Poincaré functional equations, harmonic measures on Julia sets, and fractal zeta functions

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1. Introduction

Connections between the analysis on fractals and the iteration of rational functions were discovered in the earliest publications on diffusion processes on certain self-similar sets, such as the Sierpiński gasket (see, for instance [3, 38]). The connection stems from the fact that time on the successive approximations of the fractal is modelled by a branching process. The relation of branching processes to the iteration of holomorphic functions is known for a long time (see [19]).

More precisely, in order to obtain a diffusion on a fractal, define a sequence of random walks on approximating graphs and synchronise time so that the limiting process is non-constant and continuous. This was the first approach to the diffusion process on the Sierpiński gasket given in [3, 14, 26] and later generalised to other "nested fractals" in [29]. In our description we will follow the lines of definition of self-similar graphs given in [24, 25] and adapt it for our purposes.

We consider a graph $G = (V(G), E(G))$ with vertices $V(G)$ and undirected edges $E(G)$ denoted by $\{x, y\}$. We assume throughout that G does not contain multiple edges nor loops. For $C \subset V(G)$ we call ∂C the vertex boundary, which is given by the set of vertices in $V(G) \setminus C$, which are adjacent to a vertex in C. For $F \subset V(G)$ we define the reduced graph G_F by $V(G_F) = F$ and $\{x, y\} \in E(G_F)$, if x and y are in the boundary of the same component of $V(G) \setminus F$. This requires that removing the set F disconnects the graph G into different components.

The following definition is taken from [25]. It is motivated by the properties of the infinite Sierpiński gasket (see Figure 1). Furthermore, it will turn out that this definition of self-similarity of a graph is reflected by according functional equations for the Green function (the generating function of the transition probabilities) and by rational function relations between the eigenvalues of the transition Laplace operator, which will be exploited later.

Definition 1. A connected infinite graph G is called self-similar with respect to $F \subset V(G)$ and $\varphi : V(G) \to V(G_F)$, if

1. no vertices in F are adjacent in G,
2. the intersection of the boundaries of two different components of $V(G) \setminus F$ does not contain more than one point,
3. φ is an isomorphism of G and G_F.

A random walk on G is given by transition probabilities $p(x, y)$, which are positive, if and only if $\{x, y\} \in E(G)$. For a trajectory $(Y_n)_{n\in\mathbb{N}_0}$ of this random walk with $Y_0 = x \in F$ we define stopping times recursively by

$$T_{m+1} = \min\{k > T_m \mid Y_k \in F \setminus \{Y_{T_m}\}\}, \quad T_0 = 0.$$

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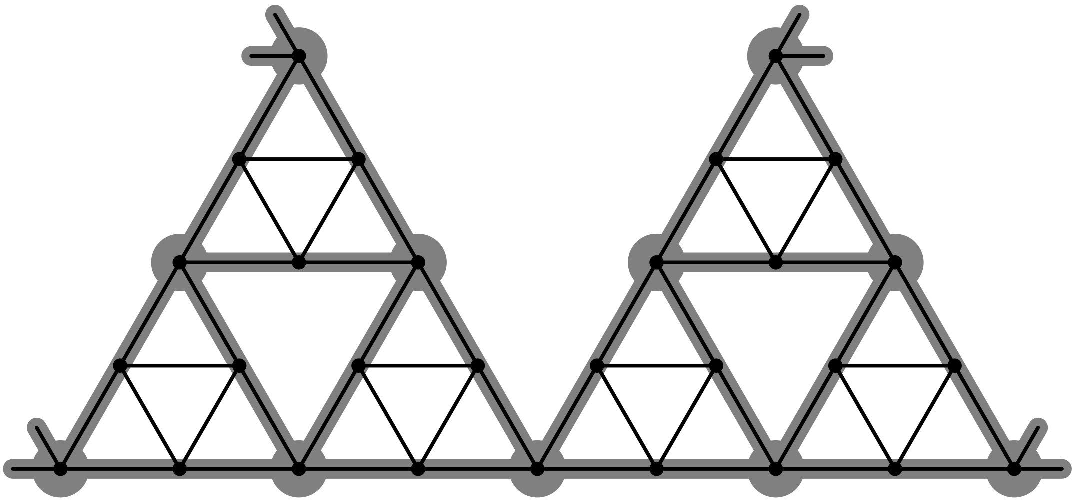

FIGURE 1. The Sierpiński graph G in black with the graph G_F in grey. The corresponding set F consists of the grey vertices.

Then $(Y_{T_m})_{m\in\mathbb{N}_0}$ is a random walk on G_F. Since the underlying graphs G and G_F are isomorphic, it is natural to require that $(\varphi^{-1}(Y_{T_m}))_{m\in\mathbb{N}_0}$ is the same stochastic process as $(Y_n)_{n\in\mathbb{N}_0}$. This requires the validity of equations for the basic transition probabilities

$$\mathbb{P}\left(Y_{T_{n+1}}=\varphi(y)\mid Y_{T_n}=\varphi(x)\right)=\mathbb{P}\left(Y_{n+1}=y\mid Y_n=x\right)=p(x,y). \tag{1.1}$$

These are usually non-linear rational equations for the transition probabilities $p(x,y)$. The existence of solutions of these equations has been the subject of several investigations, and we refer to [31, 32, 33, 41] for further details.

The process $(Y_n)_{n\in\mathbb{N}_0}$ on G and its "shadow" $(Y_{T_n})_{n\in\mathbb{N}_0}$ on G_F are equal, but they are on a different time scale. Every transition $Y_{T_n}\to Y_{T_{n+1}}$ on G_F comes from a path $Y_{T_n}\to Y_{T_n+1}\cdots\to Y_{T_{n+1}-1}\to Y_{T_{n+1}}$ in a component of $V(G)\setminus F$. The *time scaling factor* between these processes is given by

$$\lambda=\mathbb{E}(T_{n+1}-T_n)=\mathbb{E}(T_1).$$

This factor is ≥ 2 by assumption (1) on F. More precisely, the relation between the transition time on G_F and the transition time on G is given by a super-critical ($\lambda>1$) branching process, which replaces an edge $\{\varphi(x),\varphi(y)\}\in G_F$ by a path in G connecting the points x and y without visiting a point in $V(G)\setminus F$ (except for x, and for y in the last step).

