FUNCTIONAL ITERATIONS AND STOPPING TIMES FOR BROWNIAN MOTION ON THE SIERPIŃSKI GASKET

Peter J. Grabner

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Abstract. We investigate the distribution of the hitting time T defined by the first visit of the Brownian motion on the Sierpiński gasket at geodesic distance r from the origin. For this purpose we perform a precise analysis of the moment generating function of the random variable T. The key result is an explicit description of the analytic behaviour of the Laplace-Stieltjes transform of the distribution function of T. This yields a series expansion for the distribution function and the asymptotics for $t \to 0$.

1. Introduction

The Sierpiński gasket (cf. [Fa]) is a well known planar fractal which has been studied from different points of view since 1915, when Sierpiński introduced it as an example of a curve all of whose points are ramification points (cf. [Si]). Since the 1980's a notion of Brownian motion on this fractal (and later other "nested fractals", cf. [Li], [D-K]) has been developed. For an excellent introduction to this subject we refer to [B-P]. The diffusion process X_t on the Sierpiński gasket is defined as the weak limit of properly chosen rescalings of the simple random walk on the "Sierpiński graph" $\mathcal{G}$.

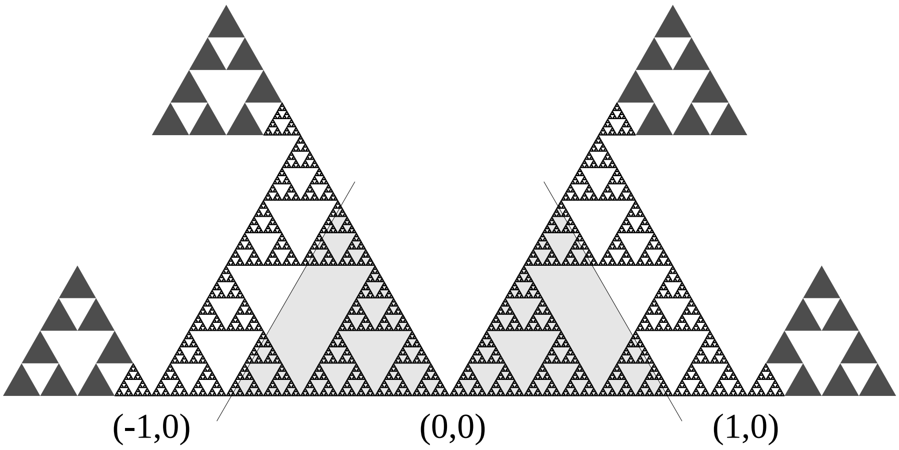

Figure 1.

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Besides the study of Brownian motion on this fractal a notion of calculus (cf. [Ki]) has been developed for functions on the Sierpiński gasket. The Laplacian as the infinite generator of the diffusion on nested fractals (cf. [Li]) has been studied extensively, see e. g. [E-I]. An equivalent approach to study this operator is to consider the limit of transition operators on graphs approximating the fractal (cf. [F-S], [Sh]).

This paper will be devoted to the study of of the stopping times defined by the first hitting of the boundary of the geodesic ball. The geodesic distance $d(x, y)$ is defined as the length of the shortest path between the points x and y through points of the gasket. An alternative way of introducing this distance is via proper rescalings of distances in the approximating graphs. The Sierpiński gasket and the geodesic metric $d(x, y)$ on it have been defined in a rigorous way in [B-P] and [G-T]. The light gray area in Figure 1 indicates the geodesic ball of radius 0.732. We note here that an exact formula for the Hausdorff-measure of the geodesic ball is given in [Gr].

Furthermore, we will give precise analytic information on the moment generating function

$$\phi(z) = \mathbb{E}e^{-zT}, \tag{1.1}$$

where T is the time of the first hitting of the boundary of the geodesic unit ball. This function is known to satisfy the functional equation (cf. [B-P])

$$\phi(5z) = \frac{\phi(z)^2}{4 - 3\phi(z)}, \quad \phi(0) = 1, \quad \phi'(0) = -1. \tag{1.2}$$

This function turns out to have a meromorphic continuation to the whole complex plane, and we will give precise information on the location of its singularities. As a consequence of the distribution of the poles we explain the fluctuating behaviour of its logarithm encountered in [B-P] and give an exact formula for the distribution function of the random variable T:

$$\mathbb{P}\left(T < t\right) = 1 - \sum_{n=1}^{\infty} \mu_n e^{-\lambda_n t}, \tag{1.3}$$

where μ_n and λ_n will be given later. This formula has the same structure as the corresponding distribution functions in the Euclidian case, where the λ's form a subsequence of the eigenvalues of the Laplacian. We will prove that the series (1.3) is uniformly convergent for $t \geq 0$ and determine the exact asymptotic behaviour for $t \to 0$. This (once again) exhibits fluctuating behaviour also described in [Bi] and [B-P].

We remark here that there is a vast literature on such periodicity phenomena in the theory of branching processes. In [F-O] a more general concept of polynomial iterations is studied, which includes branching processes with finite maximal family size. [B-B1] and [Du2] give an explanation for the near-constancy (the above mentioned fluctuations are in fact very small) of the periodic function in the asymptotics of $\phi(z)$. Since the existence of a meromorphic analytic continuation of $\phi(z)$ (which exists in our special case) could not be exploited there, the implications for the process are somewhat weaker than Theorem 2.

Furthermore, the properties of the distribution function of W, the limit of a supercritical branching process, have been studied extensively. There is an intimate connection between the analytic behaviour of $\phi(z)$ and the behaviour of the distribution for $t \to 0+$ and $t \to \infty$, which will be used in the proofs of Theorems 4 and 5. We mention the study of the tail behaviour of the distribution of W in the case of finite maximal family size in [B-B2], where again periodicity phenomena occur. This is in contrast to (1.3), which implies purely exponential decay (in the case of unbounded family size). The behaviour of the distribution function for $t \to +$ is studied in [Du1] and [B-B2].

2. Stopping times and Branching Processes

The idea of using branching processes for modelling stopping times for the Brownian motion on nested fractals was introduced in [B-P]. In order to use this idea in the study of stopping times defined by the first hit at a given geodesic distance, we have to extend this idea slightly to branching processes with more than one type (cf. [Ha]).

We consider three different types of edges for the simple random walk on the Sierpiński graph $\mathcal{G}$ truncated at distance R. Firstly, there are those edges connecting a vertex at distance $R-1$ to a vertex at distance R. These will be marked by the variable z in the generating functions. Clearly only one of these edges will be traversed in a random walk stopping at distance R. Secondly, there are the directed edges connecting a vertex at distance $R-1$, which is connected to a vertex at distance R, with a vertex at distance $R-1$ or $R-2$. These will be marked by the variable y in the generating functions. Finally, there are all the other edges, which will be marked by x in the generating functions.

We introduce the probability generating function (PGF)

$$F_R(x,y,z) = z \sum_{j,k=0}^{\infty} p_R(j,k) x^j y^k, \tag{2.1}$$

where $p_R(j,k)$ denotes the probability that the random walk starting at the origin reaches distance R for the first time after j steps of type x, k steps of type y (and one step of type z).

Proposition 1. *The function $F_R(x,y,z)$ satisfies the functional equations*

$$\begin{aligned} F_{2R}(x,y,z) &= F_R\left(f_1(x), f_2(x,y), f_3(x,y,z)\right) \quad \textit{for } R \geq 2 \\ F_{2R+1}(x,y,z) &= F_{R+1}\left(f_1(x), f_4(x,y), f_5(x,y,z)\right) \quad \textit{for } R \geq 1 \\ F_2(x,y,z) &= \frac{2xz}{4-y-xy}, \end{aligned}$$

where the functions f_i, $i = 1, \cdots, 4$ are given by

$$
\begin{aligned}
f_1(x) &= \frac{x^2}{4-3x}\\
f_2(x,y) &= \frac{2(2+x)x^2(4-y)}{64-16x-16y-16x^2-4xy+6x^2y+x^3y}\\
f_3(x,y,z) &= \frac{4(4+x)(2-x)xz}{64-16x-16y-16x^2-4xy+6x^2y+x^3y}\\
f_4(x,y) &= \frac{(2+x)xy}{8-2x-x^2-xy}\\
f_5(x,y,z) &= \frac{(2-x)(4+x)z}{8-2x-x^2-xy}.
\end{aligned}
$$

Proof. We will use elementary path arguments to derive the functional equation for F_R. A *path* in $\mathcal{G}$ is a sequence of vertices $\omega = [a_0, a_1, \ldots, a_n]$ such that two consecutive vertices are neighbours in the graph $\mathcal{G}$. The x-length of ω, $|\omega|_X$, is the number edges marked with x traversed by the path; similarly we define the y-length $|\omega|_Y$. The z-length is always 1. For every path we define its *weight* $W(\omega \mid x,y,z) = x^{|\omega|_X} y^{|\omega|_Y} z$ and for every set of paths Ω its weight is given by $W(\Omega \mid x,y,z) = \sum_{\omega\in\Omega} W(\omega \mid x,y,z)$. Notice that

$$F_R(x,y,z) = W\left(\Omega_R \mid \frac{x}{4}, \frac{y}{4}, \frac{z}{4}\right),$$

where Ω_R denotes the set of all paths starting at the origin, which end at distance R (and reach distance R only once).

