Constructive Approximation **21** (2005), 149–179

DIGITAL SUMS AND DIVIDE-AND-CONQUER RECURRENCES: FOURIER EXPANSIONS AND ABSOLUTE CONVERGENCE

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Abstract. We propose means for computing the Fourier expansions of periodic functions appearing in higher moments of the sum-of-digits function and in the solutions of some divide-and-conquer recurrences. The expansions are shown to be absolutely convergent. We also give a new approach to efficiently compute numerically the coefficients involved to high precision.

1. Introduction

Let $\nu(n)$ denote the number of 1's in the binary representation of n. Properties of this function have been extensively studied in the literature due partly to its natural and frequent appearance in many concrete problems in diverse fields; see [16] and [42] and the references therein. For more examples, see [1], [2], [5], [7], [8], [12], [20], [34].

The well-known Trollope-Delange formula (see [13], [46]) for the sum function of $\nu(n)$ has attracted much attention in the literature since it represents one of the most concrete examples of producing continuous but nowhere differentiable functions in analysis: for $n \geq 1$,

$$\begin{aligned} n^{-1}S(n) &:= n^{-1} \sum_{0\leq k<n} \nu(k) \\ &= \frac{1}{2}\log_2 n + F_1(\log_2 n), \end{aligned}$$

where $F_1(x)$ is a continuous, nowhere differentiable periodic function with period 1 whose Fourier expansion can be written as

$$F_1(x) = \frac{\omega \log \pi}{2} - \frac{\omega}{2} - \frac{1}{4} - \omega \sum_{k\neq 0} \frac{\zeta(\chi_k)}{\chi_k(\chi_k+1)}\, e^{2k\pi i x} \qquad (x \in \mathbb{R}),$$

where, *here and throughout this paper*, $\omega := 1/\log 2$, $\zeta(s)$ denotes Riemann's zeta function and $\chi_k := 2k\pi i\omega$.

If we assume that the first n nonnegative integers are equally likely, then $n^{-1}S(n)$ represents the mean value of the random variable X_n, counting the number of 1's in the binary representation of a random integer. The m-th moment of X_n is then given by $n^{-1}S_m(n)$, where

$$S_m(n) := \sum_{0\leq k<n} \nu^m(k) \qquad (m \geq 1).$$

For more information on probabilistic models for digital arithmetic functions, see [33].

Coquet [11] showed that for $n, m \geq 1$,

$$n^{-1}S_m(n) = 2^{-m}(\log_2 n)^m + \sum_{0\leq j<m} (\log_2 n)^j G_{m,j}(\log_2 n), \tag{1.1}$$

Date: February 14, 2008.

2000 *Mathematics Subject Classification.* Primary: 11K16; Secondary: 11A63, 11M41, 30B50.

Key words and phrases. Sum of digits, divide-and-conquer recurrences, Mellin-Perron formula.

† This author is supported by the START project Y96-MAT of the Austrian Science Fund.

where $G_{m,j}(x)$ are periodic functions with period unity and satisfy the recurrence

$$\sum_{j\le \ell<m}\left(2\binom{\ell}{j-1}G_{m,\ell}(x)-\binom{m}{\ell}G_{\ell,j-1}(x)\right)=\left(2^{1-j}-2^{1-m}\right)\binom{m}{j-1},$$

for $1\le j<m$. This inductive formula makes it possible to express all functions $G_{m,j}(x)$ in terms of $F_j(x):=G_{j,0}(x)$ for $j=1,\dots,m$. In particular,

$$\begin{aligned}
G_{m,m-1}(x)&=m2^{-m-1}\left(m-1+4F_1(x)\right) \qquad (m\ge 1),\\
G_{m,m-2}(x)&=\binom{m}{2}2^{-m-2}\left(m^2-5m+6+8(m-2)F_1(x)+16F_2(x)\right) \qquad (m\ge 2),\\
G_{m,m-3}(x)&=\binom{m}{3}2^{-m-3}\Big((m-3)(m^2-9m+16)+12(m-3)(m-4)F_1(x)\\
&\qquad +48(m-3)F_2(x)+64F_3(x)\Big) \qquad (m\ge 3),
\end{aligned}$$

so that

$$\mathrm{Var}(X_n)=\frac{1}{4}\log_2 n+F_2(\log_2 n)-F_1^2(\log_2 n).$$

See [11], [30], and [38] for more details.

Formula (1.1) was also derived using different approaches by several authors; see [14], [22], [36], and [37]. Continuity and non-differentiability of the $G_{m,j}$'s and similar periodic functions occurring in the study of digital arithmetic functions are discussed in [44]. Although many properties of the $G_{m,j}$'s are known, the Fourier expansions of the $G_{m,j}$'s remain open. It is the purpose of this paper to propose an analytic approach to derive the Fourier expansions of $G_{m,j}$. The approach is based on Perron-Mellin integrals and differencing argument similar to those used in [16] and [27]; the hard parts are the detailed estimates of the associated exponential sums in order to prove absolute convergence of the Fourier series.

The approach is best described by the second moment for which we present the proof in some detail; the extension to higher moments is straightforward and is only sketched.

Define

$$\xi(x):=\lfloor 2x\rfloor-2\lfloor x\rfloor-\frac{1}{2}=\begin{cases}-\frac{1}{2}, & \lfloor x\rfloor\le x<\lfloor x\rfloor+\frac{1}{2};\\ \frac{1}{2}, & \lfloor x\rfloor+\frac{1}{2}\le x<\lfloor x\rfloor+1,\end{cases}$$

$\xi(x)$ being a periodic step function of period 1. Let

$$\begin{aligned}
V_m(s)&=\frac{1}{2}\sum_{n\ge 1}\nu^m(n)\left(\frac{1}{(2n)^s}-\frac{2}{(2n+1)^s}+\frac{1}{(2n+2)^s}\right)\\
&=-\frac{s}{2^s}\int_1^\infty \frac{\nu^m(x)}{x^{s+1}}\,\xi(x)\,\mathrm{d}x \qquad (m=0,1,\dots),
\end{aligned} \tag{1.2}$$

for $\Re(s)>-1$, where $\nu(x):=\nu(\lfloor x\rfloor)$.

Theorem 1. *For $n\ge 1$*

$$n^{-1}S_2(n)=\frac{1}{4}(\log_2 n)^2+\left(\frac{1}{4}+F_1(\log_2 n)\right)\log_2 n+F_2(\log_2 n),$$

where $F_2(x)=\sum_{k\in\mathbb{Z}}p_{2,k}e^{2k\pi i x}$ with

$$p_{2,0}=\frac{\omega^2}{2}\left(1-\log\pi-\zeta''(0)\right)-\frac{\omega}{2}\left(\log\pi+4V_1'(0)\right)-\frac{19}{24}, \tag{1.3}$$

and for $k\ne 0$

$$p_{2,k}=\frac{\omega}{\chi_k(\chi_k+1)}\left(\omega\left(\frac{1}{\chi_k}+\frac{1}{\chi_k+1}-\zeta'(\chi_k)\right)+\zeta(\chi_k)-2V_1(\chi_k)\right). \tag{1.4}$$

The Fourier series is absolutely convergent.

Kirschenhofer [30] derived the following expression for the periodic function F_2

$$F_2(x) = 2^{1-\{x\}} \sum_{\substack{j,k\geq 0\\ j\neq k}} \int_0^{2^{\{x\}-1}} \left(\left\lfloor 2^{j+1}u\right\rfloor - 2\left\lfloor 2^j u\right\rfloor - \frac{1}{2}\right)\left(\left\lfloor 2^{k+1}u\right\rfloor - 2\left\lfloor 2^k u\right\rfloor - \frac{1}{2}\right) \mathrm{d}u, \tag{1.5}$$

(see also [38]) which is of a similar form as Delange's formula for F_1 (see [13])

$$F_1(x) = \frac{1-\{x\}}{2} + 2^{1-\{x\}} \sum_{j\geq 0} \int_0^{2^{\{x\}-1}} \left(\left\lfloor 2^{j+1}u\right\rfloor - 2\left\lfloor 2^j u\right\rfloor - \frac{1}{2}\right) \mathrm{d}u. \tag{1.6}$$

In contrast to F_1 whose Fourier expansion can be derived from (1.6) by straightforward calculations, it is not obvious how to derive the Fourier coefficients (1.3) and (1.4) of $F_2(x)$ from (1.5). However, our analytic approach avoids such calculations completely.

While the Fourier coefficients of $F_2(x)$ may seem "recursive" in some sense, the mean value of F_2 gives the first approximation to F_2; see Figure 1. Also in Section 4 we provide new means of computing the Fourier coefficients numerically to high precision, which is also of interests *per se*.

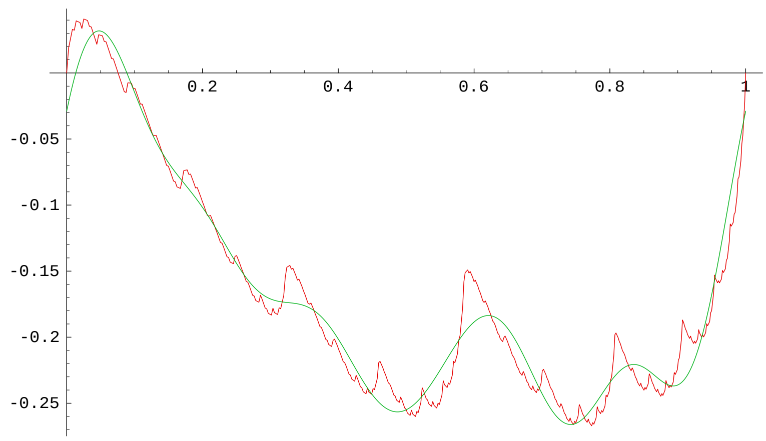

FIGURE 1. *The function $F_2(x)$ compared with the trigonometric polynomial formed with the first six Fourier coefficients.*

We use the following notations for forward and backward differences throughout this paper

$$\nabla f_n = f_n - f_{n-1},$$
$$\Delta f_n = f_{n+1} - f_n.$$

The method of proof starts from the Mellin-Perron approach used in [16] and [27], which consists in first computing the backward differences of $\nu(n)$ and then applying the summation formula (defining $f_0 := 0$)

$$\begin{aligned} n^{-1}\sum_{1\leq k<n} f_k &= n^{-1}\sum_{1\leq k<n}(n-k)\nabla f_k \\ &= \frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} \frac{n^s}{s(s+1)} \sum_{j\geq 1}\nabla f_j j^{-s}\,\mathrm{d}s, \end{aligned} \tag{1.7}$$

for any sequence f_n, where $c_1 > \max\{\sigma_1, 0\}$, σ_1 being the abscissa of absolute convergence of the Dirichlet series $\sum_{j\geq 1}\nabla f_j j^{-s}$. The major difficulty lies in proving the estimate of the

Dirichlet series $V_m(\sigma \pm it)$ for $|t| \to \infty$ because the *a priori* bound $V_m(it) = O(|t|^{1+\varepsilon})$ is not sufficient to guarantee the absolute convergence of the Fourier series and the integral in (1.7).

