QUASIMODULAR FORMS AS SOLUTIONS OF MODULAR DIFFERENTIAL EQUATIONS

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Abstract. We study quasimodular forms of depth ≤ 4 and determine under which conditions they occur as solutions of modular differential equations. Furthermore, we study which modular differential equations have quasimodular solutions. We use these results to investigate extremal quasimodular forms as introduced by M. Kaneko and M. Koike [17] further. Especially, we prove a conjecture stated by these authors concerning the divisors of the denominators occurring in their Fourier expansion.

1. Introduction

The notion of "quasimodular form" was coined by M. Kaneko and D. Zagier in [19]. Since then quasimodular forms have gained increasing attention as they have intrinsic connections to very different fields of mathematics and beyond. For two excellent introductions to the subject we refer to [32, 39].

There has been an extensive study of linear differential equations, whose solution set is invariant under modular transformations (see [1,10,15,16,18,20,27]). Such differential equations can be used to study families of modular forms and quasimodular forms. The question, under which conditions such equations have modular or quasimodular solutions, is very prominent in many of these papers. The present paper will shed some new light on that and gives a unified view on the subject.

Quasimodular forms gained new interest since they occurred prominently in the construction of certain Fourier eigenfunctions with prescribed zeros. These were used in the proof that in dimensions 8 and 24 the E_8 and the Leech lattice achieve the best packing (see [5, 36]), as well as in the proof of universal optimality of these lattices (see [6]). For a survey on the construction of such Fourier eigenfunctions we refer to [8]. In this paper modular differential equations are used to

The author is supported by the Austrian Science Fund FWF project F5503 (part of the Special Research Program (SFB) "Quasi-Monte Carlo Methods: Theory and Applications").

encode the asymptotic behaviour of quasimodular forms in a concise and tractable way. The differential equations are then used to derive linear recurrence relations for the quasimodular forms of interest.

In a series of papers M. Knopp and many coauthors have introduced and studied vector valued modular forms [9,10,21–24,26–29]. Modular differential equations also play a role in this context, as they are a method to capture properties of the components of a vector valued form in a concise way (see [10,27]). These give rise to representations of the modular group. In this context the action of the map $T : z \mapsto z + 1$ is always diagonisable. We will study a similar concept for quasimodular forms, where the action of T will turn out not to be diagonisable.

The paper is organised as follows. In Section 2 we recall some basic facts and definitions about modular forms and quasimodular forms. We shortly recall the Frobenius ansatz method for finding holomorphic solutions of differential equations.

In Section 3 we introduce and study quasimodular vectors as an analogue to vector valued modular forms. We derive several properties that have been known for the modular case and will be used later in this paper.

In Section 4 we introduce the notion of *balanced* quasimodular forms. These are forms f, which exhibit certain patterns for the vanishing orders at $i\infty$ of the quasimodular forms occurring in the transformation behaviour of f under the modular group. We find estimates for the vanishing orders of such forms and prove that they are solutions of modular differential equations if their depth is ≤ 4. There is only one degenerate exception to this rule, namely powers of Δ, which occur as solutions of modular differential equations of any order.

In Section 5 we study modular differential equations which have (balanced) quasimodular solutions. We give a full description of all differential equations of order ≤ 5 with quasimodular solutions.

In Section 6 we use the results of Section 5 to derive differential recursions for extremal quasimodular forms. Such forms are defined by the property that they have maximal possible order of vanishing at $i\infty$ (see [17]). The recursions obtained are then used to prove a conjecture stated in [17] concerning the divisors of the denominators of the Fourier coefficients of such forms of depth ≤ 4. Recently, a different proof of this conjecture for extremal quasimodular forms of depth 1 and weight divisible by 6 was given by F. Pellarin and G. Nebe [31]. A. Mono [30] gave an independent proof for depth 1, which could also cover weights in the residue classes $\equiv 2, 4 \pmod 6$.

In an Appendix we collect some huge expressions for polynomials and modular forms that occur in Section 6 for the case of depth 4.

2. Basics

In this section we collect some basic facts about modular and quasimodular forms and give a short exposition of the Frobenius ansatz method to solve linear differential equations.

2.1. **Modular forms.** The modular group Γ is the group of 2×2-matrices with integer entries and determinant 1

$$\Gamma = \mathrm{PSL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z}, ac - bd = 1 \right\} / \{\pm I\}.$$

The group Γ is generated by

$$Sz = -\frac{1}{z} \quad Tz = z + 1, \tag{2.1}$$

which satisfy the relations $S^2 = \mathrm{id}$ and $(ST)^3 = \mathrm{id}$. It acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$ by Möbius transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *weakly holomorphic modular form of weight* w, if it satisfies

$$(cz + d)^{-w} f\left(\frac{az + b}{cz + d}\right) = f(z) \tag{2.2}$$

for all $z \in \mathbb{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

The vector space of weakly holomorphic modular forms is denoted by $\mathcal{M}_w^!(\Gamma)$. This space is non-trivial only for even values of w. A form f is called holomorphic, if

$$f(i\infty) := \lim_{\Im z \to +\infty} f(z)$$

exists. The subspace $\mathcal{M}_w(\Gamma)$ of holomorphic modular forms is non-trivial only for even $w \geq 4$. Its dimension equals

$$\dim \mathcal{M}_w(\Gamma) = \begin{cases} \left\lfloor \frac{w}{12} \right\rfloor & \text{for } w \equiv 2 \pmod{12} \\ \left\lfloor \frac{w}{12} \right\rfloor + 1 & \text{otherwise.} \end{cases}$$

The most prominent examples of modular forms are the Eisenstein series

$$E_{2k}(z) = \frac{1}{2\zeta(2k)} \sum_{(m,n) \in \mathbb{Z} \setminus \{(0,0)\}} \frac{1}{(mz + n)^{2k}} \tag{2.3}$$

for $k \geq 2$, which are modular forms of weight $2k$. They admit a Fourier expansion (setting $q = e^{2\pi iz}$ as usual in this context)

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \tag{2.4}$$

where $\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$ denotes the divisor sum of order $2k-1$ and B_{2k} denote the Bernoulli numbers. The defining series (2.3) does not converge for $k = 1$ in the given form. Nevertheless, the series (2.4) converges for $k \geq 1$. This entails a slightly more complicated transformation behaviour under the action of S

$$z^{-2} E_2(Sz) = E_2(z) + \frac{6}{\pi iz}. \tag{2.5}$$

Every modular form can be expressed as a polynomial in E_4 and E_6. Furthermore, by the invariance under T, every holomorphic modular form f has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi nz} = \sum_{n=0}^{\infty} a_f(n) q^n.$$

In the sequel we will follow the convention to freely switch between dependence on z and q.

A holomorphic form f is called a *cusp form*, if $f(i\infty) = 0$. The prototypical example of a cusp form is

$$\Delta = \frac{1}{1728} \left(E_4^3 - E_6^2 \right). \tag{2.6}$$

The space of cusp forms is denoted by $\mathcal{S}_w(\Gamma)$. Since we only deal with modular forms for the full modular group Γ, we will omit reference to the group in the sequel.

For a detailed introduction to the theory of modular forms we refer to [2, 3, 7, 14, 25, 33, 34].

2.2. **Quasimodular forms.** The vector space of quasimodular forms of weight w and depth $\leq r$ is given by

$$\mathcal{QM}_w^r = \bigoplus_{\ell=0}^{r} E_2^{\ell} \mathcal{M}_{w-2\ell}. \tag{2.7}$$

Quasimodular forms occur naturally as derivatives of modular forms (see [4, 32, 39]).

The dimension of the space $\mathcal{QM}_w^r$ will play an important role in this paper, so we give a general formula. In the following the notation $[P]$ means 1, if the condition P is satisfied and 0 otherwise.

Proposition 1. *Let* $w \equiv 0 \pmod 2$ *and* $r \geq 0$*, then the dimension of the space of quasimodular forms equals*

(2.8)
$$\begin{aligned}\dim \mathcal{QM}_w^r = &\left\lfloor \frac{w(r+1)}{12} \right\rfloor - \left\lfloor \frac{r+1}{6} \right\rfloor \left(r - 3\left\lfloor \frac{r+1}{6} \right\rfloor - 1 \right) + \left\lfloor \frac{r}{6} \right\rfloor \\ &+ 1 - [w(r+1) \equiv 2 \pmod{12}] .\end{aligned}$$

In the special case $r \leq 4$ *and* $w(r+1) \equiv 0 \pmod{12}$ *this simplifies to*

$$\dim \mathcal{QM}_w^r = \frac{w(r+1)}{12} + 1. \tag{2.9}$$

Proof. We start with

$$\dim \mathcal{QM}_w^r = \sum_{k=0}^{r} \dim \mathcal{M}_{w-2k}$$

and split the range of summation in intervals of length 6 to obtain

$$\dim \mathcal{QM}_w^r = \frac{w}{2}\left\lfloor \frac{r}{6} \right\rfloor + \dim \mathcal{QM}_w^{r \pmod 6} - \left\lfloor \frac{r}{6} \right\rfloor \left(r - 3\left\lfloor \frac{r}{6} \right\rfloor - 2 \right). \tag{2.10}$$

The values of the dimension for $r = 0, \ldots, 5$ are given by

$$\begin{aligned}\dim \mathcal{QM}_w^0 &= \left\lfloor \frac{w}{12} \right\rfloor + 1 - [w \equiv 2 \pmod{12}] \\ \dim \mathcal{QM}_w^1 &= \left\lfloor \frac{w}{6} \right\rfloor + 1 \\ \dim \mathcal{QM}_w^2 &= \left\lfloor \frac{w}{4} \right\rfloor + 1 \\ \dim \mathcal{QM}_w^3 &= \left\lfloor \frac{w}{3} \right\rfloor + 1 \\ \dim \mathcal{QM}_w^4 &= \left\lfloor \frac{5w}{12} \right\rfloor + 1 - [w \equiv 10 \pmod{12}] \\ \dim \mathcal{QM}_w^5 &= \frac{w}{2}.\end{aligned}$$

From (2.10) and the dimension formulas for $r = 0, \ldots, 5$ the general formula (2.8) can be obtained by a case distinction $r \pmod 6$ and $w \pmod{12}$. □

We will follow the convention to denote the derivative by

$$f' = \frac{1}{2\pi i}\frac{df}{dz} = q\frac{df}{dq}.$$

With this notation Ramanujan's identities read

$$
\begin{aligned}
E_2' &= \frac{1}{12}\left(E_2^2 - E_4\right)\\
E_4' &= \frac{1}{3}\left(E_2E_4 - E_6\right)\\
E_6' &= \frac{1}{2}\left(E_2E_6 - E_4^2\right).
\end{aligned}
\tag{2.11}
$$

These give rise to the definition of the Serre derivative

$$\partial_w f = f' - \frac{w}{12}E_2 f,$$

where w is (related to) the weight of f. We will use the product rule

$$\partial_{w_1+w_2}(fg) = \left(\partial_{w_1} f\right) g + f\left(\partial_{w_2} g\right)$$

and also make frequent use of the following immediate consequences of (2.11)

$$
\begin{aligned}
\partial_1 E_2 &= -\frac{1}{12}E_4\\
\partial_4 E_4 &= -\frac{1}{3}E_6\\
\partial_6 E_6 &= -\frac{1}{2}E_4^2\\
\partial_{12}\Delta &= 0.
\end{aligned}
\tag{2.12}
$$

From the second and third equation together with the fact that every holomorphic form is a polynomial in E_4 and E_6, it follows immediately that for a form $f \in \mathcal{M}_w$ we have $\partial_w f \in \mathcal{M}_{w+2}$, and for $f \in \mathcal{S}_w$ we have $\partial_w f \in \mathcal{S}_{w+2}$.

The following lemma is a special case of [17, Proposition 3.3], where a similar result for a certain family of operators $\theta_k^{(r)}$ of order r is proved.

