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ROBERT F. TICHY: 50 YEARS — THE UNREASONABLE EFFECTIVENESS OF A NUMBER THEORIST

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FIGURE 1. Climbing in the Gesäuse.

The present volume of UDT is devoted to Robert F. Tichy on the occasion of his 50th birthday. We cordially congratulate him on this occasion and wish him the best for the future.

In this short note we collect highlights of his scientific work. Together with more than 70 coauthors he has written over 200 papers so far, with topics that range from number theory to applications in actuarial mathematics and also mathematical chemistry.

†The title alludes to E. Wigner's 1960 article "The unreasonable effectiveness of mathematics in the natural sciences", and to S. A. Burr's 1992 book "The unreasonable effectiveness of number theory". Both touch aspects of mathematics and number theory which are in Robert Tichy's spirit.

But, of course, the scientific work is only one facet of Robert Tichy's personality. It is remarkable how many students graduated under his supervision since 1984. At least 25 PhD students are known to us, including the five authors of the present article. The majority of his former students are still active researchers in mathematics, 9 with habilitation, three of them in the position of a full professor.

Such quantities, though very impressive, do not explain the individuality of an academic personality. We remember the time when we were students and Robert Tichy was a very young docent of less than 30 years in Vienna and then, with 33, professor in Graz. He was one of those academic teachers who regularly offered special courses on many diverse and challenging mathematical topics. They were popular not only because of the interesting content but also because of Robert Tichy's qualities as an excellent and inspiring lecturer with a very characteristic and appealing sense of humour. Some of his courses were closely connected to his own very diverse mathematical research. So he attracted a considerable part of the interested students of our generation to scientific activity in mathematics.

As Tichy's students (and coauthors) we are always impressed by his sense for good problems and his intuition how to tackle them. None of the problems he posed were unsolvable. But for each of them one had to study and further develop advanced methods from analysis and other parts of mathematics, thus becoming familiar with new and substantial parts of mathematics. At the same time one had the chance to develop one's own mathematical personality by emphasising favourite aspects of a given topic. In face of his extraordinary drive and energy Robert Tichy responds to the growth of young mathematical personalities sensitively and in a very inspiring way. He knows the strengths and weaknesses of each of his students. Thus, whenever he suggested a research topic to one of us, we still could be sure that the topic would fit perfectly to this person's individual mathematical taste and, due also to Robert Tichy's clear-sighted mathematical judgement and taste for fruitful problems, that the topic would open up new perspectives. Particularly motivating for his students is his astonishing ability to combine high exigence with human understanding – aspects of personality which often seem contradictory. Another pleasant result of this individual mentoring is the amicable atmosphere among his former students, where individuality counts more than competition.

Anybody who knows Robert Tichy as a researcher and as a teacher will find it even more unbelievable that he finds the time for his extremely effective activities in science administration: head of department, faculty dean, president of the Austrian Mathematical Society and representative of pure mathematics in the Austrian Science Foundation FWF. Even more impressing: Whenever he meets one of us he takes the time for a personal chat, maybe about his favourite leisure occupation: mountaineering. It is a matter of interpretation whether this affection for nature, physical activity and overcoming difficulties is a compensation for his mathematical passion or part of it.

Definitely part of his mathematical passion are uniform distribution, digital expansions, Diophantine equations and actuarial mathematics. We add some remarks on Robert Tichy's work in these areas.

1. Uniform Distribution Modulo 1

Robert Tichy's first major scientific interest was the theory of uniformly distributed sequences. After his PhD thesis "Gleichverteilung von Mehrfachfolgen und

Ketten" (University of Vienna, 1979) under the supervision of Edmund Hlawka and motivated by the study of the monographs by Kuipers–Niederreiter and Hlawka he substantially contributed to this theory and several of his very first PhD students (Gerhard Turnwald, Michael Drmota, Martin Goldstern, Reinhard Winkler, Peter Grabner) wrote their theses in this area. It is impossible to summarise all his contributions in uniform distribution theory. We thus concentrate only on one aspect, namely on metric discrepancy theory. He also worked, for example, on special sequences, on generalised notions of discrepancy, on uniformly distributed functions and on discrete versions of the theory. Further, together with Michael Drmota he also wrote a Lecture Notes volume "Sequences, Discrepancies and Applications" [9] that covers the 20 years development of the theory after Kuipers' and Niederreiter's monograph from 1974. It has become a standard reference in this field.

