [3] and the book by the same authors [4] for an overview of the now very substantial physics literature and bibliography.)

Motivated by these works, mathematicians became interested in developing the 'analysis on fractals'. For instance, in order to analyse how heat diffuses in a material with fractal structure, one needs to define a 'heat equation' and a 'Laplacian' on a fractal. The problem contains somewhat contradictory factors. Indeed, fractals like the Sierpiński gasket, or the von Koch curve, do not have any smooth structures and one cannot define differential operators on them directly.

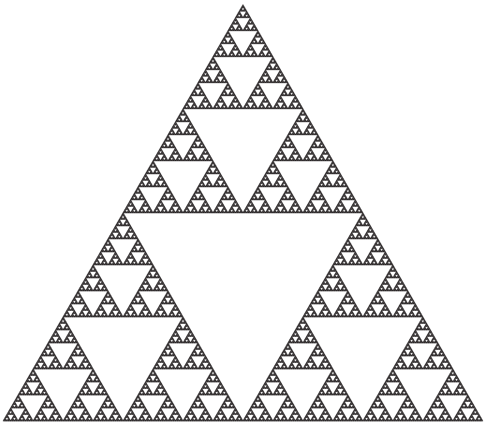

Figure 1. The Sierpiński gasket; the points of V_0 are the vertices of the triangle.

The fractals studied in the context of analysis on fractals are all *self-similar*. Moreover, we deal here mainly with *finitely ramified* fractals, i.e. fractals that can be disconnected by removing a specific finite number of points. As general references for fractals, we refer the reader to the books by Mandelbrot [5] and Falconer [6].

Objectives. There is now a number of excellent books, lecture notes and surveys on different aspects of analysis and probability on fractals [7–14, 4]. It is next to impossible to describe all activities in this area. Therefore, we restrict ourselves to a brief overview of various approaches in the study of the Laplacian and its spectral properties on certain self-similar fractals. On the other hand, we put much emphasis on the deep connection between the latter problem and the *functional equations with rescaling*, such as the classical *Poincaré* and *renewal equations*, in particular. Moreover, we concentrate mainly on those problems of diffusion on fractals, where this connection plays a keynote.

The spectral theory of the Laplace operator on a Riemannian manifold is very well understood and reveals beautiful connections between analysis, geometry and differential equations. It is one of the driving forces in the analysis on fractals to obtain a comparable understanding in the fractal situation (cf [8, 9]). Thus, we give comparative descriptions of the Euclidean and the fractal case, wherever possible. However, in some instances, a very detailed comparison would take us too far away from our main road. In such cases, we were bounded to restrict ourselves to a short informal discussion only. For a detailed description of the Euclidean situation, we refer the reader to the books by Kirsten [15], Berline *et al* [16] and Rosenberg [17]. Note also that the tight link between the spectral properties of the Laplacian and the functional equations is a specific feature of the fractal case only. It has no analogies in the Euclidean case.

Probabilistic approach. In the mid 1980s probabilists constructed 'Brownian motion' on the Sierpiński gasket (see figure 1). Goldstein [18], Kusuoka [19] and, a bit later, Barlow and Perkins [20] independently took the first step in the mathematical development of the theory. Their method of construction is now called the *probabilistic approach*. Namely, they considered a sequence of random walks $X^{(n)}$ on graphs G_n, which approximate the Sierpiński gasket G and showed that by taking an appropriate time scaling factor, those random walks

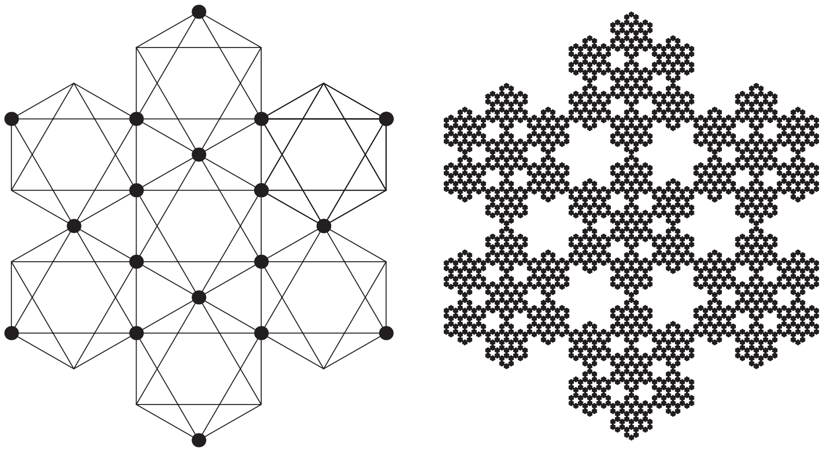

Figure 2. The Lindstrøm snowflake with the corresponding set V_0.

converge to a diffusion process X_t on the Sierpiński gasket. The Laplace operator on the fractal is then defined as the infinitesimal generator of the process X_t (see sections 3.2 and 3.3).

Lindstrøm [21] extended the construction of the Brownian motion from the Sierpiński gasket to more general *nested fractals* (which are finitely ramified self-similar fractals with strong symmetry). The Lindstrøm *snowflake* is a typical example of a nested fractal (see figure 2). The reader may refer to Barlow's lecture notes [7] for a self-contained survey of this approach.

Barlow and Perkins [20] made the following important observation: let Z_n be the *first hitting* time by $X^{(n)}$ on G_0. Then, Z_n is a *simple branching process*. Its off-spring distribution has the generating function $q(z) = z^2/(4-3z)$ and, in particular, $q'(1) = 5$. Thus, Z_n is a *super-critical branching process*. It is known (see [22]) that in this case $5^{-n}Z_n$ tends to a limiting random variable Z_∞. The moment-generating function $f(z) = \mathbb{E}\mathrm{e}^{-zZ_\infty}$ of this random variable satisfies the functional equation $f(\lambda z) = q(f(z))$, the *Poincaré equation* (see also section 5), that will play an important role throughout this paper.

Anomalous diffusion. It has been discovered in an early stage already (see [1, 2, 18–20]) that diffusion on fractals is *anomalous*. For a regular diffusion, or (equivalently) a simple random walk in all integer dimensions d, the mean-square displacement is proportional to the number of steps n: $\mathbb{E}^x\|X_n - x\|^2 = cn$ (Fick's law, 1855). On the other hand, in the case of the Sierpiński gasket, $\mathbb{E}^x\|X_n - x\|^2 \asymp n^{2/\beta}$, where $\beta = \lg 5/\lg 2$, is called the *walk dimension* ($\asymp$ means that the ratio between the two sides is bounded above and below by positive constants). This slowing down of the diffusion is caused, roughly speaking, by the removal of large parts of the space.

De Gennes [23] was amongst the first to realize the broad importance of anomalous diffusion and coined the suggestive term 'the ant in the labyrinth', describing the meandering of a random walker in percolation clusters.

Analytic approach. The second approach, based on difference operators, is due to Kigami [24]. Instead of a sequence of random walks, one can consider a sequence of discrete Laplacians on a sequence of graphs, approximating the fractal. It is possible to prove that under a proper scaling these discrete Laplacians would converge to a 'well-behaved' operator with dense domain, called the Laplacian on the Sierpiński gasket. This alternative approach is usually called the *analytic approach* (see section 3.4).

Later, it was extended by Kigami [8, 25, 26] to a more general class of fractals—*post-critically finite (p.c.f.) self-similar sets*, which roughly correspond to finitely ramified self-similar fractals.

The two approaches described above are complementary to each other. The advantage of the analytic approach is that one gets concrete and direct description of harmonic functions, Laplacians, Dirichlet forms, etc. (See also [27, 28] and section 3.5.)

On the other hand, however, the probabilistic approach is better suited for the study of heat kernels. Moreover, this approach can be applied to *infinitely ramified* self-similar fractals, which include the Sierpiński carpet, as a typical example (cf [29]).

The Poincaré equation in the analysis on fractals. In the course of his studies on the theory of automorphic functions, Poincaré introduced [30] the nonlinear functional equation

$$f(\lambda z) = R(f(z)), \quad z \in \mathbf{C},$$

where $R(z)$ is a rational function (or polynomial) and $\lambda \in \mathbf{C}$. Remarkably, nowadays the Poincaré equation finds numerous applications in different mathematical areas very distant from the original one. Among them are applications in the analysis on fractals, the subject of this paper.

One such example, already has been mentioned: the description of a *super-critical branching process* (for the first hitting moment of the random walk on the Sierpiński gasket).

Further applications of functional equations in this field are related to the phenomenon of *spectral decimation* for spectral zeta function ζ_Δ of the Sierpiński gasket and other (more general) fractals. This phenomenon was first observed and studied by Bellissard [31, 32] and Fukushima and Shima [27, 33, 34], and further progress has been made by Malozemov and Teplyaev [35] and Strichartz [36].

The precise definition of spectral decimation is given in section 4.2. It implies, in particular, that eigenvalues of the Laplacian Δ on a fractal, which admits spectral decimation, can be calculated by computing iterated preimages of a certain polynomial $p(z)$, or rational function $R(z)$. Hence, the spectral zeta function ζ_Δ may be defined by means of iterations $p^{(n)}$ of p, or $R^{(n)}$ of R (see [37]).

The above-mentioned iteration process, as is well known in iteration theory, may be conveniently described by the corresponding Poincaré equation:

$$\Phi(\lambda z) = p(\Phi(z)) \text{ and } \Phi(0) = 0, \tag{1}$$

where $\lambda = p'(0) > 1$ and $p(0) = 0$ (see, e.g., [38] or [39]).

Using that, in section 4.4, we obtain the meromorphic continuation of the zeta function ζ_Δ to the whole complex plane. The method we use for that is based on the precise knowledge of the asymptotic behaviour of the Poincaré function Φ in certain angular regions. The poles of the spectral zeta function are called the *complex dimensions* (see [40, 41, 10]). For the physical consequences of complex dimensions of fractals, see [42, 43].

In section 4.5.1, we use the Poincaré function Φ for the calculation of *Casimir (or vacuum) energy* on the Sierpiński gasket. For the underlying physical theory, we refer the reader to the monographs by Elizalde [44] and Elizalde *et al* [45], the paper by Elizalde *et al* [46] and the minicourse by Elizalde [47].

Logarithmic periodicity phenomena are quite common in the study of self-similar structures, especially if the ratios of the self-similarities form a discrete subgroup of the positive reals. In the context discussed in this paper the sources for this behaviour are functional equations like the Poincaré and renewal equations, which exhibit such behaviour of their solutions. Let us mention here that the question about log periodic oscillations of first-passage observables in fractals is intensively discussed in the physics literature (see [48]).

In section 4.6, we discuss the approach of Kigami and Lapidus [49] to the asymptotic behaviour of the eigenvalue counting function for general p.c.f. fractals using the classical renewal equation. This yields a different method to derive an analytic continuation of the zeta function to the half-plane $\Re s > 0$.

In section 5, we collected some basic facts about the Poincaré equation scattered in the literature. Since the equations occurring in the applications in the analysis on fractals are purely real, we restrict our discussion to that case. We discuss the special case of the Poincaré equation with a *quadratic* polynomial on the right-hand side. Such equations arise in several applications related to the diffusion on fractals and we give a criterion for the reality of the Julia set (see section 5.1) in this case.

Finally, one can expect that functional equations with rescaling naturally come about from problems, where renormalization-type arguments are used to study self-similarity. Furthermore, functional equations, in contrast to differential equations, do not require any *smoothness* of solutions; they are well suited to describe non-smooth (e.g., nowhere differentiable) solutions (see, e.g., [50])

In this overview, we do not or only cursorily touch the following important topics.

- The analysis on infinitely ramified fractals such as the Sierpiśki carpet [29, 51]. The very recent progress that has been made in proving the uniqueness of the diffusion on the Sierpiński carpets [52] provided a unification of the different approaches to diffusion on this class of fractals (cf [29, 53]).
- Heat kernel long-time behaviour and Harnack inequalities on general underlying spaces, including Riemannian manifolds, graphs and fractals, as special cases [54–57]
- The analysis on fractals by means of potential theory and function space (Sobolev or Besov spaces) techniques [11, 58, 61–63].