An essentially equivalent way for handling $S_m(n)$ is to consider the second difference by observing that $S_m(n)$ satisfies a recurrence of the type

$$f_n = f_{\lfloor n/2 \rfloor} + f_{\lceil n/2 \rceil} + g_n \qquad (n \geq 2),$$

with f_1 given, where g_n is some given sequence, and then to apply the summation formula (see [15])

$$\begin{aligned} \frac{f_n}{n} &= f_1 + n^{-1} \sum_{1 \leq k < n} (n-k) \nabla\Delta f_k \\ &= f_1 + \frac{1}{2\pi i} \int_{c_2 - i\infty}^{c_2 + i\infty} \frac{n^s}{s(s+1)} \sum_{j \geq 1} \nabla\Delta f_j j^{-s} \mathrm{d}s, \end{aligned} \tag{1.8}$$

where $c_2 > \max\{\sigma_2, 0\}$, σ_2 being the abscissa of absolute convergence of the Dirichlet series $\sum_{j\geq 1} \nabla\Delta f_j j^{-s}$.

An advantage of this second difference approach is that it admits an extension to more general recurrences of the type

$$f_n = \alpha f_{\lfloor n/2 \rfloor} + \beta f_{\lceil n/2 \rceil} + g_n, \tag{1.9}$$

with suitable initial conditions. Such recurrences appeared often in diverse problems; concrete examples of such recurrences include:

(1) odd numbers in Pascal triangle and more generally the probability generating function of X_n: $n^{-1} \sum_{0\leq k<n} \theta^{\nu(k)}$; see [16], [37], and Section 3.2;
(2) number of comparators used in Bose-Nelson sorting networks; see [4] and Section 3.6;
(3) number of comparators used in Batcher's sorting networks; see [27];
(4) Karatsuba multiplications; see [32];
(5) number of AND-OR gates to simulate AND; see [31], [6], and Section 3.6;
(6) Euclidean matching heuristic; see [39];
(7) period of regularity of 1-additive sequences generated by $(4, v)$, etc; see [5];
(8) recurrences with minimization or maximization; see [20], [28];
(9) Steinhaus problem; [12].

We will discuss some of these in detail. In particular, two different proofs are given of the absolute convergence of the Fourier series of the periodic function $M_\theta(x)$, where $M_\theta(\log_2 n) = n^{-\log_2(1+\theta)} \sum_{0\leq k<n} \theta^{\nu(k)}$, for the range $\sqrt{2} - 1 < \theta < \sqrt{2} + 1$, $\theta \neq 1$.

Apart from the natural application of our approach to the moments of the sum-of-digits function of numbers in q-ary expansion, the associated recurrence being of the form (see [13], [21])

$$f_n = \sum_{0 \leq j < q} f_{\lfloor (n+j)/q \rfloor} + g_n,$$

our approach is also suitable for moments of the number of 1's $g(n)$ in Gray code representation of integers whose underlying recurrence is of the form

$$g(n) = g(\lfloor n/2 \rfloor) + \frac{1}{2}\left(1 - (-1)^{\lceil n/2 \rceil}\right);$$

see [16], [17], [40] for more information. With more calculation, it is also applicable to the Newman-Coquet sequence and other digital sums; see [10], [16], [21], [35].

2. Moments of the sum-of-digits function

2.1. Integral representations and recurrences. Define

$$Y_m(s) := \sum_{n\geq 1} \nabla\nu^m(n) n^{-s} \qquad (m \geq 0).$$

Proposition 1. *For $m, n \geq 1$,*

$$n^{-1} S_m(n) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{n^s}{s(s+1)}\, Y_m(s)\, \mathrm{d}s. \tag{2.1}$$

Proof. Apply (1.7). □

Observe that $V_0(s) = 2 - 2^{-s} - 2(1-2^{1-s})\zeta(s)$ and $Y_0 = 1$.

Lemma 1. *For $m \geq 1$ and $\Re(s) > -1$*

$$Y_m(s) = \frac{2^s - 2}{2^s - 1}\zeta(s) + \frac{1}{2(2^s-1)} \sum_{1\leq j<m} \binom{m}{j} Y_j(s) - \frac{1}{1-2^{-s}} \sum_{1\leq j<m} \binom{m}{j} V_j(s).$$

Proof. By the recurrences

$$\begin{aligned} \nu(2n) &= \nu(n) \\ \nu(2n+1) &= \nu(n) + 1 \end{aligned} \qquad (n \geq 1), \tag{2.2}$$

with $\nu(0) = 0$, we obtain

$$\left.\begin{aligned} \nabla\nu^m(2n) &= \nabla\nu^m(n) - 1 - \sum_{1\leq j<m} \binom{m}{j} \nu^j(n-1) \\ \nabla\nu^m(2n+1) &= \sum_{0\leq j<m} \binom{m}{j} \nu^j(n) \end{aligned}\right. \qquad (n \geq 1);$$

and the Dirichlet series Y_m can be written as

$$\begin{aligned} Y_m(s) = 2^{-s} \sum_{n\geq 1} \nabla\nu^m(n) n^{-s} &+ \sum_{n\geq 0} \left((2n+1)^{-s} - (2n+2)^{-s}\right) \\ &+ \sum_{1\leq j<m} \binom{m}{j} \sum_{n\geq 1} \nabla\nu^j(n) \left((2n+1)^{-s} - (2n+2)^{-s}\right), \end{aligned}$$

which gives

$$Y_m(s) = \frac{1}{1-2^{-s}} \left((1-2^{1-s})\zeta(s) + \sum_{1\leq j<m} \binom{m}{j} \sum_{n\geq 1} \nu^j(n) \left((2n+1)^{-s} - (2n+2)^{-s}\right) \right).$$

The required expression for Y_m then follows from writing

$$\begin{aligned} &(2n+1)^{-s} - (2n+2)^{-s} \\ &\quad = \frac{1}{2}\left((2n)^{-s} - (2n+2)^{-s}\right) - \frac{1}{2}\left((2n)^{-s} - 2(2n+1)^{-s} + (2n+2)^{-s}\right). \end{aligned}$$

□

From the above lemma, we can express $Y_m(s)$ completely in terms of $V_j(s)$ ($j = 0, \dots, m-1$) by considering the exponential generating function of Y_m.

Lemma 2. *For $m \geq 1$*

$$Y_m(s) = 2(2^s - 2)\zeta(s) \sum_{1\leq k\leq m} \frac{k!\mathbf{S}(m,k)}{2^k(2^s-1)^k} - 2^{s+1} \sum_{1\leq h<m} \binom{m}{h} V_{m-h}(s) \sum_{1\leq k\leq h} \frac{k!\mathbf{S}(h,k)}{2^k(2^s-1)^k},$$

where the $\mathbf{S}(n,k)$ denote the Stirling numbers of the second kind.

2.2. **Growth properties of Dirichlet series.** To evaluate the integral in (2.1), we need more analytic properties of $Y_m(s)$. Lemma 2 gives the required analytic continuations of $Y_m(s)$, so we concentrate in this section on growth orders of $Y_m(\sigma \pm i\infty)$, or, equivalently, those of $V_m(\sigma \pm i\infty)$ since the growth order of $\zeta(s)$ at $s = \sigma \pm i\infty$ is well known.

Recall (see (1.2)) that for $\Re(s) > -1$

$$V_m(s) = \frac{1}{2}\sum_{n\geq 1} \nu^m(n) \left(\frac{1}{(2n)^s} - \frac{2}{(2n+1)^s} + \frac{1}{(2n+2)^s}\right).$$

Proposition 2. *For $m \geq 1$*

$$V_m(-\sigma + it) = \begin{cases} O\left(|t|^{1/2}(\log|t|)^{2m+1}\right) & \text{if } \sigma = 0; \\ O\left(|t|^{\sigma+1/2}(\log|t|)^{2m}\right) & \text{if } 0 < \sigma < 1, \end{cases} \tag{2.3}$$

for $|t| \geq t_0$.

Proof. We prove the case $m = 1$ in detail and then sketch the proof in the general case.

By symmetry, assume that $t > t_0 > 0$. Take $L = \left\lfloor \frac{3-2\sigma}{2-2\sigma} \log_2 t \right\rfloor - 1$ and $N = 2^{L+1}$. Write, for notational convenience,

$$\Delta^2(2n)^{-s} := (2n)^{-s} - 2(2n+1)^{-s} + (2n+2)^{-s}.$$

Then

$$\begin{aligned} V_1(-\sigma + it) &= \frac{1}{2}\sum_{1\leq n<N} \nu(n)\Delta^2(2n)^{\sigma-it} + \frac{1}{2}\sum_{n\geq N} \nu(n)\Delta^2(2n)^{\sigma-it} \\ &=: \Upsilon_1 + \Upsilon_2. \end{aligned}$$

By the estimates

$$1 - 2\left(1 + \frac{1}{2n}\right)^{\sigma-it} + \left(1+\frac{1}{n}\right)^{\sigma-it} = O\left(\frac{t^2}{n^2}\right) \qquad (n \gg t),$$

and

$$\sum_{1\leq n\leq x} \nu(n) = O\left(x \log x\right),$$

we have

$$\begin{aligned} \Upsilon_2 &= O\left(t^2 \sum_{n\geq N} \nu(n) n^{\sigma-2}\right) \\ &= O\left(t^2 N^{\sigma-1}\log N\right) \\ &= O\left(t^{\sigma+1/2}\log t\right). \end{aligned}$$

For Υ_1, we use the following expression

$$2\Upsilon_1 = \frac{1}{2\pi}\int_0^{2\pi} V(-\sigma+it, v)\left(\sum_{0\leq n<N} \nu(n)e^{-inv}\right) \mathrm{d}v,$$

where

$$V(s,v) = \frac{1}{2}\sum_{n\ge 1} e^{inv}\left(\frac{1}{(2n)^s} - \frac{2}{(2n+1)^s} + \frac{1}{(2n+2)^s}\right). \tag{2.4}$$