We note first that the graph $2\mathcal{G}$ obtained by multiplying all coordinates of the vertices by 2 is isomorphic to $\mathcal{G}$. Furthermore, the vertices of $2\mathcal{G}$ are all vertices of $\mathcal{G}$. Let now u be a point at distance $2R$ from the origin and consider a path $\omega = [0 = a_0, a_1, \ldots, a_n = u]$. Define $\tau_j(\omega)$ by

$$\tau_0 = 0 \quad \text{and for } j \geq 1 \quad \tau_j = \{i > \tau_{j-1} \mid a_i \in 2\mathcal{G}, \quad a_i \neq a_{\tau_{j-1}}\}, \tag{2.2}$$

for $0 \leq j \leq k$, where $k = k(\omega)$ is the maximal index for which the last set is non-empty; we have either $\tau_k = n$ and $u = 2v$ or $\tau_k < n$ and $u \notin 2\mathcal{G}$. In the second case there is a unique $u' \in 2\mathcal{G}$ which is a neighbour of u and a_{n-1}. We define the *shadow* to be the path

$$\sigma(\omega) = \begin{cases} [0 = \frac{1}{2}a_{\tau_0}, \frac{1}{2}a_{\tau_1}, \ldots, \frac{1}{2}a_{\tau_k} = \frac{1}{2}u] & \text{in the first case}\\ [0 = \frac{1}{2}a_{\tau_0}, \frac{1}{2}a_{\tau_1}, \ldots, \frac{1}{2}a_{\tau_k}, \frac{1}{2}u'] & \text{in the second case.} \end{cases} \tag{2.3}$$

Take now a path $\omega_R = [0, a_1, \ldots, a_n]$ hitting distance R for the first time after k x-steps and ℓ y-steps ($n = k + \ell + 1$).

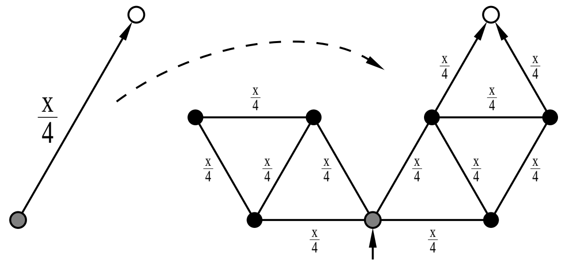
Then we have

$$W(\sigma^{-1}(\omega_R) \mid x,y,z) = f_1(4x)^k f_2(4x,4y)^\ell f_3(4x,4y,4z), \tag{2.4}$$

since every path ω_{2R} in $\sigma^{-1}(\omega_R)$ can be decomposed as

$$\omega_{2R} = [0, \ldots, 2a_1] \circ [2a_1, \ldots, 2a_2] \circ \cdots \circ [2a_{n-1}, \ldots, T],$$

where $\circ$ denotes concatenation of paths, where the end-point of the first path coincides with the initial point of the second one. T is the terminal point indicated in Figure 4. Each sub-path in ω_{2R} $[2a_j, \dots, 2a_{j+1}]$ "replaces" an edge $[a_j, a_{j+1}]$ in ω_R, and according to which type this edge belonged to, $[2a_j, \dots, 2a_{j+1}]$ has to be a path in Figure 2 for an x-edge and a path in Figure 3 for a y-edge. $[a_{n-1}, a_n]$ (which had been marked by z) has to be replaced by a path in Figure 4. (Notice that the "replacing" described above is just the familiar substitution construction used in combinatorics, cf. [G-J].) The functions f_1, f_2 and f_3 are just the PGF's of the paths joining the initial point and the terminal point in the different graphs, where the edges are marked according to the pictures. The computation of the functions f_i is an elementary matrix inversion exercise, which was done by *Maple*.



$$f_1(x) = \frac{x^2}{4-3x}$$

FIGURE 2.

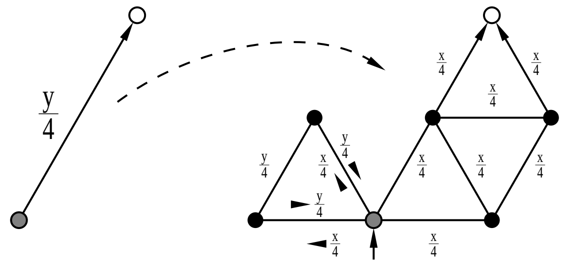


FIGURE 3.

$$f_2(x,y) = \frac{2(2+x)x^2(4-y)}{64-16x-16y-16x^2-4xy+6x^2y+x^3y}$$

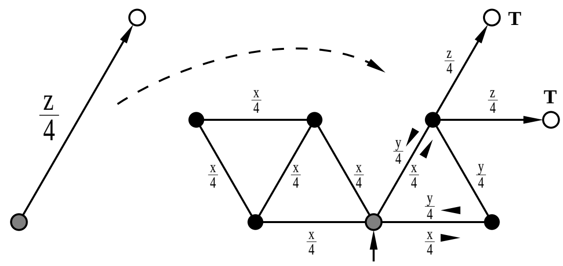
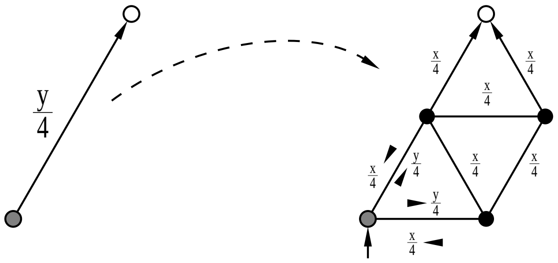


FIGURE 4.

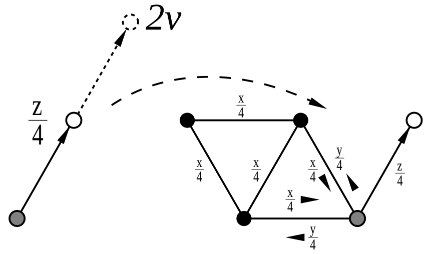
$$f_3(x,y,z) = \frac{4(4+x)(2-x)xz}{64-16x-16y-16x^2-4xy+6x^2y+x^3y}$$

For the second functional equation we extend the last edge of a path joining the origin with a point at distance $2R+1$ to obtain a point $2v$ at distance $2R+2$ (cf. Figure 6). Then we apply the same ideas as in the proof of the first equation.



$$f_4(x,y) = \frac{(2+x)xy}{8-2x-x^2-xy}$$

FIGURE 5.



$$f_5(x,y,z) = \frac{(2-x)(4+x)z}{8-2x-x^2-xy}$$

FIGURE 6.

$\square$

Theorem 1. *The moment generating function* $\phi_r(z) = \mathbb{E}e^{-zT_r}$ *of the random variable*

$$T_r = \{t \mid d(0, X_t) = r\}$$

is $\phi_r(z) = \phi(g(r)z)$, *where* $\phi(z)$ *is the function given by* (1.2). *For* $\frac{1}{2} < r < 1$ *given by the binary representation*[1] $r = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k}$ *define*

$$y_{k+1,n} = \begin{cases} \frac{3y_{k,n}}{5-y_{k,n}} & \text{for } \varepsilon_{n+1-k} = 0 \\ 6\frac{4-y_{k,n}}{32-13y_{k,n}} & \text{for } \varepsilon_{n+1-k} = 1 \end{cases} \qquad y_{1,n} = 1 \tag{2.5}$$

for $1 \le k < n$ *and*

$$t_k = \lim_{n\to\infty} y_{n-k,n}.$$

Then the function $g(r)$ *is given by*

$$g(r) = \frac{1}{5}\left(\frac{4-t_0}{2(2-t_0)}, \frac{1}{2-t_0}, \frac{1}{2-t_0}\right) \cdot \prod_{k=1}^{\infty} M_{\varepsilon_{k+1}}(t_k) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{2.6}$$

where

$$\begin{aligned} M_0(y) &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{y(32-y)}{(5-y)^2} & \frac{3}{(5-y)^2} & 0 \\ \frac{9y(2-y)}{5(5-y)^2} & \frac{2-y}{(5-y)^2} & \frac{1}{5-y} \end{pmatrix} \\ M_1(y) &= \begin{pmatrix} 1 & 0 & 0 \\ \frac{8(4-y)(92-31y)}{5(32-13y)^2} & \frac{24}{(32-13y)^2} & 0 \\ \frac{272(4-y)(2-y)}{5(32-13y)^2} & \frac{52(2-y)}{(32-13y)^2} & \frac{4}{32-13y} \end{pmatrix}. \end{aligned} \tag{2.7}$$

[1] In case of ambiguity we choose the infinite representation.

Furthermore,

$$g(1) = \frac{4}{7}, \quad g\left(\frac{1}{2}+\right) = \frac{1}{5}$$

and g can be continued to all other values of r by the relation $g(2r) = 5g(r)$. The function g is continuous for all real $r > 0$ except for dyadic rationals, where it is left continuous and has a jump discontinuity. This mirrors the structure of the gasket.