We refer the interested reader to the above-mentioned original papers.

2. Fractals and iterated function systems

Let us first recall that for any finite set of linear contractions on $\mathbb{R}^d$

$$F_i(\mathbf{x}) = \mathbf{b}_i + \mathbf{A}_i(\mathbf{x} - \mathbf{b}_i), \quad i = 1, \ldots, m,$$

with fixed points $\mathbf{b}_i$ and contraction matrices $\mathbf{A}_i$ ($\|\mathbf{A}_i\| < 1$, $i = 1, \ldots, m$), there exists a unique compact set $K \subset \mathbb{R}^d$ satisfying

$$K = \bigcup_{i=1}^{m} F_i(K). \tag{2}$$

In general, the set K obtained as a fixed point in (2) is a 'fractal' set (cf [64, 65]). If the matrices $\mathbf{A}_i$ are only assumed to be contracting, finding the Hausdorff dimension of K is a rather delicate problem (cf [66, 67]).

For this paper, we will make the following additional assumptions on the family of contractions $F = \{F_i \mid i = 1, \ldots, m\}$.

(i) The matrices $\mathbf{A}_i$ are similitudes with factors $\alpha_i < 1$:

$$\forall\, \mathbf{x} \in \mathbb{R}^d : \|\mathbf{A}_i\mathbf{x}\| = \alpha_i\|\mathbf{x}\|.$$

(ii) F satisfies the open-set-condition (cf [64, 68]), namely there exists a bounded open set $\mathcal{O}$ such that

$$\bigcup_{i=1}^{m} F_i(\mathcal{O}) \subset \mathcal{O}$$

with the union disjoint.

Assuming that the maps in F are similitudes and satisfy the open-set-condition, the Hausdorff dimension $\dim_H(K)$ of the compact set K given by (2) equals the unique positive solution $s = \rho$ of the equation

$$\sum_{i=1}^{m} \alpha_i^s = 1. \tag{3}$$

The projection map

$$\begin{aligned} \pi : \quad & \{1, \ldots, m\}^{\mathbb{N}} && \to K \\ & (\varepsilon_1, \varepsilon_2, \ldots) && \mapsto \lim_{n\to\infty} F_{\varepsilon_1} \circ F_{\varepsilon_2} \circ \cdots \circ F_{\varepsilon_n}(\mathbf{x}) \end{aligned} \tag{4}$$

(the limit is independent of $\mathbf{x} \in \mathbb{R}^d$) defines a 'parametrization' of K. If $\{1, \ldots, m\}^{\mathbb{N}}$ is endowed with the infinite product measure μ given by

$$\mu\left(\{(\varepsilon_1, \varepsilon_2, \ldots) \mid \varepsilon_1 = i_1, \varepsilon_2 = i_2, \ldots, \varepsilon_k = i_k\}\right) = (\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k})^{\rho}, \tag{5}$$

and then π is a μ-almost sure bijection. The normalized ρ-dimensional Hausdorff measure on K is then given by

$$\mathcal{H}_K^{\rho}(E) := \frac{\mathcal{H}^{\rho}(E \cap K)}{\mathcal{H}^{\rho}(K)} = \pi_*(\mu)(E) = \mu(\pi^{(-1)}(E)). \tag{6}$$

This is the unique normalized measure satisfying (cf [68, theorem 28])

$$\mathcal{H}_K^{\rho}(E) = \sum_{i=1}^{m} \alpha_i^{\rho} \mathcal{H}_K^{\rho}\big(F_i^{(-1)}(E)\big). \tag{7}$$

The set K together with the address space $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$ and the maps F_i $(i = 1, \ldots, m)$ defines a *self-similar structure* $(K, \Sigma, (F_i)_{i=1}^m)$. For K fixed and $\rho = \dim_H(K)$, we write $\mathcal{H}$ for $\mathcal{H}_K^{\rho}$.

In order to allow a sensible analysis on the fractal set K, we need some further properties. In particular, since we will later study diffusion processes, we need K to be connected. On the other hand, the techniques introduced later require a finite ramification property usually called post-critical finiteness (p.c.f.).

Connectivity of K is characterized by the fact that for any pair (i, j), there exist $i_1, i_2, \ldots, i_n \in \{1, \ldots, m\}$ with $i = i_1$ and $j = i_n$ such that $F_{i_\ell}(K) \cap F_{i_{\ell+1}}(K) \neq \emptyset$ for $\ell = 1, \ldots, n-1$ (cf [69]).

Definition 1. *Let $(K, \Sigma, (F_i)_{i=1}^m)$ be a self-similar structure. Then, the set*

$$C = \pi^{-1}\left(\bigcup_{i\neq j} F_i(K) \cap F_j(K)\right)$$

is called the critical set of K. The post-critical set of K is defined by

$$P = \bigcup_{n=1}^{\infty} \sigma^n(C),$$

where $\sigma : \Sigma \to \Sigma$ denotes the shift map on the address space Σ. If P is a finite set, then $(K, \Sigma, (F_i)_{i=1}^m)$ is called post-critically finite (p.c.f.). This is equivalent to the finiteness of C together with the fact that all points of C are ultimately periodic.

The following sequence V_m of finite sets will be used in section 3.5 to define a sequence of electrical networks giving a harmonic structure on K. For more details, we refer the reader to [8, chapter 1].

Definition 2. *Let $(K, \Sigma, (F_i)_{i=1}^m)$ be a p.c.f. self-similar structure and P its post-critical set. Let $V_0 = \pi(P)$ and define V_m iteratively by*

$$V_{n+1} = \bigcup_{i=1}^{m} F_i(V_n).$$

The sets V_n are then finite, increasing ($V_n \subset V_{n+1}$) and

$$K = \overline{\bigcup_{n\geqslant 0} V_n}.$$

3. Laplace operators on fractals

Before we introduce the Laplace operator on certain classes of self-similar fractals, let us briefly discuss the situation on manifolds, because this gives the motivation for the different approaches in the case of fractals.

3.1. Laplace operators on compact manifolds

Let M be a compact Riemannian manifold with a Riemannian metric g given as a quadratic form g_x on the tangent space T_xM for $x \in M$. As usual, we assume that the dependence of g_x on x is differentiable. Then, the quadratic form g_x defines an isomorphism α_g between the tangent space T_xM and its dual T_x^*M (and thus on the tangent bundle TM and the cotangent bundle T^*M) by $\alpha_g(v)w = g_x(v, w)$ for $v, w \in T_xM$. This defines the gradient of a function f as $\operatorname{grad} f = \alpha_g^{-1}(df)$. Define the divergence of a vector field X as the negative formal adjoint of grad with respect to the scalar product $\langle X, Y\rangle_{L^2(M)} = \int_M g_x(X, Y)\,\mathrm{d}\,\mathrm{vol}(x)$:

$$\langle X, \operatorname{grad} f\rangle_{L^2(M)} = -\langle \operatorname{div} X, f\rangle_{L^2(M)}.$$

The Laplace operator is then defined as (cf [17, 70])

$$\triangle f = \operatorname{div}\operatorname{grad} f. \tag{8}$$

By definition, this operator is self-adjoint and, thus, has only non-positive real eigenvalues by $\langle \triangle f, f\rangle_{L^2(M)} = -\langle \operatorname{grad} f, \operatorname{grad} f\rangle_{L^2(M)} \leqslant 0$.

Based on the above approach, a corresponding energy form ('Dirichlet form', cf [71]) can be defined

$$\mathcal{E}(u, v) = \int_M g_x(\operatorname{grad} u, \operatorname{grad} v)\,\mathrm{d}\,\mathrm{vol}(x) = \int_M g_x(\mathrm{d}u, \mathrm{d}v)\,\mathrm{d}\,\mathrm{vol}(x),$$

which lends itself to a further way of defining a Laplace operator via the relation

$$\mathcal{E}(u, v) = -\langle \triangle u, v\rangle_{L^2(M)}. \tag{9}$$

Geometrically, the Laplace operator measures the deviation of the function f from the mean value. More precisely, let $S(x, r) = \{y \in M \mid d(x, y) = r\}$ denote the ball of radius r in M (in the Riemannian metric). Then,

$$\triangle f = 2n \lim_{r\to 0+} \frac{1}{r^2\sigma(S(x, r))} \int_{S(x,r)} (f(y) - f(x))\,\mathrm{d}\sigma(y), \tag{10}$$

where n denotes the dimension of the manifold M and σ is the surface measure on $S(x, r)$. This also motivates the definition of the Laplace operator as a limit of finite difference operators

$$\triangle f(x) = \lim_{r\to 0} \frac{1}{r^2}\left(\sum_{p\in N_r(x)} w_p f(p) - f(x)\right),$$

where $N_r(x)$ is a finite set of points at distance r from x, and w_p are suitably chosen weights. Such approximations to the Laplace operator are the basis of the method of finite differences in numerical mathematics.

The Laplace operator can then be used to define a diffusion on M via the heat equation $\triangle u = \partial_t u$. The solution $u(t, x)$ of the initial value problem $u(0, x) = f(x)$ defines a semi-group of operators P_t by

$$P_t f(x) = u(t, x).$$

The semi-group property $P_{s+t} = P_s P_t$ comes from the uniqueness of the solution u and translation invariance with respect to t of the heat equation. From the heat semi-group P_t, the Laplace operator can be recovered as the *infinitesimal generator*

$$\triangle f = \lim_{t\to 0+} \frac{P_t f - f}{t}, \tag{11}$$

which exists on a dense subspace of $L^2(M)$ under suitable continuity assumptions on the semi-group $(P_t)_{t\geqslant 0}$ (cf [72]).

In the fractal situation, none of the above approaches can be used directly to define a Laplace operator. The main reason for this is that there is no natural definition of derivative on a fractal. But the above approaches to the Laplace operator on a manifold can be used in the opposite direction.

- Starting from a diffusion process that can be defined on fractals by approximating random walks. Then, the Laplace operator can be defined as the infinitesimal generator. This is described in sections 3.2 and 3.3.
- Taking the limit of finite-difference operators on graphs approximating the fractal gives a second possible approach to the Laplace operator, which is explained in section 3.4.
- Starting with a Dirichlet form $\mathcal{E}$ gives a third possible approach, which is presented in section 3.5.

3.2. Random walks on graphs and diffusion on fractals

The first idea to obtain a diffusion on a fractal was to define a sequence of random walks on approximating graphs and to synchronize time so that the limiting process is non-constant and continuous. This was the first approach to the diffusion process on the Sierpiński gasket given in [18–20] and later generalized to other 'nested fractals' in [21]. Because of its importance for our exposition, we explain it in some detail in this section. We will follow the lines of definition of self-similar graphs given in [73, 74] and adapt it for our purposes.

We consider a graph $G = (V(G), E(G))$ with vertices $V(G)$ and undirected edges $E(G)$ denoted by $\{x, y\}$. We assume throughout that G does not contain multiple edges nor loops. For $C \subset V(G)$, we call ∂C the vertex boundary, which is given by the set of vertices in $V(G) \setminus C$, which are adjacent to a vertex in C. For $F \subset V(G)$, we define the reduced graph G_F by $V(G_F) = F$ and $\{x, y\} \in E(G_F)$, if x and y are in the boundary of the same component of $V(G) \setminus F$.

Definition 3. *A connected infinite graph G is called self-similar with respect to $F \subset V(G)$ and $\varphi : V(G) \to V(G_F)$, if*

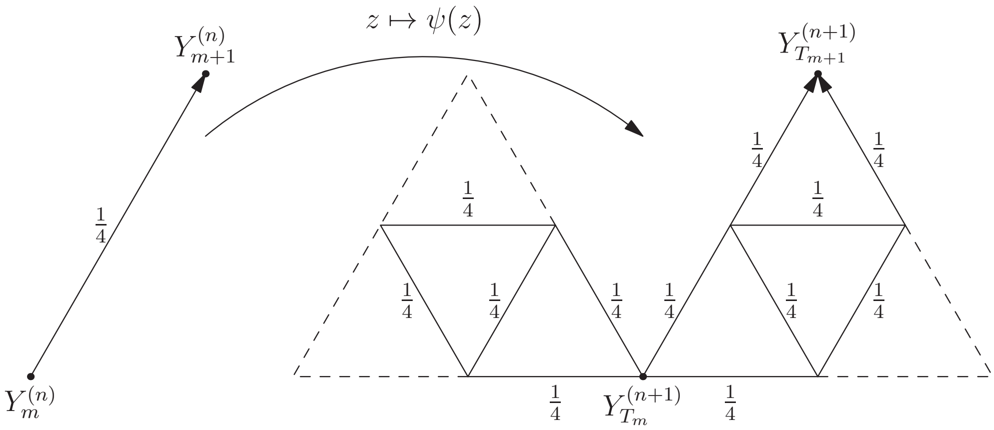

Figure 3. Transition between $Y_k^{(n)}$ and $Y_k^{(n+1)}$.