It is well known that the theory of uniformly distributed sequences is closely related to that of random numbers. In volume 2 of the monograph series "The art of computer programming" D. Knuth had posed the problem whether there exists a real number $x > 1$ such that the sequence x^{a_n} is u.d. mod 1 for all algorithmically computable sequences a_n of different natural numbers. This problem was solved by Niederreiter and Tichy [23] (and then substantially generalised by Tichy [32]) by proving that almost all $x > 1$ have this property. In [32] Tichy also shows that the discrepancy D_N of this sequence is bounded above by $C \cdot N^{-\frac{1}{2}+\eta}$ (for arbitrarily small $\eta > 0$). Later Losert, Nowak and Tichy [22] considered powers of matrices and proved corresponding results. One should also mention that it is still an unsolved problem whether $\left(\frac{3}{2}\right)^n$ or e^n are u.d. mod 1.

The situation is a little bit different if one considers sequences of the form αx^n. For example, if $x = q \geq 2$ is an integer then the sequence αq^n is u.d. mod 1 if and only if α is normal with respect to base q. Although it is unknown whether $\sqrt{2}$, π or e are normal it is relatively easy to provide explicit normal numbers; e.g. the Champernowne number $\alpha = 0.123456789101112131415\ldots$ is normal in base 10. More generally sequences of the kind αa_n have been considered. For example, H. Weyl proved in his fundamental paper from 1916 that for any increasing sequence a_n of integers it follows that αa_n is u.d. mod 1 for almost all real α. Later J.W.S. Cassels and P. Erdős-J. F. Koksma proved independently that the discrepancy can be bounded by $C \cdot N^{-\frac{1}{2}}(\log N)^{\frac{5}{2}+\eta}$. Another highlight was W. Philipp's 1977 result on lacunary sequences, that is, $a_{n+1} \geq (1+\rho)a_n$ with $\rho > 0$. He showed that for almost all α

$$\frac{1}{4} \leq \limsup_{N\to\infty} \frac{N\, D_N(\alpha a_n)}{\sqrt{N \log\log N}} \leq C. \tag{1.1}$$

Recently, Berkes, Philipp and Tichy [7] provided a very strong generalisation of (1.1) for sequences a_n that are only weakly increasing in the sense that

$$a_{k+k^{1-\eta}} \geq k\, a_k \quad (k \geq k_0(\eta))$$

for some $\eta > 0$. Interestingly, the authors also have to assume two Diophantine conditions on a_n. First, the number of solutions (h, n) with $1 \leq h \leq R$ of the equation $ha_n = b$ shall not exceed $C\, R^\gamma$ for some $\gamma < \frac{1}{2}$ and all natural numbers b. Second, the number of solutions (n_1, n_2, n_3, n_4) with $1 \leq n_i \leq N$ of the equation $h_1 a_{n_1} + h_2 a_{n_2} + h_3 a_{n_3} + h_4 a_{n_4} = 0$ shall not exceed $C_0 N^{1+\beta}$ for some $\beta < \frac{1}{2}$ and for all fixed integers h_i with $0 < |h_i| \leq N^3$ provided that there are no proper subsums that vanish. The proof of this result uses sophisticated methods from probability theory such as martingale embeddings of empirical processes. Furthermore it is an interesting number-theoretic question to show that the Diophantine conditions are satisfied for special sequences. For example, one has to use recent versions of the subspace theorem of J.-H. Evertse, H. P. Schlickewei and W. Schmidt to verify that the so-called Hardy-Littlewood-Pólya sequence a_n, that is, the sequence $q_1^{e_1} \cdots q_r^{e_r}$ arranged in increasing order where q_j are coprime integers, meets these properties.

2. Digital Expansions and Number Theory

In the late 1980s Robert Tichy became interested in number theoretical problems related to digital expansions. This led to a series of contributions to various aspects of this subject and to three PhD-theses on related topics.

It was observed by a French school of mathematicians around J. Coquet, H. Delange, H. Faure, P. Liardet, M. Mendès-France, G. Rauzy, G. Rhin, and many others that digital representations of the positive integers and additive functions given by these representations can be used to construct uniformly distributed sequences. Ergodic theory gives a different point of view on such constructions, which serve as interesting and diverse examples for dynamical systems.

Motivated by the study of uniformly distributed sequences his first contribution was a detailed study of the discrepancy of the sequence $(\alpha s_q(n))_{n\in\mathbb{N}}$ for irrational α in [21], where $s_q(n)$ denotes the q-adic sum-of-digits function. Best possible upper and lower bounds for the discrepancy as well as the uniform discrepancy of such sequences depending on Diophantine approximation properties of α were also given.

