PURITY RESULTS FOR SOME ARITHMETICALLY DEFINED MEASURES

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Abstract. We study measures that are obtained as push-forwards of measures of maximal entropy on sofic shifts under digital maps $(x_k)_{k\in\mathbb{N}} \mapsto \sum_{k\in\mathbb{N}} x_k\beta^{-k}$, where $\beta > 1$ is a Pisot number. We characterise the continuity of such measures in terms of the underlying automaton and show a purity result.

1. Introduction

Digital representations of real numbers by infinite series

$$\sum_{k=1}^{\infty} \frac{x_k}{\beta^k} \tag{1}$$

with $x_k \in \mathcal{A}$, a finite alphabet, and $\beta > 1$ have attracted attention from different points of view. The underlying dynamical system given by the map $T_\beta : x \mapsto \beta x \mod 1$ has been studied extensively since the seminal papers [24, 30]. For an overview of the development we refer to [3, 7, 13, 14]. The original study was carried out for the "canonical" digit set $\mathcal{A} = \{0, 1, \ldots, \lceil\beta\rceil-1\}$, but many variations have been studied. It turned out in [24] that Pisot numbers β play a very important rôle in that context, as for these β the transformation T_β has especially nice properties. In this case the set of representations of all real numbers in $[0, 1]$ obtained by iteration of T_β is a sofic shift (see [24]); the definition of a sofic shift will be given in Section 2. A Pisot number is an algebraic integer all of whose Galois conjugates have modulus < 1 (see [4]). Pisot numbers have the nice property that their powers are "almost integers", meaning that $(\beta^n \mod 1)_{n\in\mathbb{N}}$ tends to 0.

In the present paper we will change the point of view starting with a one-sided sofic shift space $\mathcal{K}^+ \subset \mathcal{A}^{\mathbb{N}}$, where $\mathcal{A} \subset \mathbb{Z}$ is the underlying set of digits. Then we consider a map $\phi^+ : \mathcal{K}^+ \to \mathbb{R}$ mapping $(x_k)_{k\in\mathbb{N}}$ to the series (1) for β a Pisot number. Of course, in general nothing can

2010 *Mathematics Subject Classification.* 11A63, 68Q45, 11K55, 37A45.

* The author is supported by the Austrian Science Fund FWF project F5503 (part of the Special Research Program (SFB) "Quasi-Monte Carlo Methods: Theory and Applications").

be said about injectivity of this map, or even the structure of the image. Even for the full shift $\mathcal{A}^{\mathbb{N}}$ the structure of the image can be intricate (see [38]). If $\mathcal{K}^+$ is equipped with a shift invariant measure, then this measure is pushed forward to $\mathbb{R}$ by the map ϕ^+. The properties of measures obtained in this way are our object of study. For the measure on $\mathcal{K}^+$ we will take the measure of maximal entropy, or Parry measure, see [25, 26]. This will be discussed in Section 3.

Measures of this kind occur in different contexts. Possibly, the earliest occurrence was in two papers by Erdős [11,12], where he proved the singularity of this measure for $\mathcal{K}$ being the full shift. We will discuss that further in Section 4. In the context of studying redundant expansions of integers in the context of fast multiplication algorithms used in cryptography, the precise study of the number of representations of an integer n in the form

$$n = \sum_{k=0}^{K} x_k 2^k \quad \text{with } x_k \in \{0, \pm 1\}$$

and minimising

$$\sum_{k=0}^{K} |x_k|$$

led to singular measure on $[-1, 1]$ (see [15]). Further results in this direction were obtained in [16, 17]. A more general point of view replacing the powers of 2 by the solution of a linear recurrence has been taken in [19]. Furthermore, such measures occur as spectral measures of dynamical systems related to numeration systems [18], in the study of diffraction patters of tilings (see [1, 2]), and as spectral measures of substitution dynamical systems (see [28]). In Section 4 we will present two main results, namely the fact that the measure is pure (meaning that the Lebesgue decomposition only has one term), and a characterisation of continuity of the measure in terms of properties of the underlying automaton.

Erdős' proof of the singularity of the measure in [11,12] uses the fact that the Fourier transform of these measures does not tend to 0 at ∞, and this method was used in many other cases. This motivates the study of the Fourier transform of the measures under consideration. The transform can be expressed in terms of infinite matrix products, which allow the computation of limits

$$\lim_{k\to\infty} \widehat{\nu}(z\beta^k) \tag{2}$$

for $z \in \mathbb{Z}[\beta]$. In Section 5 we find an interpretation of these limits as Fourier coefficients of a measure on the torus given by a two sided version of the map ϕ. This will be used to show that the vanishing of the limits (2) for all $z \in \mathbb{Z}[\beta]$, $z \neq 0$ is equivalent to absolute continuity for these measures.

In a final Section 6 we exhibit several simple examples as applications of our results.