Proof. The functional equation for ϕ_r is an immediate consequence of the considerations in [B-P, pp. 571ff]. It only remains to prove that $g(r)$ can be computed as described in the theorem. We notice that by the definition of the process X_t and the fact that $g(r) = \mathbb{E}T_r$ we have

$$g(r) = \lim_{n\to\infty} 5^{-n}\left(\frac{\partial F_{[2^n r]}}{\partial x}(1,1,1) + \frac{\partial F_{[2^n r]}}{\partial y}(1,1,1) + \frac{\partial F_{[2^n r]}}{\partial z}(1,1,1)\right), \tag{2.8}$$

which yields $g(2r) = 5g(r)$ immediately (we will prove the existence of this limit later). Thus we can restrict our considerations to $\frac{1}{2} < r < 1$.

Introducing the temporary notation

$$\Phi_0(x,y,z) = (f_1(x), f_2(x,y), f_3(x,y,z)), \ \Phi_1(x,y,z) = (f_1(x), f_4(x,y), f_5(x,y,z))$$

we have

$$F_{[2^n r]}(x,y,z) = F_2 \circ \Phi_{1-\varepsilon_2} \circ \Phi_{1-\varepsilon_3} \circ \cdots \circ \Phi_{1-\varepsilon_{k-1}} \circ \Phi_{\varepsilon_k} \circ \cdots \circ \Phi_{\varepsilon_n}(x,y,z),$$

if $\varepsilon_{k+1} = \cdots = \varepsilon_n = 0$ and $\varepsilon_k = 1$. This is due to the fact that the transformation

$$n \mapsto \begin{cases} \frac{n}{2} & \text{for } n \quad \text{even} \\ \frac{n+1}{2} & \text{for } n \quad \text{odd} \end{cases}$$

interchanges the digits 0 and 1 except for a block of 0's starting with the least significant digit and the first 1 (reading from right to left).

We are interested in the orbit of the point $(1,1,1)$ under the transformations

$$\Phi_{1-\varepsilon_2} \circ \Phi_{1-\varepsilon_3} \circ \cdots \circ \Phi_{1-\varepsilon_{k-1}} \circ \Phi_{\varepsilon_k} \circ \cdots \circ \Phi_{\varepsilon_n}. \tag{2.9}$$

We notice that

$$f_1(1) = 1, \quad f_2(1,y)+f_3(1,y,2-y) = 2 \quad \text{and} \quad f_4(1,y)+f_5(1,y,2-y) = 2, \tag{2.10}$$

which implies that for all applications of a mapping (2.9) to $(1,1,1)$ $y+z$ keeps the value 2. Furthermore, the mappings $h_2(y) = f_2(1,y) = 6\frac{4-y}{32-13y}$ and $h_4(y) = f_4(1,y) = \frac{3y}{5-y}$ are contractions for $0 \le y \le 1$ with $h_2'(y), h_4'(y) \le \frac{15}{16}$. Thus we have

(2.11)

$$|\Phi_{1-\varepsilon_2} \circ \cdots \circ \Phi_{1-\varepsilon_k}(1,u,2-u) - \Phi_{1-\varepsilon_2} \circ \cdots \circ \Phi_{1-\varepsilon_k}(1,v,2-v)| \le \left(\frac{15}{16}\right)^k |u-v|,$$

which implies uniform convergence of

$$t_k = \lim_{n\to\infty} y_{n-k,n} \quad \text{for} \quad 0 \le k \le \frac{n}{2}.$$

To conclude convergence of (2.5) we notice that the matrices $5M_0(y)$ and $5M_1(y)$ are the Jacobians of Φ_1 and Φ_0, respectively, and their entries (except for the $[1,1]$-entry) are ≤ 0.9, which implies convergence of the infinite matrix product

$$\prod_{k=1}^{\infty} M_{\varepsilon_{k+1}}(t_k).$$

The values at $r = 1$ and the limit for $r \to \frac{1}{2}+$ can be computed as the limits

$$\begin{aligned} g(1) &= \lim_{n\to\infty} 5^{-n} \left(\frac{\partial F_{2^n-1}}{\partial x}(1,1,1) + \frac{\partial F_{2^n-1}}{\partial y}(1,1,1) + \frac{\partial F_{2^n-1}}{\partial z}(1,1,1) \right) \\ g\left(\frac{1}{2}+\right) &= \lim_{n\to\infty} 5^{-n} \left(\frac{\partial F_{2^{n-1}+1}}{\partial x}(1,1,1) + \frac{\partial F_{2^{n-1}+1}}{\partial y}(1,1,1) + \frac{\partial F_{2^{n-1}+1}}{\partial z}(1,1,1) \right). \end{aligned}$$

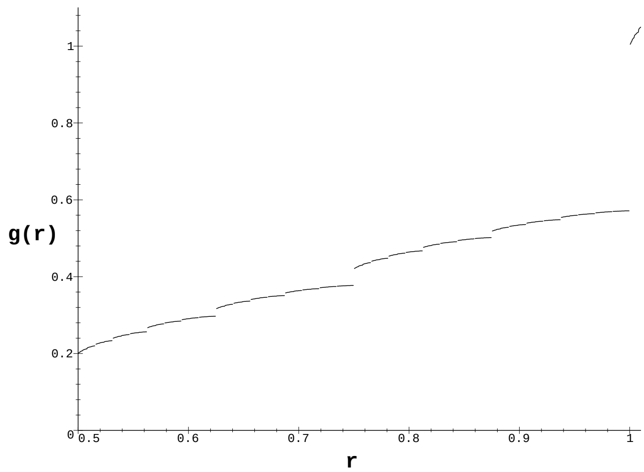
The assertion concerning continuity follows from (2.11). The discontinuity at dyadic rationals comes from the fact that they are the only numbers with two different binary expansions, and those two expansion yield different values for g (corresponding to the left and right limit). □

Corollary 1. *For a dyadic rational* $\frac{1}{2} < r < 1$ $r = \sum_{k=1}^{K} \frac{\varepsilon_k}{2^k}$, $\varepsilon_1 = \varepsilon_K = 1$, *the value is given by*

$$\begin{aligned} &\left(\frac{4 - y_{K,K}}{2(2 - y_{K,K})}, \frac{1}{2 - y_{K,K}}, \frac{1}{2 - y_{K,K}} \right) \cdot \\ &\cdot M_{\varepsilon_2}(y_{K-1,K}) \cdots M_{\varepsilon_{K-1}}(y_{2,K}) \cdot M_1(y_{1,K}) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \end{aligned} \tag{2.12}$$

where the $y_{k,K}$ *are defined as in* (2.5) *but with* $y_{1,K} = \frac{12}{13}$.

Proof. We write $r = \sum_{k=1}^{K-1} \frac{\varepsilon_k}{2^k} + \sum_{k=K+1}^{\infty} \frac{1}{2^k}$. Notice that the iterations of h_2 tend to $\frac{12}{13}$. The result follows by applying Theorem 1. □



THE FUNCTION $g(r)$

3. Analytic Study of the Function ϕ

In this section of the paper we will gather information on the analytic behaviour of the function $\phi(z)$. This includes an asymptotic expansion for $|z| \to \infty$ in $|\arg z| < \pi$, which contains a periodic fluctuating term, that can be related to the distribution of the poles of $\phi(z)$. This periodic term (the "Karlin-McGregor function") has been investigated by several authors (cf. [K-M1], [K-M2], [Du2], [B-B1], [Du3]). It turns out that the near-constancy of this function is a general phenomenon in the theory of branching processes (cf. [B-B1]), which can be explained by the exponential decay of its Fourier-coefficients. In section 5 we will use this information to prove the series expansion (1.3) and to study its asymptotic behaviour for $t \to 0$.

Since $\psi(z) = \frac{1}{\phi(z)}$ turns out to be an entire function, we will formulate our results for this function. $\psi(z)$ satisfies the functional equation

$$\psi(5z) = 4\psi(z)^2 - 3\psi(z), \quad \psi(0) = 1, \quad \psi'(0) = 1. \tag{3.1}$$

Theorem 2. *$\psi(z)$ is an entire function of order $\alpha = \frac{\log 2}{\log 5}$, which has all its zeros on the negative real axis. The asymptotic expansion*

$$\psi(z) = \exp\left(z^\alpha G\left(\frac{\log z}{\log 5}\right) + A\log z + H\left(\frac{\log z}{\log 5}\right)\right)\left(1 + \mathcal{O}_\theta\left(\frac{1}{z^{\frac{1}{2}-\alpha-\varepsilon}}\right)\right) \tag{3.2}$$

is valid for $|\arg z| \le \theta < \pi$, where $G(s)$ and $H(s)$ are periodic functions of period 1, which are holomorphic in the strip $|\Im s| < \frac{\pi}{\log 5}$. The Fourier series of G is given by

$$G(s) = \sum_{k=-\infty}^{\infty} g_k e^{2k\pi i s},$$

where

$$g_k = \frac{\pi}{2(-\log 2 + 2k\pi i)\sin\frac{\pi}{\log 5}(-\log 2 + 2k\pi i)} M\left(\alpha - \frac{2k\pi i}{\log 5}\right). \tag{3.3}$$

The function $M(s)$ is given as the Mellin transform of a measure defined in terms of the preimages of 0 *under the mapping $z \mapsto 4z^2 - 3z$ (more details will be given in the proof). This implies that*

$$|g_k| \le \frac{11}{20|k|} M(\alpha)\exp\left(-\frac{2\pi^2}{\log 5}|k|\right), \tag{3.4}$$

where $M(\alpha) = 0.8440757\ldots$. Similar expressions and estimates can be given for the Fourier coefficients of $H(s)$. An expression for the constant A will be given in the proof; its numerical value is $A = 0.48098676974525901234\ldots$.