Lemma 1. *The Serre derivative ∂_{w-r} maps quasimodular forms of weight w and depth $\leq r$ to quasimodular forms of weight $w+2$ and depth $\leq r$.*

Proof. A quasimodular form of weight w and depth $\leq r$ can be written as

$$f_w = \sum_{k=0}^{r} A_{w-2k}E_2^k \tag{2.13}$$

with $A_{w-2k} \in \mathcal{M}_{w-2k}$. Then

$$\partial_{w-r} f_w = \sum_{k=0}^{r}\left(\left(\partial_{w-2k}A_{w-2k}\right)E_2^k + A_{w-2k}\left(\partial_{2k-r}E_2^k\right)\right).$$

Inserting

$$\partial_{2k-r}E_2^k = -\frac{k}{12}E_4E_2^{k-1} + \frac{r-k}{12}E_2^{k+1}$$

yields

$$\begin{aligned}\partial_{w-r}f_w &= \partial_w A_w - \frac{1}{12}E_4A_{w-2} \\ &+ \sum_{k=1}^{r-1}\left(\partial_{w-2k}A_{w-2k} - \frac{k+1}{12}E_4A_{w-2k-2} + \frac{r-k+1}{12}A_{w-2k+2}\right)E_2^k \\ &\qquad + \left(\partial_{w-2r}A_{w-2r} + \frac{1}{12}A_{w-2r+2}\right)E_2^r,\end{aligned}$$

which is a quasimodular form of weight $w+2$ and depth $\leq r$. □

Lemma 2. *Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic. Then*

$$\partial_w\left(z^{-w}f(Sz)\right) = z^{-w-2}\left(\partial_w f\right)(Sz).$$

Proof. We compute

$$\begin{aligned}&\partial_w\left(z^{-w}f(Sz)\right) \\ =&z^{-w-2}f'(Sz) - \frac{w}{2\pi i}z^{-w-1}f(Sz) - \frac{w}{12}E_2(z)z^{-w}f(Sz) \\ =&z^{-w-2}f'(Sz) - \frac{w}{2\pi i}z^{-w-1}f(Sz) - \frac{w}{12}\left(z^{-2}E_2(Sz) - \frac{12}{2\pi iz}\right)z^{-w}f(Sz) \\ =&z^{-w-2}\left(f'(Sz) - \frac{w}{12}E_2(Sz)f(Sz)\right) = z^{-w-2}\left(\partial_w f\right)(Sz).\end{aligned}$$

□

We will use the following convention for iterated Serre derivatives throughout the paper:

$$\partial_w^0 f = f, \quad \partial_w^{k+1} = \partial_{w+2k}\left(\partial_w^k f\right).$$

We will consider differential equations of the form

$$K_{\mathbf{B}}f = B_m\partial_{w-r}^{r+1}f + B_{m+2}\partial_{w-r}^r f + \cdots + B_{m+2r+2}f = 0, \tag{2.14}$$

where $\mathbf{B} = (B_m, \ldots, B_{m+2r+2})$ are modular forms of respective weights $m, m+2, \ldots, m+2r+2$ with $B_m(i\infty) = 1$.

Lemma 3. *For every holomorphic solution $f : \mathbb{H} \to \mathbb{C}$ of the differential equation (2.14), $f(Tz)$ and $z^{r-w}f(Sz)$ are also solutions. Thus, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $(cz+d)^{r-w}f(\gamma z)$ is also a solution.*

Proof. Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic solution of (2.14). Then by Lemma 2 we have

$$K_{\mathbf{B}}(z^{r-w}f(Sz)) = z^{-r-w-m-2}K_{\mathbf{B}}(f)(Sz) = 0.$$

Similarly, since all coefficient functions and all Serre derivatives are invariant under T, we have

$$K_{\mathbf{B}}(f(Tz)) = K_{\mathbf{B}}(f)(Tz) = 0.$$

The last assertion follows from the fact that S and T generate Γ. □

We will be mostly interested in the case that $m = 0$ and thus $B_m = 1$, in which we call the corresponding equation *normalised*.

2.3. **The Frobenius ansatz method.** F. G. Frobenius [11] devised a method to find holomorphic (=power series) solutions of differential equations of the form

$$f^{(n)}(z) + a_{n-1}(z)f^{(n-1)}(z) + \cdots + a_0(z)f(z) = 0, \tag{2.15}$$

where $a_0, \ldots, a_{n-1}$ are meromorphic functions on some $U \subset \mathbb{C}$. Under the condition that a_k has a pole of order at most $n-k$ at $z_0 \in U$ (this is called a *regular singularity* in this context), there exists a solution of (2.15) of the form

$$f(z) = (z - z_0)^\lambda \sum_{n=0}^{\infty} f_n (z - z_0)^n,$$

where λ is a solution of the so called indicial equation, a polynomial equation arising from inserting this ansatz into (2.15) and requiring $f_0 \neq 0$. Originally, the method was developed for equations of degree $n = 2$.

In our case the situation is slightly different, since we are interested in power series in $q = e^{2\pi i z}$, the derivatives still being with respect to z. Furthermore, we have expressed all derivatives in terms of Serre derivatives. We are looking for solutions of (2.14) of the form

$$q^\lambda \sum_{n=0}^{\infty} a(n) q^n.$$

For such a solution to exist λ has to be a root of the *indicial equation*

$$p_{\mathbf{B}}(x) = \sum_{\ell=0}^{r+1} B_{m+2\ell}(i\infty) q_{r+1-\ell}(x, w) = 0, \tag{2.16}$$

where $q_0(x, w) = 1$ and

$$q_\ell(x, w) = \left(x - \frac{w-r}{12}\right)\left(x - \frac{w-r+2}{12}\right) \cdots \left(x - \frac{w-r+2\ell-2}{12}\right).$$

Then $p_{\mathbf{B}}(x)$ is a polynomial of degree $r+1$ with roots $\lambda_0, \ldots, \lambda_r \in \mathbb{C}$ called the *Frobenius exponents* of (2.14). As long as these exponents are pairwise different and none of the pairwise differences $\lambda_k - \lambda_\ell$ $(k \neq \ell)$ is an integer, the ansatz method immediately gives $r+1$ linearly independent solutions of (2.14) by successively solving for $a(1), a(2), \ldots$. In our case we are especially interested in the opposite situation, namely that all exponents are positive integers and thus all the pairwise differences are integers. In the classical situation this is the case, where the monodromy representation associated to a fundamental system of solutions is not diagonisable in general.

We order the exponents in decreasing order $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$. Then we start with the solution

$$f_0(z) = q^{\lambda_0} \sum_{n=0}^{\infty} a_0(n) q^n.$$

A second solution can then be found using the ansatz

$$f_1(z) = Czf_0(z) + q^{\lambda_1} \sum_{n=0}^{\infty} a_1(n) q^n,$$

where C has to be chosen so that the computation of the coefficient $a_1(\lambda_0 - \lambda_1)$ is possible. Further solutions can be found by making an ansatz

$$f_\ell(z) = C_\ell^{(\ell)} z^\ell f_0(z) + C_{\ell-1}^{(\ell)} z^{\ell-1} q^{\lambda_1} \sum_{n=0}^{\infty} a_1(n) q^n + \cdots + q^{\lambda_\ell} \sum_{n=0}^{\infty} a_\ell(n) q^n,$$

where the constants $C_\ell^{(\ell)}, \ldots, C_1^{(\ell)}$ have to be chosen so that the computation of the coefficients $a_\ell(\lambda_{\ell-1} - \lambda_\ell), \ldots, a_\ell(\lambda_0 - \lambda_\ell)$ is possible. For more details on the method we refer to [12, 13, 35].

3. Quasimodular vectors

In this section we will use the transformation behaviour of quasimodular forms to define vector valued functions that encode this transformation behaviour. This has some analogy to vector valued modular forms as studied in [9, 10, 21, 23, 24, 26, 27, 29], but also exhibits some differences.

Let f be a holomorphic quasimodular form of weight w and depth s. Then f can be written as

$$f(z) = \sum_{\ell=0}^{s} E_2(z)^\ell h_\ell(z), \tag{3.1}$$

where h_ℓ ($\ell = 0, \dots, s$) are modular forms of weight $w - 2\ell$. Define quasimodular forms g_ℓ ($\ell = 0, \dots, s$) of weight $w - 2\ell$ and depth $s - \ell$ by

$$\binom{s}{\ell} g_\ell(z) = \left(\frac{6}{\pi i}\right)^\ell \sum_{m=0}^{s-\ell} \binom{\ell+m}{m} E_2(z)^m h_{\ell+m}(z); \tag{3.2}$$

notice that $f = g_0$. Then we have

$$f(Tz) = f(z) \quad \text{and } z^{-w} f(Sz) = \sum_{\ell=0}^{s} \binom{s}{\ell} \frac{1}{z^\ell} g_\ell(z), \tag{3.3}$$

which follows from (2.5).

Definition 1. Let f be a quasimodular form of weight w and depth $s \leq r$ given by (3.1). Let then the forms g_ℓ ($\ell = 0, \dots, s$) be given by (3.2). Use these to define

$$f_k(z) = \sum_{\ell=0}^{\min(k,s)} \binom{k}{\ell} z^{k-\ell} g_\ell(z)$$

for $k = 0, \dots, r$. A holomorphic vector valued function $\vec{F} : \mathbb{H} \to \mathbb{C}^{r+1}$ is called quasimodular, if there is a quasimodular form f such that $\vec{F}$ is given by

$$\vec{F}(z) = (f_0(z), f_1(z), \dots, f_r(z))^\mathsf{T}.$$

If $s < r$, we call $\vec{F}$ *degenerate*.

Proposition 2. *Let $\vec{F} = (f_0, \dots, f_r)^\mathsf{T}$ be a holomorphic vector function on $\mathbb{H}$. Then $\vec{F}$ is quasimodular, if and only if it has the following behaviour under the generators S and T of Γ:*

$$z^{r-w} \vec{F}(Sz) = \rho(S) \vec{F}(z) \tag{3.4}$$

$$\vec{F}(Tz) = \rho(T) \vec{F}(z) \tag{3.5}$$

with

$$\rho(S) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & \iddots & 0 & 0 \\ 0 & \iddots & & 0 & 0 \\ (-1)^r & 0 & \dots & 0 & 0 \end{pmatrix} \tag{3.6}$$

and

$$\rho(T)=\begin{pmatrix}1&0&0&\dots&0&0\\1&1&0&\dots&0&0\\1&2&1&\dots&0&0\\1&3&3&\ddots&0&0\\\dots&\dots&\dots&\dots&\ddots&0\\1&\binom{r}{1}&\binom{r}{2}&\dots&\binom{r}{r-1}&1\end{pmatrix}. \tag{3.7}$$

Proof. Let

$$f(z)=\sum_{\ell=0}^{r}E_2^{\ell}h_\ell$$

be a quasimodular form of weight w and depth $s\le r$; the case $s<r$ is included by setting $h_\ell=0$ for $\ell>s$. Then h_ℓ $(\ell=0,\dots,r)$ is a modular form of weight $w-2\ell$. The transformation behaviour of f under S and T is given by (3.3).

The forms g_ℓ given by (3.2) transform under S by

$$z^{2\ell-w}g_\ell(Sz)=\sum_{m=0}^{r-\ell}\binom{r-\ell}{m}\frac{1}{z^m}g_{m+\ell}(z).$$

Using this we obtain

$$\begin{aligned}z^{r-w}f_k(Sz)&=\sum_{\ell=0}^{k}\left(-\frac{1}{z}\right)^{\ell}z^{r-w}g_{k-\ell}(Sz)\\&=(-1)^k\sum_{m=0}^{r}z^{r-k-m}g_m(z)\sum_{\ell=0}^{m}(-1)^\ell\binom{k}{\ell}\binom{r-\ell}{r-m}.\end{aligned}$$

The inner sum equals $\binom{r-k}{m}$, which gives

$$z^{r-w}f_k(Sz)=(-1)^k\sum_{m=0}^{r-k}\binom{r-k}{m}z^{r-k-m}g_m(z)=(-1)^kf_{r-k}(z).$$

Similarly, we have for the transformation behaviour under T

$$\begin{aligned}f_k(Tz)&=\sum_{\ell=0}^{k}\sum_{m=0}^{\ell}\binom{\ell}{m}\binom{k}{\ell}z^m g_{k-\ell}(z)\\&=\sum_{p=0}^{k}\binom{k}{p}\sum_{m=0}^{p}\binom{p}{m}z^m g_{p-m}(z)=\sum_{p=0}^{k}\binom{k}{p}f_p(z).\end{aligned}$$

Assume now that $\vec{F}=(f_0,\dots,f_r)^{\mathsf{T}}$ is an $(r+1)$-dimensional vector valued function satisfying (3.4) and (3.5). Then f_0 is T-invariant, thus admits a power series representation in q, which we denote by g_0 for simplifying the notation in the following argument. By (3.7) the second coordinate f_1 satisfies

$$f_1(Tz) = f_1(z) + f_0(z);$$

we consider the function

$$g_1(z) = f_1(z) - zf_0(z),$$

which is T-invariant. Thus we can write

$$f_1(z) = zg_0(z) + g_1(z),$$

where g_1 is a power series in q. Assume now by induction that we have already shown that

$$f_m(z) = \sum_{\ell=0}^{m} \binom{m}{\ell} z^\ell g_{m-\ell}(z)$$

for $0 \le m < k$ where each of the functions $g_0, \dots, g_{k-1}$ is a power series in q. We define the function

$$g_k(z) = f_k(z) - \sum_{\ell=1}^{k} \binom{k}{\ell} z^\ell g_{k-\ell}(z).$$

Then we have

$$g_k(Tz) = \sum_{\ell=0}^{k} \binom{k}{\ell} f_\ell(z) - \sum_{\ell=1}^{k} \binom{k}{\ell} \sum_{m=0}^{\ell} z^m g_{k-\ell}(z)$$
$$= f_k(z) + \sum_{\ell=0}^{k-1} \binom{k}{\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} z^m g_{\ell-m}(z) - \sum_{\ell=1}^{k} \binom{k}{\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} z^m g_{k-\ell}(z).$$