In order to obtain a process on a fractal in $\mathbb{R}^d$, we assume further that G is embedded in $\mathbb{R}^d$ (i.e. $V(G)\subset\mathbb{R}^d$). The self-similarity of the graph is carried over to the embedding by assuming that there exists a $\beta>1$ (the *space scaling factor*) such that $F=V(G_F)=\beta V(G)$. The fractal limiting structure is then given by

$$Z_G=\overline{\bigcup_{n=0}^{\infty}\beta^{-n}V(G)}.$$

Iterating this graph decimation we obtain a sequence $G_k=(\beta^{-k}V(G),E(G))$ of (isomorphic) graphs on different scales. The random walks $(Y_n^{(k)})_{n\in\mathbb{N}_0}$ on G_k are connected by time scales with the scaling factor λ. From the theory of branching processes (cf. [19]) it follows that the time on level k scaled by λ^{-k} tends to a random variable. From this it follows that $\beta^{-k}Y_{\lfloor t\lambda^k\rfloor}$ weakly tends to a (continuous time) stochastic process $(X_t)_{t\geq 0}$ on the fractal Z_G. Notice, that β has to be chosen so that the limiting process $(X_t)_{t\geq 0}$ is continuous and not constant; thus there is of course only one (intrinsic) choice for β.

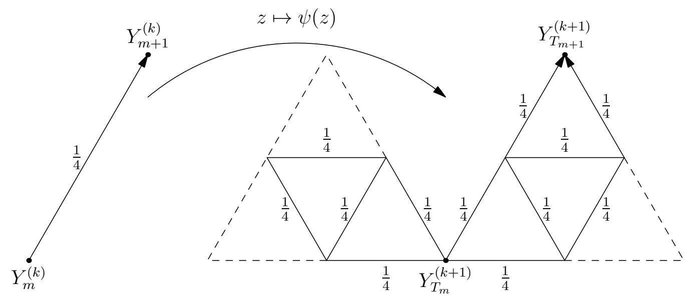

FIGURE 2. Transition between $Y^{(k)}_m$ and $Y^{(k+1)}_{T_m}$

On the level of generating functions, the transition between the random walks on the graphs G_k and G_{k+1} is encoded by the relation

$$G(x, y \mid z) = f(z)G(\varphi(x), \varphi(y) \mid \psi(z)) \tag{1.2}$$

for the Green function

$$G(x, y \mid z) = \sum_{n=0}^{\infty} p_n(x, y)z^n,$$

where $p_n(x, y)$ denotes the n-step transition probability between x and y (cf. [15, 16, 25]). The generating function f encodes paths starting and ending in x without visiting any other point of F, whereas $\psi(z)$ is the probability generating function of all paths starting in a point of $a \in F$, ending in a point of $b \in F$, $b \neq a$, without visiting any point of F different from a, except for the last step.

The Laplace operator on Z_G is then defined as the infinitesimal generator of the semigroup of operators given by

$$A_t f(x) = \mathbb{E}_x f(X_t),$$

namely

$$\triangle f = \lim_{t \to 0+} \frac{A_t f - f}{t}, \tag{1.3}$$

defined for functions f, for which the limit exists.

It has been first observed by Fukushima and Shima [13, 42, 43] that the eigenvalues of the Laplacian on the Sierpiński gasket and its higher dimensional analogues exhibit the phenomenon of *spectral decimation* (see also earlier work by Bellissard [5, 6]). Later on, spectral decimation for more general fractals has been studied by Malozemov, Strichartz, and Teplyaev [1, 2, 30, 45, 47, 48].

Definition 2 (Spectral decimation). The Laplace operator on a p. c. f. self-similar fractal Z_G admits *spectral decimation*, if there exists a rational function R, a finite set A and a constant $\lambda > 1$ such that all eigenvalues of $\triangle$ can be written in the form

$$\lambda^m \lim_{n \to \infty} \lambda^n R^{(-n)}(\{w\}), \quad w \in A, \quad m \in \mathbb{N} \tag{1.4}$$

where the preimages of w under n-fold iteration of R have to be chosen such that the limit exists. Furthermore, the multiplicities $\beta_m(w)$ of the eigenvalues depend only on w and m, and the generating functions of the multiplicities are rational.

The fact that all eigenvalues of $\triangle$ are negative real implies that the Julia set of R has to be contained in the negative real axis. We will exploit this fact later.

The function R occurring in the definition of spectral decimation is conjugate to the function ψ occurring in (1.2) by a linear fractional transformation ξ, *i. e.* $R = \xi \circ \psi \circ \xi^{-1}$. In some cases such as the higher dimensional Sierpiński gaskets, the rational function R is a polynomial. This is the case that will be discussed further in this paper.

2. Polynomial iteration

In order to discuss the consequences of spectral decimation further, we need to introduce some concepts and notation from the iteration theory of polynomials. Throughout, we will denote by $p^{(n)}$ the n-fold iterate of the (polynomial) function p, *i. e.*

$$p^0(z) = z, \quad p^{(n+1)}(z) = p(p^{(n)}(z)). \tag{2.1}$$

Let p be a real polynomial of degree d. We always assume that $p(0) = 0$ and $p'(0) = a_1 = \lambda$ with $|\lambda| > 1$. We refer to [4, 34] as general references for complex dynamics.