Remark. The presence of the function $G(s)$ in the asymptotic expansion of $\psi(z)$ explains the fluctuating behaviour of the function $\log\phi(z)$ encountered in [B-P]. The first values of the Fourier coefficients

$$g_0 = 1.95910517961221301415572878\ldots$$
$$g_1 = -0.149367294300291232109\ldots \cdot 10^{-5} - 0.677320321830852396 48\ldots \cdot 10^{-7} \cdot i$$
$$g_2 = 0.200188518722683018 64\ldots \cdot 10^{-13} - 0.837099460184543737 57\ldots \cdot 10^{-12} \cdot i$$

and the exponential estimate for the other values imply,

$$|G(s)-g_0| \leq 0.3123 \cdot 10^{-5}$$
$$|G'(s)| \leq 0.1963 \cdot 10^{-4}.$$

Proof. We first prove that the Laplace-Stieltjes transform $\phi(z)$ (which is holomorphic in $\Re z > 0$ by the general theory of branching processes, cf. [Ha, Theorem 8.2]) has an analytic continuation into a neighbourhood of $z=0$. For this purpose we consider the sequence $\phi_0(z) = e^{-z}$, $\phi_{n+1}(5z) = \frac{\phi_n(z)^2}{4-3\phi_n(z)}$. For $-\frac{1}{2} \leq x = \Re z$ it can be proved by induction that

$$|\phi_n(z)| \leq \frac{1}{1+x},$$

which implies the existence of a holomorphic limit by Montel's theorem. This limit is the function $\phi(z)$ for $\Re z > 0$ and this function therefore has an analytic continuation to $\Re z > -\frac{1}{2}$.

Thus $\psi(z) = \frac{1}{\phi(z)}$ and $\phi(0)=1$ imply that $\psi(z)$ is holomorphic in some neighbourhood of $z=0$ and therefore has a power series expansion $\psi(z) = \sum_{n=0}^{\infty} \psi_n z^n$. Equation (3.1) implies

$$\psi_n = \frac{4}{5^n-5} \sum_{k=1}^{n-1} \psi_k \psi_{n-k} \quad \text{for} \quad n \geq 2, \tag{3.5}$$

which follows by an application of Taylor's theorem. We now proceed by induction to prove that

$$\psi_n \leq \frac{1}{n!}. \tag{3.6}$$

This estimate is trivially satisfied for $n=0,1$. Assume that the estimate is true for $k \leq n-1$ and $n \geq 2$. Then we have

$$\psi_n \leq \frac{4}{5^n-5} \sum_{k=1}^{n-1} \frac{1}{k!} \frac{1}{(n-k)!} \leq 4 \frac{2^n-2}{5^n-5} \frac{1}{n!} \leq \frac{1}{n!}.$$

This implies that $\psi(z)$ is an entire function of order at most 1 (cf. [Bo]).

In order to derive more precise information on the analytic behaviour of $\psi(z)$, we note that the zeros of $\psi(z)$ can be given by

$$-5^n \cdot 5^m \cdot \xi_{m,j} \quad \text{with} \quad n \geq 1, \quad m \geq 0, \quad 1 \leq j \leq \max(1, 2^{m-1}), \tag{3.7}$$

where the numbers $\xi_{m,j}$ are given as follows: consider

$$P_m = \left\{ z \mid g^{(m)}(z) = \frac{3}{4} \right\},$$

where $g^{(m)}(z)$ denotes the m-th functional iterate of $g(z) = 4z^2 - 3z$. Then $\#P_m = 2^m$ and $P_m \subset [-\frac{1}{4}, \frac{3-\sqrt{5}}{8}] \cup [\frac{3+\sqrt{5}}{8}, 1]$, which is an immediate consequence of the computation of the Julia set of $g(z)$ in [G-W]. We note here, that the Julia set of $g(z)$

is given by the closure of $\bigcup_m P_m$. The numbers $-\xi_{m,j}$ for $m \geq 1$ are given by the preimages of $P_m \cap [-\frac{1}{4}, \frac{3-\sqrt{5}}{8}]$ (this set has 2^{m-1} elements) under ψ in the interval $[-3, 0]$ (numerical studies with *Maple* show that $\psi'(z) > 0$ for $z \in [-3, 0]$[2] and therefore the preimage is unique), $\xi_{0,1} = -\psi^{(-1)}(\frac{3}{4}) = 0.26366111924136772879\ldots$. There can be no non-real zeros of ψ, since all the iterated preimages of g of 0 are real. A complex zero of ψ would therefore yield complex values z_n arbitrarily close to 0 with $\psi(z_n) \in \mathbb{R}$. This is a contradiction to the fact that $\psi'(0) \neq 0$.

Since the order of ψ is finite we can consider the Dirichlet series

$$\nu(s) = \sum_{\psi(-\lambda)=0} \frac{1}{\lambda^s} = \sum_{n=1}^{\infty} 5^{-ns} \left(\xi_{0,1}^{-s} + \sum_{m=1}^{\infty} 5^{-ms} \sum_{j=1}^{2^{m-1}} \xi_{m,j}^{-s} \right), \tag{3.8}$$

which is convergent for $\Re s > \beta$ for some $\beta \leq 1$. In order to find an analytic continuation of $\nu(s)$, we investigate

$$M_k(s) = \frac{1}{2^{k-1}} \sum_{j=1}^{2^{k-1}} \xi_{k,j}^{-s}. \tag{3.9}$$

Lemma 1. *There exists a measure μ supported on*

$$= \overline{\bigcup_{m \geq 1} \{\xi_{m,j} \mid j = 1, \ldots, 2^{m-1}\}}$$

such that

$$T_k(f) = \frac{1}{2^{k-1}} \sum_{j=1}^{2^{k-1}} f(\xi_{k,j}) = \int_f (x)\, d\mu(s) + \mathcal{O}\left(\|f'\|_\infty \frac{1}{\sqrt{5}^k} \right) \tag{3.10}$$

for any differentiable function f, where $\|f'\|_\infty$ denotes the supremum of the derivative of f on $[, \max]$. The Hausdorff dimension of satisfies $\alpha \leq \dim() \leq 2\alpha$. Furthermore, for any interval I_ε of length $\varepsilon > 0$ we have

$$\mu(\cap I_\varepsilon) \leq C_1 \varepsilon^\alpha \tag{3.11}$$

for some positive constant C_1.

Proof of the Lemma. Let $\eta_{m,1} < \eta_{m,2} < \cdots < \eta_{m,2^m}$ be the elements of P_m. Then by the monotonicity of the two branches of $g^{(-1)}$

$$g_1(x) = \frac{3 - \sqrt{9 + 16x}}{8} \quad \text{and} \quad g_2(x) = \frac{3 + \sqrt{9 + 16x}}{8}$$

we have

$$\eta_{k+1,j} = \begin{cases} g_1(\eta_{k,j}) & \text{for } 1 \leq j \leq 2^k \\ g_2\left(\eta_{k,2^{k+1}+1-j}\right) & \text{for } 2^k + 1 \leq j \leq 2^{k+1}. \end{cases}$$

[2] Clearly such estimates could also be obtained by using (3.6) and estimating power series.

Induction proves that

$$\eta_{k+1,2j-1} < \eta_{k,j} < \eta_{k+1,2j} \quad \text{for} \quad 1 \leq j \leq 2^k$$

and

$$\eta_{k+1,2j} - \eta_{k+1,2j-1} = \mathcal{O}\left(\frac{1}{\sqrt{5}^k}\right)$$

uniformly in j. We note that $\sqrt{5}$ is the lower bound for the absolute value of the derivative of g in $[-\frac{1}{4}, \frac{3-\sqrt{5}}{8}] \cup [\frac{3+\sqrt{5}}{8}, 1]$. Since the derivative of ψ does not vanish in $[-3, 0]$, we have

$$\xi_{k,j} - \xi_{k+1,2j-1} = \mathcal{O}\left(\frac{1}{\sqrt{5}^k}\right) \quad \text{and} \quad \xi_{k+1,2j} - \xi_{k,j} = \mathcal{O}\left(\frac{1}{\sqrt{5}^k}\right) \tag{3.12}$$

uniformly in $1 \leq j \leq 2^{k-1}$ (where we assume the $\xi_{k,j}$ to be ordered).