(i) *no vertices in F are adjacent in G,*
(ii) *the intersection of the boundaries of two different components of $V(G) \setminus F$ does not contain more than one point,*
(iii) *φ is an isomorphism of G and G_F.*

A random walk on G is given by the transition probabilities $p(x, y)$, which are positive, if and only if $\{x, y\} \in E(G)$. For a trajectory $(Y_n)_{n\in\mathbb{N}_0}$ of this random walk with $Y_0 = x \in F$, we define stopping times recursively by

$$T_{m+1} = \min\left\{k > T_m \mid Y_k \in F \setminus \{Y_{T_m}\}\right\}, \quad T_0 = 0.$$

Then, $(Y_{T_m})_{m\in\mathbb{N}_0}$ is a random walk on G_F. Since the underlying graphs G and G_F are isomorphic, it is natural to require that $(\varphi^{-1}(Y_{T_m}))_{m\in\mathbb{N}_0}$ is the same stochastic process as $(Y_n)_{n\in\mathbb{N}_0}$. This requires the validity of equations for the basic transition probabilities

$$\mathbb{P}(Y_{T_{n+1}} = \varphi(y) \mid Y_{T_n} = \varphi(x)) = \mathbb{P}(Y_{n+1} = y \mid Y_n = x) = p(x, y). \tag{12}$$

These are usually nonlinear rational equations for the transition probabilities $p(x, y)$. The existence of solutions of these equations has been the subject of several investigations, and we refer the reader to [75–78].

The process $(Y_n)_{n\in\mathbb{N}_0}$ on G and its ‘shadow’ $(Y_{T_n})_{n\in\mathbb{N}_0}$ on G_F are equal, but they are on a different time scale. Every transition $Y_{T_n} \to Y_{T_{n+1}}$ on G_F comes from a path $Y_{T_n} \to Y_{T_n+1} \cdots \to Y_{T_{n+1}-1} \to Y_{T_{n+1}}$ in a component of $V(G) \setminus F$ (see figure 3). The *time scaling factor* between these processes is given by

$$\lambda = \mathbb{E}(T_{n+1} - T_n) = \mathbb{E}(T_1).$$

This factor is $\geqslant 2$ by assumption (i) on F. More precisely, the relation between the transition time on G_F and the transition time on G is given by a super-critical ($\lambda > 1$) branching process, which replaces an edge $\{\varphi(x), \varphi(y)\} \in G_F$ by a path in G connecting the points x and y without visiting a point in $V(G) \setminus F$ (except for x and for y in the last step).

In order to obtain a process on a fractal in $\mathbb{R}^d$, we assume further that G is embedded in $\mathbb{R}^d$ (i.e. $V(G) \subset \mathbb{R}^d$). The self-similarity of the graph is carried over to the embedding by assuming that there exists $\beta > 1$ (the *space scaling factor*) such that $F = V(G_F) = \beta V(G)$. The fractal limiting structure is then given by

$$Y_G = \overline{\bigcup_{n=0}^{\infty} \beta^{-n} V(G)}.$$

Iterating this graph decimation, we obtain a sequence of (isomorphic) graphs $G_n = (\beta^{-n}V(G), E(G))$ on different scales. The random walks $(Y_k^{(n)})_{k\in\mathbb{N}_0}$ on G_n are connected by time scales with the scaling factor λ. From the theory of branching processes (cf [22]), it follows that the time on level n scaled by λ^{-n} tends to a random variable. From this, it follows that $\beta^{-n}Y_{[t\lambda^n]}$ weakly tends to a (continuous time) stochastic process $(X_t)_{t\geqslant 0}$ on the fractal Y_G.

Under the assumption that the graph G has an automorphism group acting doubly transitively on the points of the boundary of every component of $G \setminus F$ (cf [73, 74]), we consider the *generating function* of the transition probabilities $p_n(x, y) = \mathbb{P}(Y_n = y \mid Y_0 = x)$, the so-called *Green function* of the random walk

$$G(z \mid x, y) = \sum_{n=0}^{\infty} p_n(x, y)z^n,$$

which can also be seen as the resolvent $(I-zP)^{-1}$ of the transition operator P. The replacement rules connecting the random walks on the graphs G and G_F result in a functional equation for the Green function

$$G(z \mid \varphi(x), \varphi(y)) = f(z)G(\psi(z) \mid x, y), \tag{13}$$

where the rational function $\psi(z)$ is the probability-generating function of the paths connecting two points $v_1, v_2 \in \varphi(V(G))$ without reaching any point in $\varphi(V(G)) \setminus \{v_1\}$. Then, the time scaling factor is the expected number of steps needed for the paths counted by ψ, so we have

$$\lambda = \psi'(1).$$

The rational function $f(z)$ is the probability-generating function of the paths starting and ending in $v_1 \in \varphi(V(G))$ without reaching any other point in $\varphi(V(G))$. Equation (13) becomes especially simple for a fixed point x of the map φ:

$$G(z \mid x, x) = f(z)G(\psi(z) \mid x, x). \tag{14}$$

It was proved in [74] that under our conditions on the set F the map φ can have at most one fixed point. Equation (14) can be used to obtain the asymptotic behaviour of $G(z \mid x, x)$ for $z \to 1$ (cf [74, 79]). From this, the asymptotic behaviour of the transition probabilities $p_n(x, x)$ can be derived. In many examples, these transition probabilities exhibit periodic fluctuations

$$p_n(x, x) \sim n^{-\gamma}(\sigma(\log_\lambda n) + \mathcal{O}(n^{-1})),$$

where σ is a continuous, periodic, non-constant function of period 1 with $\gamma = 1 - \log f(1)/\log\lambda$ (cf [74]).

A first example of such graphs is the Sierpiński graph studied as an approximation to the fractal Sierpiński gasket. In this case, we have for the probability-generating function $\psi(z) = \frac{z^2}{4-3z}$ and $\lambda = 5$. The random walk on this graph was studied in [20] in order to define a diffusion on a fractal set. Self-similarity of the graph and the fractal have been exploited further, to give a more precise description of the random walk [80] and the diffusion [81]. In [74], a precise description of the class of graphs in terms of their symmetries is given, which allow a similar construction. In [82], this analysis was carried further to obtain results for the transition probabilities under less symmetry assumptions using multivariate generating functions.

In some examples (for instance the Sierpiński graph) it occurred that the function ψ was conjugate to a polynomial p, i.e.

$$\psi(z) = \frac{1}{p\left(\frac{1}{z}\right)},$$

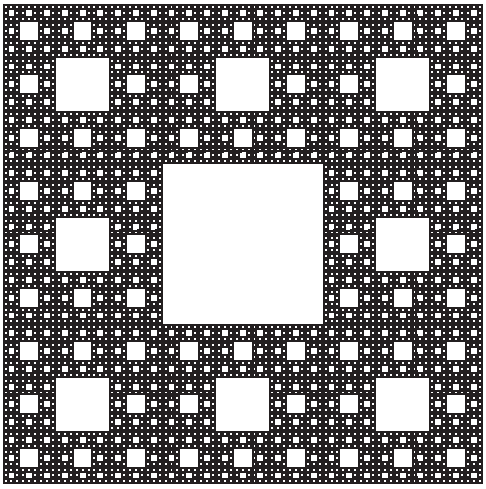

Figure 4. The Sierpiński carpet, an example of an infinitely ramified fractal.

which allowed a study of the properties of the random walk by referring to the classical Poincaré equation. Properties of the Poincaré equation have been used in [83] to study the analytic properties of the zeta function of the Laplace operator given by the diffusion on certain self-similar fractals. For more details, we refer the reader to section 4.4.

Remark 3.1. The Sierpiński carpet (see figure 4) is a typical example of an infinitely ramified fractal. In [53], approximations by graphs are used to define a diffusion on this fractal. By the infinite ramification, this approach is more intricate than the procedure described here. By the very recent result on the uniqueness of Brownian motion on the Sierpiński carpet (cf [52]), this yields the same process as constructed by rescaling the classical Brownian motion restricted to finite approximations of the Sierpiński carpet in [29, 51].

3.3. The Laplace operator as the infinitesimal generator of a diffusion

Given a diffusion process $(X_t)_{t\geqslant 0}$ on a fractal K, we can now define a corresponding Laplace operator. At first we define a semi-group of operators A_t by

$$A_t f(x) = \mathbb{E}^x f(X_t)$$

for functions $f \in L^2(K)$. The semi-group property

$$A_s A_t = A_{s+t}$$

of the operators comes from the Markov property of the underlying stochastic process X_t.

By [72, chapter 9], this semi-group has an infinitesimal generator given as

$$\Delta f = \lim_{t\to 0+} \frac{A_t f - f}{t}.$$

This limit exists on a dense subspace $\mathcal{F}$ of $L^2(K)$ and is called Laplace operator on K. This name comes from the fact that for the usual Brownian motion on a manifold this procedure yields the classical Laplace–Beltrami operator (see section 3.1). The function $u(x,t) = A_t f(x)$ satisfies the heat equation

$$\triangle u = \partial_t u, \quad u(x,0) = f(x).$$

For a comprehensive survey of the probabilistic approach to the heat equation and properties of the heat kernel, we refer the reader to [84].

It was observed in the early beginnings of the development of the theory of diffusion of fractals that the domain of Δ does not contain the restriction of any non-constant differentiable function (cf [85]).

3.4. The Laplace operator as a limit of difference operators on graphs

A totally different and more direct approach to the Laplace operator on self-similar fractals has been given by Kigami in [24]. The operator Δ is approximated by difference operators on the approximating graphs G_n. The graph Laplacians are given by

$$\Delta_n f(x) = \sum_{y\sim x, y\in G_n} p(x,y)f(y) - f(x)$$

as a weighted sum over the neighbours of x in G_n ($y \sim x$ describes the neighbourhood relation in the graph G_n). In order to make this construction compliant with the approach via stochastic processes, these operators have to be rescaled appropriately. The correct rescaling is then given by the time scaling factor λ introduced before, namely

$$\Delta f(x) = \lim_{n\to\infty} \lambda^n \Delta_n f(x).$$

3.5. Laplace operators via Dirichlet forms

Following the exposition in [8, chapters 2 and 3], we define a sequence of quadratic forms on the finite sets V_m given in definition 2.

Definition 4. *Let V be a finite set. Then, a bilinear form $\mathcal{E}$ on $\ell(V)$, the real functions on V, is called a Dirichlet form, if the following conditions hold:*

(i) $\forall u \in \ell(V) : \mathcal{E}(u,u) \geqslant 0$,
(ii) $\mathcal{E}(u,u) = 0$ *implies that u is constant on V,*
(iii) for $u \in \ell(V)$ and $\overline{u}(x) = \max(0, \min(u(x), 1))$, the inequality $\mathcal{E}(u,u) \geqslant \mathcal{E}(\overline{u},\overline{u})$ holds.

Definition 5. *Let $\mathcal{E}$ be a Dirichlet form on the finite set V and let U be a proper subset of V. Then, the restriction of $\mathcal{E}$ to U is defined as*

$$\mathcal{R}_{V,U}(\mathcal{E})(u,u) = \min\left\{\mathcal{E}(v,v) \mid v \in \ell(V), v|_U = u\right\}. \tag{15}$$

On the level of the coefficient matrices of the Dirichlet forms, the operation of restriction is given by the Schur complement (cf [8]).