Obviously,

$$2|\Upsilon_1| \le T \max_{0\le v\le 2\pi} |V(-\sigma+it,v)|, \tag{2.5}$$

where

$$T := \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{0\le n<N}\nu(n)e^{-inv}\right| \mathrm{d}v.$$

Using the identity

$$\nu(n) = \sum_{0\le j\le\lfloor\log_2 n\rfloor}\left(\left\lfloor\frac{n}{2^j}\right\rfloor - 2\left\lfloor\frac{n}{2^{j+1}}\right\rfloor\right),$$

we have

$$\begin{aligned}\sum_{0\le n<N}\nu(n)e^{-inv} &= \sum_{0\le j\le L}\sum_{0\le n<N}\left(\left\lfloor\frac{n}{2^j}\right\rfloor - 2\left\lfloor\frac{n}{2^{j+1}}\right\rfloor\right)e^{-inv}\\ &= \sum_{0\le j\le L}\sum_{\substack{0\le n<N\\ n\equiv\{2^j,2^j+1,\dots,2^{j+1}-1\} \bmod 2^{j+1}}} e^{-inv}\\ &= \sum_{0\le j\le L}\frac{e^{-2^j iv}(1-e^{-2^{L+1}iv})}{(1-e^{-iv})(1+e^{-2^j iv})}.\end{aligned}$$

Thus

$$T \le \sum_{0\le j\le L} T_j, \quad \text{where } T_j := \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1-e^{-2^{L+1}iv}}{(1-e^{-iv})(1+e^{-2^j iv})}\right| \mathrm{d}v.$$

The integrals T_j are estimated as follows.

$$\begin{aligned}T_j &= \frac{1}{\pi}\int_0^{2\pi}\left|\frac{\sin(2^L v)}{\sin v\cos(2^{j-1}v)}\right| \mathrm{d}v = \frac{1}{2^{j-1}\pi}\int_0^{2^j\pi}\left|\frac{\sin(2^{L-j+1}v)}{\sin(v/2^j)\cos v}\right| \mathrm{d}v\\ &= \frac{1}{2^{j-1}\pi}\sum_{0\le\ell<2^j}\int_{\ell\pi}^{(\ell+1)\pi}\left|\frac{\sin(2^{L-j+1}v)}{\sin(v/2^j)\cos v}\right| \mathrm{d}v\\ &= \frac{1}{2^{j-1}\pi}\sum_{0\le\ell<2^j}\int_0^{\pi}\left|\frac{\sin(2^{L-j+1}v)}{\sin((v+\ell\pi)/2^j)\cos v}\right| \mathrm{d}v\\ &=: \frac{1}{2^{j-1}\pi}\sum_{0\le\ell<2^j} I_\ell.\end{aligned}$$

For I_0, we have the upper bounds:

$$\begin{aligned}I_0 = O\Bigg(&\int_0^{2^{-L}} 2^L\,\mathrm{d}v + \int_{2^{-L}}^{\pi/2-2^{-L+j}}\frac{\mathrm{d}v}{v(\pi/2-v)}\\ &+ \int_{\pi/2-2^{-L+j}}^{\pi/2+2^{-L+j}} 2^{L-j}\,\mathrm{d}v + \int_{\pi/2+2^{-L+j}}^{\pi}\frac{\mathrm{d}v}{\pi/2-v}\Bigg)\\ = O(L).&\end{aligned}$$

Similarly,

$$I_\ell = O(L-j) = O(L),$$

for $1 \le \ell < 2^j$.

Thus

$$T_j = O(L),$$

uniformly in j, $0 \le j \le L$. It follows that

$$T = O(L^2) = O\left((\log t)^2\right). \tag{2.6}$$

It remains to estimate the function $V(s,v)$ on vertical lines. For that purpose we observe that $V(s,v)$ can be written in terms of the periodic (or Lerch) zeta function (see [29]) $\varphi(s,v) := \sum_{n\ge1} e^{inv} n^{-s}$ as follows.

$$V(s,v) = \frac{1}{2}\left(2^{-s}\left(1-e^{-iv/2}\right)^2 \varphi(s,v) + e^{iv/2}\varphi(s,v/2) - 1\right).$$

From the functional equation of $\varphi(s,v)$ (see [29]) and Stirling's formula, it follows that $\varphi(-\sigma+it,v) = O(|t|^{\sigma+1/2})$ for $\sigma > 0$ and $\varphi(it,v) = O(|t|^{1/2}\log|t|)$. Thus we have the estimate

$$V(-\sigma+it,v) = \begin{cases} O\left(|t|^{1/2}\log|t|\right) & \text{if } \sigma = 0; \\ O\left(|t|^{\sigma+1/2}\right) & \text{if } \sigma > 0, \end{cases} \tag{2.7}$$

uniformly for $v \in \mathbb{R}$. Inserting (2.6) and (2.7) into (2.5) yields the desired estimate for $V_1(-\sigma+it)$.

The general case when $m \ge 2$ can be proved similarly as above using the formula

$$\begin{aligned} \sum_{n<N} \nu^m(n)\Delta^2(2n)^{-s} &= \frac{1}{(2\pi)^m}\int_0^{2\pi}\!\!\cdots\!\int_0^{2\pi} V(s, v_1+\cdots+v_m) \\ &\qquad \times \left(\sum_{0\le n<N}\nu(n)e^{-inv_1}\right)\cdots\left(\sum_{0\le n<N}\nu(n)e^{-inv_m}\right) \mathrm{d}v_1\cdots\mathrm{d}v_m \\ &\le T^m \max_{\substack{0\le v_1\le 2\pi\\ \cdots \\ 0\le v_m\le 2\pi}} \left|V(s, v_1+\cdots+v_m)\right|. \end{aligned}$$

□

2.3. Evaluation of the Mellin-integral.

By (2.1) with $m = 2$, we have

$$n^{-1}S_2(n) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{n^s}{s(s+1)}\, Y_2(s)\,\mathrm{d}s, \tag{2.8}$$

where

$$Y_2(s) = 2^s\frac{2^s-2}{(2^s-1)^2}\,\zeta(s) - \frac{2V_1(s)}{1-2^{-s}}.$$

Note that by (1.2), $V_1(0) = 0$, so that the integrand on the right-hand side of (2.8) has a triple pole at $s = 0$, and double poles at $s = \chi_k$, $k \ne 0$. By the growth property (2.3), we can (by absolute convergence) shift the line of integration to, say $\Re(s) = -1/4$ and by taking into account all residues encountered on the imaginary axis. The result is

$$n^{-1}S_2(n) = \frac{1}{4}(\log_2 n)^2 + \left(\frac{1}{4} + F_1(\log_2 n)\right)\log_2 n + F_2(\log_2 n) + J_n,$$

where

$$J_n := \frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} \frac{n^s}{s(s+1)}\, Y_2(s)\,\mathrm{d}s.$$

We show that the integral is identically zero by applying the following result.

Proposition 3. *Let $q > -1$. Let $U(s) := s2^{-s}\int_1^\infty u(x)\xi(x)x^{-s-1}\,\mathrm{d}x$ for some non-negative, real arithmetic function $u(x) = u_{\lfloor x\rfloor}$. If $U(s)$ satisfies the two conditions:*
(i) $U(s)$ converges for $\Re(s) \geq q-\varepsilon$, where $\varepsilon > 0$;
(ii) $|U(q-\varepsilon+it)| = O(|t|^\delta)$, where $0 < \delta < 1$.
Then the integral

$$\frac{1}{2\pi i}\int_{q-\varepsilon-i\infty}^{q-\varepsilon+i\infty} \frac{n^s}{s(s+1)}\frac{U(s)}{(1-2^{q-s})^k}\,\mathrm{d}s$$

is identically zero for all integers $n, k \geq 1$ and for $\varepsilon > 0$.

Proof. See [27] for the proof of the case $k = 1$; the same proof can be rephrased for any $k \geq 1$. □

2.4. **Higher moments.** Using Proposition 1 and Lemma 2, we can express the Fourier coefficients of the functions $F_m(x)$ in terms of $\zeta^{(k)}(0)$'s and of $V_k^{(j)}(0)$'s, $1 \leq k \leq m$ and $1 \leq j \leq m-k$. The expressions become rather messy for larger values of m; thus we only state the results for the mean values of the functions F_3 and F_4:

$$\begin{aligned} p_{3,0} = &-\frac{\omega^3}{4}\left(3 - 3\log\pi - 3\zeta''(0) + \zeta'''(0)\right) \\ &+ \frac{\omega^2}{8}\left(3 + 3\log\pi + 3\zeta''(0) + 24V_1'(0) - 12V_1''(0)\right) \\ &+ \omega\left(\frac{11}{8} - \log\pi - 3V_1'(0) - 3V_2'(0)\right) - \frac{31}{32}; \end{aligned} \tag{2.9}$$

$$\begin{aligned} p_{4,0} = &\frac{\omega^4}{8}\left(12 - 12\log\pi - 12\zeta''(0) + 4\zeta'''(0) - \zeta^{(4)}(0)\right) \\ &- \omega^3\left(\frac{3}{2} + 6V_1'(0) - 3V_1''(0) + V_1'''(0)\right) \\ &- \omega^2\left(\frac{11}{4} - \frac{11}{4}\log\pi - \frac{11}{4}\zeta''(0) - 9V_1'(0) - 6V_2'(0) + \frac{9}{2}V_1''(0) + 3V_2''(0)\right) \\ &+ \omega\left(\frac{13}{4} - \frac{\log\pi}{2} - 4V_1'(0) - 6V_2'(0) - 4V_3'(0)\right) - \frac{83}{160}. \end{aligned} \tag{2.10}$$

In Section 4 we will obtain numerical approximations to these values.

3. Divide-and-conquer recurrences

The tools used above are applicable to recurrences of the type (1.9); in this section we study some concrete examples.

3.1. **Integral representation.** Recall that $\nabla\Delta f_n := f_{n+1} - 2f_n + f_{n-1}$.