We denote the Riemann sphere by $\mathbb{C}_\infty$ and consider p as a map on $\mathbb{C}_\infty$. We recall that the Fatou set $\mathcal{F}(p)$ is the set of all $z \in \mathbb{C}_\infty$ which have an open neighbourhood U such that the sequence $(p^{(n)})_{n\in\mathbb{N}}$ is equicontinuous on U in the chordal metric on $\mathbb{C}_\infty$. By definition $\mathcal{F}(p)$ is open. We will especially need the component of ∞ of $\mathcal{F}(p)$ given by

$$\mathcal{F}_\infty(p) = \left\{z \in \mathbb{C} \mid \lim_{n\to\infty} p^{(n)}(z) = \infty\right\}, \tag{2.2}$$

as well as the basin of attraction of a finite attracting fixed point w_0 ($p(w_0) = w_0$, $|p'(w_0)| < 1$)

$$\mathcal{F}_{w_0}(p) = \left\{z \in \mathbb{C} \mid \lim_{n\to\infty} p^{(n)}(z) = w_0\right\}. \tag{2.3}$$

The complement of the Fatou set is the Julia set $\mathcal{J}(p) = \mathbb{C}_\infty \setminus \mathcal{F}(p)$.

The filled Julia set is given by

$$\mathcal{K}(p) = \left\{z \in \mathbb{C}_\infty \mid (p^{(n)}(z))_{n\in\mathbb{N}} \text{ is bounded}\right\} = \mathbb{C}_\infty \setminus \mathcal{F}_\infty(p). \tag{2.4}$$

Furthermore, it is known that (cf. [12])

$$\partial\mathcal{K}(p) = \partial\mathcal{F}_\infty(p) = \mathcal{J}(p). \tag{2.5}$$

This relation only holds for polynomials; for the iteration of general rational functions the situation is much more complicated.

3. Poincaré's functional equation

We now want to analyse equation (1.4) further, assuming that $R = p$, a polynomial of degree d with a fixed point at 0 with $p'(0) = \lambda > 1$. Let z be a complex number obtained as a limit

$$\lim_{n\to\infty} \lambda^n p^{(-n)}(\{w\}); \tag{3.1}$$

this means that

$$\lim_{n\to\infty} p^{(n)}(\lambda^{-n} z) = w. \tag{3.2}$$

It is a well known fact from the iteration theory of polynomials that the function sequence $(p^{(n)}(\lambda^{-n}z))_n$ converges uniformly on compact sets to an entire function $\Phi(z)$. This function satisfies the Poincaré functional equation

$$\Phi(\lambda z) = p(\Phi(z)), \quad \Phi(0) = 0, \quad \Phi'(0) = 1. \tag{3.3}$$

The function Φ provides a linearisation of the action of p around 0 and was studied intensively since the fundamental work of H. Poincaré [36, 37]. The order of this function and precise asymptotic information about its maximal function

$$M_\Phi(r) = \max_{|z|=r} |\Phi(z)| \tag{3.4}$$

were derived in [49, 50]. In [8, 9] a complete asymptotic expansion valid in certain angular regions of the complex plane could be obtained. This was used in [10] to give an analytic continuation of the spectral ζ-function

$$\zeta_\triangle(s) = \sum_{-\triangle u = \mu u} \mu^{-s} \tag{3.5}$$

of the Laplace operator to the whole complex plane. For future reference, we denote the abscissa of convergence of this Dirichlet series by $\frac{1}{2}d_S$, the *spectral dimension.* The factor $\frac{1}{2}$ is added by convention so that the classical result of H. Weyl [51] for the asymptotic expansion of the eigenvalue counting function on a compact d-dimensional manifold Ω

$$N_{\triangle}(x) = \sum_{\substack{-\triangle u = \mu u \\ \mu < x}} 1, \tag{3.6}$$

namely (ω_d is the volume of the d-dimensional unit ball)

$$N_{\triangle}(x) \sim \frac{\omega_d}{(2\pi)^d} \mathrm{vol}(\Omega) x^{\frac{d}{2}}$$

is reproduced as a special case.

The values z that can be obtained by (3.1) are exactly the solutions of the equation $\Phi(z) = w$. As is well known from the theory of entire functions (see [7]), the behaviour of the counting function of the number of solutions of $\Phi(z) = w$ in a circle of radius r is directly connected to the growth order of Φ, or more precisely, the maximal function $M_\Phi(r)$ in (3.4).

4. Böttcher's functional equation

As was pointed out in Section 3, the Poincaré-function $\Phi(z)$ given by (3.3) provides a local linearisation of the polynomial function p around its fixed point $z = 0$. The construction of this function as the limit (3.1) depends heavily on the fact that $|\lambda| > 1$ (*repelling fixed point*), where $\lambda = p'(0)$. A similar linearisation can be found for $0 < |\lambda| < 1$ (*attracting fixed point*); the case of an *indifferent fixed point* ($|\lambda| = 1$) is much more delicate and the existence of a local linearisation depends heavily on Diophantine conditions on the argument of λ (see [4,34]). The case of vanishing derivative $\lambda = 0$ (*hyper-attracting fixed point*) leads to a different kind of linearisation, which shall be the subject of this section. Notice, that $z = \infty$ is such a fixed point for a polynomial of degree $d \geq 2$, if considered as a function on the Riemann sphere.

The Böttcher functional equation associated to the hyper-attracting fixed point ∞ of a polynomial $p(z) = a_d z^d + \cdots + a_0$ of degree $d \geq 2$ is given by

$$a_d (g(z))^d = g(p(z)). \tag{4.1}$$

The solution of this equation exists in some neighbourhood of ∞ and can be expressed as a Laurent series around ∞

$$g(z) = z + \sum_{n=0}^{\infty} \frac{c_n}{z^n}.$$

Furthermore, the sequence of functions $(a_d^{-\frac{1}{d-1}} (p^{(n)}(z))^{d^{-n}})_n$ converges uniformly to g on compact subsets of $\mathbb{C}_\infty$ contained in the domain of g (if the branches of the d-th roots are chosen accordingly).