Definition 6. *Let $(V_n, \mathcal{E}_n)_n$ be a sequence of increasing finite sets V_n and Dirichlet forms $\mathcal{E}_n$ on V_n. The sequence is called compatible, if*

$$\mathcal{R}_{V_{n+1},V_n}(\mathcal{E}_{n+1}) = \mathcal{E}_n$$

holds for all n.

For a compatible sequence $(V_n, \mathcal{E}_n)_n$ and a function f on K, the sequence $(\mathcal{E}_n(f|_{V_n}, f|_{V_n}))_n$ is increasing by definition and thus converges to a value in $[0,\infty]$. This makes the following definition natural.

Definition 7. *Let* $(V_n, \mathcal{E}_n)_n$ *be a compatible sequence of Dirichlet forms. Let*

$$\mathcal{D} = big\{f : K \to \mathbb{R} \mid \lim_{n\to\infty} \mathcal{E}_n(f|_{V_n}, f|_{V_n}) < \infty\big\}.$$

Then, for all $f \in \mathcal{D}$,

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \mathcal{E}_n(f|_{V_n}, f|_{V_n})$$

defines a Dirichlet form on K*, and* $\mathcal{D}$ *is its domain.*

In order to make the sequence of Dirichlet forms coherent with the self-similar structure of K, we require the following self-similarity condition for $\mathcal{E}_n$:

$$\mathcal{E}_{n+1}(f, f) = \lambda \sum_{i=1}^{m} r_i^{-1} \mathcal{E}_n(f \circ F_i, f \circ F_i),$$

where r_i $(i = 1, \ldots, m)$ are positive weights and λ is a proportionality factor. Furthermore, the sequence of forms has to be compatible, which amounts to the equation

$$\lambda \mathcal{R}_{V_1,V_0}\left(\sum_{i=1}^{m} r_i^{-1} \mathcal{E}_0(\cdot \circ F_i, \cdot \circ F_i))\right) = \mathcal{E}_0(\cdot, \cdot), \tag{16}$$

which comes as a solution of a nonlinear eigenvalue equation. This equation plays the same role in the Dirichlet form approach as equations (12) play for the approach via random walks. The Dirichlet form $\mathcal{E}$ on K is then defined as

$$\mathcal{E}(f, f) = \lim_{n\to\infty} \lambda^n \sum_{w\in S^n} r_w^{-1} \mathcal{E}_0(f \circ F_w|_{V_0}, f \circ F_w|_{V_0}),$$

where $S = \{1, \ldots, m\}$, $r_w = r_{w_1} \cdots r_{w_n}$ for $w = w_1 \cdots w_n \in S^n$ and $F_w = F_{w_1} \circ \cdots \circ F_{w_n}$.

Remark 3.2. There are some additional technical problems concerning this construction of Dirichlet forms, which arise from the fact that in general the form is supported only on a proper subset of K. In [8, chapter 3], sufficient conditions for the weights r_i and the form $\mathcal{E}_0$ are given, which ensure that the form $\mathcal{E}$ is supported on the whole set K.

Given a Dirichlet form $\mathcal{E}$ together with its domain $\mathcal{D}$ and a measure μ on K, we can now define the associated Laplace operator on K by

$$\forall v \in \mathcal{D} \cap L^2(\mu) : \mathcal{E}(u, v) = -\langle \Delta_\mu u, v\rangle_{L^2(\mu)}, \tag{17}$$

which defines Δ_μ, if $\mathcal{D}$ is dense in $L^2(\mu)$. Note that this is the same as equation (9) in the manifold case. In the case of a self-similar fractal K as described in section 2, the 'natural' measure on K is the according Hausdorff measure $\mathcal{H}^\rho_K$. In this case, we omit the subscript μ.

For more information on Dirichlet forms in general and their applications to the description of diffusion processes, we refer the reader to the monograph [71]. For the specifics of Dirichlet forms on fractals, we refer the reader to [8, 86].

4. Spectral analysis on fractals

Let us start with a short discussion of the manifold case, which is again somehow complementary to the fractal case.

4.1. Spectral analysis on manifolds

As described in section 3.1, the Laplace operator on an n-dimensional manifold M is defined via a Riemannian metric g. If M is compact with smooth (or empty) boundary (for simplicity), the Laplace operator has pure point spectrum. The eigenvalues $-\lambda_k$ are all real (by self-adjointness) and non-positive (by definition (8)). Denote the normalized eigenfunctions by ψ_k. Then, the heat kernel can be written as

$$p_t(x, y) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda_k t}\psi_k(x)\psi_k(y). \tag{18}$$

This expression yields the trace of the heat kernel

$$K(t) = \mathrm{Tr}\,\mathrm{e}^{t\,\triangle} = \int_M p_t(x, x)\,\mathrm{d}\,\mathrm{vol}(x) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda_k t}. \tag{19}$$

The precise order of magnitude of the heat kernel $p_t(x, y)$ and especially $K(t)$ for $t \to 0+$ has been the object of intensive research in the middle of the 20th century. We refer the reader to [87–90] for derivations of this asymptotic expansion by techniques using partial differential equations; for a probabilistic approach, we refer the reader to [84]. The reason for the interest in these quantities was their close connection to invariants of the underlying manifold and to index theorems for differential equations (see, e.g., [91, 92, 88, 89]).

The final form of the asymptotic expansion of $K(t)$ is given by

$$K(t) \sim (4\pi t)^{-\frac{n}{2}} \sum_{k=0,1/2,1,\ldots}^{\infty} B_k t^k, \tag{20}$$

where the coefficients B_k are given by

$$B_k = \int_M b_k\,\mathrm{d}\,\mathrm{vol} + \int_{\partial M} c_k\,\mathrm{d}\sigma,$$

where σ denotes the surface measure on the boundary ∂M.

The functions b_k depend on the metric tensor of M and its derivatives and vanish for $k \in \frac{1}{2} + \mathbb{N}_0$; the functions c_k encode information about the curvature of the boundary and the boundary conditions and vanish for $k \in \mathbb{N}_0$. Schemes for the computation of the b_k have been developed (see, e.g., [93]). The computation of the c_k is much more complicated; for explicit computation of the functions c_k for small k, we refer the reader to [15] and [94]. It should be mentioned that the coefficients B_k with half-integer k vanish for a closed manifold M.

From the first-order asymptotic relation $K(t) \sim \mathrm{vol}(M)/(4\pi t)^{\frac{n}{2}}$, the asymptotic behaviour of the counting function

$$N(x) = \sum_{\lambda_k < x} 1 \tag{21}$$

can be obtained by a Tauberian argument giving Weyl's [95] classical asymptotic relation

$$N(x) = \frac{\mathrm{vol}(M)}{B_n} x^{\frac{n}{2}} + o(x^{\frac{n}{2}}), \tag{22}$$

where B_n denotes the volume of the n-dimensional unit ball. The eigenvalues of $\triangle$ constitute the frequencies of the oscillations in the solutions of the wave equation $\triangle u = u_{tt}$. This led to Kac's famous question 'can one hear the shape of a drum?' (cf [96]).

More precise information on the eigenvalues is contained in the spectral zeta function

$$\zeta_\triangle(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s} = \mathrm{Tr}(-\triangle)^{-s}, \tag{23}$$

the Dirichlet generating function of $(\lambda_k)_k$. This series converges for $\Re s > \frac{n}{2}$ (the 'abscissa of convergence'). For a detailed discussion of complex powers of elliptic operators and their analytic continuation, we refer the reader to [97]. The zeta function is connected to the trace of the heat kernel by a Mellin transform

$$\zeta_\triangle(s)\Gamma(s) = \int_0^\infty K(t)t^{s-1}\,\mathrm{d}t, \qquad \text{for } \Re s > \frac{n}{2}. \tag{24}$$

The right-hand side has an analytic continuation to the whole complex plane, which can be found by using the asymptotic expansion (20) by the following standard computation:

$$t\zeta_\triangle(s)\Gamma(s) = \int_0^1 \left(K(t) - (4\pi t)^{-\frac{n}{2}}\sum_{k<L} B_k t^k\right) t^{s-1}\mathrm{d}t \\ +(4\pi)^{-\frac{n}{2}}\sum_{k<L}\frac{B_k}{s-\frac{n}{2}+k} + \int_1^\infty K(t)t^{s-1}\mathrm{d}t.$$

The first integral converges for $\Re s > \frac{n}{2} - L$, the second integral converges for all $s \in \mathbb{C}$ by the exponential decay of $K(t)$.

As a consequence of the procedure of analytic continuation and the properties of the Γ-function, some special values of $\zeta_\triangle$ can be computed ($k \in \frac{1}{2}\mathbb{N}_0$, cf [44]):

$$\operatorname*{Res}_{s=\frac{n}{2}-k} \zeta_\triangle(s) = \frac{B_k}{(4\pi)^{n/2}\Gamma(\frac{n}{2}-k)} \qquad \text{for } \frac{n}{2} - k \in \left(\frac{1}{2}\mathbb{Z}\right)\setminus \mathbb{Z}_{\leqslant 0}$$

and

$$\zeta_\triangle(-\ell) = (-1)^\ell \ell!\frac{B_{\frac{n}{2}+\ell}}{(4\pi)^{n/2}} \qquad \text{for } \ell \in \mathbb{N}_0.$$

The values of the analytic continuation of $\zeta_\triangle(\alpha)$ for α to the left of the abscissa of convergence of the series (23) can be interpreted as *regularized values* of the series (23) for values of s, where it *does not* converge. For $\alpha \leqslant \frac{n}{2}$, the regularized value of the (divergent!) series

$$\sum_n \lambda_n^{-\alpha}$$

is the *constant term* in the asymptotic expansion of the series

$$\sum_n \lambda_n^{-\alpha}\,\mathrm{e}^{-\lambda_n z}$$

for $z \to 0$ with $\Re z > 0$ (cf [98]), if there are no poles of $\zeta_\triangle$ on the line $\Re s = \alpha$. A further confirmation for the validity of the zeta regularization technique is given by the *zeta regularization theorem* (cf [99] and [44, section 2.2.1]). For further reading on regularization, we refer the reader to the book by Jorgenson and Lang [100], and for the physics background of regularization and various applications, we refer the reader to the books by Elizalde [44] and Elizalde *et al* [45], as well as to the instructive minicourse [47].

The value $-\zeta'_\triangle(0)$ is of special interest, as it represents the regularized value of the series

$$\sum_n \log \lambda_n,$$

which is the logarithm of the regularized determinant of the Laplace operator (the regularized product of the eigenvalues). This special value occurs *inter alia* in the study of Reidemeister torsion in [101] and in the computation of path integrals with an action given by the Laplace operator [102, 103]. In the latter context, it is only defined up to a divergent factor, which can be cancelled by forming the ratio of two determinants (cf [104, 105]). The determinant of $-\triangle$

can be expressed by

$$\det(-\triangle) = \exp(-\zeta'_{\triangle}(0)), \tag{25}$$

which holds if $\zeta_{\triangle}(s)$ has no pole on the imaginary axis.

In the case of a fractal, the above approach to the spectral zeta function, its analytic continuation and its finer properties cannot be used. The reason for this is that no asymptotic expansion of the heat kernel is known in the fractal case. The existence of an analytic continuation of the zeta function for fractals is not known in general. In section 4.5, we comment on the known upper and lower estimates for the heat kernel.

On the other hand, for fractals having spectral decimation, the eigenvalues can be described very precisely. In this case, they turn out to constitute a finite union of level sets of solutions of the Poincaré functional equation. This then allows one to obtain the analytic continuation of the spectral zeta function from the precise knowledge of the asymptotic behaviour of the Poincaré function; the asymptotic expansion of the trace of the heat kernel can then be obtained from the poles of the zeta function, reversing the argument in (24). The eigenvalue counting function can then be related to the harmonic measure on the Julia set of a the polynomial governing the spectral decimation.

4.2. Spectral decimation

It has been first observed by Fukushima and Shima [27, 33, 34] that the eigenvalues of the Laplacian on the Sierpiński gasket and its higher dimensional analogues exhibit the phenomenon of *spectral decimation* (see also earlier work by Bellissard [31, 32]). Later on, spectral decimation for more general fractals has been studied by Malozemov*et al* [35, 36, 106–108].