2. Regular languages and sofic shifts

Let $\mathcal{A}$ be a finite alphabet, and $G = (V, E)$ be a finite directed graph (see [8]) equipped with a *labelling* $\ell : E \to \mathcal{A}$. Then the pair (G, ℓ) is called a *labelled graph*. A finite automaton is a quadruple $\mathsf{A} = (G, \ell, I, T)$, where I (initial states) and T (terminal states) are subsets of the set of vertices (also called *states* in this context). A *path* of length n in the graph G is a sequence of edges $p = e_1e_2 \ldots e_n$, such that for every $j = 1, \ldots, n-1$ the edges $e_j = (\mathrm{i}(e_j), \mathrm{t}(e_j)))$ satisfy $\mathrm{t}(e_j) = \mathrm{i}(e_{j+1})$; the terminal vertex $\mathrm{t}(e_j)$ of every edge coincides with the initial vertex $\mathrm{i}(e_{j+1})$ of the consecutive edge. We say that p connects $\mathrm{i}(e_1)$ and $\mathrm{t}(e_n)$. The language $\mathcal{L} = \mathcal{L}(\mathsf{A})$ recognised by the automaton A is given by

(3)
$$\mathcal{L}_n = \{\ell(e_1)\ell(e_2)\ldots\ell(e_n) \mid e_1e_2\ldots e_n \text{ a path in } G, \mathrm{i}(e_1) \in I, \mathrm{t}(e_n) \in T\}$$
$$\mathcal{L} = \{\epsilon\} \cup \bigcup_{n=1}^{\infty} \mathcal{L}_n,$$

where ϵ denotes the *empty word*, which by definition has length 0; $\mathcal{L}_n$ is the set of words of length n. A subset of $\mathcal{A}^*$ is called a *regular language*, if it is recognised by a finite automaton A. A language $\mathcal{L}$ is called *irreducible*, if for any $w_1, w_2 \in \mathcal{L}$ there exists a $w \in \mathcal{A}^*$ such that $w_1ww_2 \in \mathcal{L}$. This is equivalent to the underlying graph to being *strongly connected*; for any two vertices $v_1, v_2 \in V$ there is a path connecting v_1 with v_2. The language is called *primitive*, if there is an $N \in \mathbb{N}$ such that for any $w_1, w_2 \in \mathcal{L}$ and any $n \geq N$ there exists a word $w \in \mathcal{A}^*$ of length n such that $w_1ww_2 \in \mathcal{L}$.

An automaton A is called *deterministic*, if there is only one initial state (i.e. $\#I = 1$) and for every vertex $v \in V$ and every letter $a \in \mathcal{A}$ there is at most one edge $e \in E$ with $\mathrm{i}(e) = v$ and $\ell(e) = a$. From now on we assume that all automata are deterministic and the languages recognised by them are primitive. For a comprehensive introduction to the theory of formal languages we refer to [9, 20, 32].

For a given automaton A we study the sets of one- and two-sided infinite words recognised by A

$$\begin{aligned} \mathcal{K}_I^+ &= \{(\ell(e_k))_{k\in\mathbb{N}} \mid \forall n\in\mathbb{N} : e_1e_2\dots e_n \text{ is a path in } G, \mathrm{i}(e_1)\in I\} \\ \mathcal{K}^+ &= \{(\ell(e_k))_{k\in\mathbb{N}} \mid \forall n\in\mathbb{N} : e_1e_2\dots e_n \text{ is a path in } G\} \qquad (4) \\ \mathcal{K} &= \{(\ell(e_k))_{k\in\mathbb{Z}} \mid \forall m<n : e_me_{m+1}\dots e_n \text{ is a path in } G\}. \end{aligned}$$

The set $\mathcal{K}$ is called the (two-sided) *sofic* shift associated to A (see [22]), $\mathcal{K}^+$ is the one-sided sofic shift, both spaces are closed under the according shift transformation

$$\begin{aligned} \sigma^+ &: \mathcal{K}^+ \to \mathcal{K}^+ : (x_k)_{k\in\mathbb{N}} \mapsto (x_{k+1})_{k\in\mathbb{N}} \\ \sigma &: \mathcal{K} \to \mathcal{K} : (x_k)_{k\in\mathbb{Z}} \mapsto (x_{k+1})_{k\in\mathbb{Z}}. \end{aligned}$$

The space $\mathcal{K}_I^+$, which can be seen as an extension of $\mathcal{L}$ to infinite words, is in general not closed under σ^+, but the relation

$$\mathcal{K}^+ = \bigcup_{n=0}^{N} (\sigma^+)^n \mathcal{K}_I^+$$

holds for some $N\in\mathbb{N}$ as a consequence of the strong connectedness of the graph underlying A. Notice that σ is bijective, whereas σ^+ is not.

3. Measures on shift spaces

We equip the spaces $\mathcal{K}_I^+$ and $\mathcal{K}^+$ with a "canonical" measure that we will define now. We define the *cylinder set*

$$[\varepsilon_1,\varepsilon_1,\dots,\varepsilon_k] = \left\{(x_n)_{n\in\mathbb{N}} \in \mathcal{K}_I^+ \mid x_1=\varepsilon_1,\dots,x_k=\varepsilon_k\right\}$$

for $\varepsilon_i\in\mathcal{A}$ for $i=1,\dots,k$. The cylinder sets generate a topology on $\mathcal{K}_I^+$ and also the σ-algebra of Borel sets for this topology. We define

$$\mu_I^+([\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k]) = \lim_{n\to\infty} \frac{\#([\varepsilon_1,\varepsilon_2,\dots,\varepsilon_k]\cap\mathcal{L}_n)}{\#\mathcal{L}_n}; \qquad (5)$$

the existence of the limit will become obvious from the following discussion.