The Böttcher function $g(z)$ admits the integral representation, which also provides an analytic continuation of g to any simply connected subset of $\mathbb{C} \setminus \mathcal{K}(p)$

$$g(z) = \exp\left(\int_{\mathcal{J}(p)} \log(z - x) \,\mathrm{d}\mu(x) \right), \tag{4.2}$$

where μ denotes the harmonic measure on $\mathcal{J}(p)$. The measure μ is the unique probability measure supported on $\mathcal{J}(p)$ minimising the logarithmic energy

$$\mathcal{E}(\nu) = \int_{\mathcal{J}(p)} \int_{\mathcal{J}(p)} \log \frac{1}{|z-w|} \,\mathrm{d}\nu(z) \,\mathrm{d}\nu(w)$$

(see [39]). For the measure μ the corresponding potential

$$U_\mu(z) = \int_{\mathcal{J}(p)} \log \frac{1}{|z-w|} \,\mathrm{d}\mu(w)$$

is constant on $\mathcal{K}(p)$; the constant equals $\frac{1}{d-1} \log a_d$, the logarithm of the capacity of $\mathcal{J}(p)$. This is also the value of the energy $\mathcal{E}(\mu)$.

The measure μ can be obtained as the weak limit of the sequence of measures

$$\frac{1}{d^n}\sum_{p^{(n)}(\xi)=x}\delta_\xi, \tag{4.3}$$

where x is an arbitrarily chosen point and δ_ξ denotes a unit point mass at ξ. The fact that (4.2) and (4.1) yield the same function, follows immediately from $p^*(\mu)=d\mu$.

Equation (4.2) can be used to obtain an analytic continuation of $g(z)$ to any simply connected subset of $\mathbb{C}_\infty\setminus\mathcal{K}(p)$. Furthermore, if $\mathcal{K}(p)$ is connected, $1/(a_d^{1/(d-1)}g(z))$ is the Riemann mapping, mapping $\mathbb{C}_\infty\setminus\mathcal{K}(p)$ to the unit circle. The function $\log|g(z)|$ is the Green function for the logarithmic potential on $\mathcal{F}_\infty(p)$ and

$$\lim_{\substack{z\to z_0\\ z\in\mathcal{F}_\infty(p)}}|g(z)|=a_d^{-\frac{1}{d-1}}\Leftrightarrow z_0\in\mathcal{J}(p).$$

In the case that $\mathcal{K}(p)$ is not connected, the mapping $g:\mathbb{C}\setminus\mathcal{K}(p)\to\mathbb{C}$ is much more complicated. For further details we refer to [28].

5. Asymptotic behaviour of Poincaré functions

Combining the solutions of the functional equations (3.3) and (4.1), we are now in the position to obtain an asymptotic expansion of the Poincaré function Φ for real values of λ inside angular regions, where Φ tends to ∞.

Consider the function $h(z)=g(\Phi(z))$ in an angular region

$$W_{\alpha,\beta}=\{z\in\mathbb{C}\setminus\{0\}\mid\alpha<\arg(z)<\beta\},$$

where Φ tends to ∞. Then h satisfies the functional equation

$$a_d h(z)^d=h(\lambda z),$$

which has the solution

$$h(z)=a_d^{-\frac{1}{d-1}}\exp\left(z^\rho F(\log_\lambda z)\right), \tag{5.1}$$

where $\rho=\log_\lambda d$, and F is a periodic function of period 1, which is holomorphic on the strip

$$\{z\in\mathbb{C}\mid\frac{\alpha}{\log\lambda}<\Im(z)<\frac{\beta}{\log\lambda}\}.$$

Furthermore, the fact that Φ tends to ∞ in $W_{\alpha,\beta}$ yields

$$\forall z\in W_{\alpha,\beta}:\Re(z^\rho F(\log_\lambda z))>0.$$

Writing

$$g^{(-1)}(w)=w+\sum_{n=0}^\infty\frac{b_n}{w^n},$$

we obtain the full asymptotic expansion

$$\Phi(z)=a_d^{-\frac{1}{d-1}}\exp\left(z^\rho F(\log_\lambda z)\right)+\sum_{n=0}^\infty b_n a_d^{\frac{n}{d-1}}\exp\left(-nz^\rho F(\log_\lambda z)\right) \tag{5.2}$$

valid for $z\in W_{\alpha,\beta}$. This derivation is the content of [9, Theorem 2.1].

Taking the logarithm of (5.1) and using the fact that $\Phi(z)=z+O(z^2)$ for $z\to0$, we obtain

$$\log g(z)=\int_{\mathcal{J}(p)}\log(z-x)\,\mathrm{d}\mu(x)\sim-\frac{1}{d-1}\log a_d+z^\rho F(\log_\lambda z)+O(z^{2\rho}) \tag{5.3}$$

for $z\to0$ in $W_{\alpha,\beta}$. On the other hand, taking the logarithm of (5.2), we get

$$\log\Phi(z)=-\frac{1}{d-1}\log a_d+z^\rho F(\log_\lambda z)+O(\exp(-z^\rho F(\log_\lambda z))) \tag{5.4}$$

for $z \to \infty$ again in $W_{\alpha,\beta}$. This means that the same periodic function F can be observed in the asymptotic behaviour of $\log g$ for $z \to 0$ and $\log \Phi$ for $z \to \infty$. The function F encodes properties of the Julia set $\mathcal{J}(p)$ in the following sense.

Theorem 3 ([9, Theorem 2.2]). *The periodic function F is constant, if and only if the polynomial is either linearly conjugate to z^d or to the Chebyshev polynomial of the first kind $T_d(z)$. In the first case the Julia set $\mathcal{J}(p)$ is a circle, in the second case the Julia set $\mathcal{J}(p)$ is a closed interval.*

Remark 4. It is known from [17] that the circle and the interval are the only cases of smooth Julia sets; these occur precisely for the polynomials described in the Theorem.