Thus we have

$$\begin{aligned} T_{k+1}(f) - T_k(f) &= \frac{1}{2^k} \sum_{j=1}^{2^{k-1}} \left(f(\xi_{k+1,2j-1}) - f(\xi_{k,j}) + f(\xi_{k+1,2j}) - f(\xi_{k,j})\right) \\ &= \mathcal{O}\left(\|f'\|_\infty \frac{1}{\sqrt{5}^k}\right) \end{aligned}$$

and (3.10) is proved.

The assertion on the Hausdorff dimension is a consequence of [Be, Thm 10.3.1] (for the lower bound) and of (3.12) for the upper bound. The estimate (3.11) follows immediately. □

Applying the result of the lemma to $f(x) = x^{-s}$ we obtain

$$M_k(s) = M(s) + R_k(s)$$

with $M(s) = \int_x^{-s} d\mu(x)$ and an entire function $R_k(s)$, which satisfies $R_k(\sigma + it) = \mathcal{O}_\sigma(|t| 5^{-\frac{k}{2}})$ and $R_m(0) = 0$. Inserting this into (3.8) we obtain

$$\begin{aligned} \nu(s) &= \frac{\xi_{0,1}^{-s}}{5^s - 1} + \frac{1}{5^s - 1} \frac{1}{5^s - 2} M(s) - \frac{1}{5^s - 1} R(s) \\ R(s) &= \sum_{m=1}^{\infty} 5^{-ms} 2^{m-1} R_m(s), \end{aligned} \tag{3.13}$$

where the sum converges for $\Re s > \alpha - \frac{1}{2}$.

Since the function $\psi(z)$ has positive power series coefficients and it is known by [B-P, Proposition 3.1] that $|\phi(z)| \geq \exp(-c|z|^\alpha)$, $\psi(z)$ is of order α. By classical results (cf. [Bo]) $\psi(z)$ can be written as a Weierstraß product

$$\psi(z) = \prod_{\psi(-\lambda)=0} \left(1 + \frac{z}{\lambda}\right). \tag{3.14}$$

We now compute the Mellin transform of the logarithm of $\psi(z)$:

$$\int_0^\infty \log\psi(x)x^{s-1}\,dx = \frac{\pi}{s\sin\pi s}\nu(-s) \quad \text{for} \quad -1 < \Re s < -\alpha.$$

The idea of studying functions of finite order by analyzing the Mellin transform of their logarithm goes back to Mellin [Me]. We now use Mellin's inversion formula (cf. [Do], [Iv, Appendix 2]) to produce an asymptotic expansion of $\log\psi(z)$ with a $o(1)$ error term to obtain (3.2). For $|\arg z| < \pi$ we have

$$\log\psi(z) = \frac{1}{2\pi i}\int\limits_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} \frac{\pi}{s\sin\pi s}\nu(-s)z^{-s}\,ds.$$

By shifting the line of integration to $\Re s = \frac{1}{2}-\alpha-\varepsilon$ and taking the residues into account (notice that the the integrand is tending to 0 like $e^{-(\pi-\theta)|\Im s|}$) we obtain

$$\begin{aligned}\log\psi(z) =& z^\alpha\sum_{k\in\mathbb{Z}} g_k e^{2k\pi i\frac{\log z}{\log 5}} + A\log z + \sum_{k\in\mathbb{Z}} h_k e^{2k\pi i\frac{\log z}{\log 5}} + \\ & \frac{1}{2\pi i}\int\limits_{\frac{1}{2}-\alpha-\varepsilon-i\infty}^{\frac{1}{2}-\alpha-\varepsilon+i\infty} \frac{\pi}{s\sin\pi s}\nu(-s)z^{-s}\,ds,\end{aligned}$$

where the terms of order z^α originate from the poles at the roots of $5^{-s}=2$ and the logarithmic terms are due to the second order pole at $s=0$. Notice that the first order poles at $s=0$ of the two first terms in (3.13) cancel since $M(0)=1$ and the third term has no pole in $s=0$ because all the R_m's vanish there. The coefficient A of $\log z$ is given by

$$1+\frac{M'(0)}{\log 5}+\frac{R'(0)}{\log 5}+\frac{\log\xi_{0,1}}{\log 5} = 0.48098676974525901234\ldots.$$

Similar expressions involving evaluations of R can be given for the Fourier coefficients of H. The remaining integral is estimated trivially to obtain the error term indicated in the theorem. Notice that this estimate depends on the value of θ. □

4. Distribution of the Zeros of ψ

By classical theorems (cf. [Bo]) it is well known that the number of zeros of ψ of modulus less than x is of order of magnitude x^α. Since the function $\psi(z)$ fails to have a proximate order (cf. [Le], [B-G-T]) the theorems of Levin-Pfluger and Valiron-Titchmarsh (cf. [B-G-T, chapter 7]), which give a precise relation between the proximate order of an entire function and the distribution of its zeros cannot be applied. Therefore the proof of the following theorem will make use of the precise description of the zeros in (3.7).

Theorem 3. *The zeros of ψ satisfy*

$$\sum_{\substack{\psi(-t)=0\\ t<x}} 1 = x^\alpha K\left(\frac{\log x}{\log 5}\right) + \mathcal{O}\left(x^{\frac{\alpha+3\alpha^2}{2+3\alpha}}\right), \tag{4.1}$$

where $K(s)$ is a periodic continuous singular function of period 1.[3] *As a consequence of this asymptotic behaviour of the counting function the n-th zero λ_n satisfies*

$$C_2 n^{\frac{1}{\alpha}} \le \lambda_n \le C_3 n^{\frac{1}{\alpha}} \tag{4.2}$$

for some positive constants C_2, C_3 and sufficiently large n.

Remark. The proof of this theorem will make use of a Berry-Esseen type inequality for the difference between the counting function of the $\xi_{k,j}$'s and the measure μ. Since μ is a probability measure supported on a fractal set, it cannot be absolutely continuous, and therefore the usual Berry-Esseen inequalities cannot be used, as they use the absolute continuity of at least one of the two measures they compare.

The more usual way to prove estimates for the growth of the zeros would be to apply Tauberian theorems to the Dirichlet generating function $\nu(s)$, which is impossible, because there are infinitely many poles on the line $\Re s = \alpha$. Applying the Mellin-Perron summation formula (cf. [Ap], [Iv], [Te]) would only yield an asymptotic formula for

$$\sum_{\substack{\psi(-t)=0\\ t<x}} \left(1 - \frac{t}{x}\right),$$

which is certainly weaker than our theorem.

Proposition 2. *Let $f_1(x)$ and $f_2(x)$ be two probability distribution functions with their Fourier-Stieltjes transforms defined by*

$$\widehat{df}_k(t) = \int_{-\infty}^{\infty} e^{-2\pi i t x}\, df_k(x), \quad k = 1, 2.$$

Suppose that $(\widehat{df}_1(t) - \widehat{df}_2(t))t^{-1}$ is integrable on a neighbourhood of zero and f_2 satisfies

$$|f_2(x) - f_2(y)| \le C_4 |x - y|^{\beta}$$

for some $0 < \beta < 1$. Then the following inequality holds for all real x and all $\delta > 0$

$$\begin{aligned} &\left| f_1(x) - f_2(x) - \int_{-\delta}^{\delta} \hat{J}(\delta^{-1}t)(2\pi i t)^{-1} \left(\widehat{df}_1(t) - \widehat{df}_2(t)\right) e^{2\pi i x t}\, dt \right| \\ &\le \left(C_4 + \frac{1}{\pi^2}\right) \delta^{-\frac{2\beta}{2+\beta}} + \frac{1}{2\delta} \int_{-\delta}^{\delta} \left(1 - \frac{|t|}{\delta}\right) \left(\widehat{df}_1(t) - \widehat{df}_2(t)\right) e^{2\pi i x t}\, dt, \end{aligned} \tag{4.3}$$

where

$$\hat{J}(t) = \pi t(1 - |t|) \cot \pi t + |t|.$$

[3] The term “singular” refers to the fact, that the function is not the integral of its derivative.

Corollary 2. *The discrepancy of the counting function*

$$A_k(x) = \frac{1}{2^{k-1}} \sum_{\substack{1 \le j \le 2^{k-1} \\ \xi_{k,j} < x}} 1$$

and the measure $\mu([0,x])$ can be estimated as follows

$$\sup_x |A_k(x) - \mu([0,x])| = \mathcal{O}\left(5^{-\frac{\alpha}{2+3\alpha}k}\right).$$

Proof of the Proposition. The proof will follow the proof of the Berry-Esseen inequality in [Va], thus we will restrict ourselves to the points, where different ideas have to be used. Vaaler's proof of the usual Berry-Esseen inequality makes use of the Beurling-Selberg extremal function

$$B(z) = \left(\frac{\sin \pi z}{\pi}\right)\left(\sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} + \frac{2}{z}\right), \tag{4.4}$$

which minimizes the integral

$$\int_{-\infty}^{\infty} (F(x) - \operatorname{sgn}(x))\, dx$$

for all entire functions $F(z)$ of exponential type 2π satisfying $F(x) \ge \operatorname{sgn}(x)$. In [Va] this function and related functions are used to construct integral kernels for Fourier analysis, which yield almost best possible approximation for characteristic functions of intervals.