A similar rearrangement as before then gives $g_k(Tz) = g_k(z)$, which shows that g_k can be expressed as a power series in q. Summing up, it follows from (3.5) that there are power series in q, $g_0, \dots, g_r$, such that each of the functions f_k, $k = 0, \dots, r$ can be expressed as

$$f_k(z) = \sum_{\ell=0}^{k} \binom{k}{\ell} z^\ell g_{k-\ell}(z).$$

Assume now that in addition (3.4) holds. Then we have

$$z^{r-w} f_0(Sz) = f_r(z) = \sum_{\ell=0}^{r} \binom{r}{\ell} z^\ell g_{r-\ell}(z).$$

Now we have (recall that $f_0 = g_0$)

$$\begin{aligned} z^{r-w} f_1(Sz) &= z^{r-w}\left(\left(-\frac{1}{z}\right) g_0(Sz) + g_1(Sz)\right) \\ &= -\frac{1}{z}\sum_{\ell=0}^{r}\binom{r}{\ell} z^{\ell} g_{r-\ell}(z) + z^{r-w} g_1(Sz) \\ &= -f_{r-1}(z) = -\sum_{\ell=0}^{r-1}\binom{r-1}{\ell} z^{\ell} g_{r-1-\ell}(z), \end{aligned}$$

from which we derive

$$\begin{aligned} z^{r-w} g_1(Sz) &= \sum_{\ell=0}^{r}\binom{r}{\ell} z^{\ell-1} g_{r-\ell}(z) - \sum_{\ell=0}^{r-1}\binom{r-1}{\ell} z^{\ell} g_{r-1-\ell}(z) \\ &= \frac{1}{z} g_r(z) + \sum_{\ell=0}^{r-1}\left[\binom{r}{\ell+1} - \binom{r-1}{\ell}\right] z^{\ell} g_{r-1-\ell}(z) \\ &= \sum_{\ell=0}^{r-1}\binom{r-1}{\ell} z^{\ell-1} g_{r-\ell}(z), \end{aligned}$$

which gives

$$z^{2-w} g_1(Sz) = \sum_{\ell=0}^{r-1}\binom{r-1}{\ell}\frac{1}{z^{\ell}} g_{\ell+1}(z).$$

Assume now by induction that we have already shown

$$z^{2m-w} g_m(Sz) = \sum_{\ell=0}^{r-m}\binom{r-m}{\ell}\frac{1}{z^{\ell}} g_{m+\ell}(z)$$

for $0 \le m < k$.

Applying S to f_k gives

$$\begin{aligned} z^{r-w} f_k(Sz) &= z^{r-w}\sum_{\ell=0}^{k}\binom{k}{\ell}\left(-\frac{1}{z}\right)^{\ell} g_{k-\ell}(Sz) \\ &= z^{r-w} g_k(Sz) + \sum_{\ell=1}^{k}(-1)^{\ell}\binom{k}{\ell} z^{r+\ell-2k}\sum_{m=0}^{r+\ell-k}\binom{r+\ell-k}{m}\frac{1}{z^m} g_{k-\ell+m}(z) \\ &= z^{r-w} g_k(Sz) + \sum_{m=0}^{r} z^{m-k} g_{r-m}(z)\sum_{\ell=1}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{r+\ell-k}{m}. \end{aligned}$$

The inner sum evaluates to

$$\sum_{\ell=1}^{k}(-1)^{\ell}\binom{k}{\ell}\binom{r+\ell-k}{m}=(-1)^{k}\binom{r-k}{m-k}-\binom{r-k}{m},$$

which gives

$$z^{r-w}f_k(Sz)=z^{r-w}g_k(Sz)+(-1)^k\sum_{m=k}^{r}\binom{r-k}{m-k}z^{m-k}g_{r-m}(z)\\ -\sum_{m=0}^{r-k}\binom{r-k}{m}z^{m-k}g_{r-m}(z). \tag{3.8}$$

On the other hand we have

$$z^{r-w}f_k(Sz)=(-1)^kf_{r-k}(z)=(-1)^k\sum_{\ell=0}^{r-k}\binom{r-k}{\ell}z^{\ell}g_{r-k-\ell}(z),$$

which equals the first sum after shifting the index of summation. Inserting this into (3.8) gives

$$z^{r-w}g_k(Sz)=\sum_{m=0}^{r-k}\binom{r-k}{m}z^{m-k}g_{r-m}(z).$$

Thus $f_0=g_0$ has the transformation behaviour of a quasimodular form under the action of S and T. Since S and T generate Γ, this together with the assumed holomorphy implies that f_0 is a quasimodular form. □

A similar reasoning was used in [6] in the case of $r=2$ to show that the solutions of certain functional equations were quasimodular.

In order to get a better understanding of quasimodular vectors, we study the modular Wronskian of a quasimodular vector $\vec{F}$:

$$W(z)=W_{\vec{F}}(z)=\det\left(\vec{F},\partial_{w-r}\vec{F},\dots,\partial_{w-r}^{r}\vec{F}\right). \tag{3.9}$$

For vector valued modular forms this has been studied in [27]. The most important property of W is its modularity.

Proposition 3. *Let $\vec{F}$ be a quasimodular vector of weight w and dimension $r+1$. Then the corresponding modular Wronskian $W_{\vec{F}}$ is a modular form of weight $w(r+1)$.*

Proof. Let $\vec{F}$ be a quasimodular vector. Then $\vec{F}(Tz)=\rho(T)\vec{F}(z)$ and therefore $W(Tz)=W(z)$.

For the transformation behaviour under S, we recall that by Lemma 2 and Proposition 2

$$\rho(S)\partial_{w-r}^{\ell}\vec{F}(z)=\partial_{w-r}^{\ell}z^{r-w}\vec{F}(Sz)=z^{r-w-2\ell}\left(\partial_{w-r}^{\ell}\vec{F}\right)(Sz),$$

from which we derive

$$z^{-w(r+1)}W(Sz)=\det(\rho(S))W(z).$$

Since $\det(\rho(S))=1$, this gives the assertion. □

For the fundamental system of a normalised modular differential equation we have a far more precise statement. This is the analogue to [27, Theorem 4.3].

Proposition 4. *Let $f_0,f_1,\dots,f_r$ be a fundamental system of solutions of the normalised modular differential equation*

$$\partial_{w-r}^{r+1}f+B_4\partial_{w-r}^{r-1}f+\cdots+B_{2r+2}f=0,$$

where $B_4,B_6,\dots,B_{2r+2}$ are modular form of respective weights $4,6,\dots,2r+2$. Assume further that the solutions of the indicial equation $\lambda_0\geq\lambda_1\geq\cdots\geq\lambda_r\geq 0$ are all integers. Then the modular Wronskian of $f_0,\dots,f_r$ equals $c\Delta^{\frac{w(r+1)}{12}}$ for some constant $c\neq 0$.

Proof. Without loss of generality, we take $f_0,f_1,\dots,f_r$ as the solutions obtained by the Frobenius ansatz in this order. Notice that $\lambda_0+\cdots+\lambda_r=\frac{w(r+1)}{12}$. Let $\vec{F}=(f_0,\dots,f_r)^{\mathsf{T}}$. Then there exist matrices $\rho(S)$ and $\rho(T)$, such that

$$\vec{F}(Tz)=\rho(T)\vec{F}(z)\quad\text{and } z^{r-w}\vec{F}(Sz)=\rho(S)\vec{F}(z).$$

By the construction of $\vec{F}$ from the Frobenius ansatz it follows that $\rho(T)$ is lower triangular with entries 1 on the diagonal, which gives $\det(\rho(T))=1$. On the other hand we have

$$\rho(S)^2=(-1)^r\mathrm{id},$$

from which we derive $\det(\rho(S))^2=1$. Furthermore, we have

$$\left(\rho(S)\rho(T)\right)^3=(-1)^r\mathrm{id},$$

which implies $\det(\rho(S))^3=1$. Thus we finally have $\det(\rho(S))=1$.

Applying Lemma 2 we obtain

$$\partial_{w-r}^{k}\left(z^{r-w}\vec{F}(Sz)\right)=z^{r-w-2k}\left(\partial_{w-r}^{k}\vec{F}\right)(Sz).$$

The Wronskian $W=\det(\vec{F},\partial_{w-r}\vec{F},\dots,\partial_{w-r}^r\vec{F})$ then satisfies

$$\begin{aligned}
&z^{-w(r+1)}W(Sz)\\
&=\det\left(z^{r-w}\vec{F}(Sz),z^{r-w-2}\left(\partial_{r-w}\vec{F}\right)(Sz),\dots,z^{-r-w}\left(\partial_{r-w}^r\vec{F}\right)(Sz)\right)\\
&\qquad=\det\left(\rho(S)\vec{F}(z),\rho(S)\partial_{w-r}\vec{F}(z),\dots,\rho(S)\partial_{w-r}^r\vec{F}(z)\right)\\
&\qquad\qquad=\det(\rho(S))W(z)=W(z).
\end{aligned}$$

Similarly, we have $W(Tz)=W(z)$. Thus W is a modular form with weight $w(r+1)$. From the vanishing orders of f_i we obtain that W vanishes to order (at least) $\frac{w(r+1)}{12}$ at $i\infty$. Since $W(z)\neq 0$ for $z\in\mathbb{H}$, this implies that W has to be a non-zero multiple of $\Delta^{\frac{w(r+1)}{12}}$, thus proving the assertion. □

Proposition 5. *Let f be a quasimodular form of depth $\leq r$ and weight w and assume that f is a solution of a differential equation* (2.14)*. Let then g_ℓ ($\ell=0,\dots,r$) be given by* (3.2)*. Then the functions*

$$f_\ell(z)=\sum_{m=0}^{\ell}\binom{\ell}{m}z^m g_{\ell-m}(z)\tag{3.10}$$

form a fundamental system of (2.14)*.*

Proof. Let $f=f_0$ be a solution of (2.14). Then by S-invariance of the differential equation, also the function

$$z^{r-w}f(Sz)=\sum_{\ell=0}^{r}\binom{r}{\ell}z^\ell g_{r-\ell}(z)=f_r(z)$$

is a solution of (2.14). By the invariance under T of the differential equation also the functions

$$f_r(T^kz)=\sum_{\ell=0}^{r}\binom{r}{\ell}k^{r-\ell}f_\ell(z)$$

are solutions for $k\in\mathbb{Z}$. Here we have used Proposition 2. This gives

$$\begin{pmatrix}f_r(z)\\ f_r(Tz)\\ \vdots\\ f_r(T^rz)\end{pmatrix}=\begin{pmatrix}0&0&\dots&0&1\\ 1&1&\dots&1&1\\ 2^{r+1}&2^r&\dots&2&1\\ \vdots&\vdots&\ddots&\vdots&1\\ r^{r+1}&r^r&\dots&r&1\end{pmatrix}\begin{pmatrix}\binom{r}{0}f_0(z)\\ \binom{r}{1}f_1(z)\\ \vdots\\ \binom{r}{r}f_r(z)\end{pmatrix},$$

which shows that the functions $f_\ell(z)$ ($\ell=0,\dots,r$) can be expressed as linear combinations of $f_r(T^kz)$ ($k=0,\dots,r$) and therefore are again solutions of (2.14). Notice that the matrix is a Vandermonde matrix and thus invertible.

Since the functions f_ℓ ($\ell = 0, \dots, r$) are linearly independent they form a fundamental system of (2.14). □

4. Balanced quasimodular forms

We start with a proposition that clarifies the possible orders of vanishing of quasimodular vectors and the underlying quasimodular forms. This answers a question posed in [17]. The assertion of the proposition is the analogue to [27, Theorem 3.7 and Corollary 3.8]. A similar inequality for the case of extremal quasimodular forms is given in [31].

Proposition 6. *Let f be a quasimodular form of weight w and depth r given by*

$$f(z) = \sum_{\ell=0}^{r} E_2^\ell h_\ell,$$

where h_ℓ ($\ell = 0, \dots, r$) are modular forms of weight $w - 2\ell$. Let the functions g_ℓ given by (3.2) *have vanishing orders $\lambda_\ell < \dim \mathcal{QM}_{w-2\ell}^{r-\ell}$ ($\ell = 0, \dots, r$) with $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_r \geq 0$, i.e.*

$$g_\ell(z) = q^{\lambda_\ell} \sum_{n=0}^{\infty} a_\ell(n) q^n, \quad \text{with } a_\ell(0) \neq 0.$$

Then

$$\lambda_0 + \ldots + \lambda_r \leq \frac{w(r+1)}{12}. \tag{4.1}$$

Proof. Under the assumptions of the proposition the term in the definition of f_k, which does not carry a positive power of z, has vanishing order λ_k and by the ordering of the exponents λ_ℓ, this is the order of vanishing of f_k at $i\infty$ (except possibly for terms multiplied by z). The same holds for the derivatives $\partial_{w-r}^\ell f_k$. Thus the Wronskian W vanishes at least to order $\lambda_0 + \dots + \lambda_r$ (the terms carrying a z are eliminated by the determinant by Proposition 3). Thus it can be written as

$$W = \Delta^{\lambda_0 \dots + \lambda_r} H$$

for a holomorphic form H. Comparing the weights gives (4.1). □

Definition 2. Let f be a quasimodular form of weight w and depth r; thus there are quasimodular forms g_ℓ ($\ell = 0, \dots, r$) of weights $w - 2\ell$ and depth $r - \ell$ such that

$$z^{-w} f(Sz) = f(z) + \sum_{\ell=1}^{r} \binom{r}{\ell} \frac{1}{z^\ell} g_\ell(z).$$

The form f is called *balanced*, if there are non-negative integers

$$\lambda_\ell < \dim \mathcal{QM}_{w-2\ell}^{r-\ell} \quad \text{for } \ell = 0, \dots, r \tag{4.2}$$

and

$$\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_r, \tag{4.3}$$

such that

$$g_\ell(z) = q^{\lambda_\ell} \sum_{n=0}^{\infty} a_\ell(n) q^n$$

with $a_\ell(0) \neq 0$ for $\ell = 0, \dots, r$ and furthermore

$$\lambda_0 + \dots + \lambda_r = \dim \mathcal{QM}_w^r - 1 \tag{4.4}$$

holds.