Proposition 4. *Let α and β be two positive constants. Consider the recurrence*

$$f_n = \alpha f_{\lfloor n/2\rfloor} + \beta f_{\lceil n/2\rceil} + g_n, \qquad (n \geq 2), \tag{3.1}$$

with f_1 and the sequence $\{g_n\}_{n\geq 2}$ given. Let the abscissa of convergence of the Dirichlet series $W(s) := \sum_{n\geq 1}\nabla\Delta f_n\, n^{-s}$ be σ_f. Suppose that $c > \max\{0, \sigma_f, \log_2(\alpha+\beta)-1\}$. Then the solution of (3.1) satisfies

$$\frac{f_n}{n} = f_1 + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{n^s}{s(s+1)}\frac{W(s)}{1-(\alpha+\beta)2^{-s-1}}\,\mathrm{d}s, \tag{3.2}$$

where ($\Delta f(x) := \Delta f(\lfloor x \rfloor)$, $f_0 = g_0 = g_1 = 0$*)*

$$W(s) = (\alpha+\beta-2) f_1 \left(1-2^{-s-1}\right) + \sum_{n\ge1} \frac{\nabla\Delta g_n}{n^s} + \frac{(\alpha-\beta)s}{2^s} \int_1^\infty \frac{\Delta f(x)}{x^{s+1}}\, \xi(x)\,\mathrm{d}x. \tag{3.3}$$

Proof. Define $f(z) := \sum_{n\ge1} f_n z^n$ and $g(z) := \sum_{n\ge2} g_n z^n$. Then the relation (3.1) translates into

$$\begin{aligned} f(z) &= \frac{1}{z}(\beta + (\alpha+\beta)z + \alpha z^2) f(z^2) + g(z) + (1-\beta) f_1 z \\ &= \frac{\alpha+\beta}{2}\frac{(1+z)^2}{z} f(z^2) + \frac{\beta-\alpha}{2}\frac{1-z^2}{z} f(z^2) \\ &\quad + g(z) + (1-\beta) f_1 z. \end{aligned}$$

Multiplying now both sides by $(1-z)^2/z$ yields

$$\begin{aligned} \frac{(1-z)^2}{z} f(z) &= \frac{\alpha+\beta}{2}\frac{(1-z^2)^2}{z^2} f(z^2) + \frac{\beta-\alpha}{2}(1-z)^2\frac{1-z^2}{z^2} f(z^2) \\ &\quad + \frac{(1-z)^2}{z} g(z) + (1-\beta) f_1 (1-z)^2, \end{aligned}$$

or in terms of coefficients

$$\begin{aligned} \sum_{n\ge1} \nabla\Delta f_n z^n &= \frac{\alpha+\beta}{2} \sum_{n\ge1} \nabla\Delta f_n z^{2n} + \frac{\beta-\alpha}{2} \sum_{n\ge1} \Delta f_n z^{2n} (1-z)^2 \\ &\quad + \sum_{n\ge1} \nabla\Delta g_n z^n + \frac{\alpha+\beta-2}{2} f_1 z(2-z). \end{aligned}$$

Translating into Dirichlet series, or using the transformation $z \mapsto e^{-t}$ and then taking the Mellin transform of both sides, we obtain

$$\begin{aligned} W(s) &= \frac{\alpha+\beta}{2^{s+1}} W(s) + \frac{\beta-\alpha}{2} \sum_{n\ge1} \Delta f_n \Delta^2 (2n)^{-s} \\ &\quad + \sum_{n\ge1} \nabla\Delta g_n n^{-s} + (\alpha+\beta-2) f_1 (1-2^{-s-1}). \end{aligned}$$

Solving this equation and then applying (1.8), we obtain (3.2). This completes the proof. □

Note that

$$\begin{aligned} \frac{s}{2^s}\int_1^\infty \frac{\Delta f(x)}{x^{s+1}}\,\xi(x)\,\mathrm{d}x &= \frac{f_1}{2^{s+1}}\left(1 + \frac{1}{2^s} - \frac{2^{s+1}}{3^s}\right) \\ &\quad - \frac{1}{2}\sum_{n\ge2} f_n \left(\frac{1}{(2n-2)^s} - \frac{2}{(2n-1)^s} + \frac{2}{(2n+1)^s} - \frac{1}{(2n+2)^s}\right), \end{aligned}$$

the series on the right-hand side usually having a wider half-plane of convergence. Also note that when $\alpha = \beta = 1$, the solution (3.2) reduces to that given in [15].

The integral representation (3.2), coupling with analytic properties of $Y(s)$ and Proposition 3, can be applied to a variety of problems. We discuss the odd numbers in Pascal triangle and exponential sums of $\nu(n)$ in some detail, and indicate other digital problems.

3.2. **Odd numbers in Pascal triangle.** The number of odd entries in the j-th row of the Pascal triangle is given by $2^{\nu(j)}$; thus the number of odd binomial coefficients in the first n rows of the Pascal triangle is enumerated by

$$\Phi(n) := \sum_{0\le k<n} 2^{\nu(k)};$$

see [16] and the references therein. In particular, Flajolet et al. [16] showed that

$$\Phi(n) = n^\rho M_2(\log_2 n) \qquad (\rho := \log_2 3), \tag{3.4}$$

where M_2 is continuous of period 1 and satisfies a Lipschitz condition of order $\rho - 1 \approx 0.58$. A $(C,1)$-summable Fourier series for M_2 is also computed in [16]. Note that the uniqueness property is lacking for series being merely $(C,1)$-summable, e.g., $1/2+\sum_{n\ge1}\cos nx = 0$ $(C,1)$, for all x not an even multiple of π.

We propose two different approaches to derive absolutely convergent Fourier expansion for M_2. The first approach is based on the arguments used above and theory of transfer operators, and the second relies on the pseudo-Tauberian argument used in [16] and Bernstein's theorem for Fourier series (see [47, p. 240] or Proposition 6 below).

3.2.1. **Absolutely convergent Fourier series for $M_2(x)$: A purely analytic approach.** We proceed along the same line of arguments used above for $S_m(n)$ to derive an absolutely convergent Fourier series for $M_2(x)$.

Theorem 2. *The periodic function M_2 in* (3.4) *has the Fourier expansion*

$$M_2(x) = \omega \sum_{k\in\mathbb{Z}} \frac{A_2(\rho-1+\chi_k)}{(\rho-1+\chi_k)(\rho+\chi_k)}\, e^{2k\pi i x}, \tag{3.5}$$

where for $\Re(s) > \rho - 2$, A_2 is defined by

$$A_2(s) = 1 - 2^{-1-s} - \frac{1}{2}\sum_{j\ge1} 2^{\nu(j)}\left((2j)^{-s} - 2(2j+1)^{-s} + (2j+2)^{-s}\right),$$

and

$$A_2(\rho-1+\chi_k) := O\left(|k|^{0.9961}\right). \tag{3.6}$$

The Fourier series (3.5) *is absolutely convergent.*

Proof. We start from the obvious recurrence

$$\begin{cases} \Phi(n) = 2\Phi(\lfloor n/2\rfloor) + \Phi(\lceil n/2\rceil), & (n\ge 2);\\ \Phi(1) = 1. \end{cases} \tag{3.7}$$

We assume at the moment the validity of (3.6) whose proof is more technical and is given below. Applying Proposition 4, we obtain

$$\begin{aligned}\Phi(n) &= n + \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{n^{s+1}}{s(s+1)}\frac{A_2(s)}{1-3\cdot 2^{-s-1}}\,\mathrm{d}s\\ &= n + n^\rho M_2(\log_2 n) + I_1 + I_2,\end{aligned}$$

where (for some $\varepsilon > 0$ small enough)

$$I_1 = \frac{1}{2\pi i}\int_{\rho-1-\varepsilon-i\infty}^{\rho-1-\varepsilon+i\infty} \frac{n^{s+1}}{s(s+1)}\frac{1-2^{-1-s}}{1-3\cdot 2^{-s-1}}\,\mathrm{d}s;$$

and

$$I_2 = \frac{1}{2\pi i}\int_{\rho-1-\varepsilon-i\infty}^{\rho-1-\varepsilon+i\infty} \frac{n^{s+1}}{s(s+1)}\frac{A_2(s)-1+2^{-s-1}}{1-3\cdot 2^{-s-1}}\,\mathrm{d}s.$$

Now by successively moving the line of integration to the left, we obtain $I_1 = -n$ (the residue of the integrand at $s = 0$). Observe that

$$A_2(s) - 1 + 2^{-s-1} = s2^{-s} \int_1^\infty 2^{\nu(x)} x^{-s-1} \xi(x) \, \mathrm{d}x.$$

Thus the contribution of I_2 is zero by applying Proposition 3. □

Growth order of $A_2(s)$. We still need to prove (3.6). The proof lies much deeper than that for $V_m(s)$ and relies on tools from transfer operators.

We will actually prove more, namely,

$$A_2(\rho - 1 + it) = O\left(|t|^{c_3} \log |t|\right),$$

for $|t| \geq t_0$, where

$$c_3 := \frac{3 \log \lambda_2}{2\rho(\log_2 \lambda_2 + (2-\rho)\log 2)} + \frac{1}{2\rho} < 0.99602.$$

Here $\lambda_2 \approx 2.08852$ is given in (3.11) below. By convexity of the order function of Dirichlet series (see [23, Chapter III] [43, Part II] [45, §9.41]), it suffices to show that

$$A_2(it) = O\left(|t|^{c_4} \log |t|\right), \tag{3.8}$$

where $c_4 := \rho c_3 < \rho$. To that purpose, write

$$B_2(s) = \sum_{n\geq 1} 2^{\nu(n)} \Delta^2(2n)^{-s},$$

so that $A_2(s) = 1 - 2^{-s-1} - B_2(s)/2$. Take $L := \lfloor c_5 \log |t| \rfloor$, where c_5 will be specified later. Let $N := 2^L$ and decompose the series into two parts:

$$B_2(s) = \left(\sum_{n<N} + \sum_{n\geq N}\right) 2^{\nu(n)} \Delta^2(2n)^{-s}.$$

The constant c_5 will be chosen so that the two parts roughly satisfy the same estimate.

The second part is easily estimated as follows.

$$\begin{aligned}
\sum_{n\geq N} 2^{\nu(n)} \Delta^2(2n)^{-it} &= O\left(t^2 \sum_{n\geq N} 2^{\nu(n)} n^{-2}\right) \\
&= O\left(t^2 \int_N^\infty x^{\rho-3} \, \mathrm{d}x\right) \\
&= O\left(t^2 N^{\rho-2}\right) \\
&= O\left(|t|^{2-c_5(2-\rho)\log 2}\right). \qquad (3.9)
\end{aligned}$$

We apply the same argument as that for $V_m(it)$ to the first part:

$$\begin{aligned}
\sum_{n<N} 2^{\nu(n)} \Delta^2(2n)^{-it} &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{1\leq n<N} e^{inv} \Delta^2(2n)^{-it}\right) \left(\sum_{0\leq n<N} 2^{\nu(n)} e^{-inv}\right) \mathrm{d}v \\
&\leq T_2 \max_{0\leq v\leq 2\pi} |V(it, v)| \\
&= O\left(T_2 |t|^{1/2} \log |t|\right),
\end{aligned}$$

where $V(s,v)$ is given in (2.4) and

$$T_2 = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{0\le n<N} 2^{\nu(n)} e^{-inx}\right| \mathrm{d}x$$

$$= \int_0^1 \prod_{1\le j\le L}\left|1+2e^{-2^j\pi ix}\right| \mathrm{d}x. \tag{3.10}$$

We will show that

$$T_2 = O\left(\lambda_2^L\right), \tag{3.11}$$

where $\lambda_2 \approx 2.08852$. Assuming that this estimate holds, we can now choose c_5 such that $2 - c_5(2-\rho)\log 2 = c_5 \log\lambda_2 + 1/2$, giving

$$c_5 := \frac{3}{2(\log_2\lambda_2 + (2-\rho)\log 2)},$$

which yields (3.8).