6. Fractal zeta functions

We now return to the study of the spectrum of the Laplacian $\triangle$ on a fractal admitting spectral decimation with the polynomial p in the sense of Definition 2. In this case the Julia set $\mathcal{J}(p)$ is contained in the negative real axis, which implies that $\rho = \log_\lambda d \le \frac{1}{2}$ by [9, Theorem 4.1]. The Poincaré function Φ is thus an entire function of order $\rho \le \frac{1}{2}$. Here, the case $\rho = \frac{1}{2}$ can only occur, if $\mathcal{J}(p)$ is an interval, or equivalently p is a Chebyshev polynomial. In the context of fractals with spectral decimation, this occurs, if the fractal is a compact interval viewed as a self-similar fractal. If this case is ruled out, we have $\rho < \frac{1}{2}$. Functions of order $< \frac{1}{2}$ are unbounded on every ray (see [7]). Furthermore, this together with the fact that Φ attains values in $\mathcal{J}(p) = \mathcal{K}(p)$ only for negative real arguments yields that

$$\lim_{\substack{z\to\infty\\ z\in W_{-\pi,\pi}}} \Phi(z) = \infty. \tag{6.1}$$

Especially, this implies that $\lim_{x\to+\infty} \Phi(x) = \infty$ and thus (5.2) holds for $z \to +\infty$ along the positive real axis.

Let $-\xi_\ell(w)$ ($\ell = 1, 2\ldots$) denote the solutions of $\Phi(z) = w$; for $w = 0$, we set $\xi_0(0) = 0$ and $\xi_\ell(0) \ne 0$ for $\ell = 1, 2, \ldots$. Define

$$\Phi_0(z) = \frac{1}{z}\Phi(z) \quad \text{and} \quad \Phi_w(z) = 1 - \frac{1}{w}\Phi(z).$$

Then we have the following Hadamard product expansion

$$\Phi_w(z) = \prod_{\ell=1}^{\infty} \left(1 + \frac{z}{\xi_\ell(w)}\right). \tag{6.2}$$

Taking the Mellin transform of the logarithm of (6.2) yields

$$M_w(s) = \int_0^\infty \log(\Phi_w(x)) x^{s-1}\,\mathrm{d}x = \frac{\pi}{s \sin \pi s} \sum_{\ell=1} \xi_\ell(w)^s \tag{6.3}$$

for $-1 < \Re(s) < -\rho$. The left inequality comes from the fact that $\log(\Phi_w(x)) = O(x)$ for $x \to 0$, whereas the right inequality comes from the behaviour of Φ for $x \to \infty$ given in (5.2): $\log(\Phi_w(x)) = O(x^\rho)$.

The functions

$$\zeta_{\Phi,w}(s) = \sum_{\ell=1} \xi_\ell(w)^{-s} \tag{6.4}$$

will be used to derive an expression for $\zeta_\triangle$ later. In order to obtain an analytic continuation of $\zeta_\triangle$ to the whole complex plane, we will need analytic continuations of the functions $\zeta_{\Phi,w}$. We will follow the lines of [11]; similar, but slightly different ideas were used in [10].

We consider the function

$$\Psi_w(z) = \frac{p(\Phi(z)) - w}{a_d(\Phi(z) - w)} = \frac{\Phi_w(\lambda z)}{a_d(-w)^{d-1}\Phi_w(z)^d}$$

for $w \ne 0$. Taking the logarithm, we obtain

$$\log \Psi_w(z) = \log \Phi_w(\lambda z) - d \log \Phi_w(z) - \log a_d - (d-1)\log(-w);$$

this function tends to 0 like $\exp(-cz^\rho)$ for $z \to +\infty$. Taking the Mellin transform and using standard methods to obtain analytic continuations of such transforms, we obtain (we indicate the region of validity of the equation in every line)

$$\begin{aligned}
&(\lambda^{-s} - d)M_w(s) \\
&= \int_0^\infty (\log \Phi_w(\lambda x) - d \log \Phi_w(x))\, x^{s-1}\, \mathrm{d}x \quad (\text{for } -1 < \Re s < -\rho) \\
&= \int_0^1 (\log \Phi_w(\lambda x) - d \log \Phi_w(x))\, x^{s-1}\, \mathrm{d}x - (\log a_d + (d-1)\log(-w))\frac{1}{s} \\
&\quad + \int_1^\infty (\log \Phi_w(\lambda x) - d \log \Phi_w(x) - \log a_d - (d-1)\log(-w))\, x^{s-1}\, \mathrm{d}x \quad (\text{for } \Re s > -1) \\
&= \int_0^\infty \log(\Psi_w(x)) x^{s-1}\, \mathrm{d}x \quad (\text{for } \Re s > 0).
\end{aligned}$$

The above computation shows that $M_w(s)$ has a simple pole at $s = 0$ with residue

$$\operatorname*{Res}_{s=0} M_w(s) = \frac{\log a_d}{d-1} + \log(-w).$$

Furthermore, it provides an analytic continuation of $M_w(s)$ to the half-plane $\Re s > 0$; the second line also gives the analytic continuation to the half-plane $\Re s > -1$. Using (6.3) gives an analytic continuation of $\zeta_{\Phi,w}(s)$ to the half-plane $\Re s < 0$

$$\zeta_{\Phi,w}(s) = \frac{s \sin \pi s}{\pi(\lambda^s - d)} \int_0^\infty \log(\Psi_w(x)) x^{-s-1}\, \mathrm{d}x.$$

From this we derive the existence of "trivial zeros" $\zeta_{\Phi,w}(-m) =$ (for $m \in \mathbb{N}_0$). Notice, that the simple pole of $M_w(s)$ at $s = 0$ is cancelled by the double zero of $s \sin \pi s$. Observing this, we also obtain

$$\zeta'_{\Phi,w}(0) = -\frac{\log a_d}{d-1} - \log(-w).$$

Similar computations yield the analytic continuation of $\zeta_{\Phi,0}$ to the whole complex plane; this function has "trivial" zeros $\zeta_{\Phi,0}(-m) = 0$ (for $m \in \mathbb{N}$) and

$$\zeta_{\Phi,0}(0) = 1, \quad \zeta'_{\Phi,0}(0) = -\frac{\log a_d}{d-1}.$$

Simple poles of $\zeta_{\Phi,w}(s)$ can occur only at the solutions of $\lambda^s = d$, namely $s = \rho + 2k\pi i/\log \lambda$ ($k \in \mathbb{Z}$). These poles are in correspondence with the growth order of Φ, which implies that there is a pole at $s = \rho$. The other poles for $k \neq 0$ only occur, if the periodic function F in (5.2) is not constant. Theorem 3 characterises the polynomials, for which the periodic function F is constant.