Definition 8 (spectral decimation)**.** *The Laplace operator on a p.c.f. self-similar fractal G admits spectral decimation, if there exists a rational function R, a finite set A and a constant $\lambda > 1$, such that all eigenvalues of Δ can be written in the form*

$$\lambda^m \lim_{n\to\infty} \lambda^n R^{(-n)}(\{w\}), \quad w \in A, \quad m \in \mathbb{N}, \tag{26}$$

where the preimages of w under n-fold iteration of R have to be chosen such that the limit exists. Furthermore, the multiplicities $\beta_m(w)$ of the eigenvalues depend only on w and m, and the generating functions of the multiplicities are rational.

The fact that all eigenvalues of Δ are negative real implies that the Julia set of R has to be contained in the negative real axis. We will exploit this fact later.

In some cases, such as the higher dimensional Sierpiński gaskets, the rational function R is conjugate to a polynomial. The method for meromorphic continuation of ζ_{Δ} presented in section 4.4 makes use of this assumption.

Recently, Teplyaev [106] showed under the assumption of spectral decimation that the zeta function of the Laplacian admits a meromorphic continuation to $\Re s > -\varepsilon$ for some $\varepsilon > 0$ depending on properties of the Julia set of the polynomial given by spectral decimation. His method uses ideas similar to those used in [81] for the meromorphic continuation of a Dirichlet series attached to polynomial iteration. Complementary to the ideas used here, Teplyaev's method carries over to rational functions R.

4.3. Eigenvalue counting

As in the Euclidean case, the eigenvalue counting function

$$N(x) = \sum_{\lambda_k < x} 1$$

measures the number of eigenvalues less than x. In a physical context, this quantity is often referred to as the 'integrated density of states'. It turns out that $N(x)$ in many cases does not exhibit a pure power law as in the Euclidean case, but shows periodic fluctuations. One source of this periodicity phenomenon is actually spectral decimation, especially the high multiplicities of eigenvalues, as will become clear in section 4.4.

Recall the definition of harmonic measure on the Julia set of a polynomial p of degree d (cf [109]): the sequence of measures

$$\mu_n = \frac{1}{d^n} \sum_{p^{(n)}(x)=\xi} \delta_x$$

converges weakly to a limiting measure μ, the harmonic measure on the Julia set of p. The point ξ can be chosen arbitrarily.

Assume now that the Laplace operator on the fractal K admits spectral decimation with a polynomial p. Then, the relation $\lim_{n\to\infty} \lambda^n p^{(-n)}(\{w\}) \in B(0,x)$ can be translated into

$$\lim_{n\to\infty} p^{(n)}(\lambda^{-n}z) = w \text{ and } |z| < x.$$

Here, $B(0,x)$ denotes the ball of radius x around 0. By general facts about the polynomial iteration (cf [39]), the limit exists and defines an entire function of z, the Poincaré function $\Phi(z)$ satisfying (1). Let

$$N_w(x) = \#\{z \in B(0,x) \mid \Phi(z) = w\}.$$

Then, the following relation can be obtained from the definition of harmonic measure:

$$\lim_{n\to\infty} d^{-n} N_w(\lambda^n x) = \mu(\Phi^{-1}(B(0,x)));$$

this holds for all x small enough to ensure the existence of the inverse function Φ^{-1} on $B(0,x)$.

In [110, theorem 5.2], we could prove a relation between the asymptotic behaviour of the partial counting functions $N_w(x)$ and the harmonic measure of balls $\mu(B(0,x))$. The existence of the two limits ($\rho = \log_\lambda d$)

$$\lim_{x\to\infty} x^{-\rho} N_w(x)$$

$$\lim_{t\to 0} t^{-\rho} \mu(B(0,t))$$

is equivalent. We conjectured there that these limits can only exist, if p is either a Chebyshev polynomial or a monomial. These are the only cases of polynomials with smooth Julia sets (cf [111]).

Summing up the above discussion, the eigenvalue counting function can be written as

$$N(x) = \sum_{w\in A} \sum_{m=0}^{\infty} \beta_m(w) N_w(\lambda^{-m}x). \tag{27}$$

Note that for fixed x, these sums are actually finite. In the known cases, such as the Sierpiński gasket, the growth of $\beta_m(w)$ is stronger than d^m, which implies that the terms for large m (with $N_w(\lambda^{-m}x)$ still positive) become dominant in this sum (using that $N_w(x)$ grows like x^ρ). This shows that the multiplicity of the eigenvalues has the main influence on the asymptotic behaviour of $N(x)$. Furthermore, this explains the presence of an oscillating factor in the asymptotic main term of $N(x)$. We will discuss that in more detail in section 4.4.

4.4. Spectral zeta functions

As in the Euclidean case, the eigenvalues of the Laplace operator $\triangle$ can be put into a Dirichlet generating function. This will later allow us to use methods and ideas from analytic number theory to obtain more precise asymptotic information on $N(x)$. The zeta function is again given as the Dirichlet generating function of the sequence of eigenvalues $(-\lambda_k)_k$:

$$\zeta_\triangle(s) = \sum_{\lambda_k \neq 0} \lambda_k^{-s},$$

where all eigenvalues are counted with their multiplicity. This series converges for $\Re s > \frac{d_S}{2}$; the value d_S is called the *spectral dimension* of the underlying fractal. As opposed to the case of manifolds, this dimension usually differs from the (Hausdorff) dimension of the space. This fact has been realized in the early studies of diffusion on fractals; see, e.g., [21, 20]. The zeta function is related to the eigenvalue counting function by

$$\zeta_\triangle(s) = \int_0^\infty x^{-s}\,\mathrm{d}N(x) = s\int_0^\infty N(x)x^{-s-1}\,\mathrm{d}x. \tag{28}$$

The second relation identifies $\zeta_\triangle$ as the Mellin transform of the counting function $N(x)$. This relation also gives a more intuitive description of the spectral dimension by the fact that $N(x)$ is of order $x^{\frac{d_S}{2}}$.

For the spectral zeta function on fractals, analytic continuations beyond the abscissa of convergence are known to exist only in specific cases: if the Laplace operator is defined via a self-similar Dirichlet form, an analytic continuation to the half-plane $\Re s > 0$ can be given (see remark 4.3), spectral decimation in general allows for an analytic continuation to a half-plane $\Re s > -\varepsilon$ for some $\varepsilon > 0$ (cf [37, 106, 112]); the case of spectral decimation given by a polynomial is rather specific, as it allows for an analytic continuation to the whole complex plane, as we will explain in what follows.

In the following, we will exploit the consequences of spectral decimation. Not too surprisingly after definition 8, iteration of polynomials will play an important role in this discussion. Furthermore, since relation (26) can be expressed in terms of the Poincaré function Φ, properties of this function will be used to derive the meromorphic continuation of $\zeta_\triangle$ to the whole complex plane.

Under the assumptions of spectral decimation, the Julia set of the polynomial $p(x) = a_d x^d + \cdots + a_1 x$ is a subset of the non-positive reals, which contains 0. By [110, theorem 4.1], this implies that $\lambda = p'(0) \leqslant d^2$. By [110, theorem 4.1], the equality can only occur, if p is a Chebyshev polynomial, which would correspond to spectral decimation on the unit interval (viewed as a self-similar fractal). Thus, in the cases of interest, we have that $\rho = \log_\lambda d < \frac{1}{2}$. The Poincaré function Φ is then an entire function of order ρ.

In order to find the analytic continuation of $\zeta_\triangle(s)$ to the whole complex plane, we analyse the *partial zeta functions*

$$\zeta_{\Phi,w}(s) = \sum_{\Phi(-\mu)=w,\mu\neq 0} \mu^{-s}. \tag{29}$$

Since $\Phi_w = 1 - \frac{1}{w}\Phi$ is a function of order $\rho = \log_\lambda d < \frac{1}{2}$, it can be expressed as a Hadamard product (cf [113])

$$1 - \frac{1}{w}\Phi(z) = \prod_{\Phi(-\mu)=w}\left(1 + \frac{z}{\mu}\right);$$

for $w = 0$, we have the slightly modified expression

$$\Phi_0(z) = \frac{1}{z}\Phi(z) = \prod_{\Phi(-\mu)=0,\mu\neq 0}\left(1 + \frac{z}{\mu}\right).$$

Taking the Mellin transform of $\log \Phi_w$, which exists for $-1 < \Re s < -\rho$, we obtain

$$M_w(s) = \int_0^\infty (\log \Phi_w(x))\, x^{s-1}\,\mathrm{d}x = \frac{\pi}{s\sin\pi s}\zeta_{\Phi,w}(-s). \tag{30}$$

Thus, to find the analytic continuation of $\zeta_{\Phi,w}(s)$ to the left of its abscissa of convergence, it suffices to find the analytic continuation of $M_w(s)$ for $\Re s > -\rho$. Following slightly different lines as in [83], we consider the function

$$\Psi_w(z) = \frac{p(\Phi(z)) - w}{a_d(\Phi(z) - w)^d} = \frac{\Phi_w(\lambda z)}{a_d(-w)^{d-1}\Phi_w(z)^d}.$$

Then, we have

$$\log \Psi_w(z) = \log \Phi_w(\lambda z) - d \log \Phi_w(z) - \log a_d - (d-1)\log(-w)$$

and this function tends to 0 exponentially for $z \to +\infty$. Taking the Mellin transform, we obtain

$$\begin{aligned}
(\lambda^{-s} - d)M_w(s) &= \int_0^\infty (\log \Phi_w(\lambda x) - d\log \Phi_w(x))\, x^{s-1}\, \mathrm{d}x \quad (\text{for } -1 < \Re s < -\rho)\\
&= \int_0^1 (\log \Phi_w(\lambda x))\, x^{s-1}\mathrm{d}x - (\log a_d + (d-1)\log(-w))\frac{1}{s} \qquad (*)\\
&\quad + \int_1^\infty (\log \Phi_w(\lambda x) - d\log \Phi_w(x) - \log a_d - (d-1)\log(-w))\, x^{s-1}\mathrm{d}x\\
&\quad (\text{for } \Re s > -1)\\
&= \int_0^1 (\log \Psi_w(x))\, x^{s-1}\mathrm{d}x\\
&\quad + \int_1^\infty (\log \Phi_w(\lambda x) - d\log \Phi_w(x) - \log a_d - (d-1)\log(-w))\, x^{s-1}\mathrm{d}x\\
&\quad (\text{for } \Re s > 0)\\
&= \int_0^\infty (\log \Psi_w(x))\, x^{s-1}\, \mathrm{d}x.
\end{aligned}$$

Reading the line marked $(*)$ of this computation shows that $M_w(s)$ has a simple pole at $s = 0$ with the residue

$$\operatorname*{Res}_{s=0} M_w(s) = \frac{\log a_d}{d-1} + \log(-w).$$

Furthermore, this computation shows that $M_w(s)$ is holomorphic in the half-plane $\Re s > 0$. Using (30) gives the analytic continuation of $\zeta_{\Phi,w}(s)$ for $\Re s < 0$:

$$\zeta_{\Phi,w}(s) = \frac{s \sin \pi s}{\pi(\lambda^s - d)} \int_0^\infty (\log \Psi_w(x))\, x^{-s-1}\, \mathrm{d}x. \tag{31}$$

This shows that $\zeta_{\Phi,w}(-m) = 0$ for $m \in \mathbb{N}_0$ (for $s = 0$ the double zero of $s \sin \pi s$ cancels the simple pole of $M_w(-s)$). These could be called the 'trivial zeros' as in the case of the Riemann zeta function. Furthermore, we obtain

$$\zeta'_{\Phi,w}(0) = -\frac{\log a_d}{d-1} - \log(-w).$$

Equation (31) even lends itself to the numerical computation of values of $\zeta_{\Phi,w}(s)$ for $\Re s < 0$, as we will see in section 4.5.2.