Transfer operators. In order to derive a better estimate for T_2 than the crude bound $T_2 = O(3^L)$, we apply transfer operators as studied in [9] and [24]. Our application is similar to that used in Fouvry and Mauduit [19].

Consider the operator P_2 defined by

$$P_2[f](x) := |1+2e^{-2\pi ix}| f(2x \bmod 1),$$

for f in the space Lip$[0,1]$ of Lipschitz functions from $[0,1]$ to itself with the norm

$$|||f||| = \sup_{0\le x\le 1}|f(x)| + \sup_{0\le x<y\le 1}\left|\frac{f(x)-f(y)}{x-y}\right|.$$

With this operator, we can write

$$\prod_{1\le j\le L}\left|1+2e^{-2^j\pi ix}\right| = P_2^L[\mathbf{1}](x),$$

where $\mathbf{1}(x)\equiv 1$ for $0\le x\le 1$, and the integral (3.10) can be written as

$$\int_0^1 \prod_{1\le j\le L}\left|1+2e^{-2^j\pi ix}\right| \mathrm{d}x = \langle P_2^L[\mathbf{1}],\mathbf{1}\rangle = \langle \mathbf{1}, Q_2^L[\mathbf{1}]\rangle,$$

where Q_2 denotes the adjoint operator of P_2. This operator is given by

$$Q_2[f](x) = \frac{1}{2}\left(\left|1+2e^{-\pi ix}\right| f\left(\frac{x}{2}\right) + \left|1+2e^{-\pi i(x+1)}\right| f\left(\frac{x+1}{2}\right)\right),$$

and is the transfer operator associated with the positive function $|1+e^{-2\pi ix}|$. We need the spectral decomposition of Q_2 on Lip$[0,1]$.

Clearly, Q_2 is a positive linear operator. It follows from the theory of quasi-compactness of transfer operators (see [9] and [25]) that Q_2 has a positive dominating eigenvalue λ_2 (which is simple and also the spectral radius) and a unique corresponding eigenfunction ψ_2. From positivity of the operator, it follows that, for any positive function f, we have

$$\min_{x\in[0,1]}\frac{Q_2[f](x)}{f(x)} \le \lambda_2 \le \max_{x\in[0,1]}\frac{Q_2[f](x)}{f(x)}. \tag{3.12}$$

This estimate makes it possible to derive precise numerical estimates for λ_2; see below.

From the fact that the dominating eigenvalue is simple, it follows that

$$Q_2^L[\mathbf{1}](x) = C\lambda_2^L\psi_2(x) + O((\lambda_2-\varepsilon)^L) \quad \text{for some } \varepsilon>0,$$

from which we can conclude that

$$T_2 = \lambda_2^L \int_0^1 \psi_2(x)\,\mathrm{d}x + O((\lambda_2-\varepsilon)^L) \asymp \lambda_2^L.$$

Numerical approximations to λ_2. The proof given above is not self-contained and left open the computation of the exact value of λ_2. We give a simple, straightforward, and self-contained way to compute λ_2 numerically as follows (see [19]).

Let

$$\mu_k := \left(\max_{0\le x\le 1} Q_2^k[\mathbf{1}](x)\right)^{1/k} \qquad (k=1,2,\dots).$$

We can write $T_2 = T_2(L)$ and, by splitting the integral at 1/2 and making the change of variables $x \mapsto 2x$, we have

$$\begin{aligned} T_2(L) &= \int_0^1 Q_2[\mathbf{1}](x) \prod_{1\le j<L} \left|1+2e^{-2^j\pi i x}\right| \mathrm{d}x\\ &= \int_0^1 Q_2^k[\mathbf{1}](x) \prod_{1\le j\le L-k} \left|1+2e^{-2^j\pi i x}\right| \mathrm{d}x\\ &\le \mu_k^k T_2(L-k)\\ &\le \mu_k^L, \end{aligned}$$

for any $k\ge 1$. This gives a way to approximate from above the value of λ_2. For $k=1$,

$$\mu_1 = \sqrt{5} \approx 2.236\ldots$$

For $k=2$,

$$\mu_2 = \sqrt{\frac{\sqrt{5}}{2}\left(\sqrt{5+2\sqrt{2}}+\sqrt{5-2\sqrt{2}}\right)} \approx 2.185\ldots.$$

And for $k\ge 3$, this gives successively (up to 10^{-3})

$$\{\mu_k\}_{k\ge 3} = \{2.157, 2.140, 2.130, 2.123, 2.118, 2.114, 2.111, 2.109, \cdots\}.$$

Note that the value $\mu_7 \approx 2.118 < 3/\sqrt{2} \approx 2.121$ is already sufficient for our uses (for bounding the growth magnitude of $A_2(\rho-1\pm it) = O\left(|t|^{0.9997}\right)$ to conclude absolute convergence).

A better but not independent means of approximating the value of λ_2 is to use the property that (see [25])

$$\lim_{k\to\infty} \frac{Q_2^{k+1}[\mathbf{1}](x)}{Q_2^k[\mathbf{1}](x)} = \lambda_2,$$

for all $0\le x\le 1$. This corresponds to inserting the function $f(x) = Q_2^k[\mathbf{1}](x)$ into (3.12). Numerical studies of the above functions for $k=1,\dots,10$ show that the maximum of $Q_2^k[\mathbf{1}](x)$ is attained for $x=\frac12$ and the minimum is attained for $x=0$. These values yield effective upper and lower bounds for λ_2 (truncated to the digits different from the next approximation):

$$\left\{\frac{Q_2^{k+1}[\mathbf{1}](\frac12)}{Q_2^k[\mathbf{1}](\frac12)} \mid k=1,\dots,10\right\} = \{2.13, 2.101, 2.092, 2.089, 2.0888, 2.0886, 2.08855, 2.08853, 2.088528, 2.088527\}$$

$$\left\{\frac{Q_2^{k+1}[\mathbf{1}](0)}{Q_2^k[\mathbf{1}](0)} \mid k=1,\dots,10\right\} = \{2.05, 2.079, 2.086, 2.087, 2.0883, 2.0884, 2.08851, 2.088521, 2.088524, 2.088525\}.$$

Thus $\lambda_2 \approx 2.08852$, as required. This completes the proof of (3.11) and that of (3.5).

3.2.2. **Absolutely convergent Fourier series for $M_2(x)$: Bernstein's theorem.** The approach is modified from that used by Flajolet et al. [16]. Instead of considering $\Phi(n)$ and then applying (1.7), we start from the representation (see also [26])

$$\begin{aligned}\sum_{1\le k<n} \Phi(k) &= \sum_{0\le k<n} (n-1-k)2^{\nu(k)} \\ &= \frac{n^2}{2} - \frac{\Phi(n)}{2} + \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{n^{s+2}}{s(s+1)(s+2)} \sum_{j\ge 1} \nabla\Delta\Phi(j) j^{-s}\, \mathrm{d}s. \end{aligned} \tag{3.13}$$

By (3.3),

$$\sum_{j\ge1} \nabla\Delta\Phi(j)j^{-s} = \frac{A_2(s)}{1-3\cdot 2^{-s-1}} = \frac{1-2^{-s-1}-B_2(s)/2}{1-3\cdot 2^{-s-1}}.$$

To derive the growth rate of $B_2(\sigma \pm it)$ for large $|t|$, we use the following simple estimate: for $\rho - 2 < \sigma \le \rho$ and large $|t|$,

$$\begin{aligned} B_2(\sigma+it) &= \left(\sum_{1\le k\le |t|} + \sum_{k>|t|}\right) 2^{\nu(k)}\Delta^2(2k)^{-\sigma-it} \\ &= O\left(\sum_{k\le|t|} 2^{\nu(k)}k^{-\sigma} + t^2 \sum_{k>|t|} 2^{\nu(k)}k^{-\sigma-2}\right) \\ &= O\left(|t|^{\rho-\sigma}\right). \end{aligned}$$

In particular, $B_2(\rho - 1 + \chi_k) = O(|k|)$. We then obtain by (3.13)

$$\sum_{1\le k<n} \Phi(k) \sim \omega n^{\rho+1} \sum_{k\in\mathbb{Z}} \frac{A_2(\rho-1+\chi_k)}{(\rho-1+\chi_k)(\rho+\chi_k)(\rho+1+\chi_k)} n^{\chi_k},$$

where the Fourier series is absolutely convergent.

To get the Fourier expansion for $M_2(x)$, we use the following pseudo-Tauberian argument.

Proposition 5 (Flajolet et al. [16])**.** *Let f be a continuous, periodic function of period 1, and let τ be a complex number with $\Re(\tau) > 0$. Then there exists a continuously differentiable function g of period 1, such that*

$$\frac{1}{N^{\tau+1}} \sum_{n<N} n^\tau f(\log_2 n) = g(\log_2 N) + o(1),$$

$$\int_0^1 g(x)\,\mathrm{d}x = \frac{1}{\tau+1}\int_0^1 f(x)\,\mathrm{d}x.$$

Applying this Proposition with $\tau = \rho - 1 + \chi_k$ yields

$$M_2(x) = \omega \sum_{k\in\mathbb{Z}} \frac{A_2(\rho-1+\chi_k)}{(\rho-1+\chi_k)(\rho+\chi_k)} e^{2k\pi i x}; \tag{3.14}$$

convergence of the series on the right-hand side needs other theories, which in turn demand more functional properties of $M_2(x)$.