Remark 1. Let $\lambda_0 \geq \lambda_{\geq} \dots \geq \lambda_r$ be integers satisfying (4.2) and (4.4). Define a linear map $\Phi : \mathcal{QM}_w^r \to \mathbb{C}^{\dim \mathcal{QM}_w^r - 1}$, which maps f to $(b_0(0), \dots, b_0(\lambda_0 - 1), b_1(0), \dots, b_1(\lambda_1 - 1), \dots, b_r(0), \dots, b_r(\lambda_r - 1))$, where $b_\ell(n)$ is given by

$$g_\ell(z) = \sum_{n=0}^{\infty} b_\ell(n) q^n$$

for the forms g_ℓ associated to f by (3.2). This map has a non-trivial kernel by dimension considerations. Thus for w large enough quasimodular forms f of depth r exist such that the corresponding forms g_ℓ have *at least* vanishing orders λ_ℓ at $i\infty$ ($\ell = 0, \dots, r$) under the restrictions (4.2), (4.3), and (4.4).

Remark 2. Extremal quasimodular forms as studied in [17] are special cases, namely $\lambda_1 = \dots = \lambda_r = 0$, which gives the maximum possible order of vanishing of a quasimodular form of weight w obtained by the argument given in Remark 1.

Remark 3. Notice that for $r \leq 4$ and $w(r+1) \equiv 0 \pmod{12}$

$$\dim \mathcal{QM}_w^r - 1 = \frac{w(r+1)}{12}.$$

Thus there is equality in (4.1) for balanced forms of depth ≤ 4.

Remark 4. Notice that as opposed to the situation studied in [27] the assumption (4.3) on the ordering of the vanishing orders is a restriction in our case. Nevertheless, this restrictive condition will be satisfied in our later applications.

Remark 5. Combining Proposition 6 and Remarks 1 and 3 shows that balanced quasimodular forms exist for $r \leq 4$ for any choice of $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0$ satisfying (4.2) and (4.4).

Theorem 1. *Every balanced quasimodular form of depth* $r \leq 4$ *and weight* w *with* $(r+1)w \equiv 0 \pmod{12}$ *is a solution of a modular differential equation of the form*

$$\partial_{w-r}^{r+1} f + a_4 E_4 \partial_{w-r}^{r-1} f + \cdots + a_{2r+2} E_{2r+2} f = 0 \tag{4.5}$$

with $a_4, a_6, \ldots, a_{2r+2} \in \mathbb{Q}$.

Proof. Let f be a form satisfying the assumptions of the theorem. Then choose $a_4, \ldots, a_{2r+2}$ so that the indicial equation of (4.5)

$$\begin{aligned}\left(\lambda - \frac{w+r}{12}\right)\left(\lambda - \frac{w+r-2}{12}\right)\cdots\left(\lambda - \frac{w-r}{12}\right)& \\ + a_4\left(\lambda - \frac{w+r-4}{12}\right)\cdots\left(\lambda - \frac{w-r}{12}\right) + \cdots + a_{2r+2} &= 0\end{aligned} \tag{4.6}$$

has solutions $\lambda_0, \ldots, \lambda_r$ (counted with multiplicity). The λ^r-term comes from the first summand and has coefficient $-\frac{w(r+1)}{12}$, which is an integer by assumption. Thus we have

$$\lambda_0 + \cdots + \lambda_r = \frac{w(r+1)}{12} = \dim \mathcal{QM}_w^r - 1$$

by (2.9). The coefficients $a_4, \ldots, a_{2r+2}$ are then all rational.

With this choice of coefficients define the differential operator

$$K = \partial_{w-r}^{r+1} + a_4 E_4 \partial_{w-r}^{r-1} + \cdots + a_{2r+2} E_{2r+2}.$$

By Lemma 1, $\phi = Kf$ is then a quasimodular form of weight $w+2r+2$ and depth $\leq r$. Following Definition 1 we define the functions f_ℓ ($\ell = 0, \ldots, r$) from f and set

$$\phi_\ell = K f_\ell.$$

Then we have

$$z^{-r-w-2}\phi_\ell(Sz) = K\left(z^{r-w} f_\ell(Sz)\right) = (-1)^\ell K f_{r-\ell}(z) = (-1)^\ell \phi_{r-\ell}(z)$$

and

$$\phi_\ell(Tz) = K\left(f_\ell(Tz)\right) = \sum_{k=0}^{\ell} \binom{\ell}{k} K f_k(z) = \sum_{k=0}^{\ell} \binom{\ell}{k} \phi_k(z),$$

which shows that $(\phi_0, \ldots, \phi_r)^\mathsf{T}$ satisfies (3.4) and (3.5) and is thus a quasimodular vector of weight $w + 2r + 2$.

From the fact that the indicial equation of K has the roots $\lambda_0, \dots, \lambda_r$ it follows that

$$\phi_\ell(z) = K f_\ell(z) = \mathcal{O}(z^{\mu-1} q^{\lambda_\ell+1}),$$

where μ is the multiplicity of λ_ℓ. Thus ϕ is a quasimodular form of weight $w+2r+2$ and depth $\leq r$, such that the corresponding orders of vanishing sum up to

$$(\lambda_0+1)+\cdots+(\lambda_r+1) = \frac{(w+12)(r+1)}{12} > \frac{(w+2r+2)(r+1)}{12}.$$

By Proposition 6 this shows that ϕ has to vanish identically and f is a solution of $Kf=0$. □

Remark 6. Since for every r the sum of the solutions of the indicial equation equals $\frac{w(r+1)}{12}$, which is larger than the dimension of the space $\dim \mathcal{QM}_w^r$ for $r \geq 5$, the assertion of the theorem is false for $r \geq 5$: no quasimodular form of depth ≥ 5 is the solution of a normalised differential equation of the form (4.5).

Remark 7. The functions $f, f_1, \dots, f_r$ are the functions that would be obtained by solving (4.5) using the Frobenius ansatz (see [12, 13, 35]) in this order.

Theorem 1 has an obvious converse.

Theorem 2. *Let $r \leq 4$ be a natural number, w such that $w(r+1) \equiv 0 \pmod{12}$, and $a_4, a_6, \dots, a_{2r+2}$ be rational numbers such that the equation (4.6) has only non-negative integer solutions $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_r$. Then the solution of the differential equation*

$$\partial_{w-r}^{r+1} f + a_4 E_4 \partial_{w-r}^{r-1} f + \cdots + a_{2r+2} E_{2r+2} f = 0 \tag{4.7}$$

with q-expansion

$$f(z) = q^{\lambda_0} \sum_{n=0}^{\infty} a_0(n) q^n, \quad a_0(0) = 1 \tag{4.8}$$

is a balanced quasimodular form of weight w and depth r, if at least one of the inequalities in (4.3) is strict. If $\lambda_0 = \cdots = \lambda_r$ (which implies that $w \equiv 0 \pmod{12}$), then the functions $z^\ell \Delta^{\frac{w}{12}}$ ($\ell = 0, \dots, r$) form a fundamental system of (4.7).

Proof. Define g to be the balanced quasimodular form with exponents $\lambda_0 \geq \cdots \geq \lambda_r$. Such a form exists by Remark 5. Then by Theorem 1 g is the solution of the differential equation (4.5). Since the exponents uniquely define the coefficients $a_4, \dots, a_{2r+2}$ this is the same equation as (4.7). Then g is the solution of (4.7) characterised by vanishing order λ_0 at $i\infty$. Thus $g = cf$ for some $c \in \mathbb{C} \setminus \{0\}$ and f is a quasimodular form. From Proposition 5 we know that $f_0 = f, f_1, \ldots, f_r$ (with the notation of Definition 1) form a fundamental system of (4.5). This shows the first assertion of the Theorem.

It only remains to show that $\Delta^{\frac{w}{12}}$ is the solution of a normalised quasimodular differential equation for any $r \geq 0$ with coefficients chosen so that $\lambda_0 = \frac{w}{12}$ is an $(r+1)$-fold zero of the indicial equation. For this purpose we observe that

$$\partial_{w-r}^{k} \Delta^{\frac{w}{12}} = \Delta^{\frac{w}{12}} Q_k^{(r)},$$

where $Q_k^{(r)}$ is a quasimodular form of weight $2k$ and depth $\leq k$. These forms satisfy the recurrence relation

$$Q_k^{(r)} = \frac{r+1-k}{12} E_2 Q_{k-1}^{(r)} + \partial_{k-1} Q_{k-1}^{(r)}$$

with initial condition $Q_0^{(k)} = 1$. This recursion shows that the depth increases with k, except for $k = r+1$, where the first term vanishes. Thus $Q_{r+1}^{(r)}$ has depth $\leq r$; indeed it has depth $r-1$, since there is no quasimodular form of weight $2r+2$ and depth r. By successively subtracting the highest power of E_2 we obtain modular forms $B_4, B_6, \ldots, B_{2r}$ such that

$$Q_{r+1}^{(r)} + B_4 Q_{r-1}^{(r)} + \cdots + B_{2r} Q_1^{(r)} + B_{2r+2} Q_0^{(r)} = 0.$$

Now take $\mathbf{B} = (1, 0, B_4, \ldots, B_{2r})$. The corresponding modular differential operator $K_{\mathbf{B}}$ then annihilates $\Delta^{\frac{w}{12}}$. A fundamental system of solutions of $K_{\mathbf{B}} f = 0$ is given by

$$\Delta^{\frac{w}{12}}, z\Delta^{\frac{w}{12}}, \ldots, z^r \Delta^{\frac{w}{12}},$$

which can be derived from the fact that $z^{r-w} \Delta^{\frac{w}{12}}(Sz) = z^r \Delta^{\frac{w}{12}}$ is also a solution by the invariance properties of $K_{\mathbf{B}}$. The other elements of the fundamental system can be found by applying T and taking differences. □

Remark 8. The proof shows that for every $r \geq 0$ there is a linear differential equation (4.7) such that the functions $z^{\ell} \Delta^{\frac{w}{12}}$, $\ell = 0, \ldots, r$ form a fundamental system of solutions.

5. Modular differential equations for quasimodular forms of depth ≤ 4

In this section we discuss the consequences of Theorems 1 and 2 for finding balanced quasimodular forms of depths $r \leq 4$ also for weights w, which do not satisfy $w(r+1) \equiv 0 \pmod{12}$. We include the case $r = 0$ for completeness, even if the results are rather trivial (see Table 1).

In this section we will use the notation

$$(\lambda_0, \lambda_1, \dots, \lambda_r) \ominus 1 \tag{5.1}$$

for to denote a new set of exponents satisfying the order condition (4.3), but with one exponent diminished by 1.

w	f	differential equation
$0 \pmod{12}$	$\Delta^{\frac{w}{12}}$	$\partial_w f = 0$
$2 \pmod{12}$	$E_4^2 E_6 \Delta^{\frac{w-14}{12}}$	$E_4 E_6 \partial_w f + \frac{1}{6}(3E_4^3 + 4E_6^2) f = 0$
$4 \pmod{12}$	$E_4 \Delta^{\frac{w-4}{12}}$	$E_4 \partial_w f + \frac{1}{3} E_6 f = 0$
$6 \pmod{12}$	$E_6 \Delta^{\frac{w-6}{12}}$	$E_6 \partial_w f + \frac{1}{2} E_4^2 f = 0$
$8 \pmod{12}$	$E_4^2 \Delta^{\frac{w-8}{12}}$	$E_4 \partial_w f + \frac{2}{3} E_6 f = 0$
$10 \pmod{12}$	$E_4 E_6 \Delta^{\frac{w-10}{12}}$	$E_4 E_6 \partial_w f + \frac{1}{6}(3E_4^3 + 2E_6^2) f = 0$

TABLE 1. The forms and differential equations for $r = 0$

5.1. **Depth 1.** In this case the dimension of the space of quasimodular forms is given by

$$\dim \mathcal{QM}_w^1 = \left\lfloor \frac{w}{6} \right\rfloor + 1. \tag{5.2}$$

Theorem 3. *Let $w \equiv a \pmod 6$ ($a = 0, 2, 4$) and let $\frac{w-a}{12} \le \lambda \le \frac{w-a}{6}$. Let*

$$f(z) = q^\lambda \sum_{n=0}^{\infty} a(n) q^n.$$

<u>$a = 0$</u>**:** *If f is a solution of*

$$\partial_{w-1}^2 f_w - \frac{(12\lambda - w - 1)(12\lambda - w + 1)}{144} E_4 f_w = 0, \tag{5.3}$$

then f is a balanced quasimodular form of weight w and depth 1, if $\lambda > \frac{w}{12}$. If $\lambda = \frac{w}{12}$, the functions $\Delta^{\frac{w}{12}}$ and $z\Delta^{\frac{w}{12}}$ form a fundamental system of (5.3).