The proof uses the fact that the functions

$$J_\delta(z) = \int_{-\delta}^{\delta} \hat{J}(\delta^{-1}t) e^{2\pi i z t}\, dt$$

$$K_\delta(z) = \left(\frac{\sin \pi\delta z}{\pi\delta z}\right)^2$$

have the property that for any increasing function f we have

$$f(x) \le f * J_\delta(x) + \frac{1}{2\delta}(df) * K_\delta(x), \tag{4.5}$$

where $*$ denotes convolution. This yields the inequality

$$\begin{aligned} f_1(x) - f_2(x) \le & (f_1 - f_2) * J_\delta(x) + \frac{1}{2\delta}(df_1 - df_2) * K_\delta(x) + \\ & f_2 * J_\delta(x) + \frac{1}{2\delta}(df_2) * K_\delta(x) - f_2(x). \end{aligned} \tag{4.6}$$

The first two terms on the right hand side yield the two Fourier inversion integrals in (4.3) by the same arguments as used in [Va]. The three remaining terms have to be estimated by different means.

As in [Va] we have

$$f_2 * J_\delta(x) + \frac{1}{2\delta}(df_2) * K_\delta(x) - f_2(x) =$$
$$\frac{1}{2}\int\limits_{-\infty}^{\infty} (B(\delta(x-\xi)) - \operatorname{sgn}(\delta(x-\xi)))\, df_2(\xi).$$

Using the fact that $B(x) - \operatorname{sgn}(x) \leq K(x)$ for all real x [Va, Lemma 5], we estimate the last term by

$$\int\limits_{-\infty}^{\infty} K(\delta(x-\xi))\, df_2(\xi). \tag{4.7}$$

We now insert the estimate

$$K(\delta x) \leq \begin{cases} 1 & \text{for } |x| \leq \delta^{-\frac{2}{2+\beta}} \\ \frac{1}{\delta^2\pi^2|x|^2} & \text{for } |x| > \delta^{-\frac{2}{2+\beta}} \end{cases}$$

into (4.7) to obtain the upper bound

$$\int\limits_{|x-\xi|\leq\delta^{-\frac{2}{2+\beta}}} df_2(\xi) + \int\limits_{|x-\xi|>\delta^{-\frac{2}{2+\beta}}} \frac{1}{\delta^2\pi^2(x-\xi)^2}\, df_2(\xi). \tag{4.8}$$

The first integral is estimated by $C_4\delta^{-\frac{2\beta}{2+\beta}}$ using our assumptions on the f_2-measure of short intervals. The second integral is estimated by $\frac{1}{\pi^2\delta^2\cdot\delta^{-\frac{4}{2+\beta}}} = \frac{1}{\pi^2}\delta^{-\frac{2\beta}{2+\beta}}$, which proves the proposition. □

Proof of the Corollary. We will use the estimates provided by Lemma 1 for the difference of the Fourier-Stieltjes transforms

$$\left|\widehat{dA_k}(t) - \widehat{d\mu}(t)\right| = \mathcal{O}\left(|t|5^{-\frac{k}{2}}\right).$$

We now set $f_1(x) = A_k(x)$ and $f_2(x) = \mu([0,x])$. The assumption on the distribution function f_2 with $\beta = \alpha$ is satisfied by (3.11). Inserting these estimates into Proposition 2 we derive

$$\sup_x |A_k(x) - \mu([0,x])| = \mathcal{O}\left(5^{-\frac{k}{2}}\right) + \mathcal{O}\left(\delta 5^{-\frac{k}{2}}\right) + \mathcal{O}\left(\delta^{-\frac{2\alpha}{2+\alpha}}\right).$$

Choosing $\delta = 5^{\frac{2+\alpha}{4+6\alpha}k}$ we derive the desired estimate. □

We have now completed all preparations for the proof of Theorem 3.

Proof of the theorem. Throughout this prove we will use $[x]$ to denote the integer part of x and $\{x\}$ to denote the fractional part of x, such that $x = [x] + \{x\}$. Set $m_1 =^{(-1)} (\frac{3-\sqrt{5}}{8})$ and $m_2 = \max = \psi^{(-1)}(-\frac{1}{4})$. These values can be easily computed by *Maple*:

$$m_1 = 1.14909680368976601320 2\ldots, \quad m_2 = 1.86286091513982018242\ldots.$$

We set $X = \frac{\log x - \log m_2}{\log 5}$ and use the description of the zeros by (3.7) to derive
(4.9)

$$S(x) = \sum_{\substack{t<x \\ \psi(-t)=0}} 1 = \left[\frac{\log x - \log \xi_{0,1}}{\log 5}\right] + \sum_{n+m\le X} 2^{m-1} + \sum_{n+m=[X]+1} \sum_{j=1}^{2^{m-1}} \chi_{[0,x\cdot 5^{-T-1})}(\xi_{m,j}),$$

where the first term on the right hand side originates from the zeros given by $-\xi_{0,1}\cdot 5^n$, the first sum collects the values of n and m such that $\xi_{m,j} \le m_2 \le x\cdot 5^{-n-m}$, where j can take all possible values, and the second sum is extended over those values of n and m, where $x\cdot 5^{-n-m} < m_2$. Notice that all the summands in the last sum are zero, if $x \cdot 5^{-[X]-1} < m_1$. We now sum up the first sum and use Corollary 2 to obtain
(4.10)

$$S(x) = \left[\frac{\log \frac{x}{\xi_{0,1}}}{\log 5}\right] - [X] + 2^{[X]} - 1 + \sum_{m=1}^{[X]} 2^{m-1}\left(\mu([0, m_2 5^{\{X\}-1})) + \mathcal{O}\left(5^{-\frac{\alpha}{2+3\alpha}m}\right)\right).$$

The first two terms differ at most by 2 and the basis in the $\mathcal{O}$-term is $0.8101\ldots$. Thus the error term is $\mathcal{O}(5^{\frac{\alpha+3\alpha^2}{2+3\alpha}m})$ and we have

$$\begin{aligned} S(x) &= 2^{[X]}\cdot\left(1+\mu([0,m_2 5^{\{X\}-1}))\right) + \mathcal{O}\left(5^{\eta T}\right) \\ &= 2^X \cdot 2^{-\{X\}}\cdot\left(1+\mu([0,m_2\cdot 5^{\{X\}-1}))\right) + \mathcal{O}\left(x^{\eta}\right) \\ &= m_2^{-\alpha} x^{\alpha}\cdot 2^{-\{X\}}\cdot\left(1+\mu([0,m_2\cdot 5^{\{X\}-1}))\right) + \mathcal{O}\left(x^{\eta}\right), \end{aligned} \tag{4.11}$$

where $\eta = \frac{\alpha+3\alpha^2}{2+3\alpha}$. Notice that the measure μ satisfies (3.11), therefore its distribution function is continuous. The derivative exists and vanishes almost everywhere by the fact, that the measure is supported on a Cantor set; this is a consequence of [Be, Thm 9.8.1]. The function

$$K\left(\frac{\log x}{\log 5}\right) = m_2^{-\alpha}\cdot 2^{-\{X\}}\cdot\left(1+\mu([0,m_2\cdot 5^{\{X\}-1}))\right) \tag{4.12}$$

therefore is proved to be continuous except for possible jump discontinuities in the points, where $\{X\} = 0$. But, in these points the jumps of the first factor are compensated by the jumps of the second factor. Thus the theorem is proved. □

Remark. Numerical experiments (see the plot of $K(t)$ in Figure 7. below) suggest that

$$0.470951109\ldots = \frac{1}{2m_1^{\alpha}} \le K(t) \le \frac{1}{m_2^{\alpha}} = 0.764961577\ldots,$$

which would imply that the constants in (4.2) could be chosen as

$$C_2 < m_2 = 1.8628609\ldots \quad C_3 > 5m_1 = 5.745484018\ldots$$

for n sufficiently large.

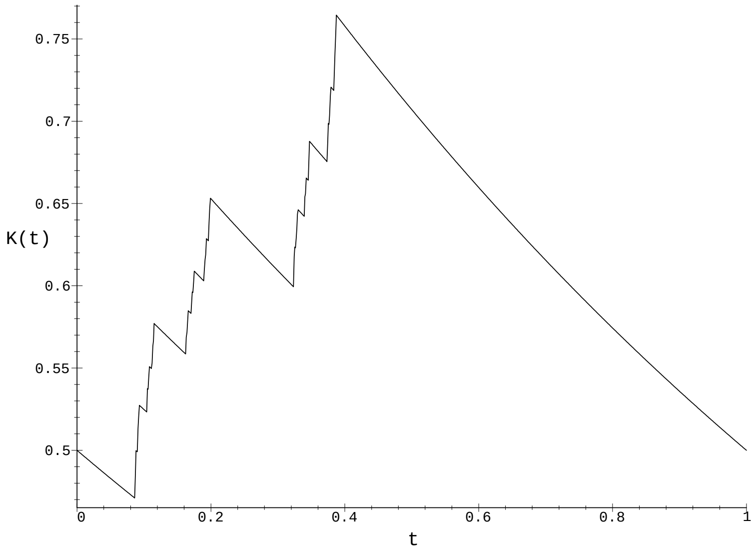


FIGURE 7.