By the recurrence (3.7), we obtain $\Phi(2^k) = 3^k$ and then deduce that (see [42], [41])

$$\Phi(n) = \sum_{1\le j\le k} 2^{j-1}3^{b_j},$$

where $n = 2^{b_1} + \cdots + 2^{b_k}$ with $b_1 > \cdots > b_k \geq 0$. Define a function w in $[0,1]$ as follows (see [16])

$$w\left(\sum_{j\geq 0} 2^{-d_j}\right) = \sum_{j\geq 0} 2^j 3^{-d_j} \qquad (0 = d_0 < d_1 < \cdots).$$

Then

$$M_2(x) = 3^{-\{x\}} w(2^{\{x\}}).$$

With this representation, it is easily seen that $M_2(x)$ is continuous, implying that the series in (3.14) is at least $(C,1)$-summable.

Also it can be checked that $M_2(x)$ satisfies a Lipschitz condition of order $\rho - 1 \approx 0.58$. We recall a classical result of Bernstein (see [47, p. 240]).

Proposition 6. *If f is a real-valued function defined on $[0,1]$ and satisfies a Lipschitz condition of order $\lambda > 1/2$, namely,*

$$|f(x) - f(y)| \leq K|x-y|^\lambda \qquad (x, y \in [0,1]),$$

for some positive constant K, then the Fourier series of f converges absolutely and uniformly.

Applying this result, we conclude that the series in (3.14) is absolutely and uniformly convergent.

A comparative discussion of the two approaches is given below.

3.3. **General exponential sums of $\nu(n)$.** We consider more general sums of the type

$$\Phi_\theta(n) := \sum_{0\leq k<n} \theta^{\nu(k)}, \tag{3.15}$$

which is essentially the probability generating function of X_n. Such a consideration offers not only absolutely convergent Fourier series for the periodic functions $\Phi_\theta(n)/n^{\log_2(1+\theta)}$ for some values of θ, but also gives a rather informative comparison of both approaches used above.

Okada et al. [37] showed that

$$\Phi_\theta(n) = n^{\log_2(1+\theta)} M_\theta(\log_2 n) \qquad (\theta > 0), \tag{3.16}$$

where $M_\theta(x)$ is continuous and 1-periodic.

Theorem 3. *For $\sqrt{2}-1 < \theta < \sqrt{2}+1$, $\theta \neq 1$ the sum function $\Phi_\theta(n)$ given by* (3.15) *satisfies* (3.16)*, where the periodic function M_θ is given by the absolutely and uniformly convergent Fourier series*

$$M_\theta(x) = \omega \sum_{k\in\mathbb{Z}} \frac{A_\theta(\rho_\theta - 1 + \chi_k)}{(\rho_\theta - 1 + \chi_k)(\rho_\theta + \chi_k)}\, e^{2\pi i k x},$$

where $\rho_\theta := \log_2(1+\theta)$ and A_θ is defined by

$$A_\theta(s) := (\theta - 1)\left(1 - 2^{-1-s} - \frac{1}{2}\sum_{n\geq 1} \theta^{\nu(n)} \Delta^2 (2n)^{-s}\right). \tag{3.17}$$

We first prove the theorem using the second approach based on Lipschitz condition and Bernstein's theorem. For more methodological interests, we also apply the first approach, which covers only the interval $0.471316 < \theta < 2.12173$, $\theta \neq 1$. Although the second approach applies to a wider range, it relies the absolute convergence on a Lipschitz condition that is not necessarily available or easily proved in general. In contrast, the first approach does not need any *a priori* functional properties of the periodic function.

3.3.1. *Lipschitz condition and Bernstein's theorem.* The approach that we applied for $\Phi(n)$ can be easily amended for $\Phi_\theta(n)$ for general values of θ; it suffices to replace 2 by θ there. It is easily checked that the periodic function M_θ satisfies a Lipschitz condition of order $\rho_\theta - 1$ if $\theta < 1$, which is greater than 1/2 if $\theta > \sqrt{2}-1$, and it is Lipschitz continuous of order $\log_2(1+\theta^{-1})$ if $\theta > 1$, which is greater than 1/2 if $\theta < \sqrt{2}+1$. This completes the proof.

3.3.2. $1 < \theta < 2.12173$. Since $\Phi_\theta(n)$ satisfies (3.1), we can apply (3.2); the only missing part is the growth order of $A_2(s)$ at $\sigma \pm i\infty$.

The same reasoning as described above for the case $\theta = 2$ can be applied for $\theta < 3$, since the sum

$$B_\theta(s) = \sum_{n\geq 1} \theta^{\nu(n)} \Delta^2(2n)^{-s}$$

converges for $\Re(s) = 0$ for these values of θ. Then we use the same arguments as above to derive the estimates

$$\sum_{n\geq N} \theta^{\nu(n)} \Delta^2(2n)^{-it} = O\left(t^2 \int_N^\infty x^{\rho_\theta - 3}\,\mathrm{d}x\right) = O\left(t^2 N^{\rho_\theta-2}\right), \tag{3.18}$$

where N will be specified below, and

$$\sum_{n<N} \theta^{\nu(n)} \Delta^2(2n)^{-it} = O\left(T_\theta |t|^{1/2} \log|t|\right),$$

where

$$T_\theta = \int_0^1 \prod_{1\leq j\leq L} \left|1 + \theta e^{-2^j \pi i x}\right| \mathrm{d}x. \tag{3.19}$$

By the same transfer operator approach, we obtain

$$T_\theta = O\left(N^{\log_2 \lambda_\theta} |t|^{1/2} \log|t|\right); \tag{3.20}$$

Figure 2 shows the graph of λ_θ as a function of θ.

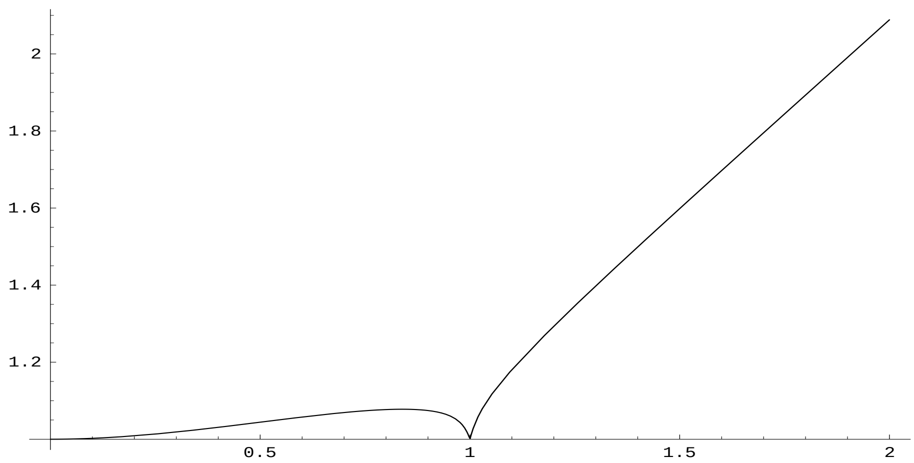

FIGURE 2. *The dominating eigenvalue λ_θ as a function of θ.*

We now take $N = 2^{\lfloor c_6 \log|t| \rfloor}$ and choose the value c_6 such that the two estimates (3.20) and (3.18) have the same order of magnitude. This yields the estimate

$$B_\theta(it) = O\left(|t|^{e_\theta}\right) \quad \text{with } e_\theta = \frac{1}{2} + \frac{3\log_2 \lambda_\theta}{2(2 + \log_2 \lambda_\theta - \rho_\theta)}, \tag{3.21}$$

from which we conclude, again by convexity of the order function of Dirichlet series, that

$$B_\theta(\rho-1+it)=O\left(|t|^{e_\theta/\rho_\theta}\right). \tag{3.22}$$

Using the numerical estimates for λ_θ we obtain that

$$\frac{e_\theta}{\rho_\theta}<1 \quad \text{for } 1<\theta<2.12173.$$

3.3.3. $0.471316<\theta<1$. In this case $\rho_\theta<1$ and therefore the zeros of $1-(\theta+1)2^{-s-1}$ have real part $\rho_\theta-1<0$. By the definition of the transfer operator P_θ in Section 3.3.2, we see that the identity $\theta P_{1/\theta}=P_\theta$ holds for $\theta>0$. From this it follows immediately that $\theta\lambda_{1/\theta}=\lambda_\theta$ holds for the dominant eigenvalue. Thus we have the estimate

$$B_\theta(\rho_\theta-1+it)=O\left(|t|^{3/2-\rho_\theta}T_\theta+t^2\sum_{n\geq N}\theta^{\nu(n)}n^{-1-\rho_\theta}\right) \tag{3.23}$$

$$=O\left(|t|^{3/2-\rho_\theta}T_\theta+t^2N^{-1}\right), \tag{3.24}$$

where T_θ is given in (3.19) and $N=2^L$. We choose L to minimize the right-hand side; this gives

$$L=\left\lfloor\frac{1/2+\log_2(\theta+1)}{1+\log_2\lambda_\theta}\log_2|t|\right\rfloor.$$

From this estimate we get

$$B_\theta(\rho_\theta-1+it)=O\left(|t|^{2-\frac{1+2\rho_\theta}{2+2\log_2\lambda_\theta}}\right).$$

For $\theta=1/2$ we have $\lambda_{1/2}<1.04426319$ and therefore the exponent equals 0.9788. This case was encountered in [5]. We note here that the exponent in the estimate for B_θ is less than 1 for $\theta>0.471316$.

3.4. **Newman-Coquet formula.** Newman [35] first observed that the sum

$$N_3(n)=\sum_{k<n}(-1)^{\nu(3k)}$$

is always positive. Coquet [10] gave the exact formula

$$N_3(n)=n^{\log_4 3}H_3(\log_4 n)+\begin{cases}0 & \text{if } n \text{ is even;}\\ \dfrac{(-1)^{\nu(3n-1)}}{3} & \text{if } n \text{ is odd,}\end{cases}$$

and showed that H_3 satisfies a Lipschitz condition of order $\log_4 3>1/2$. The study of the sum $S_3(n)$ leads to the Dirichlet series

$$\Psi_k(s)=\sum_{n\geq1}(-1)^{\nu(3n+k)}\left(\frac{1}{n^s}-\frac{1}{(n+1)^s}\right) \quad \text{for } k=0,1,2. \tag{3.25}$$

In order to obtain the analytic continuations of Ψ_k, we split the range of summations into odd and even summands as before, obtaining the system of equations

$$\begin{cases}\Psi_0(s)= & 2^{-s-1}\Psi_0(s)-2^{-s-1}\Psi_1(s)+1-2^{-s}+Y_0(s)+Y_1(s),\\ \Psi_1(s)= & -2^{-s-1}\Psi_0(s)+2^{-s-1}\Psi_2(s)-1+2^{-s}-Y_0(s)-Y_2(s),\\ \Psi_2(s)= & 2^{-s-1}\Psi_1(s)-2^{-s-1}\Psi_2(s)+1-2^{-s}+Y_1(s)+Y_2(s),\end{cases} \tag{3.26}$$

where

$$Y_k(s)=\frac12\sum_{n\geq1}(-1)^{\nu(3n+k)}\left(\frac{1}{(2n)^s}-\frac{2}{(2n+1)^s}+\frac{1}{(2n+2)^s}\right) \qquad (k=0,1,2). \tag{3.27}$$

Solving (3.26), we obtain

$$\Psi_0(s) = \frac{1}{4^{s+1}-3}\left(4^{s+1} - 5 + 2^{-s} + 4(4^s + 2^s)Y_0(s) + (4^{s+1} + 2^{s+1} - 2)Y_1(s) + 2^{s+1}Y_2(s)\right)$$

and similar expressions for $\Psi_1(s)$ and $\Psi_2(s)$. This provides an analytic continuation of $\Psi_0(s)$ for $\Re(s) > \log_4 3 - 2$, with single poles at the solutions of $4^{s+1} = 3$.