<u>$a = 2$</u>**:** *If f is a solution of*

$$E_4 \partial_{w-1}^2 f_w + \frac{1}{3} E_6 \partial_{w-1} f_w - \frac{(12\lambda - w + 1)(12\lambda - w + 3)}{144} E_4^2 f_w = 0 \tag{5.4}$$

then f is a balanced quasimodular form of weight w and depth 1, if $\lambda > \frac{w-2}{12}$. If $\lambda = \frac{w-2}{12}$, the functions $E_2 \Delta^{\frac{w-2}{12}}$ and $zE_2\Delta^{\frac{w-2}{12}} + \frac{6}{\pi i}\Delta^{\frac{w-2}{12}}$ form a fundamental system of (5.4).

<u>$a=4$</u>: *If f is a solution of*

$$E_4^2\partial_{w-1}^2 f_w + \frac{2}{3}E_4E_6\partial_{w-1}f_w - \left(\left(\frac{(12\lambda-w+3)(12\lambda-w+5)}{144}-\frac{1}{18}\right)E_4^3+384\Delta\right)f_w=0 \tag{5.5}$$

then f is a balanced quasimodular form of weight w and depth 1, if $\lambda>\frac{w-4}{12}$. If $\lambda=\frac{w-4}{12}$, the functions $E_4\Delta^{\frac{w-4}{12}}$ and $zE_4\Delta^{\frac{w-4}{12}}$ form a fundamental system of (5.5).

Proof. For $a=0$ this is the statement of Theorem 2.

For $a=2$, we take g to be solution of (5.3) for $w-2$. Then $f=\partial_{w-3}g$ is a solution of (5.4). Similarly, for $a=4$ we take g to be a solution of (5.3) for $w-4$. Then E_4g is a solution of (5.5). In both cases, neither application of the Serre derivative, nor multiplication by E_4 change the vanishing orders. Thus the resulting forms are still balanced. □

5.2. **Depth** 2. In this case the dimension of the space $\mathcal{QM}_w^2$ is given by

$$\dim\mathcal{QM}_w^2=\left\lfloor\frac{w}{4}\right\rfloor+1. \tag{5.6}$$

Theorem 4. *Let $w\equiv a \pmod 4$ ($a=0,2$) and let*

$$\lambda_0\geq\lambda_1\geq\lambda_2 \tag{5.7}$$

be positive integers with $\lambda_0+\lambda_1+\lambda_2=\frac{w-a}{4}$ and set

$$A=\frac{1}{144}\left(4-3(w-a)^2\right)+\lambda_0\lambda_1+\lambda_0\lambda_2+\lambda_1\lambda_2$$
$$B=-\left(\lambda_0-\frac{w-a-2}{12}\right)\left(\lambda_1-\frac{w-a-2}{12}\right)\left(\lambda_2-\frac{w-a-2}{12}\right)$$

Let

$$f(z)=q^{\lambda_0}\sum_{n=0}^{\infty}a(n)q^n.$$

<u>$a=0$</u>: *If f is a solution of*

$$\partial_{w-2}^3 f+AE_4\partial_{w-2}f+BE_6f=0 \tag{5.8}$$

then f is a balanced quasimodular form of weight w and depth 2, if there is at least one strict inequality in (5.7). *If $\lambda_0=\lambda_1=\lambda_2=\frac{w}{12}$ the functions $\Delta^{\frac{w}{12}}$, $z\Delta^{\frac{w}{12}}$, and $z^2\Delta^{\frac{w}{12}}$ form a fundamental system of* (5.8).

<u>$a=2$</u>**:** *If f is a solution of*

(5.9)
$$E_6\partial_{w-2}^3 f+\frac12 E_4^2\partial_{w-2}^2 f+AE_4E_6\partial_{w-2}f+\left(\frac12 AE_4^3+\frac13(3B-A)E_6^2\right)f=0$$

then f is a balanced quasimodular form of weight w and depth 2, if there is at least one strict inequality in (5.7). If $\lambda_0=\lambda_1=\lambda_2=\frac{w-2}{12}$, the functions $E_2\Delta^{\frac{w-2}{12}}$, $zE_2\Delta^{\frac{w-2}{12}}+\frac{3}{\pi i}\Delta^{\frac{w-2}{12}}$, and $z^2E_2\Delta^{\frac{w-2}{12}}+\frac{6z}{\pi i}\Delta^{\frac{w-2}{12}}$ form a fundamental system of (5.9).

Proof. The case $a=0$ is covered by Theorem 2. For $a=2$ we observe that if g is a solution of (5.8) for $w-2$, then $\partial_{w-4}g$ is a solution of (5.9). □

5.3. **Depth 3.** In this case the dimension of the space $\mathcal{QM}_w^3$ is given by

$$\dim\mathcal{QM}_w^3=\left\lfloor\frac{w}{3}\right\rfloor+1. \tag{5.10}$$

We denote

$$\begin{aligned}\sigma_2&=\lambda_0\lambda_1+\cdots+\lambda_2\lambda_3\\ \sigma_3&=\lambda_0\lambda_1\lambda_2+\cdots+\lambda_1\lambda_2\lambda_3\end{aligned}$$

the elementary symmetric functions in $\lambda_0,\dots,\lambda_3$. Using this notation we set

$$\begin{aligned}A_0&=-\frac{w^2}{24}+\sigma_2+\frac{5}{72}\\ B_0&=-\frac{w^3}{216}+\frac{w^2}{72}+\frac{w-2}{6}\sigma_2-\sigma_3-\frac{5}{216}\\ C_{0,2,4}&=\left(\lambda_0-\frac{w-3}{12}\right)\left(\lambda_1-\frac{w-3}{12}\right)\left(\lambda_2-\frac{w-3}{12}\right)\left(\lambda_3-\frac{w-3}{12}\right),\end{aligned}$$

and

$$\begin{aligned}A_2&=-\frac{(w-2)^2}{24}+\sigma_2+\frac{5}{72}\\ B_2&=-\frac{(w-2)^3}{216}+\frac{w-2}{6}\sigma_2-\sigma_3\\ D_2&=\frac{16}{3}(w-2)^3-16(w-2)^2+\frac{80}{3}-192(w-4)\sigma_2+1152\sigma_3,\end{aligned}$$

and

$$A_4 = -\frac{(w-1)^2}{24} + \frac{1}{36} + \sigma_2$$
$$B_4 = -\frac{2w^3 - 9w^2 + 12w - 3}{432} - \sigma_3 + \frac{w-2}{6}\sigma_2$$
$$D_4 = \frac{4}{3}\left(2w^3 - 9w^2 + 12w - 3\right) - 96(w-2)\sigma_2 + 576\sigma_3.$$

Theorem 5. *Let* $w \equiv 0, 2, 4 \pmod 6$ *and let*

$$\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \tag{5.11}$$

be positive integers with $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = \lfloor \frac{w}{3} \rfloor$ *and not all equal. Let* A_0, B_0, C_0, $A_2, \ldots, D_2$, *and* $A_4, \ldots, D_4$ *be given as above and let*

$$f(z) = q^{\lambda_0} \sum_{n=0}^{\infty} a(n) q^n. \tag{5.12}$$

$w \equiv 0 \pmod 6$: *If* f *is a solution of*

$$\partial_{w-3}^4 f + A_0 E_4 \partial_{w-3}^2 f + B_0 E_6 \partial_{w-3} f + C_0 E_4^2 f = 0, \tag{5.13}$$

then f *is a balanced quasimodular form of weight* w *and depth* 3.

$w \equiv 2 \pmod 6$: *If* f *is a solution of*

$$E_4 \partial_{w-3}^4 f + \frac{2}{3} E_6 \partial_{w-3}^3 f + A_2 E_4^2 \partial_{w-3}^2 f + B_2 E_4 E_6 \partial_{w-3} f + \left(C_2 E_4^3 + D_2 \Delta\right) f = 0, \tag{5.14}$$

then f *is a balanced quasimodular form of weight* w *and depth* 3.

$w \equiv 4 \pmod 6$: *If* f *is a solution of*

$$E_4 \partial_{w-3}^4 f + \frac{1}{3} E_6 \partial_{w-3}^3 f + A_4 E_4^2 \partial_{w-3}^2 f + B_4 E_4 E_6 \partial_{w-3} f + \left(C_4 E_4^3 + D_4 \Delta\right) f = 0, \tag{5.15}$$

then f *is a balanced quasimodular form of weight* w *and depth* 3.

Proof. For $w \equiv 0 \pmod 6$ this is the assertion of Theorem 2. For $w \equiv 2 \pmod 6$ let g be the solution of the form (5.12) of (5.13) for $w-2$. Then $\partial_{w-5} g$ is a balanced quasimodular form of weight w and depth 3 satisfying (5.14).

For $w \equiv 4 \pmod 6$ we have to take into account that $\lfloor \frac{w}{3} \rfloor = \lfloor \frac{w-4}{3} \rfloor + 1$, which means that the total order of vanishing for a balanced quasimodular form of weight w is one higher than that of a form of weight $w-$

4. We take g as the solution of (5.13) for the exponents $(\lambda_0,\ldots,\lambda_3)\ominus 1$, where we call λ the exponent that has been decreased. We then set

$$f=\partial_{w-7}^2 g-\left(\lambda-\frac{w+5}{12}\right)\left(\lambda-\frac{w+7}{12}\right)E_4 g. \tag{5.16}$$

Then f is a balanced quasimodular form of weight w. Notice that g has order $\lambda-1$ for the corresponding vanishing order, whereas f has again λ by the choice of the operator applied to g. Then f is a solution of (5.15). □

5.4. **Depth** 4. In this case the dimension of the space $\mathcal{QM}_w^4$ is given by

$$\dim\mathcal{QM}_w^4=\left\lfloor\frac{5w}{12}\right\rfloor+\begin{cases}1 & \text{if } w\equiv 0,2,4,6,8 \pmod{12}\\ 0 & \text{if } w\equiv 10 \pmod{12}.\end{cases} \tag{5.17}$$

In principle, a similar theorem to Theorems 3–5 could be given. For depth 4 it would give six modular differential equations according to the residue classes $0,2,4,6,8,10 \pmod{12}$. The equation for $w\equiv$ (mod 12) can just be determined from $\lambda_0,\ldots,\lambda_4$ by requiring that these are the solutions of the indicial equation. For $w\equiv 2 \pmod{12}$ the balanced quasimodular form of weight w can be obtained from the form of weight $w-2$ by applying ∂_{w-6}. Similarly, for $w\equiv 4 \pmod{12}$ the balanced form of weight w can be obtained from the form of weight $w-4$ by a similar operator as (5.16). For $w\equiv 6,8 \pmod{12}$ operators of orders 3 and 4 have to be used, which are determined by two respectively three roots of their indicial equation. For $w\equiv 10 \pmod{12}$ the form can again be obtained by applying ∂_{w-6} to the form obtained for $w\equiv 8 \pmod{12}$.

6. Recursions for extremal quasimodular forms and a conjecture of Kaneko and Koike

In this section we use the ideas developed so far to obtain differential recursions for extremal quasimodular forms as introduced and studied by M. Kaneko and M. Koike in [17]. In [17] a conjecture about the possible prime divisors of the denominators of the Fourier coefficients of normalised extremal quasimodular forms. Recently, this conjecture was proved by F. Pellarin and G. Nebe [31] for extremal quasimodular forms of depth 1 and weight $\equiv 0 \pmod 6$. Later, A. Mono [30] extended their proof for extremal quasimodular forms of depth 1 and weight $\equiv 2,4 \pmod 6$. We will provide a proof for quasimodular forms of depths ≤ 4 as a consequence of the explicit form of the differential recursions.

We adopt the notation used in [15] that an equation number with subscript $w + a$ means that the parameter w in this equation is replaced by $w + a$. Also, throughout this section we take f_w to denote a normalised form of weight w; recall that we call a form normalised, if its leading coefficient equals 1.

6.1. **Depth 1.** As stated in [15, 17, 37] extremal quasimodular forms of depth 1 satisfy the differential equation

$$\partial_{w-1}^2 f_w - \frac{w^2-1}{144} E_4 f_w = 0 \tag{6.1}$$

for $w \equiv 0 \pmod 6$, which we assume throughout this subsection. We observe that

$$K_w^{\mathrm{up}} f_w = E_4 \partial_{w-1} f_w - \frac{w+1}{12} E_6 f_w \tag{6.2}$$

is a solution of $(6.1)_{w+6}$, if f_w is a solution of $(6.1)_w$. This can be seen from the fact that (6.2) is a quasimodular form of weight $w+6$ of depth 1 with one order of vanishing higher than f_w, thus an extremal quasimodular form of weight $w + 6$.