5. Properties of the Distribution Function of T

In this section we will prove that the distribution function can be given by the infinite sum (1.3). In the classical case of Brownian motion on the unit interval this expansion can be derived from the eigenfunction expansion of the transition density (cf. [I-M, p.31]). This makes it plausible that the values $-\lambda_n$ are (a subset of) the eigenvalues of the Laplacian on the geodesic unit ball on the gasket (corresponding to eigenfunctions with boundary values 0).

Although the eigenfunction expansion

$$p_t(x,y)=\sum_{\mu} e^{-\mu t}\phi_\mu(x)\phi_\mu(y)$$

(μ running through all the eigenvalues of the Laplacian, and ϕ_μ being the corresponding normalized eigenfunctions) exists as a consequence of the general theory of semigroups of operators (cf. [Ru], [Yo]) and the Hilbert-Schmidt theory of symmetric integral operators (cf. [R-N]), current knowledge of the properties of the eigenfunctions does not allow to conclude (1.3). Furthermore, we will prove an asymptotic expression for $\mathbb{P}(T<t)$ for $t\to 0$, which refines the upper and lower estimates for this quantity given in [Bi] and [B-P].

Theorem 4. *The distribution function of the random variable T is given by*

$$1-\sum_{\psi(-\lambda)=0}\frac{1}{\lambda\psi'(-\lambda)}e^{-\lambda t}. \tag{5.1}$$

The series is uniformly convergent for $t\geq 0$.

The proof will make use of several lemmas.

Lemma 2. *The function ψ satisfies*

$$|\psi(-\sigma+it)|\geq|\psi(-\sigma)|\exp\left(C_5t^2\sigma^{-2+\alpha}\right)$$

for $|t|\leq C_6\sigma$, where the constant $C_5>0$ only depends on C_2, C_3 and C_6.

Proof. We apply the following estimates to the product expansion (3.14)

$$\left|1-\frac{\sigma}{\lambda}+\frac{it}{\lambda}\right| \geq \begin{cases} \left|1-\frac{\sigma}{\lambda}\right| & \text{for } \lambda < \sigma \\ \left|1-\frac{\sigma}{\lambda}\right|\sqrt{1+\frac{t^2}{\lambda^2}} & \text{for } \lambda \geq \sigma. \end{cases}$$

Choose C_5' such that

$$1+t^2 \geq \exp(C_5' t^2) \quad \text{for } 0 \leq |t| \leq C_6$$

and insert the two inequalities into (3.14) to obtain

$$|\psi(-\sigma+it)| \geq |\psi(-\sigma)| \exp\left(\frac{C_5'}{2} t^2 \sum_{\lambda>\sigma} \frac{1}{\lambda^2}\right). \tag{5.2}$$

Applying (4.2) we have

$$\sum_{\lambda>\sigma} \frac{1}{\lambda^2} \geq \sum_{n>\left(\frac{\sigma}{C_2}\right)^\alpha} \frac{1}{C_3^2 n^{\frac{2}{\alpha}}} \geq C_5'' \sigma^{-2+\alpha} \tag{5.3}$$

and putting (5.2) and (5.3) together and setting $C_5 = \frac{C_5' C_5''}{2}$ yields the desired estimate. $\square$

Lemma 3. *For $|\arg z| \leq \frac{3\pi}{4}$ the following inequality holds for some positive constant C_7*

$$\Re\left(z^\alpha G\left(\frac{\log z}{\log 5}\right)\right) \geq C_7 |z|^\alpha.$$

Proof. We write $z = re^{i\varphi}$ and insert this into the Fourier series for $G(s)$ to obtain

$$\Re\left(e^{i\alpha\varphi} G\left(\frac{\log r}{\log 5}+\frac{i\varphi}{\log 5}\right)\right) = g_0 \cos\alpha\varphi + \sum_{k\in\mathbb{Z}\setminus\{0\}} \Re\left(g_k e^{i\alpha\varphi} e^{2k\pi i \frac{\log r}{\log 5}}\right) e^{-\frac{2k\pi}{\log 5}\varphi}$$

$$\geq g_0 \cos\alpha\varphi - \frac{3\pi}{2} \cdot M(-\alpha) \left(\sum_{k=1}^{\infty} e^{-\frac{2\pi}{\log 5}(\pi-\varphi)k} + e^{-\frac{2\pi}{\log 5}(\pi+\varphi)k}\right)$$

$$\geq 1.959 \cdot \cos\alpha\varphi - 3.978 \cdot \left(\frac{1}{e^{\frac{2\pi}{\log 5}(\pi-\varphi)}-1} + \frac{1}{e^{\frac{2\pi}{\log 5}(\pi+\varphi)}-1}\right).$$

The last expression is easily shown by *Maple* to be greater than 0.8 in the interval $-\frac{3\pi}{4} \leq \varphi \leq \frac{3\pi}{4}$. $\square$

Lemma 4. *The function $\psi(t)$ attains a value of modulus $\geq \frac{9}{16}$ in every interval between two consecutive zeros.*

Proof. Clearly, the derivative of $\psi(z)$ vanishes between two consecutive zeros by Rolle's theorem. Thus we differentiate (3.1) and ask for conditions for the vanishing of the derivative of ψ. The relation

$$5\psi'(5z) = \psi'(z)\,(8\psi(z)-3) \tag{5.4}$$

shows that the derivative of ψ at $5z$ can only vanish, if either $\psi(z) = \frac{3}{8}$, or $\psi'(z) = 0$. The first possibility implies that $\psi(5z) = -\frac{9}{16}$ and we are done. In the second case we iterate this argument to obtain

$$\psi'(5z) = \psi'(z) = \cdots = \psi'(5^{-k+1}z) = 0, \quad \psi(5^{-k}z) = \frac{3}{8},$$

for some $k \geq 1$. This implies that $\psi(5z) = g^{(k+1)}(\frac{3}{8})$, and it is an easy exercise to show that $|g^{(k+1)}(\frac{3}{8})| > \frac{9}{16}$. $\square$

Proof of the theorem. We first notice that $\phi(z) = \frac{1}{\psi(z)}$ is the Laplace transform of the density function of the random variable T. Thus $\frac{1}{z\psi(z)}$ is the Laplace transform of the distribution function of T. By the Laplace inversion theorem (cf. [Do], [Wi]) we have

$$\Phi(t) = \mathbb{P}(T < t) = \frac{1}{2\pi i} \int\limits_{1-i\infty}^{1+i\infty} \frac{e^{st}}{s\psi(s)}\, ds. \tag{5.5}$$

We shift the line of integration to the left to $\Re s = -\sigma$ and collect residues at the zeros of ψ. Thus we have

$$\Phi(t) = 1 - \sum_{\substack{\lambda<\sigma \\ \psi(-\lambda)=0}} \frac{1}{\lambda\psi'(-\lambda)} e^{-\lambda t} + \frac{1}{2\pi i} \int\limits_{-\sigma-i\infty}^{-\sigma+i\infty} \frac{e^{st}}{s\psi(s)}\, ds. \tag{5.6}$$

We choose σ such that $|\psi(-\sigma)| \geq \frac{9}{16}$ and the sum is not altered. This is possible by Lemma 4. We split the range of integration into two parts: for $\tau = \Im s$ we estimate the integral by

$$\left| \frac{1}{2\pi i} \int\limits_{|\tau|\leq\sigma} \frac{e^{st}}{s\psi(s)}\, ds \right| + \left| \frac{1}{2\pi i} \int\limits_{|\tau|>\sigma} \frac{e^{st}}{s\psi(s)}\, ds \right|. \tag{5.7}$$

We use Lemma 2 and Lemma 4 to estimate the first integral by

$$e^{-\sigma t} \frac{16}{9} \frac{1}{2\pi\sigma} \int_{-\sigma}^{\sigma} \exp\left(-C_5 \tau^2 \sigma^{-2+\alpha}\right)\, d\tau \leq e^{-\sigma t} \sqrt{\frac{\pi}{C_5}} \sigma^{-\frac{\alpha}{2}}. \tag{5.8}$$

An application of Lemma 3 to the second integral yields the upper estimate

$$\frac{e^{-\sigma t}}{\pi} \int_{\sigma}^{\infty} \exp\left(-C_7 \tau^{\alpha}\right) \frac{d\tau}{\tau} \leq e^{-\sigma t} \frac{1}{C_7 \pi \alpha \sigma^{\alpha}} \exp\left(-C_7 \sigma^{\alpha}\right) \tag{5.9}$$

for the second integral.