For $a\in\mathcal{A}$ define the *a-transition matrix* by

$$(M_a)_{ij} = \begin{cases} 1 & \text{if } (i,j)\in E \text{ and } \ell((i,j))=a \\ 0 & \text{otherwise} \end{cases} \qquad (6)$$

and

$$M = \sum_{a\in\mathcal{A}} M_a. \qquad (7)$$

Furthermore, set $\mathbf{v}_I$ and $\mathbf{v}_T$ the indicator vectors of the sets I and T, respectively. Then for $n \geq k$ we have

$$\#([\varepsilon_1, \varepsilon_1, \ldots, \varepsilon_k] \cap \mathcal{L}_n) = \mathbf{v}_I^\mathsf{T} M_{\varepsilon_1} \cdots M_{\varepsilon_k} M^{n-k} \mathbf{v}_T.$$

By the assumption that the language $\mathcal{L}$ is primitive which is equivalent to the fact that M is primitive, the Perron-Frobenius theorem (see [33]) implies that there is a dominating eigenvalue $\lambda > 0$ such that

$$M^n = \lambda^n \mathbf{v}_R \mathbf{v}_L^\mathsf{T} + o(\lambda^n), \tag{8}$$

where $\mathbf{v}_L^\mathsf{T}$ is a left eigenvector of M for the eigenvalue λ, and $\mathbf{v}_R$ is a right eigenvector with

$$\mathbf{v}_L^\mathsf{T} \mathbf{v}_R = 1.$$

With this we can write the quantity under the limit in (5) as

$$\frac{\mathbf{v}_I^\mathsf{T} M_{\varepsilon_1} \cdots M_{\varepsilon_k} \left(\lambda^{n-k} \mathbf{v}_R \mathbf{v}_L^\mathsf{T} + o(\lambda^n)\right) \mathbf{v}_T}{\mathbf{v}_I^\mathsf{T} \left(\lambda^n \mathbf{v}_R \mathbf{v}_L^\mathsf{T} + o(\lambda^n)\right) \mathbf{v}_T},$$

which shows that the limit exists and equals

$$\mu_I^+([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) = \lambda^{-k} \frac{\mathbf{v}_I^\mathsf{T} M_{\varepsilon_1} \cdots M_{\varepsilon_k} \mathbf{v}_R}{\mathbf{v}_I^\mathsf{T} \mathbf{v}_R}. \tag{9}$$

On $\mathcal{K}^+$ we define the measure μ^+ by

$$\mu^+([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) = \lambda^{-k} \mathbf{v}_L^\mathsf{T} M_{\varepsilon_1} \cdots M_{\varepsilon_k} \mathbf{v}_R. \tag{10}$$

This measure is shift invariant by the observation

$$\begin{aligned} \mu^+((\sigma^+)^{-1}[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) &= \sum_{a \in \mathcal{A}} \mu^+([a, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]) \\ = \sum_{a \in \mathcal{A}} \lambda^{-k-1} \mathbf{v}_L^\mathsf{T} M_a M_{\varepsilon_1} \cdots M_{\varepsilon_k} \mathbf{v}_R &= \lambda^{-k-1} \mathbf{v}_L^\mathsf{T} M M_{\varepsilon_1} \cdots M_{\varepsilon_k} \mathbf{v}_R \\ &= \lambda^{-k} \mathbf{v}_L^\mathsf{T} M_{\varepsilon_1} \cdots M_{\varepsilon_k} \mathbf{v}_R = \mu^+([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k]). \end{aligned}$$

Assuming that $\mathcal{L}$ is primitive, the dynamical system $(\mathcal{K}^+, \mu^+, \sigma)$ is strongly mixing and thus ergodic (see [6, 37]):

$$\begin{aligned} &\lim_{n \to \infty} \mu^+ \left([\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k] \cap T^{-n}[\delta_1, \ldots, \delta_s]\right) \\ &\quad = \lim_{n \to \infty} \lambda^{-n-s} \mathbf{v}_L^\mathsf{T} M_{\varepsilon_1} \cdots M_{\varepsilon_k} M^{n-k} M_{\delta_1} \cdots M_{\delta_s} \mathbf{v}_R \\ &\quad = \lambda^{-k} \mathbf{v}_L^\mathsf{T} M_{\varepsilon_1} \cdots M_{\varepsilon_k} \mathbf{v}_R \lambda^{-s} \mathbf{v}_L^\mathsf{T} M_{\delta_1} \cdots M_{\delta_s} \mathbf{v}_R \\ &\quad = \mu^+([\varepsilon_1, \varepsilon_1, \ldots, \varepsilon_k]) \mu^+([\delta_1, \ldots, \delta_s]). \end{aligned}$$

The measures μ_I^+ and $\mu^+|_{\mathcal{K}_I^+}$ (restriction to $\mathcal{K}_I^+$) are equivalent. The measure μ^+ is the unique measure of maximal entropy on $\mathcal{K}^+$, also called the Parry measure [25, 26].

Similarly, we define a measure on $\mathcal{K}$ by

$$\mu([\varepsilon_1,\dots,\varepsilon_k]_m)=\lambda^{-k}\mathbf{v}_L^\mathsf{T}M_{\varepsilon_1}\cdots M_{\varepsilon_k}\mathbf{v}_R, \tag{11}$$

where

$$[\varepsilon_1,\dots,\varepsilon_k]_m=\{(x_n)_{n\in\mathbb{Z}}\in\mathcal{K}\mid x_{m+1}=\varepsilon_1,\dots,x_{m+k}=\varepsilon_k\}$$

for $m\in\mathbb{Z}$. The measure μ is σ-invariant by definition.