For the growth order of $Y_k(s)$ along vertical lines, we again use the method developed in Section 3.2.1. In this case, three operators have to be studied:

$$P_k : f \mapsto \left|1 - e^{2\pi i(x+k/3)}\right| f(2x \bmod 1) \qquad (k = 0, 1, 2).$$

It turns out that the dominating eigenvalue of P_1 and P_2 equals 1.27277. The dominating eigenvalue of P_0 equals 1.32265. Proceeding as in Section 3.2 yields $Y_k(it) = O(|t|^{0.87154})$. From this we obtain that

$$Y_k(\log_4 3 - 1 + it) = O\left(|t|^{1.07906}\right) \quad (1.07906 \approx 0.87154 + 1 - \log_4 3),$$

which is too weak to conclude absolute convergence of the Fourier series directly. Thus we use the same line of arguments as in Section 3.3.1 to obtain the Fourier expansion $H_3(x) = \sum_{k\in\mathbb{Z}} h_k e^{2k\pi ix}$, where

$$h_k = \frac{1}{3\log 4}\frac{1}{z_k(z_k+1)}\Bigg(-2 + 2\frac{(-1)^k}{\sqrt{3}} + (3 + 2(-1)^k\sqrt{3})Y_0(z_k) + (1 + (-1)^k\sqrt{3})Y_1(z_k) + (-1)^k\sqrt{3}Y_2(z_k)\Bigg),$$

with $z_k = \log_4 3 - 1 + 2k\pi i/\log 4$. This Fourier series is absolutely and uniformly convergent.

3.5. **AND/OR problem.** The AND/OR problem is as follows (see [31]). Given n OR gates and some AND gates, which are indistinguishable from each other. If one is to compute the AND function of two values correctly, how many gates are necessary to construct an error-free circuit? It is shown, in the special case of a modular circuit, that (see [6]) the size f_n of the smallest circuit tolerates exactly $n-1$ OR gates satisfies $f_1 = 1$ and

$$f_n = f_{\lfloor n/2\rfloor} + 2f_{\lceil n/2\rceil} \qquad (n \geq 2),$$

or, in terms of $\nu_0(n)$, the number of 0's in the dyadic representation of n,

$$f_n = 1 + 2\sum_{1\leq k<n} 2^{\nu_0(k)}.$$

The same set of tools applies and we have

$$f_n = n^\rho Z(\log_2 n) \qquad (\rho = \log_2 3), \tag{3.28}$$

where $Z(x)$ is continuous and 1-periodic; see Figure 3 for a plot.

3.6. **Bose-Nelson sorting network.** The recurrence in question is now $f_1 = 0$ and

$$f_n = f_{\lfloor n/2\rfloor} + 2f_{\lceil n/2\rceil} + \lfloor n/2\rfloor \qquad (n \geq 2).$$

Inserting this into (3.3) gives the corresponding Dirichlet generating function

$$Y(s) = (1 - 2^{1-s})\zeta(s) + \frac{1}{2}\sum_{n\geq 2}\Delta f_n \Delta^2(2n)^{-s}.$$

To prove absolute convergence of the Fourier series, we need to estimate the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{0\leq n<N}\Delta f_n e^{-inv}\right| \mathrm{d}v = \int_0^1 \sum_{1\leq j\leq L}\left|\frac{1 - e^{-2^{L+1}\pi iv}}{1 - e^{-2^{j+1}\pi iv}}\prod_{1\leq \ell<j}\left(2 + e^{-2^\ell\pi iv}\right)\right| \mathrm{d}v. \tag{3.29}$$

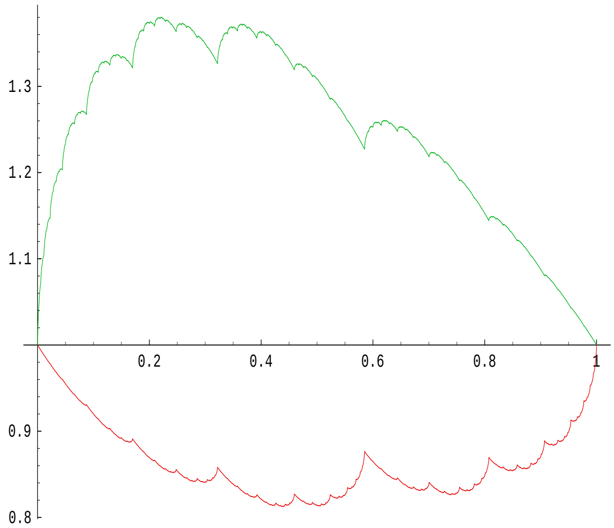

FIGURE 3. *The periodic functions $M_2(x)$ (bottom) and $2Z(x)$ (top).*

This can be done by observing that the integral

$$\int_0^1 \left| \frac{1 - e^{-2^{L+1}\pi i v}}{1 - e^{-2^{j+1}\pi i v}} \prod_{1 \le \ell < j} \left(2 + e^{-2^\ell \pi i v}\right) \right| \mathrm{d}v$$

can be estimated by $2^{L-j}\lambda_2^j$ by the discussion in Section 3.2.1. Summing up, we get an estimate of order λ_2^L for the integral in (3.29). The same arguments as in Section 3.2 yield the estimate $Y(\rho - 1 + it) = O(|t|^{0.9961})$.

Thus f_n satisfies the exact formula

$$f_n = n^\rho K(\log_2 n) - n \qquad (\rho := \log_2 3), \tag{3.30}$$

where K is given by the absolutely and uniformly convergent Fourier series

$$K(x) = \omega \sum_{k \in \mathbb{Z}} \frac{Y(\rho - 1 + \chi_k)}{(\rho - 1 + \chi_k)(\rho + \chi_k)} e^{2k\pi i x}.$$

Note that by the digital sum expression for $f_n + n$ (see [4])

$$f_n + n = \sum_{1 \le j \le k} 3^{b_j} \left(1 + \sum_{1 \le \ell < j} 2^{b_\ell - b_j + \ell - j + 1}\right),$$

where $n = 2^{b_1} + \cdots + 2^{b_k}$, $b_1 > \cdots > b_k \ge 0$, and by the same argument used for $M_2(x)$, we can show that $K(x)$ is Lipschitz continuous of order $\log_2 3 - 1 > 1/2$; thus Bernstein's theorem is also applicable to conclude absolute convergence of the Fourier series.

4. NUMERICAL COMPUTATIONS

In this section we propose a simple approach to compute the Fourier coefficients to high precision of the periodic functions studied in this paper.

4.1. A 1/2-balancing principle. We describe here our approach by Riemann's zeta function:

$$\zeta(s) = \sum_{n \ge 1} n^{-s} \qquad (\Re(s) > 1).$$

A simple way of computing the values of $\zeta(s)$ is to use the functional equation

$$\begin{aligned}\zeta(s) &= 1 + 2^{-s}\sum_{n\ge1} n^{-s} + 2^{-s}\sum_{n\ge1}(n+1/2)^{-s} \\ &= \frac{1}{1-2^{1-s}}\left(1 + 2^{-s}\sum_{m\ge1}\binom{s+m-1}{m}\frac{(-1)^m}{2^m}\,\zeta(s+m)\right), \qquad (4.1)\end{aligned}$$

obtained by expanding the factor $(n+1/2)^{-s}$ in increasing powers of n^{-1}. Note that this equation also provides an analytic continuation of $\zeta(s)$ to almost the whole plane (up to points where $2^{s+k} = 2$ for integer $k \ge 0$).

A much better way of computing $\zeta(s)$ is to use the formula

$$\zeta(s) = \frac{\Lambda(s) + 2^s}{2^s - 1} \quad \text{where } \Lambda(s) := \sum_{n\ge1}(n+1/2)^{-s}. \qquad (4.2)$$

(The function $s \mapsto \Lambda(s) + 2^s$ is sometimes called the Dirichlet lambda function.) The reason is that $\Lambda(s)$ can be computed by the functional equation

$$\Lambda(s) = \frac{1}{1-2^{1-s}}\left((2/3)^s + 2^{1-s}\sum_{m\ge1}\binom{s+2m-1}{2m}\frac{\Lambda(s+2m)}{16^m}\right), \qquad (4.3)$$

where the terms are convergent much faster (the terms behaving like $m^{\Re(s)-1}36^{-m}$ for large m and fixed s) than those in (4.1). This equation is obtained similarly as for (4.1) and has the feature that the contribution from odd summands and that from even summands are to some extent balanced.

While equation (4.2) cannot numerically compete with several known formulæ for $\zeta(k)$ for integral $k \ge 2$ with much better convergence than (4.3) (see [3]), it is very efficient for general values of s. Other features of (4.2) are: (*i*) it is an independent equation, namely, the evaluation does not rely on any known values like π or powers of π; (*ii*) the underlying principle of balancing the contribution from odd and even summands can be applied to many other Dirichlet series whose coefficients satisfy a relation of the form (3.1); (*iii*) it is computationally simpler for, say complex parameters of s, than almost all known formulæ for $\zeta(s)$; and (*iv*) unlike (4.1), the evaluation of (4.3) involves only positive terms when $s > 1$, making it less sensible to numerical errors. Indeed, we originally obtained (4.3) by trying to drop the factor $(-1)^j$ in (4.1).