We now use the assumption that the multiplicities $\beta_m(w)$ of the eigenvalues $\lambda^m \xi_\ell(w)$ have a rational generating function (see Definition 2). Let

$$B_w(x) = \sum_{m=0}^\infty \beta_m(w) x^m.$$

Then using our knowledge on the eigenvalues of $\triangle$ together with our assumptions from the definition of spectral decimation, we obtain

$$\zeta_\triangle(s) = \sum_{w \in A} B_w(\lambda^{-s}) \zeta_{\Phi,w}(s). \tag{6.5}$$

This expression provides the analytic continuation of the spectral zeta function to the whole complex plane.

If $\rho < \frac{1}{2}d_S$ then all the functions $\zeta_{\Phi,w}(s)$ are holomorphic in a half-plane $\Re s > \frac{1}{2}d_S - \varepsilon$ for some $\varepsilon > 0$. On the other hand, $\zeta_\triangle(s)$ has a simple pole at $s = \frac{1}{2}d_S$ by the fact that $N_\triangle(x) \asymp x^{\frac{1}{2}d_S}$ (see [23]). Thus at least one of the rational functions $B_w(x)$ has to have a pole at $x = \lambda^{-\frac{1}{2}d_S}$. Since all the rational functions B_w have positive power series coefficients (the multiplicities of the eigenvalues), there can be no cancellation of poles, which implies that the functions B_w can have at most a simple pole at $s = \lambda^{-\frac{1}{2}d_S}$. Let W denote the set of all $w \in A$, for which the corresponding function B_w has a (simple) pole at $x = \lambda^{-\frac{1}{2}d_S}$. Then we write the Laurent expansion of $B_w(x)$ around $x = \lambda^{-\frac{1}{2}d_S}$ in the form

$$B_w(x) = \frac{c_1(w)}{1 - x\lambda^{\frac{1}{2}d_S}} + \cdots .$$

This implies that $c_1(w) > 0$ by the combinatorial interpretation of B_w. Then the Dirichlet series

$$\eta(s) = \sum_{w \in W} c_1(w)\zeta_{\Phi,w}(s)$$

has positive coefficients. By [27, Theorem 9.5, p. 184] this implies that $\eta(\frac{1}{2}d_S + ik\tau) = 0$ cannot hold for fixed $\tau > 0$ and all $k \in \mathbb{Z} \setminus \{0\}$. Thus the function

$$\sum_{w \in W} B_w(\lambda^{-s})\zeta_{\Phi,w}(s)$$

has a simple pole at $s = \frac{1}{2}d_S$ and at least two non-real poles on the line $\Re s = \frac{1}{2}d_S$. The remaining summands in (6.5) do not have poles on the line $\Re s = \frac{1}{2}d_S$; thus the function $\zeta_\triangle(s)$ has at least two non-real poles on this line.

As a conclusion, we have reached the following theorem (see [10, Theorem 9]).

Theorem 5. *Let Z_G be a p. c. f. self-similar compact fractal, whose Laplace operator $\triangle$ admits spectral decimation in the sense of Definition 2 with a polynomial of degree d. Then the Dirichlet generating function of the eigenvalues of $\triangle$*

$$\zeta_\triangle(s) = \sum_{-\triangle u = \mu u} \frac{1}{\mu^s},$$

has a meromorphic continuation to the whole complex plane with poles contained in a finite union of sets $\{\rho_k + 2\pi i m\sigma \mid m \in \mathbb{Z}\}$, where $\sigma = \frac{1}{\log\lambda}$ and λ is the parameter coming from spectral decimation. There is a simple pole at $s = \frac{1}{2}d_S$. If $\log_\lambda d < \frac{1}{2}d_S$ then $\zeta_\triangle(s)$ has at least two non-real poles on the line $\Re s = \frac{1}{2}d_S$.

Remark 6. The case of $G = [0,1]$ which gives the Riemann zeta function and has $\log_\lambda d = \frac{1}{2}d_S$ shows that the condition $\log_\lambda d < \frac{1}{2}d_S$ is needed for the last assertion. The case $\log_\lambda d > \frac{1}{2}d_S$ cannot occur.

7. Consequences and a conjecture

We introduce one further notion in connection with diffusion on a fractal, namely the trace of the heat operator

$$P(t) = \mathrm{Tr}(A_t) = \mathrm{Tr}(e^{t\triangle}) = \sum_{-\triangle u = \mu u} e^{-\mu t}. \tag{7.1}$$

In the classical case of Riemannian manifold studied by H. Weyl [51], the behaviour of this function for $t \to 0+$ was used to prove asymptotic relations for the eigenvalue counting function $N_\triangle$. Furthermore, precise information on the asymptotic behaviour of $P(t)$ for $t \to 0+$ can be used to prove that the spectral zeta function of $\triangle$ on a Riemannian manifold has an analytic continuation to the whole complex plane (see [35, 40]). In the case of a fractal with spectral decimation, we proceed in the opposite direction; starting from precise information on the eigenvalues we derive the existence of an analytic continuation of $\zeta_\triangle$ to the whole complex plane with the location of all poles, from which we conclude asymptotic information about $N_\triangle$ and $P(t)$. We sum this up by citing the following theorem.