4. Generalised Erdős measures

In [11] Erdős studied the distribution measure of the random series

$$\sum_{k=1}^\infty\frac{X_k}{\beta^k},$$

where $(X_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables taking the values ±1 with equal probability $\frac12$, and $\beta=\frac{1+\sqrt5}{2}$. He showed that the distribution is purely singular continuous. Later [12] he extended this result for β an irrational Pisot number, a positive algebraic integer, all of whose conjugates lie strictly inside the unit circle (see [4]). Notice that integers ≥ 2 are also considered as Pisot numbers. The Pisot property plays an important rôle in the proof of singularity, as it allows to show that the Fourier transform of the measure does not tend to 0 along the sequence $(\beta^k)_{k\in\mathbb{N}}$. This argument will be elaborated later.

In the meantime the set of $\beta>1$, for which the measure constructed as above is singular continuous has been studied further. Solomyak [36] could prove that for almost all $\beta\in(1,2)$ the measure is absolutely continuous. This result was refined by Shmerkin [34], who proved that the exceptional set has Hausdorff dimension 0. It is still open, whether Pisot numbers are the only exceptions. For a survey on the development until the year 2000 we refer to [27]. For more recent developments and results in this direction we refer to [31, 35].

From now on we assume that the alphabet $\mathcal{A}$ is a subset of $\mathbb{Z}$. For an automaton A and a Pisot number β of degree $r\geq1$ we introduce the maps

$$\begin{aligned}\phi_I^+&:\mathcal{K}_I^+\to\mathbb{R}:(x_k)_{k\in\mathbb{N}}\mapsto\sum_{k=1}^\infty\frac{x_k}{\beta^k}\\ \phi^+&:\mathcal{K}^+\to\mathbb{R}:(x_k)_{k\in\mathbb{N}}\mapsto\sum_{k=1}^\infty\frac{x_k}{\beta^k}.\end{aligned}\tag{12}$$

The measures

$$\begin{aligned}\nu_I &= \left(\phi_I^+\right)_* (\mu_I^+)\\ \nu &= \left(\phi^+\right)_* (\mu^+)\end{aligned} \tag{13}$$

are analogues of the Erdős measures studied in [11, 12]. Here and throughout this paper we denote by $f_*(\mu)$ the push-forward measure on Y given by a map $f : X \to Y$ and a measure μ on X; $f_*(\mu)(A) = \mu(f^{-1}(A))$. The properties of the measures ν and ν_I will be the subject of the remaining part of this paper. By definition, ν_I is absolutely continuous with respect to ν. Furthermore, by the definition of μ_I, ν_I is given by

$$\nu_I = \lim_{n\to\infty} \frac{1}{\#\mathcal{L}_n} \sum_{w\in\mathcal{L}_n} \delta_{\phi_I^+(w)},$$

where δ_x denotes a unit point mass in x.

Theorem 1. *The measures ν_I and ν are pure in the sense that they are either absolutely continuous with respect to Lebesgue measure, purely singular continuous, or purely atomic. The last case can only occur, if the image $\phi^+(\mathcal{K}^+)$ is finite. The number of atoms is bounded by the number of vertices in* A.

Proof. The Jessen-Wintner theorem [21, Theorem 35] (for a more modern formulation see also [10, Lemma 1.22 (ii)]) is concerned with the distribution measure of a random series

$$\sum_{n=1}^{\infty} X_n\lambda_n,$$

where $(X_n)_{n\in\mathbb{N}}$ is a sequence of independent discrete random variables and the series

$$\sum_{n=1}^{\infty} \mathbb{E}(X_n)\lambda_n \text{ and } \sum_{n=1}^{\infty} \mathbb{V}(X_n)\lambda_n^2$$

converge. It states that this measure is either absolutely continuous with respect to Lebesgue measure, purely singular continuous, or purely atomic.

In [17, Lemma 9] the following extended formulation of the Jessen-Wintner theorem was given.

Lemma 1. *Let $Q = \prod_{n=0}^{\infty} Q_n$ be an infinite product of discrete spaces equipped with a measure ν, which satisfies Kolmogorov's* 0-1*-law (i.e. every tail event has either measure* 0 *or* 1*). Furthermore, let X_n be a sequence of random variables defined on the spaces Q_n, such that the* *series* $X = \sum_{n=0}^{\infty} X_n$ *converges* ν*-almost everywhere. Then the distribution of* X *is either purely discrete, or purely singular continuous, or absolutely continuous with respect to Lebesgue measure.*

The proof of this lemma is just the observation that the proof of the Jessen-Wintner theorem only uses the fact that the measure on the product space satisfies a 0-1-law. In our context the 0-1-law is ensured by the ergodicity with respect to the measure μ^+ on the shift. This proves the statement for the measure ν.

The statement for ν_I follows from the the absolute continuity of ν_I with respect to ν. The assertion about the number of atoms will be proved in the proof of Theorem 2. □

Lemma 2. *Let* $x \in \mathbb{R}$ *be such that* $\mu((\phi^+)^{-1}(\{x\})) > 0$. *Then there exists a path* $e_1 \ldots e_n$ *in* A *such that*

$$x = \frac{1}{1-\beta^{-n}} \sum_{k=1}^{n} \frac{\ell(e_k)}{\beta^k},$$

especially, $x \in \mathbb{Q}(\beta)$.