By our assumption on spectral decimation, the generating functions of the multiplicities of the eigenvalues are rational

$$R_w(z) = \sum_{m=0}^\infty \beta_m(w) z^m.$$

Thus, we can write the spectral zeta function of $\triangle$ as

$$\zeta_\triangle(s) = \sum_{w\in A}\sum_{m=0}^\infty \beta_m(w) \sum_{\Phi(-\mu)=w} (\lambda^m\mu)^{-s} = \sum_{w\in A} R_w(\lambda^{-s})\zeta_{\Phi,w}(s). \tag{32}$$

Since all functions involved in the last (finite) sum are meromorphic in the whole complex plane, we have found the meromorphic continuation of $\zeta_\triangle$ to the whole complex plane. The functions $\zeta_{\Phi,w}(s)$ have at most simple poles in the points $s = \rho + \frac{2k\pi \mathrm{i}}{\log\lambda}$ ($k \in \mathbb{Z}$). Furthermore, the residues of these poles do not depend on w. All other poles of $\zeta_\triangle$ come from the poles of the functions $R_w(\lambda^{-s})$. Since these are rational functions of λ^{-s}, their poles are equally spaced on vertical lines.

Summing up, $\zeta_\triangle(s)$ has a meromorphic continuation to the whole complex plane with possible poles in the points

$$s = -\log_\lambda \beta_{w,j} + \frac{2k\pi \mathrm{i}}{\log\lambda} \text{ with } k \in \mathbb{Z},$$

where the $\beta_{w,j}$ are the poles of the rational functions $R_w(z)$ and (at most) simple poles in the points

$$s = \rho + \frac{2k\pi \mathrm{i}}{\log\lambda} \text{ with } k \in \mathbb{Z}.$$

The *spectral dimension* d_S is then given as

$$\frac{d_S}{2} = \max\left(\rho, \max_{w,j}(-\Re\log_\lambda \beta_{w,j})\right),$$

the rightmost poles of the function $\zeta_\triangle$.

Furthermore, since the functions $\zeta_{\Phi,w}(s)$ are bounded for $\Re s \geqslant \rho + \varepsilon$ and the functions $R_w(\lambda^{-s})$ are bounded along every vertical line, which contains no poles, the function $\zeta_\triangle(s)$ is bounded along every vertical line $c + \mathrm{i}t$ for $c > \rho$, which does not contain a pole of any of the functions $R_w(\lambda^{-s})$.

In the case of the Sierpiński gasket and its higher dimensional analogues, the rightmost poles of $\zeta_\triangle$ come from the rational functions $R_w(\lambda^{-s})$. Furthermore, the relation

$$\sum_{w\in A} R_w(1/d) = 0$$

holds, which amounts to a cancellation of the poles of the functions $\zeta_{\Phi,w}$ in (32). Thus, the analytic behaviour of the function $\zeta_\triangle$ is mainly governed by the functions R_w. This means that in this respect the effects of multiplicity prevail over the individual eigenvalues. In a recent paper [112], it was proved that the spectral zeta function of *fully symmetric fractals* has an analytic continuation to the region $\Re s > -\varepsilon$ for some positive ε; furthermore, it is shown that all poles of $\zeta_\triangle$ have real part 0 or $\frac{d_S}{2}$. This explains the above-mentioned cancellation of poles in this case.

We now investigate the asymptotic behaviour of $N(x)$ under the assumption that $\beta_m(w)$ grows exponentially faster than d^m for some $w \in A$. This implies that $\frac{d_S}{2} > \rho$. We use the classical Mellin–Perron formula (cf [114]) to express $N(x)$ in terms of $\zeta_\triangle(s)$:

$$N(x) = \lim_{T\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{c-iT}^{c+iT} \zeta_\triangle(s) x^s \frac{\mathrm{d}s}{s},$$

for any $c > \frac{d_S}{2}$. Now the line of integration can be shifted to the left to $\Re(s) = c'$ for $\frac{d_S}{2} > c' > \rho$. This is justified, because $\zeta_\triangle(\sigma + \mathrm{i}t)$ remains bounded for $|t| \to \infty$ and $\sigma > \rho$. Then, we have

$$N(x) = \lim_{T\to\infty} \left(\frac{1}{2\pi \mathrm{i}} \int_{c'-\mathrm{i}T}^{c'+\mathrm{i}T} \zeta_\triangle(s) x^s \frac{\mathrm{d}s}{s} - \sum_{2\pi|k|<T\log\lambda} \operatorname*{Res}_{s=d_S/2+2k\pi \mathrm{i}/\log\lambda} \frac{\zeta_\triangle(s)x^s}{s} \right).$$

Now the limit of the integral $\int_{c'-it}^{c'+iT}$ can be shown to exist for $T \to \infty$, which shows that the limit

$$\lim_{T\to\infty} \sum_{2\pi|k|<T\log\lambda} \operatorname*{Res}_{s=d_S/2+2k\pi i/\log\lambda} \frac{\zeta_\triangle(s)x^s}{s}$$

exists. This can be rewritten as $x^{d_S/2}H(\log_\lambda x)$ for a periodic continuous function H given by its Fourier expansion

$$H(t) = \sum_{k\in\mathbb{Z}} \frac{e^{2k\pi it}}{d_S/2 + 2k\pi i/\log\lambda} \operatorname*{Res}_{s=d_S/2+2k\pi i/\log\lambda} \zeta_\triangle(s).$$

The limit of the integral can be shown to be $\mathcal{O}(x^{c'})$. Thus, we have shown

$$N(x) = x^{d_S/2}H(\log_\lambda x) + \mathcal{O}(x^{c'}). \tag{33}$$

In particular, the limit $\lim_{x\to\infty} x^{-d_S/2}N(x)$ does not exist.

We remark here that there exists a totally different approach to zeta functions of fractals due to Lapidus and his collaborators [10, 41, 115, 116]. In this geometric approach, the volume of tubular neighbourhoods of the fractal G

$$V_G(\varepsilon) = \mathrm{vol}\,(\{x \in \mathbb{R}^n \mid d(x, G) < \varepsilon\})$$

is analysed. The asymptotic behaviour of $V_G(\varepsilon)$ for $\varepsilon \to 0$ gives rise to the definition of a zeta function. In this geometric context, the complex solutions of (3) occur as poles of the zeta function; they are also called the 'complex dimensions' of the fractal G in this context. This approach is motivated by the definition of Minkowski content, which itself turned out to be too restrictive to measure (most of the) self-similar fractals.

4.5. Trace of the heat kernel

The diffusion semi-group A_t introduced in section 3.3 can be given in terms of the heat kernel $p_t(x, y)$. As opposed to the situation in the Euclidean case, the knowledge of the behaviour of the heat kernel for $t \to 0$ is by far not as precise. Kumagai [117] proved the following lower and upper estimates of the form:

$$t^{-\frac{d_S}{2}} \exp\left(-c_1\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right) \lesssim p_t(x, y) \tag{34}$$

$$p_t(x, y) \lesssim t^{-\frac{d_S}{2}} \exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right), \tag{35}$$

where d_S and d_w are the spectral and the walk dimension of the fractal, respectively. These dimensions are related to the Hausdorff dimension d_f of the fractal via the so-called Einstein relation

$$d_S d_w = 2d_f.$$

In the fractal case usually $d_w > 2$, as opposed to the Euclidean case, where $d_w = 2$, which implies $d_S = d_f$.

Only recently, a conjecture by Barlow and Perkins [20] could be proved by Kajino [118]. Namely that for any $x \in K$ the limit

$$\lim_{t\to 0} t^{\frac{d_S}{2}} p_t(x, x)$$

does *not* exist for a large class of self-similar fractals. This gives an indication as to why obtaining more precise information on the heat kernel than the estimates (34) and (35) would be very difficult.

Even if the precise behaviour of the heat kernel seems to be far from reach, the trace of the heat kernel

$$K(t) = \int_K p_t(x,x)\,\mathrm{d}\mathcal{H}(x)$$

can still be analysed in some detail under the assumption of spectral decimation. The reason for this are the two relations between $K(t)$, $N(x)$, and $\zeta_\triangle$:

$$K(t) = \int_0^\infty \mathrm{e}^{-tx}\,\mathrm{d}N(x) = t\int_0^\infty N(x)\,\mathrm{e}^{-xt}\,\mathrm{d}x, \tag{36}$$

$$\zeta_\triangle(s) = \frac{1}{\Gamma(s)}\int_0^\infty K(t)t^{s-1}\,\mathrm{d}t \tag{37}$$

valid for $t > 0$ and $\Re s > \frac{d_S}{2}$. The first expresses $K(t)$ as the Laplace transform of $N(x)$, and the second gives $\zeta_\triangle$ as the Mellin transform of $K(t)$.

Given the precise knowledge on the zeta function obtained in section 4.4, we can use the Mellin inversion formula to derive asymptotic information on $K(t)$ for $t \to 0$. We have

$$K(t) = \frac{1}{2\pi i}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \zeta_\triangle(s)\Gamma(s)t^{-s}\,\mathrm{d}s, \tag{38}$$

where integration is along the vertical line $\Re s = c > \frac{d_S}{2}$. We note that the Gamma function decays exponentially along vertical lines, whereas Dirichlet series grow at most polynomially by the general theory of Dirichlet series (cf [119]). Thus, convergence of the integral is guaranteed.

Shifting the line of integration in (38) to $\Re s = -M - \frac{1}{2}$ ($M \in \mathbb{N}$) and taking the poles of the integrand into account (note that the poles of the Gamma function are compensated by the 'trivial' zeros of $\zeta_\triangle$), we obtain for $t \to 0$

$$K(t) = t^{-\frac{d_S}{2}}H_{\frac{d_S}{2}}(\log_\lambda t) + H_0(\log_\lambda t) + \mathcal{O}(t^{M+\frac{1}{2}}). \tag{39}$$

Here, $H_{\frac{d_S}{2}}$ and H_0 are the periodic continuous functions of period 1, whose Fourier coefficients are given as the residues of $\zeta_\triangle(s)\Gamma(s)$ in the poles on the lines $\Re s = \frac{d_S}{2}$ or $\Re s = 0$, respectively (using the recent result by Steinhurst and Teplyaev [112] on the location of the poles of $\zeta_\triangle$ for fully symmetric fractals). By the strong decay of the Gamma function, these functions are even real analytic. Since M can be made arbitrarily large, the error term decays faster than any positive power of t for $t \to 0$.

The existence of complex poles of the zeta function thus implies the presence of periodically oscillating terms in the asymptotic behaviour of the trace of the heat kernel for $t \to 0$. This implies that the limit $\lim_{t\to 0} t^{d_S/2}p_t(x,x)$ does not exist on a set of positive measure for x in accordance with the above-mentioned result by Kajino [118]. The consequences of the presence of complex poles of the zeta function to properties of the heat kernel, the density of states and the partition function, as well as the physical implications of the resulting fluctuating behaviour in these quantities, have been discussed in [42, 43, 120].

4.5.1. Casimir energy on fractals. As an application of the spectral zeta function and its properties, we compute the Casimir energy of the Sierpiński gasket and similar fully symmetric fractals. We follow the lines of the exposition in [44, 121] and [15] and refer the reader to these works for further details. For a very recent discussion of Casimir physics on fractals, we refer the reader to Dunne [120].

Consider the differential operator $P = -\frac{\partial^2}{\partial\tau^2} - \triangle$ on $(\mathbb{R}/\frac{1}{\beta}\mathbb{Z}) \times G$, where G denotes a fully symmetric fractal in the sense of [112] admitting spectral decimation. As usual, we set the parameter $\beta = 1/kT$. Then, the eigenfunctions of P are the products of eigenfunctions of the operators $\partial^2/\partial\tau^2$ and $\triangle$; the eigenvalues are sums $4k^2\pi^2/\beta^2 + \lambda_n$ for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. The zeta function of the operator P is then given by (for $\Re s > \frac{d_S+1}{2}$)

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^\infty K(t) \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{4\pi^2 n^2}{\beta^2}t} t^{s-1} \mathrm{d}t.$$

Using the theta-function relation

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{4\pi^2 n^2}{\beta^2}t} = \frac{\beta}{2\sqrt{\pi t}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\frac{\beta^2 n^2}{4t}},$$

we obtain the relation

$$\zeta_P(s) = \frac{\beta}{2\sqrt{\pi}\Gamma(s)} \Gamma\left(s-\frac{1}{2}\right) \zeta_\triangle\left(s-\frac{1}{2}\right) + \frac{\beta}{\sqrt{\pi}\Gamma(s)} \int_0^\infty K(t) \sum_{n=1}^\infty \mathrm{e}^{-\frac{\beta^2 n^2}{4t}} t^{s-\frac{3}{2}}\, \mathrm{d}t \tag{40}$$

valid for $\Re s > \frac{d_S}{2} + \frac{1}{2}$. The first summand has an analytic continuation to the whole complex plane, whereas the second summand represents an entire function. This gives the analytic continuation of ζ_P to the whole complex plane, which allows us to compute $\zeta_P'(0)$, the logarithm of the regularized determinant of P.