On the other hand,

$$K_w^{\mathrm{down}} f_w = \frac{1}{\Delta} \left(E_4 \partial_{w-1} f_w + \frac{w-1}{12} E_6 f_w \right) \tag{6.3}$$

is a solution of $(6.1)_{w-6}$: it is a quasimodular form of weight $w - 6$ vanishing to one order less than f_w. The holomorphy follows from the fact that the differential operator in parenthesis applied to constants gives $\mathcal{O}(q)$. The leading coefficient of $K_w^{\mathrm{down}} f_w$ equals $\frac{w}{6}$, thus

$$K_w^{\mathrm{down}} f_w = \frac{w}{6} f_{w-6}.$$

Furthermore, we have

$$K_{w+6}^{\mathrm{down}} K_w^{\mathrm{up}} f_w = 12(w+1)(w+5) f_w \tag{6.4}$$

using (6.1). Defining c_{w+6} by

$$K_w^{\mathrm{up}} f_w = c_{w+6} f_{w+6},$$

we have

$$K_{w+6}^{\mathrm{down}} K_w^{\mathrm{up}} f_w = c_{w+6} \frac{w+6}{6} f_w,$$

which allows to compute c_{w+6} from (6.4). Summing up, we have proved the following proposition.

Proposition 7. *Let $(f_w)_{w\in 2\mathbb{N}}$ $(w \geq 6)$ denote the sequence of normalised extremal quasimodular forms of depth 1. Then for $w \equiv 0 \pmod 6$*

$$f_6 = \frac{1}{720}\left(E_2E_4 - E_6\right) \tag{6.5}$$

and

$$f_{w+6} = \frac{w+6}{72(w+1)(w+5)}\left(E_4\partial_{w-1}f_w - \frac{w+1}{12}E_6f_w\right). \tag{6.6}$$

Furthermore, we have

$$\begin{aligned} f_{w+2} &= \frac{12}{w+1}\partial_{w-1}f_w \\ f_{w+4} &= E_4f_w. \end{aligned} \tag{6.7}$$

6.2. **Depth 2.** The extremal quasimodular forms of weight w and depth 2 satisfy the differential equation

$$\partial_{w-2}^3 f - \frac{3w^2-4}{144}E_4\partial_{w-2}f - \frac{(w+1)(w-2)^2}{864}E_6f = 0 \tag{6.8}$$

for $w \equiv 0 \pmod 4$, which we assume for this subsection. If f_w satisfies $(6.8)_w$, then

$$K_w^{\mathrm{up}}f_w = \frac{w(w+1)}{36}E_4f_w - \partial_{w-2}^2f_w \tag{6.9}$$

satisfies $(6.8)_{w\,+\,4}$ and

$$K_w^{\mathrm{down}}f_w = \frac{1}{\Delta}\left(E_4\partial_{w-2}^2f_w + \frac{w-1}{6}E_6\partial_{w-2}f_w + \frac{(w-2)^2}{144}E_4^2f_w\right) \tag{6.10}$$

satisfies $(6.8)_{w\,-\,4}$. As before this can be seen from the fact that $K_w^{\mathrm{up}}f_w$ and $K_w^{\mathrm{down}}f_w$ are quasimodular forms of respective weights $w+4$ and $w-4$, which vanish to respective orders $\frac{w}{4}+1$ and $\frac{w}{4}-1$, thus being extremal. Notice that the indicial equation of $\Delta K_w^{\mathrm{down}}$ has a double root at 0, thus it maps linear functions in z to $q\times$ (linear functions in z). In [17] essentially the same operator expressed in terms of the Rankin-Cohen bracket (see [38]) has been used.

As before, define c_{w+4} by

$$K_w^{\mathrm{up}}f_w = c_{w+4}f_{w+4}$$

and observe that

$$K_w^{\mathrm{down}}f_w = \left(\frac{w}{4}\right)^2 f_{w-4}.$$

Furthermore, we have

$$K_{w+4}^{\mathrm{down}} K_w^{\mathrm{up}} f_w = \frac{1}{3}(w+1)(w+2)^2(w+3) f_w.$$

Putting these together gives

$$c_{w+4} = \frac{16(w+1)(w+3)(w+2)^2}{3w^2}.$$

Thus we have proved

Proposition 8. *Let $(f_w)_{w\in\mathbb{N}}$ $(w \geq 4)$ denote the sequence of normalised extremal quasimodular forms of depth 2. Then for $w \equiv 0 \pmod 4$*

$$f_4 = \frac{1}{288}\left(E_4 - E_2^2\right) \tag{6.11}$$

and

$$f_{w+4} = \frac{3w^2}{16(w+1)(w+2)^2(w+3)}\left(\frac{(w-1)w}{36} E_4 f_w - \partial_{w-2}^2 f_w\right). \tag{6.12}$$

Furthermore, we have

$$f_{w+2} = \frac{6}{w+1}\partial_{w-2} f_w. \tag{6.13}$$

6.3. **Depth 3.** For depth 3 the extremal quasimodular forms of weight w satisfy the differential equation

$$\begin{aligned}&\partial_{w-3}^4 f - \frac{3w^2-5}{72} E_4 \partial_{w-3}^2 f - \frac{w^3-3w^2+5}{216} E_6 \partial_{w-3} f\\ &-\frac{(w+1)(w-3)^3}{6912} E_4^2 f = 0\end{aligned} \tag{6.14}$$

for $w \equiv 0 \pmod 6$, which we assume for this subsection.

We define

$$\begin{aligned}K_w^{\mathrm{up}} f &= 48(7w^2+42w+60)\partial_{w-3}^3 f\\ &\quad - (15w^4+96w^3+151w^2-30w-116) E_4 \partial_{w-3} f\\ &\quad - \frac{1}{6}(w+1)(9w^4+45w^3+40w^2+24w+144) E_6 f\end{aligned} \tag{6.15}$$

Then $K_w^{\mathrm{up}} f_w$ is a solution of $(6.14)_{w+6}$, if f_w is a solution of $(6.14)_w$ by a similar reasoning as before.

For the corresponding operator K_w^{down} we make an ansatz $K_w^{\mathrm{down}} = \frac{1}{\Delta^2} L_w$ with

$$L_w = A_{12}\partial_{w-3}^3 + A_{14}\partial_{w-3}^2 + A_{16}\partial_{w-3} + A_{18}$$

with unknown modular forms $A_{12}, \dots, A_{18}$ of respective weights $12, \dots, 18$. This ansatz is motivated by the fact that the form $K_w^{\mathrm{down}} f_w$ has to have a vanishing order two less than the vanishing order of f_w. Thus the factor $\frac{1}{\Delta^2}$. On the other hand $K_w^{\mathrm{down}} f_w$ has to be a form of weight $w-6$, thus L_w has to add a weight of 18. We take L_w as an operator of order 3, because for higher order the order could be reduced by the fact that f_w satisfies a differential equation of order 4. The ansatz given above is the general form of an operator satisfying the described requirements.

We apply L_w to the solution f_w of (6.14) with $f_w = q^{\frac{w}{3}}(1+\mathcal{O}(q))$ (a quasimodular form of weight w and depth 3). Then $L_w f_w$ is a quasimodular form of weight $w+18$. From this form we define the quasimodular forms $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3$ as in Section 3. We define the map

$$\begin{aligned}\Phi : \mathcal{M}_{12} \oplus \cdots \oplus \mathcal{M}_{18} &\to \mathbb{C}^6\\ (A_{12}, \dots, A_{18}) &\mapsto \text{coefficients of 1 and } q \text{ of } \widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3.\end{aligned}$$

Since the dimension of $\mathcal{M}_{12} \oplus \cdots \oplus \mathcal{M}_{18}$ is 7, this map has a non-trivial kernel. We take $(A_{12}, \dots, A_{18})$ to be in this kernel. Then the functions $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3$ vanish to order 2, $L_w f_w$ vanishes to order $\frac{w}{3}$, thus $\frac{1}{\Delta^2} L_w f_w$ is a holomorphic form of weight $w-6$ vanishing to order $\frac{w}{3}-2$ at $i\infty$, thus an extremal quasimodular form of this weight.

This gives

$$\begin{aligned}K_w^{\mathrm{down}} f = \frac{1}{\Delta^2}\Bigg(&\Big(\big(9w^2-54w+84\big)E_4^3+\big(7w^2-42w+60\big)E_6^2\Big)\partial_{w-3}^3 f\\ &+4(w-3)^2(w-1)E_4^2E_6\partial_{w-3}^2 f\\ &+\frac{1}{144}\Bigg(\big(39w^4-336w^3+1099w^2-1626w+924\big)E_4E_6^2\\ &\quad+3\big(3w^4-48w^3+231w^2-450w+316\big)E_4^4\Bigg)\partial_{w-3} f\\ &+\frac{1}{864}(w-3)^2E_6\Big(3\big(3w^3-24w^2+64w-56\big)E_4^3-\big(w^3-24w+48\big)E_6^2\Big)f\Bigg).\end{aligned} \tag{6.16}$$

Then $K_w^{\mathrm{down}} f_w$ is a holomorphic form of weight $w-6$ with vanishing order $\frac{w}{3}-2$, which solves $(6.14)_{w-6}$ for a solution f_w of $(6.14)_w$. This gives

$$K_{w+6}^{\mathrm{down}} K_w^{\mathrm{up}} f_w = 5184(w+1)(w+2)^3(w+3)^2(w+4)^3(w+5) f_w. \tag{6.17}$$

From the fact that

$$K_w^{\mathrm{down}} f_w = 16(w-3)^2\left(\frac{w}{3}\right)^3 q^{\frac{w}{3}-2} + \cdots = 16(w-3)^2\left(\frac{w}{3}\right)^3 f_{w-6},$$

we obtain with $K_w^{\text{up}} f_w = c_{w+6} f_{w+6}$ that

$$\begin{aligned} K_{w+6}^{\text{down}} K_w^{\text{up}} f_w &= 16(w+3)^2 \left(\frac{w+6}{3}\right)^3 c_{w+6} f_w \\ &= 5184(w+1)(w+2)^3(w+3)^2(w+4)^3(w+5) f_w, \end{aligned}$$

which gives c_{w+6}.

Furthermore, the Frobenius ansatz for f_w gives

$$f_w = q^{\frac{w}{3}} \left(1 + \frac{w\,(w^2+15w-18)}{(w+3)^2} q + \mathcal{O}(q^2)\right).$$

Inserting this into ∂_{w-3} and $\frac{(w+1)(3w+1)}{48} E_4 f_w - \partial_{w-3}^2 f_w$ gives

$$\begin{aligned} \partial_{w-3} f_w &= q^{\frac{w}{3}} \left(\frac{w+1}{4} + \mathcal{O}(q)\right) \\ \frac{(w+1)(3w+1)}{48} E_4 f_w - \partial_{w-3}^2 f_w &= q^{\frac{w}{3}} \left(\frac{27(w+1)(w+2)^3}{2(w+3)^2} q + \mathcal{O}(q^2)\right). \end{aligned}$$

These two forms then have respective weights $w+2$ and $w+4$ and vanishing orders $\lfloor \frac{w+2}{3} \rfloor$ and $\lfloor \frac{w+4}{3} \rfloor$ and are thus extremal. The leading coefficients can be read off.

Thus we have proved

Proposition 9. *Let $(f_w)_{w \in 2\mathbb{N}}$ ($w \geq 6$) denote the sequence of normalised extremal quasimodular forms of depth 2. Then*

$$f_6 = \frac{5E_2^3 - 3E_2E_4 - 2E_6}{51840} \tag{6.18}$$

and

$$f_{w+6} = \frac{(w+6)^3}{2^2 3^7 (w+1)(w+2)^3(w+4)^3(w+5)} K_w^{\text{up}} f_w \tag{6.19}$$

with K_w^{up} given by (6.15). Furthermore, we have

(6.20)
$$\begin{aligned} f_{w+2} &= \frac{4}{w+1} \partial_{w-3} f_w \\ f_{w+4} &= \frac{2(w+3)^2}{27(w+1)(w+2)^3} \left(\frac{(w+1)(3w+1)}{48} E_4 f_w - \partial_{w-3}^2 f_w\right). \end{aligned}$$

6.4. **Depth 4.** For depth 4 the extremal quasimodular forms of weight w satisfy the differential equation

(6.21)
$$\partial_{w-4}^5 f-\frac{5}{72}\left(w^2-2\right)E_4\partial_{w-4}^3 f-\frac{5}{432}\left(w^3-3w^2+6\right)E_6\partial_{w-4}^2 f$$
$$-\frac{15w^4-120w^3+280w^2-496}{20736}E_4^2\partial_{w-4}f-\frac{(w-4)^4(w+1)}{62208}E_4E_6 f=0$$

for $w\equiv 0 \pmod{12}$, which we assume for this subsection.