Putting (5.8) and (5.9) together and estimating the terms $e^{-\sigma t}$ by 1 we obtain the desired uniform convergence for $\sigma \to \infty$. $\square$

Finally, we prove a theorem, which gives precise information on the behaviour of the distribution function for $t \to 0+$. We note here that J.D. Biggins and N.H. Bingham gave the first term in the asymptotic expansion of $\log \Phi(t)$ for a general branching process (cf. [B-B2, Theorem 3]).

Theorem 5. *The distribution function $\Phi(t)$ has the following asymptotic expansion for $t \to 0+$*

$$\Phi(t) = \exp\left(-t^{-\frac{\alpha}{1-\alpha}} Q\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right) - B\log\frac{1}{t} - S\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right)\right)\left(1+\mathcal{O}\left(t^{\frac{1}{10}}\right)\right), \tag{5.10}$$

where Q and S are periodic continuous functions of period 1 *and*

$$B = \frac{A+1}{1-\alpha} - \frac{2-\alpha}{2(1-\alpha)} = 1.22307461365998120057\ldots,$$

where A is the constant which occurred in (3.2)*.*

Remark. Lower and upper bounds for $Q(s)$ can be given in terms of estimates for the functions $G(s)$ and $G'(s)$. Inserting those estimates into the expression for Q in (5.15) below yields

$$0.56 \le Q(s) \le 1.19.$$

Similar estimates for general branching processes are given in [B-B2, Proposition 7].

Proof. We apply the saddle point method to (5.5). We remark here that the saddle point method is one of the standard methods in the theory of large deviations (cf. [El]).

Since the asymptotic expansion (3.2) is uniform with respect to the argument and the function $z^\alpha G(\frac{\log z}{\log 5}) + A\log z + H(\frac{\log z}{\log 5})$ is holomorphic in $|\arg z| < \pi$, we can apply the Laplace inversion formula to (3.2) and then estimate the error term. This yields

(5.11)

$$\begin{aligned}\Phi(t) =& \frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \exp\left(-z^\alpha G\left(\frac{\log z}{\log 5}\right) - (A+1)\log z - H\left(\frac{\log z}{\log 5}\right) + zt\right) dz \times \\ &\times\left(1+\mathcal{O}(\sigma^{\alpha-\frac{1}{2}+\varepsilon})\right).\end{aligned}$$

For choosing the appropriate path of integration we have to solve the equation

$$\sigma^{\alpha-1}\left(\alpha G\left(\frac{\log\sigma}{\log 5}\right) + \frac{1}{\log 5}G'\left(\frac{\log\sigma}{\log 5}\right)\right) + \frac{A+1}{\sigma} + \frac{1}{\sigma}H'\left(\frac{\log\sigma}{\log 5}\right) = t, \tag{5.12}$$

which describes the location of the saddle point by the vanishing of the derivative of the argument of the exponential in (5.11). Since the two last terms on the left hand side tend to 0 and the saddle point turns out to be stable, we solve the simpler equation

$$\begin{aligned}&\sigma^{\alpha-1} L\left(\frac{\log\sigma}{\log 5}\right) = t \quad \text{with}\\ &L\left(\frac{\log\sigma}{\log 5}\right) = \alpha G\left(\frac{\log\sigma}{\log 5}\right) + \frac{1}{\log 5}G'\left(\frac{\log\sigma}{\log 5}\right).\end{aligned} \tag{5.13}$$

By differentiating the logarithm of (3.14) twice we obtain

$$\frac{d^2\log\psi(z)^{-1}}{dz^2} = \frac{d^2\log\phi(z)}{dz^2} = \sum_{\psi(-\lambda)=0}\frac{1}{(z+\lambda)^2},$$

which is positive for all positive real z. This implies that the left hand side of (5.13), which is (up to an error term) the negative first derivative of $\log\phi(z)$, is monotonically decreasing. Therefore (5.13) has exactly one solution. Using the periodicity of L and inserting 5σ into (5.13) we see that $\sigma\to 5\sigma$ corresponds to $t\to\frac{2}{5}t$. Since the solution $\sigma(t)$ depends continuously on t, we have

$$\sigma(t)=t^{-\frac{1}{1-\alpha}}\Lambda\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right) \tag{5.14}$$

for some continuous periodic function Λ. We note that $\sigma(t)$ differs only by $\mathcal{O}(t^{\frac{1}{1-\alpha}})$ from the solution of (5.12).

Observe now that the second derivative of the argument of the exponential is given by

$$z^{\alpha-2}L_1\left(\frac{\log z}{\log 5}\right)-\frac{A+1}{z^2}-\frac{1}{z^2}L_2\left(\frac{\log z}{\log 5}\right)$$

for some continuous periodic functions L_1 and L_2. This shows that the second derivative is tending to 0 like $z^{\alpha-2}$; the same argument shows that the third derivative tends to zero like $z^{\alpha-3}$. Furthermore, the second derivative is positive by the arguments given above.

We now move the line of integration to $\Re z=\sigma(t)$ and replace the argument of the exponential by its Taylor expansion to obtain

(5.15)

$$\begin{aligned}\Phi(t)=\exp&\left\{-t^{-\frac{\alpha}{1-\alpha}}\left[\left(\Lambda\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right)\right)^{\alpha}G\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}+\frac{\log\Lambda\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right)}{\log 5}\right)-\Lambda\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right)\right]\right.\\
&\left.-\frac{A+1}{1-\alpha}\log\frac{1}{t}-H\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}+\frac{\log\Lambda\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right)}{\log 5}\right)\right\}\left(1+\mathcal{O}\left(t^{\frac{1}{1-\alpha}-\frac{7}{5}}\right)\right)\times\\
&\times\frac{1}{2\pi}\left\{\int\limits_{|\tau|<t^{-\frac{7}{5}}}\exp\left(-\frac{\tau^2}{2}t^{\frac{2-\alpha}{1-\alpha}}\left(\Lambda\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right)\right)^{\alpha-2}L_1\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}+\frac{\log\Lambda\left(\frac{\log\frac{1}{t}}{\log\frac{5}{2}}\right)}{\log 5}\right)\right)d\tau\times\right.\\
&\left.\times\left(1+\mathcal{O}\left(t^{\frac{3}{10}}\right)\right)+\int_{|\tau|\geq t^{-\frac{7}{5}}}\frac{1}{(\sigma(t)+i\tau)\psi(\sigma(t)+i\tau)}e^{it\tau}\,d\tau\right\}\left(1+\mathcal{O}\left(t^{\frac{1}{10}}\right)\right).\end{aligned}$$

Notice that the errors originating from the first and third derivative are bounded by $\mathcal{O}(t^{\frac{1}{1-\alpha}-\frac{7}{5}})$ and $\mathcal{O}(t^{\frac{3-\alpha}{1-\alpha}-\frac{21}{5}})=\mathcal{O}(t^{\frac{3}{10}})$, respectively. The factor of $t^{-\frac{\alpha}{1-\alpha}}$ has to be positive, since $\Phi(t)$ must tend to zero as $t\to 0$. We equate the first integral by extending the range of integration to infinity and observing that the error is exponentially small. We put all the periodic functions together to obtain (5.10). It remains to bound the second integral.

We use Lemma 3 to estimate

$$\left|\int_{|\tau|\geq t^{-\frac{7}{5}}}\frac{1}{(\sigma(t)+i\tau)\psi(\sigma(t)+i\tau)}e^{it\tau}\,d\tau\right|\leq\int\limits_{|\tau|\geq t^{-\frac{7}{5}}}\frac{1}{|\tau|}\exp\left(-C_7|\tau|^{\alpha}\right)d\tau. \tag{5.16}$$

The integral can be bounded by

$$\frac{2}{\alpha C_7} t^{\frac{7\alpha}{5}} \exp\left(-C_7 t^{-\frac{7\alpha}{5}}\right).$$

Inserting this estimate into (5.15) we derive that the last summand in (5.15) tends to zero, as $t \to 0$. $\square$

Remark. The same procedure as above yields similar asymptotic estimates for all derivatives of $\Phi(t)$.

6. Concluding Remarks

The same ideas as described above can be used for the description of stopping times for the Brownian motion on d-dimensional Sierpiński spaces as introduced in [Ki]. The functional equation for the Laplace-Stieltjes transform of the distribution function of the hitting time of the boundary of the unit ball is given by

$$\phi\left((d+3)z\right) = \frac{\phi(z)^2}{2d - (3d-3)\phi(z) + (d-2)\phi(z)^2}, \quad \phi(0) = 1, \quad \phi'(0) = -1.$$

The corresponding function $g(z)$, whose iterations have to be studied, is given by $g(z) = 2dz^2 - (3d-3)z + d - 2$.

It was observed in [B-P] that in the case $d = 1$ (Brownian motion on the real line) the functional equation reads

$$\phi(2z) = \frac{\phi(z)^2}{2 - \phi(z)^2}, \quad \phi(0) = 1, \quad \phi'(0) = -1,$$

which has the solution $\phi(z) = \frac{1}{\cosh\sqrt{2z}}$. This is in accordance with classical results (cf. [I-M]).

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P. Grabner
Institut für Mathematik A
Technische Universität Graz
Steyrergasse 30
8010 Graz, Austria
E-mail address: grabner@weyl.math.tu-graz.ac.at