Proof. Set $A = (\phi^+)^{-1}(\{x\})$. Then $\phi^+(A) = \{x\}$ and

$$\phi^+(\sigma^{-m}(A)) \subseteq \left\{ \beta^{-m} \left(x + \sum_{k=0}^{m-1} \ell(e_{m-k})\beta^k \right) \mid e_1 \ldots e_m \text{ a path in } \mathsf{A} \right\}.$$

Since $\mu(A) > 0$ there is an $n \geq 1$ such that $A \cap \sigma^{-n}(A) \neq \emptyset$. This implies that

$$x \in \left\{ \beta^{-n} \left(x + \ell(e_n) + \cdots + \ell(e_1)\beta^{n-1} \right) \mid e_1 \ldots e_n \text{ a path in } \mathsf{A} \right\},$$

from which we derive the assertion of the Lemma. □

Lemma 3. *Let* β *be a Pisot number. Then the set of words*

$$\mathcal{L}_0 = \left\{ (x_1, x_2, \ldots, x_n) \in \mathcal{A}^* \mid \sum_{k=1}^{n} \frac{x_k}{\beta^k} = 0 \right\}$$

is recognisable by a finite automaton. As a consequence the set

$$\mathcal{K}_0 = \left\{ (x_1, x_2, \ldots) \in \mathcal{A}^{\mathbb{N}} \mid \sum_{k=1}^{\infty} \frac{x_k}{\beta^k} = 0 \right\} \tag{14}$$

is a space $\mathcal{K}_I^+$ *for that automaton and an appropriate initial state* I *(labelled by* 0*).*

Proof. We define an automaton inductively by starting with an initial state labelled 0 and adding further states by appropriate transitions. Assume that we have a state labelled x, then for every $a \in \mathcal{A}$ we have a transition $x \mapsto \beta x - a$, and a possibly new state labelled $\beta x - a$. In order to show that this gives a finite automaton, we have to show that the number of states is finite.

We think of the states as sums of the form

$$\sum_{k=1}^{m-s} \frac{x_{k+s}}{\beta^k}, \tag{15}$$

where

$$\sum_{k=1}^{m} \frac{x_k}{\beta^k} = 0.$$

Then all the values given by (15) are bounded by $\frac{M}{\beta-1}$, if $|a| \leq M$ for all $a \in \mathcal{A}$. Furthermore, by the procedure given in the first paragraph all labels of the states are algebraic integers of the form

$$-\sum_{k=0}^{m} x_k \beta^k \quad \text{with } x_0, \ldots, x_m \in \mathcal{A}.$$

Applying the Galois conjugation $\beta \mapsto \beta_q$ $(q = 2, \ldots, r)$ gives

$$\left| -\sum_{k=0}^{m} x_k \beta^k \right| \leq \frac{M}{1 - |\beta_q|}$$

(here and in the sequel we denote the Galois conjugates of β by $\beta_2, \ldots, \beta_r$, recall that $|\beta_q| < 1$). Thus the set of states is a bounded set of algebraic integers all of whose Galois conjugates are bounded. Such a set has to be finite.

Now we have a finite set of states and transitions, an automaton. The terminal state is course given again by the state 0.

All infinite words recognised by this automaton correspond to sums as in the definition of $\mathcal{K}_0$. □

Lemma 4. *Let $x \in \mathbb{R}$ be such that $\mu((\phi^+)^{-1}(\{x\})) > 0$. Then the set $(\phi^+)^{-1}(\{x\})$ has non-empty interior.*

Proof. The set $A = (\phi^+)^{-1}(\{x\})$ is recognisable by a finite automaton. To see this, we first observe that x has a periodic "β-representation"

$$x = \sum_{k=1}^{\infty} \frac{b_k}{\beta^k},$$

where $b_{k+mn} = \ell(e_k)$ for $k = 1, \dots, n$ and $m \geq 0$. By the periodicity of the representation of x and the fact that all "β-representations" of 0 are recognised by a finite automaton also A is recognisable by a finite automaton B.

If any word $\ell(e_1) \cdots \ell(e_m)$ corresponding to a path in A would not be recognised by B, then the entropy of the language corresponding to B would be strictly less than the entropy of the language recognised by A (see [5]), which would imply that $\mu(A) = 0$. Thus B recognises the same words as A, which implies the assertion. $\square$

Theorem 2. *The set $\phi^+(\mathcal{K}^+)$ is either finite or perfect and thus uncountable. In the first case the measure $\phi^+_*(\mu)$ is atomic, in the second case it is continuous.*

Proof. The set $\phi^+(\mathcal{K}^+)$ is compact as the continuous image of a compact set.

Assume that there exists a vertex $v \in V$ and two paths $e_1 e_2 \dots e_{L_1}$ and $f_1 f_2 \dots f_{L_2}$ both connecting v to itself such that

$$\frac{1}{1-\beta^{-L_1}} \sum_{k=1}^{L_1} \frac{\ell(e_k)}{\beta^k} \neq \frac{1}{1-\beta^{-L_2}} \sum_{k=1}^{L_2} \frac{\ell(f_k)}{\beta^k}. \tag{16}$$

Then we will show that the set $\phi^+(\mathcal{K}^+)$ is perfect. For this purpose we choose $x \in \phi^+(\mathcal{K}^+)$ and show that there is a sequence of points $(x_n)_{n\in\mathbb{N}}$ of points in $\phi^+(\mathcal{K}^+)$ with $x = \lim_{n\to\infty} x_n$ and $x_n \neq x$ for all n.