Theorem 7 ([10, Theorem 10]). *Let Z_G be a p. c. f. self-similar compact fractal, whose Laplace operator $\triangle$ admits spectral decimation in the sense of Definition 2. Then the following are equivalent:*

1. *$\zeta_\triangle(s)$ has at least two non-real poles in the set $\frac{1}{2}d_S + \frac{2\pi i}{\log\lambda}\mathbb{Z}$,*

2. *the limit* $\lim_{x\to\infty} x^{-\frac12 d_S} N_\triangle(x)$ *does not exist, where* $N_\triangle(x)$ *denotes the eigenvalue counting function* (3.6),
3. *the limit* $\lim_{t\to0+} P(t)t^{\frac12 d_S}$ *does not exist, where* $P(t)$ *denotes the trace of the heat kernel* (7.1).

Remark 8. Recently, N. Kajino [20, 21, 22] could prove an asymptotic expansion of the trace of the heat kernel on a p. c. f. fractal and also on the generalised Sierpiński carpet

$$P(t) = \sum_{k=0}^{n} t^{-\alpha_k} G_k(\log t) + O\left(\exp\left(-ct^{-\gamma}\right)\right) \quad \text{for } t \to 0+$$

for certain exponents $\alpha_0 > \alpha_1 > \cdots > \alpha_n \geq 0$, periodic continuous functions G_k $(k = 0, \ldots, n)$, and $c, \gamma > 0$. This result was obtained without precise knowledge of the eigenvalues and properties of the zeta function. This was used in [44] to obtain an analytic continuation of the zeta function $\zeta_\triangle$ to the whole complex plane in these cases.

Remark 9. Theorems 5 and 7 together show that the limit $\lim_{x\to\infty} x^{-\frac12 d_S} N_\triangle(x)$ does not exist for fractals admitting spectral decimation with a polynomial of degree d and $\log_\lambda d < \frac12 d_S$.

More precisely, in the case that $\log_\lambda d < \frac12 d_S$ we obtain

$$N_\triangle(x) = x^{\frac12 d_S} Q(\log_\lambda x) + o\left(x^{\frac12 d_S}\right) \quad \text{for } x \to \infty$$

and

$$P(t) = t^{-\frac12 d_S} R(\log_\lambda t) + O\left(t^{-\frac12 d_S + \varepsilon}\right) \quad \text{for } t \to 0+$$

for some $\varepsilon > 0$ and for continuous periodic functions with period 1, Q and R (see [10, 11]).

Remark 10. In [18] it was shown that there exist gaps in the spectrum of the Laplacian if and only if the Julia set of the spectral decimation function R is totally disconnected. Spectral gaps (in the sense that there exists a subsequence, along which the quotient of consecutive eigenvalues stays bounded away from 1) yields uniform convergence of the Fourier series of continuous functions along the subsequence mentioned above (see [46]).

In the context of fractals the polynomials occurring for spectral decimation have a negative real Julia set $\mathcal{J}(p)$ (which is a Cantor set, except for the case when $\mathcal{J}(p)$ is an interval; this last case only occurs, if the underlying fractal itself is an interval). Nevertheless, the Poincaré and Böttcher functions can be defined and studied for any polynomial p of degree $d \geq 2$. This was done in [9]. There the asymptotic behaviour of the zero counting function of Φ

$$N_\Phi(x) = \sum_{\substack{|\xi|<x \\ \Phi(\xi)=0}} 1 \tag{7.2}$$

could be related to the behaviour of the harmonic measure of small balls around the origin, namely

Theorem 11 ([9, Theorem 5.2]). *Let* Φ *be the entire solution of* (3.3), *and let* $\rho = \log_\lambda d$. *Then the limit*

$$\lim_{x\to\infty} x^{-\rho} N_\Phi(x) \tag{7.3}$$

exists, if and only if the limit

$$\lim_{t\to0} t^{-\rho} \mu(B(0,t)) \tag{7.4}$$

exists.

We repeat the following conjecture about the existence of the limits (7.3) and (7.4)

Conjecture ([9]). The limits (7.3) and (7.4) exist, if and only if the polynomial p is either linearly conjugate to a pure power or a Chebyshev polynomial of the first kind.

References

[1] N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst, and A. Teplyaev, *Vibration modes of* $3n$*-gaskets and other fractals*, J. Phys. A **41** (2008), no. 1, 015101, (21pp).

[2] ———, *Vibration spectra of finitely ramified, symmetric fractals*, Fractals **16** (2008), 243–258.

[3] M. T. Barlow and E. A. Perkins, *Brownian motion on the Sierpiński gasket*, Probab. Theory Relat. Fields **79** (1988), 543–623.

[4] A. F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, no. 132, Springer Verlag, 1991.

[5] J. Bellissard, *Stability and instability in quantum mechanics*, Trends and developments in the eighties (Bielefeld, 1982/1983), World Sci. Publishing, Singapore, 1985, pp. 1–106.

[6] ———, *Renormalization group analysis and quasicrystals*, Ideas and methods in quantum and statistical physics (Oslo, 1988), Cambridge Univ. Press, Cambridge, 1992, pp. 118–148.

[7] R. P. Boas, Jr., *Entire functions*, Academic Press Inc., New York, 1954.

[8] G. Derfel, P. J. Grabner, and F. Vogl, *Asymptotics of the Poincaré functions*, Probability and Mathematical Physics: A Volume in Honor of Stanislav Molchanov (Montreal) (D. Dawson, V. Jaksic, and B. Vainberg, eds.), CRM Proceedings and Lecture Notes, vol. 42, Centre de Recherches Mathématiques, 2007, pp. 113–130.

[9] ———, *Complex asymptotics of Poincaré functions and properties of Julia sets*, Math. Proc. Cambridge Philos. Soc. **145** (2008), 699–718.

[10] ———, *The Zeta function of the Laplacian on certain fractals*, Trans. Amer. Math. Soc. **360** (2008), 881–897.

[11] ———, *Laplace operators on fractals and related functional equations*, J. Phys. A **45** (2012), no. 46, 463001 (34pp).

[12] K. J. Falconer, *Fractal Geometry*, John Wiley & Sons Inc., Hoboken, NJ, 2003, Mathematical foundations and applications.