We define

$$\begin{aligned}K_w^{\mathrm{up}}f&=-p_0(w)E_4\partial_{w-4}^4 f+\frac{(w+4)^4}{12}p_1(w)E_6\partial_{w-4}^3 f\\&+\frac{1}{720}p_2(w)E_4^2\partial_{w-4}^2 f+\frac{1}{8640}p_3(w)E_4E_6\partial_{w-4}f\\&+\left(\frac{w+1}{25920}p_4(w)E_4^3+\frac{(w+1)(w+4)^4}{15}p_5(w)\Delta\right)f.\end{aligned} \tag{6.22}$$

The polynomials $p_0(w),\dots,p_5(w)$ are given in (A.1)–(A.6) in the Appendix. These polynomials are chosen so that the operator removes the powers $q^{\frac{5w}{12}},\dots,q^{\frac{5w}{12}+4}$ from the power series of f_w. Then $K_w^{\mathrm{up}}f$ is a solution of $(6.21)_{w+12}$, if f is a solution of $(6.21)_w$.

Similarly, we make an ansatz for an operator $K_w^{\mathrm{down}}=\frac{1}{\Delta^5}L_w$ with

$$L_w f=C_{40}\partial_{w-4}^4 f+C_{42}\partial_{w-4}^3 f+C_{44}\partial_{w-4}^2 f+C_{46}\partial_{w-4}f+C_{48}f, \tag{6.23}$$

where $C_{40},\dots,C_{48}$ are modular forms of weights $40,\dots,48$. This ansatz is motivated by the fact that $K_w^{\mathrm{down}}f_w$ has to have five less order of vanishing than f_w, thus the division by Δ^5. The operator should be of order 4, because otherwise it could reduced to lower order by the fact that f_w satisfies an equation of order 5. In order to make $K_w^{\mathrm{down}}f_w$ a form of weight $w-12$, L_wf_w has to have weight $w+48$, and the ansatz above gives the general form of such an operator.

First we notice that for any such operator L_w, L_wf is a quasimodular form of weight $w+48$ and depth 4. Furthermore, if f vanishes to some order at $i\infty$, then L_wf vanishes to at least this order. Now take f_w to be a solution of $(6.21)_w$ with $f_w=\mathcal{O}(q^{\frac{5w}{12}})$. Then, as in Section 3 we form the quasimodular forms $\widetilde{g}_1,\dots,\widetilde{g}_4$ from the function L_wf_w and consider the linear map

$$\begin{aligned}\Phi:\mathcal{M}_{40}\oplus\cdots\oplus\mathcal{M}_{48}&\to\mathbb{C}^{20}\\(C_{40},\dots,C_{48})&\mapsto\text{coefficients of }1,q,\dots,q^4\text{ of }\widetilde{g}_1,\dots,\widetilde{g}_4.\end{aligned}$$

The space $\mathcal{M}_{40}\oplus\cdots\oplus\mathcal{M}_{48}$ has dimension 21, thus the map Φ has a non-trivial kernel. We choose $(C_{40},\dots,C_{48})$ in the kernel to form the operator L_w (and then K_w^{down}). Then the functions $L_w f_w, \widetilde{g}_1, \ldots, \widetilde{g}_4$ all vanish to order at least 5 at $i\infty$. If we divide these functions by Δ^5, we still obtain holomorphic functions. Thus $K_w^{\mathrm{down}} f_w$ is a holomorphic quasimodular form of weight $w - 12$ and depth 4. Furthermore, it vanishes to order (at least) $\frac{5(w-12)}{12}$ at $i\infty$ and is thus extremal and solves $(6.21)_{w-12}$. The choice of the forms is given in (A.7)–(A.11) in the Appendix.

With these two operators we obtain

$$K_{w+12}^{\mathrm{down}} K_w^{\mathrm{up}} f_w = \frac{2^{40}3^{23}}{5}(w+1)(w+2)^5(w+3)^5(w+4)^5(w+5) \times (w+6)^4(w+7)(w+8)^5(w+9)^5(w+10)^5(w+11) f_w. \tag{6.24}$$

On the other hand by the construction of K_w^{down} we have

$$K_w^{\mathrm{down}} f_w = \frac{5^4}{2^4} w^4(5w-12)^4(5w-24)^4(5w-36)^4(5w-48)^4 f_{w-12}.$$

Furthermore, the Frobenius ansatz gives

$$f_w = q^{\frac{5w}{12}}\left(1 + \frac{2w\,(211w^4 + 4440w^3 + 12960w^2 - 20736)}{(5w+12)^4} q + \cdots\right)$$

with explicit coefficients for $q^2, \cdots, q^6$. From this we obtain

(6.25)

- $\partial_{w-4}f_w = \frac{w+1}{3}q^{\frac{5w}{12}} + \cdots$

- $\frac{(w+1)(2w+1)}{18}E_4 f_w - \partial_{w-4}^2 f_w = \frac{2^6 3^5 (w+1)(w+2)^5}{(5w+12)^4} q^{\frac{5w}{12}+1} + \cdots$

- $$\Big(17w^2+78w+90\Big)\partial_{w-4}^3 f_w - \frac{1}{144}\Big(191w^4+1008w^3+1504w^2+192w-576\Big)E_4\partial_{w-4}f_w$$
$$-\frac{1}{432}(w+1)\Big(81w^4+376w^3+560w^2+528w+576\Big)E_6 f_w$$
$$=\frac{2^{18}3^7(w+1)(w+2)^5(w+3)^5(w+4)(w+5)}{(5w+12)^4(5w+24)^4}q^{\frac{5w}{12}+2}+\cdots$$

- $$-\Big(1313w^6+28678w^5+255122w^4+1183008w^3+3016512w^2+4012416w+2177280\Big)\partial_{w-4}^4 f_w$$
$$+\frac{1}{144}\Big(13423w^8+295800w^7+2645368w^6+12166080w^5+29311504w^4+29020416w^3-15653376w^2$$
$$-56692224w-33094656\Big)E_4\partial_{w-4}^2 f_w$$
$$+\frac{1}{432}\Big(6561w^9+136994w^8+1139536w^7+4759344w^6+10294016w^5+11541472w^4+14671104w^3$$
$$+41398272w^2+63016704w+31974912\Big)E_6\partial_{w-4}f_w$$
$$+\frac{1}{2592}(w+1)\Big(2048w^9+38685w^8+287792w^7+1130616w^6+3110288w^5+8497968w^4$$
$$+18484992w^3+14141952w^2-20570112w-30855168\Big)E_4^2 f_w$$
$$=\frac{2^{24}3^{13}(w+1)(w+2)^5(w+3)^5(w+4)^5(w+5)(w+6)^4(w+7)}{(5w+12)^4(5w+24)^4(5w+36)^4}q^{\frac{5w}{12}+3}+\cdots$$

- $$\Big(293w^4+4332w^3+22968w^2+51192w+40824\Big)E_4\partial_{w-4}^3 f_w$$
$$-\frac{4}{3}\Big(w^5+15w^4+90w^3+270w^2+405w+243\Big)E_6\partial_{w-4}^2 f_w$$
$$-\frac{1}{144}\Big(3311w^6+51234w^5+291550w^4+731040w^3+717696w^2-2592w-256608\Big)E_4^2\partial_{w-4}f_w$$
$$-\frac{1}{432}(w+1)\Big(1313w^6+19430w^5+104354w^4+251616w^3+310464w^2+300672w+248832\Big)E_4E_6 f_w$$
$$=\frac{2^{28}3^{14}(w+1)(w+2)^5(w+3)^5(w+4)(w+5)(w+6)^4(w+7)(w+8)}{(5w+12)^4(5w+24)^4(5w+36)^4}q^{\frac{5w}{12}+3}+\cdots.$$

These are extremal quasimodular forms of respective weights $w+2, \ldots, w+10$, whose leading coefficient can be read off.

Proposition 10. *Let* $(f_w)_{w\in 2\mathbb{N}}$ $(w \geq 12)$ *denote the sequence of normalised extremal quasimodular forms of depth* 4. *Then for* $w \equiv 0 \pmod{12}$

$$f_{12} = \frac{13025E_4^3-12796E_6^2+3852E_2E_4E_6-2706E_2^2E_4^2+27500E_2^3E_6-28875E_2^4E_4}{7449432883200} \tag{6.26}$$

and

(6.27)
$$f_{w+12}=\frac{5^5(w+12)^4(5w+12)^4(5w+24)^4(5w+36)^4(5w+48)^4}{2^{44}3^{23}(w+1)(w+2)^5(w+3)^5(w+4)^5(w+5)(w+6)^4(w+7)(w+8)^5(w+9)^5(w+10)^5(w+11)}K_w^{\mathrm{up}}f_w.$$

Furthermore, the functions

$$
\begin{aligned}
&f_{w+2}=\tfrac{3}{w+1}\partial_{w-4}f_w\\
&f_{w+4}=\tfrac{(5w+12)^4}{2^6 3^5(w+1)(w+2)^5}\Big(\tfrac{(w+1)(2w+1)}{18}E_4 f_w-\partial^2_{w-4}f_w\Big)\\
&f_{w+6}=\tfrac{(5w+12)^4(5w+24)^4}{2^{18}3^7(w+1)(w+2)^5(w+3)^5(w+4)(w+5)}\\
&\qquad\times\Big(\Big(17w^2+78w+90\Big)\partial^3_{w-4}f_w-\cdots\Big)\\
&f_{w+8}=\tfrac{(5w+12)^4(5w+24)^4(5w+36)^4}{2^{24}3^{13}(w+1)(w+2)^5(w+3)^5(w+4)^5(w+5)(w+6)^4(w+7)}\\
&\qquad\times\Big(-\Big(1313w^6+\cdots+2177280\Big)\partial^4_{w-4}f_w+\cdots\Big)\\
&f_{w+10}=\tfrac{(5w+12)^4(5w+24)^4(5w+36)^4}{2^{28}3^{14}(w+1)(w+2)^5(w+3)^5(w+4)(w+5)(w+6)^4(w+7)(w+8)}\\
&\qquad\times\Big(\Big(293w^4+4332w^3+22968w^2+51192w+40824\Big)E_4\partial^3_{w-4}f_w-\cdots\Big)
\end{aligned}
\tag{6.28}
$$

are normalised extremal quasimodular forms of weights $w+2,\dots,w+10$ *(the omitted terms indicated by "…" are given in* (6.25)*).*

As an immediate consequence of Propositions 7– 10, we obtain the following fact conjectured in [17].

Theorem 6. *The denominators of the coefficients of the normalised extremal quasimodular forms of depth* ≤ 4*, are divisible only by primes* $< w$*.*

Acknowledgement. The author is grateful to an anonymous referee for her/his very helpful comments on an earlier version of this paper. The referee suggested to formulate Proposition 5 as a separate statement and provided the reference [31], which contains a closely related study.

Appendix A. Coefficients of the operators for depth 4

In this appendix we collect the more elaborate formulas used especially in Section 6.4. The polynomials $p_0,\dots,p_5$ in equations (A.1)–(A.6) and the modular forms $C_{40},\dots,C_{48}$ in equations (A.7)– (A.11) were computed with the help of `Mathematica`.

$$\begin{aligned} p_0(w) =\,&53567w^{14}+4499628w^{13}+173318340w^{12}+4055616864w^{11}+64374205218w^{10}+732790207224w^9+6165100658404w^8+38914973459904w^7\\ &+185044363180416w^6+659055640624128w^5+1729058937394176w^4+3237068849283072w^3+4084118362128384w^2+3105388005949440w\\ &+1072718335180800 \end{aligned} \tag{A.1}$$

$$\begin{aligned} p_1(w) =\,&21257w^{11}+1465884w^{10}+45186990w^9+821051740w^8+9759703548w^7+79588527156w^6+453687847200w^5\\ &+1804779218520w^4+4900200364800w^3+8628400143360w^2+8845395333120w+3990767616000 \end{aligned} \tag{A.2}$$

$$\begin{aligned} p_2(w) =\,&2662740w^{16}+224120550w^{15}+8648003840w^{14}+202621853220w^{13}+3217542322665w^{12}+36586266504480w^{11}+306658234963680w^{10}\\ &+1919356528986240w^9+8970889439482816w^8+30866477857195008w^7+75319919247624192w^6+118664936756305920w^5\\ &+83296021547483136w^4-82769401579438080w^3-258790551639293952w^2-245119018746249216w-86822757140004864 \end{aligned} \tag{A.3}$$

$$\begin{aligned} p_3(w) =\,&4272785w^{17}+351970350w^{16}+13234823080w^{15}+300533087760w^{14}+4592608729932w^{13}+49787752253076w^{12}+392868254956864w^{11}\\ &+2274866661846720w^{10}+9597118952486912w^9+28789901067644544w^8+58741997991303168w^7+79017091035181056w^6\\ &+100071999240486912w^5+278562611915587584w^4+779359222970449920w^3+1260737947219525632w^2+1054463073573666816w\\ &+355736061701259264 \end{aligned} \tag{A.4}$$

$$\begin{aligned} p_4(w) =\,&517135w^{17}+40772970w^{16}+1455719580w^{15}+31076826800w^{14}+441034824168w^{13}+4375275488634w^{12}+31084796008256w^{11}\\ &+160090786631040w^{10}+608772267089664w^9+1834128793979392w^8+5229385586024448w^7+15775977503047680w^6+40287913631023104w^5\\ &+57115900062203904w^4-19258645489385472w^3-224285038806564864w^2-343616934723452928w-182090547421249536 \end{aligned} \tag{A.5}$$