Let $x = \phi^+((\ell(a_1), \ell(a_2), \dots))$. For $n \in \mathbb{N}$ we set

$$\begin{aligned}
\xi_n &= \phi^+(\ell(a_1), \ell(a_2), \dots, \ell(a_n), \ell(b_1), \dots, \ell(b_k), \overline{\ell(e_1), \dots, \ell(e_{L_1})})\\
\eta_n &= \phi^+(\ell(a_1), \ell(a_2), \dots, \ell(a_n), \ell(b_1), \dots, \ell(b_k), \overline{\ell(f_1), \dots, \ell(f_{L_2})}),
\end{aligned}$$

where $b_1, \dots, b_k \in E$ are chosen so that $\ell(a_1) \dots \ell(a_n) \ell(b_1) \dots \ell(b_k) \in \mathcal{L}$ and $\mathrm{t}(b_k) = v$. There exists an integer $k \leq \#V$ with this property for every n. By (16) $\xi_n \neq \eta_n$, and thus at least one of these two values is different from x. We take x_n to be this value. Then $\lim_{n\to\infty} x_n = x$ showing that x is not isolated.

By Lemma 4 the preimage of any atom of the measure $\phi^+_*(\mu)$ would contain a cylinder set. This would contradict the fact that we have just proved that the images of all cylinder sets are uncountable.

Assume on the contrary that for all $v \in V$ there exits a value $c(v) \in \mathbb{R}$ such that for all paths $e_1 e_2 \dots e_n$ connecting v to itself

$$c(v) = \frac{1}{1-\beta^{-n}} \sum_{k=1}^{n} \frac{\ell(e_k)}{\beta^k}. \tag{17}$$

In this case every infinite path $e_1e_2\dots$ starting at v yields the value $c(v)$ for ϕ^+: assume that k is chosen to be the minimal index so that $w = \mathrm{t}(e_k)$ is visited infinitely often by the path. If $k = 1$, the value of ϕ^+ given by the path is $c(v)$ by definition. For $k > 1$ we decompose the path

$$e_1 \dots e_{k_1} e_{k_1+1} \dots e_{k_2} e_{k_2+1} \dots e_{k_3} \dots,$$

where $k_1 = k$ and $\mathrm{t}(e_{k_j}) = w$. Then by our assumption (17) every path $e_{k_j+1} \dots e_{k_{j+1}}$ can be replaced by a path $f_1 \dots f_m f_{m+1} \dots f_{m+q}$ with $\mathrm{i}(f_1) = w$, $\mathrm{t}(f_m) = \mathrm{i}(f_{m+1}) = v$, and $\mathrm{t}(f_{m+q}) = w$ without changing the value of ϕ^+. The new path visits v infinitely often, and thus assigns the value $c(v)$ to ϕ^+. Thus ϕ^+ only takes the values $\{c(v) \mid v \in V\}$. Each of these values is assigned a positive mass. This proves that the number of atoms of $\phi^+_*(\mu)$ is bounded by the number of vertices of A. This shows the last assertion of Theorem 1. □

The following result is a consequence of the proof of Theorem 2.

Theorem 3. *The set $\phi^+(\mathcal{K}^+)$ is finite, if and only if for every vertex $v \in V$ there exists a value $c(v) \in \mathbb{R}$ such that for all paths $e_1e_2\dots e_n$ connecting v to itself* (17) *holds.*

Remark 1. Notice that the measure $\phi^+_*(\mu)$ can only be absolutely continuous, if $\beta \le \lambda$. Otherwise the set $\phi^+(\mathcal{K}^+)$ has Hausdorff dimension $\le \frac{\log\lambda}{\log\beta} < 1$ and cannot support an absolutely continuous measure.

5. Fourier transforms and matrix products

The Fourier transforms of the measures ν and ν_I are given by

$$\begin{aligned} \widehat{\nu}_I(t) &= \int_{-\infty}^{\infty} e^{-2\pi i x t}\, d\nu_I(x) \\ \widehat{\nu}(t) &= \int_{-\infty}^{\infty} e^{-2\pi i x t}\, d\nu(x). \end{aligned} \tag{18}$$

In order to derive expressions for $\widehat{\nu}_I$ and $\widehat{\nu}$, we introduce the weighted transition matrix for ϕ^+ and the underlying automaton A

$$W(t) = \frac{1}{\lambda} \sum_{a\in\mathcal{A}} e(-at) M_a, \tag{19}$$

where we use the notation $e(x) = e^{2\pi i x}$.

Proposition 1. *The Fourier transforms of the measures ν and ν_I can be expressed as*

$$\widehat{\nu}(t)=\mathbf{v}_L^{\mathsf{T}}\prod_{n=1}^{\infty}W(\beta^{-n}t)\mathbf{v}_R \tag{20}$$

and

$$\widehat{\nu}_I(t)=\frac{1}{\mathbf{v}_I^{\mathsf{T}}\mathbf{v}_R}\mathbf{v}_I^{\mathsf{T}}\prod_{n=1}^{\infty}W(\beta^{-n}t)\mathbf{v}_R, \tag{21}$$

where the infinite matrix product is interpreted so that the factors are ordered from left to right.

Proof. We set

$$\phi_n^+((x_k)_{k\in\mathbb{N}})=\sum_{k=1}^{n}\frac{x_k}{\beta^k}.$$

Then $\lim_{n\to\infty}\phi_n^+((x_k)_{k\in\mathbb{N}})=\phi^+((x_k)_{k\in\mathbb{N}})$ holds uniformly on $\mathcal{K}^+$. We set $\nu_n=(\phi_n^+)_*(\mu^+)$ and observe that $\nu_n\rightharpoonup\nu$. Then

$$\widehat{\nu}_n(t)=\mathbf{v}_L^{\mathsf{T}}\prod_{k=1}^{n}W(\beta^{-k}t)\mathbf{v}_R,$$

where the product symbol is interpreted as multiplying the factors from left to right with increasing index. The limit relation $\lim_{n\to\infty}\widehat{\nu}_n=\widehat{\nu}$ and equation (20) then follow by weak convergence. A similar reasoning gives (21). ☐

Remark 2. As pointed out in Section 4 Erdős [11, 12] proved the singularity of the distribution measure ν of the random series

$$\sum_{n=1}^{\infty}\frac{X_n}{\beta^n},$$

by showing that $(\widehat{\nu}(\beta^k))_{k\in\mathbb{N}}$ does not tend to 0. Of course the fact that $\lim_{|t|\to\infty}\widehat{\nu}(t)=0$ does not suffice in general to prove absolute continuity, as there are singular measures, so called Rajchman measures, whose Fourier transform vanishes at ∞ (see [29]).