For the rescaled operator P/μ^2, we have

$$\zeta_{P/\mu^2}'(0) = \zeta_P'(0) + \zeta_P(0) \ln \mu^2.$$

From (40), we compute $\zeta_P(0) = 0$, where we have again used the fact from [112] that $\zeta_\triangle$ has no poles with $\Re s < 0$ as well as the existence of an analytic continuation obtained in section 4.4. Furthermore, we obtain

$$\zeta_P'(0) = -\beta\zeta_\triangle\left(-\frac{1}{2}\right) + \frac{\beta}{\sqrt{\pi}} \sum_{n=1}^\infty \sum_{j=1}^\infty \int_0^\infty \mathrm{e}^{-\frac{\beta^2 n^2}{4t} - \lambda_j t} t^{-\frac{3}{2}} \mathrm{d}t.$$

The integral and the summation over n can be evaluated explicitly, which finally gives

$$\zeta_P'(0) = -\beta\zeta_\triangle\left(-\frac{1}{2}\right) - 2\sum_{j=1}^\infty \ln(1 - \mathrm{e}^{-\beta\sqrt{\lambda_j}}).$$

The case of the Sierpiński gasket and other fully symmetric self-similar fractal shows an important contrast to the manifold case, where the zeta function (generically) has a pole at $-\frac{1}{2}$ (see section 4.1). In the manifold case, the value $\zeta_\triangle(-\frac{1}{2})$ has to be replaced by the *finite part* of $\zeta_\triangle(s)$ at $s = -\frac{1}{2}$, the function minus the principal part at the polar singularity (cf [15]). The presence of a pole of $\zeta_\triangle(s)$ at $s = -\frac{1}{2}$ amounts to a scale dependence of the value $\zeta_P'(0)$; if $\zeta_\triangle$ has no pole at $s = -\frac{1}{2}$, then $\zeta_P'(0)$ is scale invariant. This happens in the fractal case as well as for specific manifolds, as is mentioned in [45, p 86].

Now, the energy of the system is given by

$$E = -\frac{1}{2}\frac{\partial}{\partial\beta}\zeta_{P/\mu^2}'(0) = \frac{1}{2}\zeta_\triangle\left(-\frac{1}{2}\right) + \sum_{j=1}^\infty \frac{\sqrt{\lambda_j}}{\mathrm{e}^{\beta\sqrt{\lambda_j}} - 1}.$$

Letting $\beta \to \infty$, which is equivalent to letting temperature tend to 0, gives the Casimir energy

$$E_{\text{Cas}} = \frac{1}{2}\zeta_\triangle\left(-\frac{1}{2}\right).$$

4.5.2. Numerical computations. We will now describe the numerical computation of the values $\zeta_{\Phi,w}(-1/2)$, which are needed for the computation of $\zeta_{\triangle}(-1/2)$ in the case of the Sierpiński gasket. This fractal admits spectral decimation with the polynomial $p(x) = x(x+5)$ (cf [2, 27]). The corresponding Poincaré function is then given by the unique holomorphic solution of the equation

$$\Phi(5z) = \Phi(z)(\Phi(z)+5), \quad \Phi(0) = 0, \quad \Phi'(0) = 1. \tag{41}$$

We use the expression for $\zeta_{\triangle}(s)$ for Dirichlet boundary conditions derived in [83, section 7]:

$$\zeta_{\triangle}^{\mathrm{D}}(s) = 5^{-s}\zeta_{\Phi,-2}(s) + \frac{3}{5^s(5^s-1)(5^s-3)}\zeta_{\Phi,-3}(s) + \frac{2\cdot 5^s - 5}{(5^s-1)(5^s-3)}\zeta_{\Phi,-5}(s).$$

Combining this with (31) and inserting $s = -\frac{1}{2}$ gives

$$\begin{aligned}\zeta_{\triangle}^{\mathrm{D}}\left(-\frac{1}{2}\right) = \frac{1}{2\pi(\sqrt{5}-2)}\Bigg[&\sqrt{5}\int_0^\infty \log\Psi_{-2}(x)x^{-\frac{1}{2}}\,\mathrm{d}x \\ &+\frac{155+135\sqrt{5}}{88}\int_0^\infty \log\Psi_{-3}(x)x^{-\frac{1}{2}}\,\mathrm{d}x \\ &-\frac{245+47\sqrt{5}}{124}\int_0^\infty \log\Psi_{-5}(x)x^{-\frac{1}{2}}\,\mathrm{d}x\Bigg].\end{aligned} \tag{42}$$

From the general information on the asymptotic behaviour of Poincaré functions derived in [110, 122], we obtain estimates of the form

$$\exp(C_1x^\rho) \leqslant \Phi(x) \leqslant \exp(C_2x^\rho)$$

for positive constants C_1 and C_2, $\rho = \log_5 2$ and valid for $x \geqslant x_0$. Such estimates can be proved easily by first showing the estimate for an interval of the form $[x_0, 5x_0]$ and then extending it by using the functional equation (41). For instance, we used $C_1 = 1$, $C_2 = 1.08$ and $x_0 = 10$. Then, the functions $\log\Psi_w(x)$ tend to 0 like $\exp(-C_2x^\rho)$ for $x \to \infty$. Given a precision goal $\varepsilon > 0$, we choose T so large that

$$\int_T^\infty \exp(-C_2x^\rho)x^{-\frac{1}{2}}\mathrm{d}x < \frac{\varepsilon}{2}.$$

Then, the improper integrals in (42) can be replaced by $\int_0^T$. The functions $\Psi_w(x)$ can be computed to high precision by using the power series representation of $\Phi(x)$ for $|x| \leqslant 1$ and the functional equation (41) to obtain

$$\Phi(x) = p^{(k+1)}(\Phi(x/5^{k+1}))$$

for $5^k < |x| \leqslant 5^{k+1}$. This allows the computation of the remaining finite integrals up to precision $\frac{\varepsilon}{2}$. We obtained

$$E_{\mathrm{Cas}}^{\mathrm{D}} = 0.547\,469\,3544\ldots$$

for the Casimir energy of the two-dimensional Sierpiński gasket with Dirichlet boundary conditions.

Similarly, we have for Neumann boundary conditions (cf [83])

$$\zeta_{\triangle}^{\mathrm{N}}(s) = \frac{1}{(5^s-1)(5^s-3)}((2\cdot 5^s-5)\zeta_{\Phi,-3}(s) + 5^s\zeta_{\Phi,-5}(s)).$$

This gives by the same numerical estimates as before

$$E_{\mathrm{Cas}}^{\mathrm{N}} = 2.134\,394\,089\,264\ldots$$

for the Casimir energy two-dimensional Sierpiński gasket with Neumann boundary conditions.

4.6. Self-similarity and the renewal equation

Let K be a p.c.f. self-similar fractal with self-similar structure $(K, \Sigma, (F_i)_{i=1}^m)$ with contraction ratios α_i and Hausdorff dimension ρ. Let K be equipped with a Dirichlet form with parameters r_i and λ as described in section 3.5. Denote by $N_{\mathrm{D}}(x)$ and $N_{\mathrm{N}}(x)$ the eigenvalue counting functions for the Laplace operator under Dirichlet and Neumann boundary conditions, respectively. Then, the following crucial fact was observed in [49]:

$$\sum_{i=1}^{m} N_{\mathrm{D}}\left(\frac{r_i\alpha_i^{\rho}}{\lambda}x\right) \leqslant N_{\mathrm{D}}(x) \leqslant N_{\mathrm{N}}(x) \leqslant \sum_{i=1}^{m} N_{\mathrm{N}}\left(\frac{r_i\alpha_i^{\rho}}{\lambda}x\right), \tag{43}$$

where α_i are the contraction ratios introduced in section 2, ρ is the Hausdorff dimension of K, and the r_i are the weights used for the construction of the Dirichlet form $\mathcal{E}$ in section 3.5. Furthermore, the inequalities

$$N_{\mathrm{D}}(x) \leqslant N_{\mathrm{N}}(x) \leqslant N_{\mathrm{D}}(x) + \#(V_0)$$

hold, where V_0 is defined in definition 2. Setting

$$\gamma_i = \left(\frac{r_i\alpha_i^{\rho}}{\lambda}\right)^{1/2},$$

we end up with the equation

$$N_{\mathrm{D}}(x) = \sum_{i=1}^{m} N_{\mathrm{D}}(\gamma_i^2 x) + g(x), \tag{44}$$

where $g(x)$ is defined as the difference between the left-hand side and the sum on the right-hand side and remains bounded by (43). A similar equation holds for $N_{\mathrm{N}}(x)$.

Equation (44) can be transformed into the classical renewal equation occurring in probability theory (cf [123]). The asymptotic behaviour of the solutions of this equation is described by the following theorem (stated as in [8]).

Theorem 4.1 (renewal theorem)**.** *Let $t^* > 0$ and f be a measurable function on $\mathbb{R}$, such that $f(t) = 0$ for $t < t^*$. If f satisfies the renewal equation*

$$f(t) = \sum_{j=1}^{N} p_j f(t - \alpha_j) + u(t),$$

where $\alpha_1, \ldots, \alpha_N$ are positive real numbers, $p_j > 0$ for $j = 1, \ldots, N$ and $\sum_{j=1}^{N} p_j = 1$. Assume that u is non-negative and directly Riemann integrable on $\mathbb{R}$ with $u(t) = 0$ for $t < t^*$. Then, the following conclusions hold.

(i) Arithmetic (or lattice) case: if the group generated by α_j is discrete (i.e. there exist $T > 0$ and integers m_j with greatest common divisor 1 such that $\alpha_j = Tm_j$; all the ratios α_i/α_j are then rational), then $\lim_{t\to\infty} |f(t) - G(t)| = 0$, where the T-periodic function G is given by

$$G(t) = \left(\sum_{j=1}^{N} p_j m_j\right)^{-1} \sum_{k=-\infty}^{\infty} u(t + kT).$$

(ii) Non-arithmetic (non-lattice) case: if the group generated by the α_j is dense in $\mathbb{R}$ (i.e. at least one of the ratios α_i/α_j is irrational), then

$$\lim_{t\to\infty} f(t) = \left(\sum_{j=1}^{N} p_j\alpha_j\right)^{-1} \int_{-\infty}^{\infty} u(t)\,\mathrm{d}t.$$

Corollary 4.1. *Let f be a solution of the equation*

$$f(x) = \sum_{i=1}^{m} f\left(\gamma_i^2 x\right) + g(x),$$

where $0 < \gamma_i < 1$ and g is a bounded function. Let d_S be the unique positive solution of the equation

$$\sum_{i=1}^{m} \gamma_i^s = 1,$$

then the following assertions hold.

(i) Arithmetic (or lattice) case: if the group generated by the values $\log \gamma_j$ is discrete, generated by $T > 0$, then

$$f(x) = x^{d_S/2}\,(G((\log x)/2) + o(1))$$

for a periodic function G of period T.

(ii) Non-arithmetic (or non-lattice) case: if the group generated by the values $\log \gamma_j$ is dense, then

$$\lim_{x\to\infty} f(x) x^{-d_S/2}$$

exists.

The corollary is an immediate consequence of the theorem by setting $f(e^t) = \mathrm{e}^{d_S t/2} F(t)$ and applying the theorem to F and $g(e^t)\,\mathrm{e}^{-d_S t/2}$.

Summing up, we have the following theorem.