$$\begin{aligned} p_5(w) =\,&531441w^{13}+36690686w^{12}+1133566168w^{11}+20680195920w^{10}+247548700336w^9+2043291298652w^8+11897624359104w^7+49185666453888w^6\\ &+143692776009216w^5+293687697411072w^4+418695721574400w^3+426532499288064w^2+316421756411904w+135523565862912. \end{aligned} \tag{A.6}$$

$$
\begin{aligned}
C_{40} =\,& 212336640(w-9)(w-3)\Big(26359w^{14}-2214156w^{13}+85087955w^{12}-1981432728w^{11}+31214109018w^{10}-351608948568w^{9}+2918019038293w^{8}\\
&-18107342458608w^{7}+84336226011558w^{6}-293055231675096w^{5}+746938048608792w^{4}-1352381369540544w^{3}+1642191216192000w^{2}\\
&-1195772294131200w+393589456128000\Big)E_4E_6^6\\
&+1658880\Big(51900019w^{16}-4982401824w^{15}+221226817445w^{14}-6027400068900w^{13}+112718884720404w^{12}-1533303103010400w^{11}\\
&+15683910429124776w^{10}-122977017767207520w^{9}+746511198878954304w^{8}-3517545090663659520w^{7}+12814220784607361280w^{6}\\
&-35687146233130066944w^{5}+74437303893443933184w^{4}-112350543174642769920w^{3}+115665398295339663360w^{2}\\
&-72541839922746163200w+20875509918437474304\Big)E_4^4E_6^4\\
&+1244160\Big(79223933w^{16}-7605497568w^{15}+337672506115w^{14}-9198636642300w^{13}+171985015024288w^{12}-2338768976800320w^{11}\\
&+23913646971494624w^{10}-187421212329029760w^{9}+1137138517815826176w^{8}-5355376637586038784w^{7}+19499550045226841088w^{6}\\
&-54282475542685077504w^{5}+113193191971235303424w^{4}-170842339568900800512w^{3}+175944268854662529024w^{2}\\
&-110441402045312532480w+31829998802618548224\Big)E_4^7E_6^2\\
&+1296\Big(5782232065w^{16}-555094278240w^{15}+24644070643200w^{14}-671264352864000w^{13}+12548411370416640w^{12}-170603540496691200w^{11}\\
&+1743913567470202880w^{10}-13663290089279078400w^{9}+82868952111275704320w^{8}-390124920526971863040w^{7}\\
&+1419971443258483998720w^{6}-3951666980565374730240w^{5}+8238643867091026575360w^{4}-12434466652068469800960w^{3}\\
&+12809293284994355036160w^{2}-8045822842460386099200w+2321594527761517510656\Big)E_4^{10}
\end{aligned}
\tag{A.7}
$$

(A.8)

$$
\begin{aligned}
C_{42} = {} & 8847360(w-10)(w-9)(w-7)(w-4)(w-3)(w-1)\Big(21257w^{11}-1340040w^{10}+37636350w^{9}-621010700w^{8}+6681486588w^{7}-49160979324w^{6}\\
&+252123174720w^{5}-900257357160w^{4}+2191152850560w^{3}-3459266009856w^{2}+3186365921280w-1296999475200\Big)E_6^7\\
&+221184(w-1)\Big(59652285w^{16}-5727423730w^{15}+254372232445w^{14}-6933224340880w^{13}+129733172064000w^{12}-1766135709167940w^{11}\\
&+18084440432438220w^{10}-141993070201020780w^{9}+863455748585349240w^{8}-4077640101288170880w^{7}\\
&+14896222070817504960w^{6}-41630583090120877440w^{5}+87211051870194381312w^{4}-132333305508808224768w^{3}+137125544222548426752w^{2}\\
&-86677899606946971648w+25177794683564851200\Big)E_4^3E_6^5\\
&+34560(w-1)\Big(1163484751w^{16}-111693353753w^{15}+4959075896102w^{14}-135096500329618w^{13}+2526019283240820w^{12}\\
&-34353212893709184w^{11}+351290143724951040w^{10}-2753475334119164160w^{9}+16707612532291757568w^{8}-78689634714927968256w^{7}\\
&+286520082131305949184w^{6}-797543727623713492992w^{5}+1662730176637474947072w^{4}-2508547795667083984896w^{3}\\
&+2581778110323244400640w^{2}-1619027909322136879104w+465981816686361182208\Big)E_4^6E_6^3\\
&+432(w-1)\Big(28531797265w^{16}-2738717899920w^{15}+121567987037280w^{14}-3310530840743520w^{13}+61865709853233600w^{12}\\
&-840720468904181760w^{11}+8588483182402488320w^{10}-67231802818857031680w^{9}+407292115696721326080w^{8}-1914415680515583836160w^{7}\\
&+6953430231459668459520w^{6}-19296803817086754816000w^{5}+40082761664735503712256w^{4}-60204747917480347828224w^{3}\\
&+61632482296473191448576w^{2}-38403785466601122299904w+10969770793919657803776\Big)E_4^9E_6
\end{aligned}
$$

(A.9)

$$
\begin{aligned}
C_{44} = & 73728\Big(12532755w^{18}-1228914345w^{17}+55932044285w^{16}-1568787184755w^{15}+30362687859910w^{14}-430260670247520w^{13} \\
& +4622655124809540w^{12}-38470537186614840w^{11}+251211075702837014w^{10}-1295864332043809260w^{9}+5290237710120191760w^{8} \\
& -17044689685346563680w^{7}+43013914131829395456w^{6}-83886321769065465408w^{5}+123718742229427747200w^{4}-133286662841842990080w^{3} \\
& +98925565236150067200w^{2}-45228008659406118912w+9609196389953863680\Big)E_4^2E_6^6 \\
& +23040\Big(208766802w^{18}-20738706663w^{17}+957610061114w^{16}-27294230630727w^{15}+537821633898004w^{14}-7775876558499438w^{13} \\
& +85445305359966780w^{12}-729304045355384352w^{11}+4899650334544430160w^{10}-26095137555730257504w^{9}+110418717465901382976w^{8} \\
& -370315908903799143936w^{7}+977199822897307158528w^{6}-2002217065285338178560w^{5}+3117189914871002634240w^{4} \\
& -3561207870202807173120w^{3}+2814329391503205335040w^{2}-1374584400002855337984w+312699758864962682880\Big)E_4^5E_6^4 \\
& +432\Big(5640942435w^{18}-574145660265w^{17}+27210408942670w^{16}-797473001419560w^{15}+16188499652022320w^{14}-241595348932913040w^{13} \\
& +2745725576819892320w^{12}-24286261064854871040w^{11}+169405063961957458432w^{10}-938429115674081925120w^{9} \\
& +4136645160744791040000w^{8}-14470987007027825786880w^{7}+39867414308025880412160w^{6}-85316923540371471630336w^{5} \\
& +138712136173982221271040w^{4}-165366920569631924551680w^{3}+136184996014666316513280w^{2} \\
& -69179329337614286192640w+16327133797959811989504\Big)E_4^8E_6^2 \\
& +18\Big(3825382395w^{18}-417693655380w^{17}+21201009227390w^{16}-664296074843520w^{15}+14390341237827840w^{14}-228724925396964480w^{13} \\
& +2762631554345230080w^{12}-25910717672216739840w^{11}+191177029926129983488w^{10}-1117275590093461463040w^{9} \\
& +5181258594279899381760w^{8}-19011361988249752043520w^{7}+54763132291635013484544w^{6}-122130979006167868243968w^{5} \\
& +206227290808252579184640w^{4}-254460321591668401766400w^{3}+216144824557113157091328w^{2} \\
& -112869386433737376399360w+27296067396795679899648\Big)E_4^{11}
\end{aligned}
$$

(A.10)

$$
\begin{aligned}
C_{46} =\,& 12288(w-9)(w-3)\Big(21385w^{17}-3000540w^{16}+181158795w^{15}-6431059740w^{14}+152509563792w^{13}-2584774376916w^{12}+32580209973804w^{11}\\
&-312952105940940w^{10}+2324178632140120w^{9}-13441417608474240w^{8}+60599123972703360w^{7}-211935600366998400w^{6}\\
&+568296967045625280w^{5}-1144426627632504960w^{4}+1672115398874698752w^{3}-1670883597388947456w^{2}\\
&+1019823559435689984w-286322948470702080\Big)E_4E_6^{7}\\
&+1536(w-9)(w-3)\Big(43564885w^{17}-4199424540w^{16}+188153370170w^{15}-5202745378740w^{14}+99416655438692w^{13}-1392798686959116w^{12}\\
&+14810213044144264w^{11}-122061062138129400w^{10}+789187186132282416w^{9}-4024771696392895584w^{8}+16186815100766545920w^{7}\\
&-51035279993385265152w^{6}+124573554816237061632w^{5}-230424866987544400896w^{4}+311845100542176927744w^{3}\\
&-290906510930217861120w^{2}+166976811547859681280w-44392037471484641280\Big)E_4^{4}E_6^{5}\\
&+48(w-3)\Big(5935658055w^{18}-604130680965w^{17}+28644817180790w^{16}-840407590446360w^{15}+17090822555729336w^{14}\\
&-255749220384903192w^{13}+2917535433452941424w^{12}-25935973597843047168w^{11}+182093301447056161536w^{10}\\
&-1017033389895633205248w^{9}+4528828858768286662656w^{8}-16038621164741078679552w^{7}+44835034780192894009344w^{6}\\
&-97590317268828859711488w^{5}161770255432330054828032w^{4}-197074103775825302913024w^{3}+166176836346425804587008w^{2}\\
&-86567632937344309395456w+20972979080173738524672\Big)E_4^{7}E_6^{3}\\
&+3(w-3)\Big(35224547665w^{18}-3533943045270w^{17}+164951941875370w^{16}-4757220455686080w^{15}+94946288308939008w^{14}\\
&-1391879999449512576w^{13}+15524090523960134912w^{12}-134623421025963823104w^{11}+919695494473819799552w^{10}\\
&-4984015509758024933376w^{9}+21464444623798611689472w^{8}-73246940966966519660544w^{7}+196469687001100291670016w^{6}\\
&-408349070730564361519104w^{5}+642737421946789924700160w^{4}-738663893223616618168320w^{3}+583143789019164227665920w^{2}\\
&-281900069458357354758144w+62720415667791014658048\Big)E_4^{10}E_6
\end{aligned}
$$

(A.11)

$$\begin{aligned}
C_{48} =& 4096(w-10)(w-9)(w-7)(w-4)^5(w-3)(w-1)\Big(16w^{10}-22025w^9+1225150w^8-30082800w^7+422010000w^6-3708422460w^5+21084612660w^4 \\
& -77240291040w^3+175004264880w^2-221507758080w+119022943536\Big)E_6^8 \\
& +512(w-9)(w-4)^4(w-3)\Big(2457393w^{14}-206994196w^{13}+7967808727w^{12}-185579433574w^{11}+2918385315234w^{10}-32734570590132w^9 \\
& +269644653708324w^8-1653949412659908w^7+7574407203930120w^6-25704354625374456w^5+63427818462081552w^4 \\
& -109936627329417984w^3+125935036372661760w^2-84849181728168960w+25173889100697600\Big)E_4^3E_6^6 \\
& +16(w-4)^4\Big(372768821w^{16}-35518302589w^{15}+1564351553185w^{14}-42250630382915w^{13}+782728613878590w^{12}-10539973948217796w^{11} \\
& +106642134401044044w^{10}-826431392449118340w^9+4953943715462672760w^8-23029565126880689280w^7+82687822898011670016w^6 \\
& -226727406051812308224w^5+465078193737971627520w^4-689454957866260193280w^3+696182683156551475200w^2 \\
& -427583296582966296576w+120287920179051823104\Big)E_4^6E_6^4 \\
& +(w-4)^4\Big(2314158061w^{16}-225295075074w^{15}+10156109605065w^{14}-281276718421650w^{13}+5354355751604718w^{12} \\
& -74250742207449696w^{11}+775561858607479616w^{10}-6221336083117357440w^9+38716279294460726016w^8-187454904571023464448w^7 \\
& +703499359272507740160w^6-2024100636419312971776w^5+4375390173868546007040w^4-6867228800076590481408w^3 \\
& +7378429565839006236672w^2-4847796658391713579008w+1467072833894567903232\Big)E_4^9E_6^2 \\
& +\tfrac{3}{16}(w-4)^4\Big(177147w^{16}-4094579648w^{15}+374950410640w^{14}-15829704501920w^{13}+408049598642272w^{12}-7176728984678912w^{11} \\
& +91165949198565376w^{10}-863556865787120640w^9+6207504451510781952w^8-34121559189028110336w^7+143351014974408622080w^6 \\
& -456319260766953603072w^5+1080402954748938289152w^4-1841108952145673060352w^3+2131230194096535502848w^2 \\
& -1498210290755300229120w+482085103232027197440\Big)E_4^{12}
\end{aligned}$$

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