Using the Pisot property of β we define the map

$$\phi:\mathcal{K}\to\mathbb{T}^r,(x_k)_{k\in\mathbb{Z}}\mapsto\left(\sum_{k=-\infty}^{\infty}x_k\beta^{-k+m}\pmod 1\right)_{m=0}^{r-1}, \tag{22}$$

where we use the notation $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. Notice that the series for $k\leq 0$ converges $\pmod 1$ by the fact that

$$\beta^m+\beta_2^m+\cdots+\beta_r^m\in\mathbb{Z}$$

and $|\beta_2|, \ldots, |\beta_r| < 1$.

Theorem 4. *Let $z = m_0 + m_1\beta + \cdots + m_{r-1}\beta^{r-1} \in \mathbb{Z}[\beta]$ for β a Pisot number of degree r and ν be the measure given by (13). Then the limit*

$$\widehat{\psi}(m_0, \ldots, m_{r-1}) = \lim_{k\to\infty} \widehat{\nu}(z\beta^k) \tag{23}$$

exists. These values are the Fourier coefficients of the measure $\psi = \phi_*(\mu)$ on $\mathbb{T}^r$.

Proof. We define the maps

$$\phi_n((x_k)_{k\in\mathbb{Z}}) = \left(\sum_{k=-n}^{\infty} x_k\beta^{-k+m} \pmod 1\right)_{m=0}^{r-1}.$$

Then $(\phi_n)_{n\in\mathbb{N}}$ converges to ϕ uniformly on $\mathcal{K}$. Let $\psi_n = (\phi_n)_*(\mu)$. Then $\psi_n \rightharpoonup \psi$ and

$$\widehat{\psi}_n(m_0, \ldots, m_{r-1}) = \widehat{\nu}(z\beta^n).$$

The limit relation (23) then follows by weak convergence and the fact that $(\beta^n \mod 1) \to 0$. $\square$

Remark 3. The shift on $\mathcal{K}$ is conjugate via ϕ to the hyperbolic toral endomorphism

$$B : \mathbb{T}^r \to \mathbb{T}^r, \begin{pmatrix} t_0 \\ t_1 \\ \vdots \\ t_{r-1} \end{pmatrix} \mapsto \begin{pmatrix} t_1 \\ \vdots \\ t_{r-1} \\ a_0t_0 + \cdots + a_{r-1}t_{r-1} \pmod 1 \end{pmatrix},$$

where

$$\beta^r = a_{r-1}\beta^{r-1} + \cdots + a_1\beta + a_0$$

is the minimal equation of β. The measure ψ is then a B-invariant measure on $\mathbb{T}^r$, and B is ergodic with respect to ψ.

Theorem 5. *The measure ν given by (13) is absolutely continuous, if and only if for all $z \in \mathbb{Z}[\beta] \setminus \{0\}$*

$$\lim_{k\to\infty} \widehat{\nu}(z\beta^k) = 0. \tag{24}$$

Proof. Let $\beta_2, \ldots, \beta_r$ denote the Galois conjugates of β and assume that $\beta, \beta_2, \ldots, \beta_s \in \mathbb{R}$ and $\beta_{s+1}, \beta_{s+2}, \ldots, \beta_{s+t}, \overline{\beta_{s+1}}, \overline{\beta_{s+2}}, \ldots, \overline{\beta_{s+t}} \in \mathbb{C} \setminus \mathbb{R}$; then $r = s + 2t$. Then the map

$$\widetilde{\phi} : \mathcal{K} \to \mathbb{R}^s \times \mathbb{C}^t, (x_k)_{k\in\mathbb{Z}} \mapsto \left(\sum_{k=1}^{\infty} x_k\beta^{-k}, \sum_{k=0}^{\infty} x_{-k}\beta_2^k, \ldots, \sum_{k=0}^{\infty} x_{-k}\beta_{s+t}^k\right)$$

is continuous and $\widetilde{\phi}(\mathcal{K})$ is compact. Together with the map

$$\rho:\mathbb{R}^s\times\mathbb{C}^t\to\mathbb{T}^r,(y_1,\dots,y_s,z_{s+1},\dots,z_{s+t})\mapsto$$
$$\big(\beta^m y_1-(\beta_2^m y_2+\cdots+\beta_s^m y_s)-2\Re(\beta_{s+1}^m z_{s+1}+\cdots+\beta_{s+t}^m z_{s+t})\big)_{m=0}^{r-1}$$

we have

$$\phi=\rho\circ\widetilde{\phi}\pmod 1.$$

The map ρ is linear and $\rho\pmod 1$ is finite to one on $\widetilde{\phi}(\mathcal{K})$. Now the measure $\psi=(\mathrm{mod}\ 1)_*\circ\rho_*\circ\widetilde{\phi}_*(\mu)$ is equal to the Lebesgue measure, if and only if $\widehat{\psi}(m_0,\dots,m_{r-1})=0$ for all $(m_0,\dots,m_{r-1})\in\mathbb{Z}^r\setminus\{\mathbf{0}\}$. If ψ is Lebesgue measure, then the measure $\nu=P_*\circ\widetilde{\phi}_*(\mu)$ is absolutely continuous, where P denotes the projection to the first coordinate. On the other hand, if ψ is not the Lebesgue measure, then ν cannot be absolutely continuous by (23). □