Theorem 4.2 ([49, theorem 2.4])**.** *Let $(K, \Sigma, (F_i)_{i=1}^m)$ be a self-similar structure with contraction ratios α_i and Hausdorff dimension ρ. Let K be equipped with a Dirichlet form with parameters r_i and λ as described in section 3.5. Let $\frac{d_S}{2}$ be the unique positive solution of the equation*

$$\sum_{i=1}^{m} \left(\frac{r_i \alpha_i^{\rho}}{\lambda}\right)^s = 1,$$

the 'spectral dimension' of the harmonic structure of K. Then, the following assertions hold.

(i) Lattice case: if the group generated by the values $\log(r_i\alpha_i^{\rho}/\lambda)$ is discrete, then

$$N_D(x) = x^{d_S/2}\,(G((\log(x))/2) + o(1))$$

for a periodic function G, which is non-constant in general.

(ii) Non-lattice case: if the group generated by the values $\log(r_i\alpha_i^{\rho}/\lambda)$ is dense, then

$$\lim_{x\to\infty} N_D(x) x^{-d_S/2}$$

exists.

The behaviour of $N_N(x)$ for $x \to \infty$ is the same.

Remark 4.1. Similar ideas are used in [124] to study the asymptotic expansion of the eigenvalue counting function of the Laplace operator on an open set $\mathfrak{G}$, which is formed from an open set G_0 with smooth boundary as

$$\mathfrak{G} = \bigcup_{n=0}^{\infty} G_n$$

with

$$G_{n+1} = \bigcup_{i=1}^{m} F_i(G_n),$$

with similitudes F_i and all the unions assumed to be disjoint.

Remark 4.2. Very recently, Kajino [125] extended this approach to infinitely ramified fractals such as the Sierpiński carpets. He observed that in order to have an inequality of the form (43) it suffices to have the corresponding self-similarity property

$$\mathcal{E}(f, f) = \lambda \sum_{i=1}^{m} r_i^{-1} \mathcal{E}(f \circ F_i, f \circ F_i)$$

of the underlying Dirichlet form. This self-similarity together with symmetries characterizes the Dirichlet form on the Sierpiński carpet uniquely, as was shown in [52].

Remark 4.3. Taking the Mellin transform (28) of equation (44) yields an analytic continuation of $\zeta_\triangle(s)$ to the half-plane $\Re s > 0$ by

$$\zeta_\triangle(s) = \frac{s}{1 - \sum_{i=1}^{m} \gamma_i^{2s}} \int_0^\infty g(x) x^{-s-1}\, \mathrm{d}x$$

exhibiting poles at the complex solutions of the equation

$$\sum_{i=1}^{m} \gamma_i^{2s} = 1,$$

the *complex dimensions* of the fractal structure (cf [10, 41, 42, 116]).

5. Self-similarity and the Poincaré equation

As has been described in section 3.2, diffusion processes on finitely ramified fractals, which have strong self-similarity and symmetry properties, may be defined as limits of discrete simple symmetric nearest-neighbour random walks on the associated approximating graphs. The analysis is based on a renormalization-type argument, involving self-similarity and decimation invariance.

Let X_t be the Markov process defined in section 3.2. This process has continuous sample paths (in fact, it is a Feller process), which is invariant under the rescaling $x \to 2x, t \to 5t$ by definition. By the construction of the process X_t the time scaling is governed by a super-critical branching process.

By the general theory of branching processes (cf [22]), the time scaling leads to an iterative functional equation, i.e. the Poincaré equation. The fluctuating behaviour of its solution led to multiplicative periodicity phenomena in the asymptotics of the transition probabilities, the trace of the heat kernel and the eigenvalue counting function. In this section, we will give a more detailed description of the asymptotic behaviour of the Poincaré functions and relate the periodic fluctuations to the Julia set of the underlying polynomial. It should be mentioned that the fluctuations are usually very small in amplitude; this has been observed by Karlin and McGregor [126, 127] and Dubuc [128].

5.1. Historical remarks on the Poincaré equation

In the seminal papers [30, 129], Poincaré has studied the equation

$$f(\lambda z) = R(f(z)), \quad z \in \mathbf{C}, \tag{45}$$

where $R(z)$ is a rational function and $\lambda \in \mathbf{C}$. He proved that if $R(0) = 0$, $R'(0) = \lambda$, and $|\lambda| > 1$, then there exists a meromorphic (for rational R) or entire (for polynomial R) solution of (45). After Poincaré, (45) is called *the Poincaré equation* and the solutions of (45) are called *the Poincaŕe functions*. The next important step was made by Valiron [130, 131], who investigated the case, where $R(z) = p(z)$ is a polynomial, i.e.

$$f(\lambda z) = p(f(z)), \quad z \in \mathbf{C}, \tag{46}$$

and obtained conditions for the existence of an entire solution $f(z)$. Furthermore, he derived the following asymptotic formula for $M(r) = \max_{|z|\leqslant r} |f(z)|$:

$$\ln M(r) \sim r^{\rho} Q\left(\frac{\ln r}{\ln |\lambda|}\right), \quad r \to \infty. \tag{47}$$

Here, $Q(z)$ is a 1-periodic function bounded between two positive constants: $\rho = \frac{\ln d}{\ln |\lambda|}$ and $d = \deg p(z)$.

Various aspects of the Poincaré functions have been studied in [110, 122, 132–137].

5.2. Asymptotics and dynamics in the real case. Properties of the Julia set

Further refinements are possible when $\lambda > 1$ is real and $p(z) = p_d z^d + \cdots + p_1 z + p_0$ is a polynomial with real coefficients. Without loss of generality, we can also assume that

$$f(0) = p(0) = 0; \quad p'(0) = \lambda > 1 \text{ and } f'(0) = 1.$$

For the reader's convenience, we recall now some basic notions from the iteration theory and complex dynamics [38, 6]. To make things shorter, we give here definitions which slightly differ from standard ones, but are equivalent to them in the *polynomial* case. That will suffice our needs, in what follows.

We will especially need the component of ∞ of Fatou set $\mathcal{F}(p)$ given by

$$\mathcal{F}_\infty(p) = \left\{z \in \mathbb{C} \mid \lim_{n\to\infty} p^{(n)}(z) = \infty\right\}, \tag{48}$$

and the filled Julia set is given by

$$\mathcal{K}(p) = \{z \in \mathbb{C} \mid (p^{(n)}(z))_{n\in\mathbb{N}} \text{ is bounded}\} = \mathbb{C} \setminus \mathcal{F}_\infty(p). \tag{49}$$

Now, according to [6], one can define the Julia set $\mathcal{J}(p)$ as

$$\partial\mathcal{K}(p) = \partial\mathcal{F}_\infty(p) = \mathcal{J}(p). \tag{50}$$

In the case of polynomials, this can be used as an equivalent (and much simpler) definition of the Julia set.

Let $f(z)$ be an entire solution of (46). In contrast with the previous section (where the asymptotics of $|f(z)|$ is studied), here we collect some results on the asymptotics of the solution $f(z)$ *itself* (in some angular regions of the complex plane). Our presentation is based on [83, 122, 110].

Theorem 5.1 ([83, theorem 1])**.** *Let f be an entire solution of the functional equation (46). Furthermore, suppose that $\mathcal{F}_\infty(p)$ contains an angular region of the form*

$$W_\beta = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \beta\}$$

for some $\beta > 0$. Then, for any $\varepsilon > 0$ and any $M > 0$, the asymptotic relation

$$f(z) = \exp\left(z^{\rho} Q\left(\frac{\log z}{\log \lambda}\right) + o\left(|z|^{-M}\right)\right) \tag{51}$$

holds uniformly for $z \in W_{\beta-\varepsilon}$, where Q is a periodic holomorphic function of period 1 on the strip $\{w \in \mathbb{C} \mid |\Im w| < \frac{\beta}{\log \lambda}\}$. The real part of $z^{\rho} Q(\frac{\log z}{\log \lambda})$ is bounded between two positive constants; Q takes real values on the real axis.

Remark 5.1. Note that the condition on the Fatou component $\mathcal{F}_\infty$ is used in the proof of this theorem to ensure that $f(z)$ tends to infinity in the angular region W_β (cf [83]). Therefore, this condition could be replaced by

$$\lim_{z\to\infty} f(z) = \infty \text{ for } |\arg z| < \beta.$$

Yet a stronger result can be derived under the additional assumption that the Julia set $\mathcal{J}(p)$ of polynomial $p(z)$ is *real* (see corollary 5.1).

Remark 5.2. In the context of spectral decimation, the assumption that the Julia set $\mathcal{J}(p)$ is real, is rather natural: the Julia set is closely related to level sets of the Poincaré function, which in turn constitute the spectrum of the self-adjoint operator $\triangle$.

Corollary 5.1 ([110, corollary 4.1])**.** *Assume that p is a real polynomial such that $\mathcal{J}(p)$ is real and all coefficients p_i ($i \geqslant 2$) of p are non-negative. Then, $\mathcal{J}(p) \subset \mathbb{R}^- \cup \{0\}$, and therefore,*

$$f(z) \sim \exp\left(z^\rho Q\left(\frac{\log z}{\log \lambda}\right)\right) \tag{52}$$

for $z \to \infty$ and $|\arg z| < \pi$. Here, Q is a periodic function of period 1 holomorphic in the strip given by $|\Im w| < \frac{\pi}{\log \lambda}$. Furthermore, for every $\varepsilon > 0$, the real part of $z^\rho Q(\frac{\log z}{\log \lambda})$ is bounded between two positive constants for $|\arg z| \leqslant \pi - \varepsilon$.

If $p(z)$ is a quadratic polynomial (a case arising in the context of spectral decimation), it is possible to give an exact criterion for reality of $\mathcal{J}(p)$:

Lemma 5.1 ([122, lemma 6.7])**.** *Let*

$$p(z) = az(z - \omega), \qquad 0 \neq \omega \in \mathbf{R}. \tag{53}$$

Then, Julia set $\mathcal{J}(p)$ is real, if and only if the following condition is fulfilled:

$$a|\omega| \geqslant \begin{cases} 2, & \omega > 0 \\ 4, & \omega < 0 \end{cases}. \tag{54}$$

Note that in the above lemma $|p'(0)| = a|\omega|$ and $d = \deg p(z) = 2$. Therefore, (54) can be rewritten in the form

$$|p'(0)| \geqslant \begin{cases} d, & \omega > 0 \\ d^2, & \omega < 0 \end{cases}. \tag{55}$$

It turns out that the *necessity* part of latter criterion (in the form (55)) is valid for general polynomial p, with *real* Julia set $\mathcal{J}(p)$. Namely, the following result of Pommerenke–Levin–Eremenko–Yoccoz type (on inequalities for multipliers) is true

Theorem 5.2 ([110, theorem 4.1])**.** *Let p be a polynomial of degree $d > 1$ with real Julia set $\mathcal{J}(p)$. Then, for any fixed point ξ of p with $\min \mathcal{J}(p) < \xi < \max \mathcal{J}(p)$, we have $|p'(\xi)| \geqslant d$. Furthermore, $|p'(\min \mathcal{J}(p))| \geqslant d^2$ and $|p'(\max \mathcal{J}(p))| \geqslant d^2$. An equality in one of these inequalities implies that p is linearly conjugate to the Chebyshev polynomial T_d of degree d.*

Remark 5.3. This theorem can be compared to [136, 138, 133, 139] where inequalities (of the opposite direction) for the multipliers of p with *connected* Julia sets were derived.

5.3. Future perspectives

Below we present several open questions related to the material of this paper.

What can be said about the existence of analytic continuations beyond the imaginary axis of spectral zeta functions for more general classes of fractals, especially infinitely ramified fractals? (See, e.g., the conjectures stated in [112].)

Can spectral decimation in the general setting as given by (26) be understood in terms of the according Poincaré function of the rational function R? This would require knowledge of the asymptotic behaviour of meromorphic solutions of the general Poincaré equation $f(\lambda z) = R(f(z))$.

How far can the spectral properties of elliptic and more general differential operators on Riemannian manifolds be carried over to their fractal analogues? Can the special values of the fractal spectral zeta function be assigned a geometric meaning?

The asymptotic behaviour of the solutions of the Poincaré equation in the case of arbitrary complex polynomials and complex scaling factor λ has been investigated in [122]. It would be interesting to see applications of these results in physics.

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