[13] M. Fukushima and T. Shima, *On the spectral analysis for the Sierpiński gasket*, J. of Potential Analysis **1** (1992), 1–35.

[14] S. Goldstein, *Random walks and diffusions on fractals*, Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., vol. 8, Springer, New York, 1987, pp. 121–129.

[15] P. J. Grabner, *Functional iterations and stopping times for Brownian motion on the Sierpiński gasket*, Mathematika **44** (1997), 374–400.

[16] P. J. Grabner and W. Woess, *Functional iterations and periodic oscillations for the simple random walk on the Sierpiński graph*, Stochastic Processes Appl. **69** (1997), 127–138.

[17] D. H. Hamilton, *Length of Julia curves*, Pacific J. Math. **169** (1995), 75–93.

[18] K. E. Hare, B. A. Steinhurst, A. Teplyaev, and D. Zhou, *Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified fractals*, Math. Res. Lett. **19** (2012), no. 3, 537–553.

[19] T. E. Harris, *The Theory of Branching Processes*, Springer, Berlin, New York, 1963.

[20] N. Kajino, *Spectral asymptotics for Laplacians on self-similar sets*, J. Funct. Anal. **258** (2010), no. 4, 1310–1360.

[21] ———, *On-diagonal oscillation of the heat kernels on post-critically finite self-similar fractals*, Probab. Theory Related Fields **156** (2013), no. 1-2, 51–74.

[22] ———, *Log-periodic asymptotic expansion of the spectral partition function for self-similar sets*, Comm. Math. Phys. **328** (2014), no. 3, 1341–1370.

[23] J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals*, Comm. Math. Phys. **158** (1993), 93–125.

[24] B. Krön, *Green functions on self-similar graphs and bounds for the spectrum of the Laplacian*, Ann. Inst. Fourier (Grenoble) **52** (2002), 1875–1900.

[25] B. Krön and E. Teufl, *Asymptotics of the transition probabilities of the simple random walk on self-similar graphs*, Trans. Amer. Math. Soc. **356** (2004), 393–414.

[26] S. Kusuoka, *A diffusion process on a fractal*, Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), Academic Press, Boston, MA, 1987, pp. 251–274.

[27] M. L. Lapidus and M. van Frankenhuysen, *Fractal geometry and number theory,* Complex dimensions of fractal strings and zeros of zeta functions, Birkhäuser Boston Inc., Boston, MA, 2000.

[28] G. M. Levin and M. L. Sodin, *Polynomials with a disconnected Julia set, and the Green mapping*, Dynamical systems and complex analysis (Russian) (V. A. Marchenko, ed.), "Naukova Dumka", Kiev, 1992, pp. 17–24.

[29] T. Lindstrøm, *Brownian Motion on Nested Fractals*, Mem. Amer. Math. Soc., vol. 420, Amer. Math. Soc., 1990.

[30] L. Malozemov and A. Teplyaev, *Self-similarity, operators and dynamics*, Math. Phys. Anal. Geom. **6** (2003), 201–218.

[31] V. Metz, *Hilbert's projective metric on cones of Dirichlet forms*, J. Funct. Anal. **127** (1995), no. 2, 438–455.

[32] ______, *Renormalization contracts on nested fractals*, J. Reine Angew. Math. **480** (1996), 161–175.

[33] ______, *The cone of diffusions on finitely ramified fractals*, Nonlinear Anal. **55** (2003), no. 6, 723–738.

[34] J. Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, N. J, 2006.

[35] S. Minakshisundaram and Å. Pleijel, *Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds*, Canadian J. Math. **1** (1949), 242–256.

[36] H. Poincaré, *Sur une classe étendue de transcendantes uniformes*, C. R. Acad. Sci. Paris **103** (1886), 862–864.

[37] ______, *Sur une classe nouvelle de transcendantes uniformes*, J. Math. Pures Appl. IV. Ser. **6** (1890), 316–365.

[38] R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters*, J. Physique Lettres **44** (1983), L13–L22.

[39] T. Ransford, *Potential Theory in the Complex Plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995.

[40] S. Rosenberg, *The Laplacian on a Riemannian manifold*, London Mathematical Society Student Texts, vol. 31, Cambridge University Press, Cambridge, 1997.

[41] C. Sabot, *Existence and uniqueness of diffusions on finitely ramified self-similar fractals*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 5, 605–673.

[42] T. Shima, *The eigenvalue problem for the Laplacian on the Sierpiński gasket*, Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990) (K.D. Elworthy and N. Ikeda, eds.), Pitman Res. Notes Math. Ser., vol. 283, Longman Sci. Tech., Harlow, 1993, pp. 279–288.

[43] ______, *On eigenvalue problems for Laplacians on p.c.f. self-similar sets*, Japan J. Indust. Appl. Math. **13** (1996), 1–23.

[44] B. A. Steinhurst and A. Teplyaev, *Existence of a meromorphic extension of spectral zeta functions on fractals*, Lett. Math. Phys. **103** (2013), no. 12, 1377–1388.

[45] R. S. Strichartz, *Fractafolds based on the Sierpiński gasket and their spectra*, Trans. Amer. Math. Soc. **355** (2003), 4019–4043.

[46] ______, *Laplacians on fractals with spectral gaps have nicer Fourier series*, Math. Res. Lett. **12** (2005), no. 2-3, 269–274.

[47] R. S. Strichartz and A. Teplyaev, *Spectral analysis on infinite Sierpiński fractafolds*, J. Anal. Math. **116** (2012), 255–297.

[48] A. Teplyaev, *Spectral zeta functions of fractals and the complex dynamics of polynomials*, Trans. Amer. Math. Soc. **359** (2007), no. 9, 4339–4358 (electronic).

[49] G. Valiron, *Lectures on the General Theory of Integral Functions*, Private, Toulouse, 1923.

[50] ______, *Fonctions Analytiques*, Presses Universitaires de France, Paris, 1954.

[51] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung)*, Math. Ann. **71** (1912), 441–479.

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