6. Examples

Example 1. Let $\mathcal{K}\subset\{0,\pm1\}^{\mathbb{N}}$ be given by the automaton in Figure 1 and take $\beta=\frac{1+\sqrt5}{2}$. Then the map ϕ^+ takes the values

$$\begin{array}{ll} 0 & \nu(\{0\})=\dfrac{1}{\gamma^2}\\[2ex] \pm1 & \nu(\{1\})=\nu(\{-1\})=\dfrac12\left(\dfrac1\gamma-\dfrac1{\gamma^2}\right)\\[2ex] \pm\dfrac1\beta & \nu\left(\left\{\dfrac1\beta\right\}\right)=\nu\left(\left\{-\dfrac1\beta\right\}\right)=\dfrac1{\gamma^3}\end{array}$$

with the indicated probabilities, where γ is the positive solution of

$$x^3=x^2+2.$$

The value γ is the Perron-Frobenius eigenvalue of the adjacency matrix of the automaton given by Figure 1.

Example 2. Let $\mathcal{K}\subset\{0,1\}^{\mathbb{N}}$ be the set of sequences of 0 and 1 with no two consecutive 1s (given by the automaton in Figure 2). These are the digital representations of all real numbers in $[0,1]$ obtained by iteration of the β-transformation $T_\beta: x\mapsto\beta x\mod 1$ for $\beta=\frac{1+\sqrt5}{2}$ (see [24]). Then the measure $\phi^+(\mu)$ is Lebesgue measure on $[0,1]$.

Example 3. Take $\mathcal{K}=\{0,1,2,3\}^{\mathbb{N}}$ and μ the infinite product measure assigning probability $\frac14$ to each letter. Take $\beta=2$ and consider the

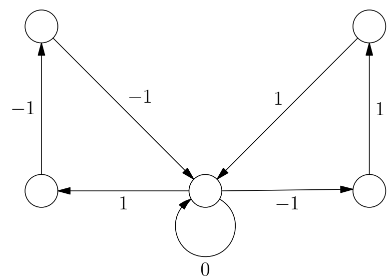

FIGURE 1. The automaton recognising all expansions of 0 in base $\frac{1+\sqrt{5}}{2}$ with digits $\{0, \pm 1\}$

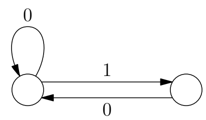

FIGURE 2. The automaton recognising all expansions in base $\frac{1+\sqrt{5}}{2}$ with digits $\{0, 1\}$

map ϕ^+ as above. Then the corresponding measure on $[0, 3]$ is given by the density

$$h(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 \leq x \leq 1 \\ \frac{1}{2} & \text{for } 1 \leq x \leq 2 \\ \frac{1}{2}(3-x) & \text{for } 2 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Projecting this measure $\mod 1$ gives Lebesgue measure, because in this case $r = 1$ and the map ϕ maps to $\mathbb{T}$.

Example 4. Let $\mathcal{K}^+ \subset \{0, 1, 2\}^{\mathbb{N}}$ be the set of sequences recognised by the automaton in Figure 4. These are all expansions of real numbers expressed in base β^2, where $\beta = \frac{1+\sqrt{5}}{2}$. This is similar to Example 3, where expansions in base β^2 are interpreted in base β. In this case the measure is singular, though. This can be seen by computing

$$\lim_{n\to\infty} \widehat{\nu}(\beta^n) = 0.0608424\ldots + 0.0208583\ldots i.$$

numerically.

Example 5. This example is taken from [15]. Take $\mathcal{K} \subset \{0, \pm 1\}^{\mathbb{N}}$ to be the set of sequences recognised by the automaton in Figure 5. The automaton with initial state I recognises all expansions of integers in

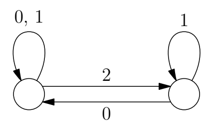

FIGURE 3. The automaton recognising all expansions in base $\beta^2 = \frac{3+\sqrt{5}}{2}$ with digits $\{0, 1, 2\}$

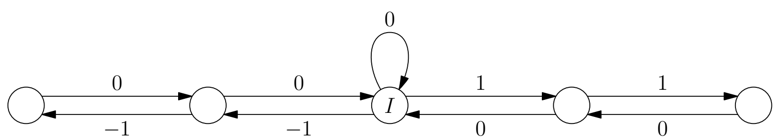

FIGURE 4. The automaton recognising expansions in base 2 with digits $\{0, \pm 1\}$

the form

$$n = \sum_{k=0}^{K} x_k 2^k \quad \text{with } x_k \in \{0, \pm 1\}$$

which minimise the weight

$$w(n) = \sum_{k=0}^{K} |x_k|.$$

This is motivated by the fact that this weight is the number of additions/subtractions in computing nP by a Horner-type scheme, where P is a point on an elliptic curve. The measures $\phi_*^+(\mu^+)$ and $\phi_*^+(\mu_I)$ are singular in this case, as was proved in [15].

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