Such a 1/2-balancing principle can be easily applied to some known constants like those discussed in [18]. For example, starting from

$$\begin{aligned}\pi &= \sum_{n\ge1}\left(\frac{1}{n-1/4} - \frac{1}{n-3/4}\right), \\ \gamma &= 1 + \sum_{n\ge1}\left(\frac{1}{n+1} + \log\frac{n}{n+1}\right),\end{aligned}$$

we have

$$\begin{aligned}\pi &= \frac{8}{3} + 8\sum_{m\ge1}\frac{\Lambda(2m)}{16^m}, \\ \gamma &= 1 + \sum_{m\ge1}\frac{2m\Lambda(2m+1)}{(2m+1)\,4^m} - 2\sum_{m\ge1}\frac{\Lambda(2m)}{4^m}.\end{aligned}$$

4.2. **Coefficients appearing in moments of $\nu(n)$.** Motivated by the 1/2-balancing principle, we consider the series

$$\psi_k(s) = \sum_{n\ge 1} \frac{\nu(n)^k}{\left(n+\frac{1}{2}\right)^s}, \tag{4.4}$$

instead of using directly $V_k(s)$ or using the seemingly more natural series $\sum_{n\ge1}\nu^k(n)n^{-s}$. We then express V_k in terms of ψ_k.

First, by splitting the sum in (4.4) into odd and even summands and using (2.2), we obtain

$$\begin{aligned}\psi_k(s) &= \sum_{n\ge1}\frac{\nu(2n)^k}{\left(2n+\frac{1}{2}\right)^s} + \sum_{n\ge0}\frac{\nu(2n+1)^k}{\left(2n+\frac{3}{2}\right)^s}\\ &= 2^{-s}\zeta\left(s,\frac{3}{4}\right) + 2^{1-s}\psi_k(s) + 2^{1-s}\sum_{m\ge1}\binom{s+2m-1}{2m}\frac{\psi_k(s+2m)}{16^m}\\ &\quad + 2^{-s}\sum_{1\le j<k}\binom{k}{j}\sum_{m\ge0}\binom{s+m-1}{m}\frac{(-1)^m}{4^m}\,\psi_j(s+m).\end{aligned}$$

Solving for $\psi_k(s)$, we then obtain

$$\begin{aligned}\psi_k(s) &= \frac{1}{2^s-2}\zeta\left(s,\frac{3}{4}\right) + \frac{2}{2^s-2}\sum_{m\ge1}\binom{s+2m-1}{m}\frac{\psi_k(s+2m)}{16^m}\\ &\quad + \frac{1}{2^s-2}\sum_{1\le j<k}\binom{k}{j}\sum_{m\ge0}\binom{s+m-1}{m}\frac{(-1)^m}{4^m}\,\psi_j(s+m).\end{aligned} \tag{4.5}$$

To compute $\psi_k(s)$ to within a given error ε, we choose $m_0\in\mathbb{N}$ so large that $|\psi_k(s+m) - (\frac{2}{3})^{s+m}| < \varepsilon$ for all $m\ge m_0$. Since $\psi_k(s)\sim(\frac{2}{3})^s$ for large s with $\Re(s)>0$, we can evaluate the values $\psi_k(s+m)$ ($m=0,\dots,m_0$) by using (4.5), approximating $\psi_k(s+m)$ for $m>m_0$ by $(2/3)^{s+m}$, and then truncating the infinite sum to obtain a numerical estimate to within an error.

Note that the Hurwitz zeta function $\zeta(s,3/4)$ can either be computed by using existing built-in functions in computer algebra softwares or be computed directly by the formula

$$\zeta(s,3/4) = (4/3)^s + \sum_{m\ge0}\binom{s+m-1}{m}\frac{(-1)^m}{4^m}\,\Lambda(s+m).$$

It turns out that especially for the derivatives, this equation is much faster than the algorithm implemented in `Mathematica` when computing the values to within the same error.

For $V_k'(0)$, we have the expression in terms of $\psi_k(2j)$'s

$$\begin{aligned}V_k'(0) &= -\frac{1}{2}\sum_{n\ge1}\nu(n)^k\left(\log(2n) - 2\log(2n+1) + \log(2n+2)\right)\\ &= \sum_{m\ge1}\frac{\psi_k(2m)}{m4^m},\end{aligned}$$

which gives the numerical approximation

$$V_1'(0) = 0.16891\ 60545\ 92381\ 08766\ 41250\ 86505\ 72086\ 21392\ 02956\ 25995\dots$$

This in turn provides a good approximation to the mean value of F_2

$$p_{2,0} = -0.16743\ 75414\ 08216\ 30925\ 51550\ 10992\ 47202\ 32933\ 06264\ 89369\dots$$

For the second and third derivatives of V_k, we have

$$V_k''(0) = \sum_{m\geq 1} \frac{1}{m4^m} \left\{ (H_{2m-1} - \log 2)\, \psi_k(2m) + \psi_k'(2m) \right\},$$
$$V_k'''(0) = \frac{3}{2} \sum_{m\geq 1} \frac{1}{m4^m} \Big\{ \Big((H_{2m-1} - \log 2)^2 - H_{2m-1}^{(2)} \Big)\, \psi_k(2m) + 2(H_{2m-1} - \log 2)\psi_k'(2m) + \psi_k''(2m) \Big\},$$

where $H_k = \sum_{1\leq m\leq k} 1/m$ and $H_k^{(2)} = \sum_{1\leq m\leq k} 1/m^2$. From these, we obtain the following numerical approximations to the values appearing in (2.9) and (2.10):

$$\begin{aligned}
V_1''(0) &= -0.40632\ 91671\ 14929\ 22563\ 37014\ 58481\ 78635\ 30386\ 92416\ 64842\ldots,\\
V_1'''(0) &= 1.12746\ 03441\ 76855\ 00723\ 94784\ 63671\ 80426\ 48344\ 45077\ 21808\ldots,\\
V_2'(0) &= 0.31047\ 16129\ 81928\ 91222\ 32068\ 52261\ 52855\ 96918\ 44215\ 57523\ldots,\\
V_2''(0) &= -1.20785\ 26305\ 05474\ 15248\ 60897\ 62038\ 67711\ 07449\ 26970\ 51090\ldots,\\
V_3'(0) &= 0.79612\ 43185\ 47763\ 30582\ 71007\ 27435\ 50514\ 41134\ 19022\ 61579\ldots
\end{aligned}$$

From these we obtain

$$\begin{aligned}
p_{3,0} &= 0.03510\ 79771\ 90647\ 59775\ 76100\ 01574\ 86700\ 21149\ 58450\ 45765\ldots,\\
p_{4,0} &= 0.31334\ 81715\ 66982\ 67450\ 76841\ 74593\ 65540\ 16102\ 28008\ 82561\ldots
\end{aligned}$$

Our approach is also suitable for computing other Fourier coefficients. For example,

$$\begin{aligned}
p_{2,1} =& 0.03625\ 04797\ 06516\ 31341\ 36434\ 95281\ 70383\ 70571\ 75744\ 31121\ldots\\
&- 0.03167\ 02979\ 13892\ 95813\ 50796\ 92403\ 02205\ 98609\ 42456\ 99190\ldots i\\
p_{2,2} =& 0.01245\ 82164\ 62996\ 68591\ 21201\ 10896\ 53268\ 30883\ 63731\ 56116\ldots\\
&- 0.02586\ 93530\ 81429\ 91501\ 58145\ 81406\ 92252\ 24536\ 92736\ 31896\ldots i.
\end{aligned}$$

4.3. **Coefficients appearing in exponential sums of $\nu(n)$.** Using similar arguments as above, we obtain the functional equation

$$\eta_\theta(s) = \frac{\theta}{2^s - \theta - 1}\left(\frac{4}{3}\right)^s + \frac{1}{2^s - \theta - 1} \sum_{m\geq 1} \binom{s+m-1}{m} \frac{1+\theta(-1)^m}{4^m}\, \eta_\theta(s+m), \tag{4.6}$$

for the series

$$\eta_\theta(s) = \sum_{n\geq 1} \theta^{\nu(n)}\, (n+1/2)^{-s}.$$

Iterating (4.6) again yields numerical approximations to $\eta_\theta(s)$. With these $\eta_\theta(s)$'s, the values of $B_\theta(s)$ can be computed by the expression

$$B_\theta(s) = 2^{1-s} \sum_{m\geq 1} \binom{s+2m-1}{2m} \frac{\eta_\theta(s+2m)}{16^m}.$$

With $s = \rho - 1$, we obtain

$$B_2(\rho - 1) = 0.22334\ 70274\ 23462\ 06739\ 82010\ 64124\ 73348\ 43558\ 40047\ 70137\ldots,$$

which yields the mean value of the periodic function M_2:

$$0.86360\ 49963\ 99079\ 60496\ 05033\ 61308\ 09499\ 10614\ 32997\ 57541\ldots;$$

this agrees with that given in [16].

4.4. **Other coefficients.** The mean value of the periodic function Z in (3.28) can be computed similarly; its numerical value is

$$1.24965\ 73149\ 83882\ 77854\ 29433\ 00014\ 57215\ 00885\ 07064\ 01518\ldots.$$

In the case of the Bose-Nelson sorting network (see Section 3.6), the function

$$\kappa(s) := \sum_{n\geq 1} \frac{\Delta f_n}{(n+1/2)^s}$$

satisfies the functional equation

$$\kappa(s) = \frac{1}{2^s-3}\left(\zeta\left(s,\frac{3}{4}\right) + \sum_{m\geq 1}\binom{s+m-1}{m}\frac{2+(-1)^m}{4^m}\kappa(s+m)\right).$$

From this equation and the relation

$$\frac{1}{2}\sum_{n\geq 1}\Delta f_n\left(\frac{1}{(2n)^{\rho-1}} - \frac{2}{(2n+1)^{\rho-1}} + \frac{1}{(2n+2)^{\rho-1}}\right)$$
$$= \frac{2}{3}\sum_{m\geq 1}\binom{\rho+2m-2}{2m}\frac{\kappa(\rho+2m-1)}{4^m},$$

we can compute the mean value of the function K in (3.30), giving the approximate value

$$1.08958\ 03027\ 44297\ 39417\ 20270\ 53669\ 38508\ 47692\ 44469\ 52816\ldots$$

Similar calculations also yield an approximation to the mean value of the periodic function H_3 discussed in Section 3.4:

$$h_0 = 1.40922\ 03477\ 84529\ 82145\ 02883\ 99558\ 66864\ 77313\ 78873\ 61184\ldots$$

Acknowledgments. The authors are indebted to Philippe Flajolet for drawing their attention to the transfer operator approach used in Section 3.2.1. They also thank Christian Mauduit for useful suggestions.

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