Digital functions such as the binary sum-of-digits are not only of number-theoretical interest, but also occur naturally in the context of average case analysis of algorithms. In [10] the classical Mellin-Perron formula was used to derive H. Delange's summation formula for the sum-of-digits function and other digital functions in a unified way. Publication in a Computer Science journal made this a standard reference for such type of results and techniques.

Digital representations with respect to a given increasing sequence of integers $G = (G_n)_{n\in\mathbb{N}_0}$ can be used to construct "adic" compactifications $\mathcal{K}_G$ of the positive integers. The addition-by-one map τ acts naturally on $\mathcal{K}_G$. In [15], Grabner, Liardet and Tichy studied the dynamical system $(\mathcal{K}_G, \tau)$ from a topological and measure theoretic point of view, and in particular investigated the continuity of τ and measure theoretic isomorphisms to group rotations. Continuing earlier work of J. C. Alexander and D. B. Zagier on the entropy of Bernoulli convolutions related to representations of real numbers to base $\frac{1+\sqrt{5}}{2}$, in [14] a description of this entropy in terms of combinatorics on words could be given. In a series of papers T. Kamae developed several techniques for proving spectral disjointness of dynamical systems related to the sum-of-digits functions $s_p(n)$ and $s_q(n)$ with respect to coprime bases p and q. In [16] finite automata and M. Queffélec's point of view based on Šreĭder-characters are used to derive spectral disjointness for skew-products given by more general additive functions with respect to multiplicatively independent bases.

The study of the arithmetic structure of sets defined by congruence relations on the sum-of-digits function or more general additive digital functions has been initiated by A. O. Gelfond and continued by É. Fouvry, C. Mauduit, J. Rivat, and A. Sárközy. The paper [28] contributes to these investigations by proving an Erdős-Kac type theorem for the distribution of the arithmetic function $\omega(n)$, where n is restricted by congruence relations for several additive digital functions with respect to pairwise coprime bases. A very involved combination of the Hardy-Littlewood circle method with sieve techniques was used in [29] to count the number of solutions of the equation

$$N = x_1^k + \cdots + x_s^k$$

under the restriction that $s_q(x_i) \equiv h \pmod m$ (for $i = 1, \ldots, s$), i.e. Waring's problem with digital restrictions.

3. Diophantine Equations

Many Diophantine approximation problems are closely related to distributional problems of special sequences. For example, the order of the discrepancy of the Weyl sequence αn depends on the continued fraction expansion of α that encodes the approximation behaviour of α by rationals. It is a long standing problem to find a corresponding property for the Kronecker sequence $(\alpha_1, \ldots, \alpha_k)n$, for example, in terms of Diophantine approximation conditions for α_j. It was therefore a natural step for Tichy to work on Diophantine problems and — as it turned out — on Diophantine equations, too.

He worked, for example, on parametrised families of Thue equations [18, 20, 19], where one can use Alan Baker's theory of linear forms in the logarithms of algebraic numbers. For several examples, Tichy and his coauthors could determine all solutions. It is a highly non-trivial problem to provide a complete solution of a Diophantine equation in 2 variables and 1 parameter; usually two variables cause enough troubles. Even if one knows that there are only finitely many solutions of the equation $F(x, y) = 0$ it is difficult (and mostly impossible) to determine all of them.

It is therefore a remarkable result that Bilu and Tichy [8] gave a complete and definite answer to the equation $f(x) = g(y)$, where f and g are rational polynomials. Before partial results have been obtained by M. D. Fried and A. Schinzel. Their main result says that the equation $f(x) = g(y)$ has infinitely many rational solutions (with a bounded denominator) if and only if there are linear polynomials $\lambda(x)$, $\mu(x) \in \mathbb{Q}[x]$, a polynomial $\varphi(x) \in \mathbb{Q}[x]$ and a pair (f_1, g_1) in an explicit set of five families of pairs of polynomials (including, for example, Dickson polynomials) with

$$f = \varphi \circ f_1 \circ \lambda, \quad g = \varphi \circ g_1 \circ \mu$$

and the equation $f_1(x) = g_1(y)$ has infinitely many rational solutions with a bounded denominator. The proof rests on arguments of Fried and Schinzel but includes several new ideas. One main point is to determine when the polynomial $f(x)-g(y)$ has an exceptional factor, that is, when there exists an irreducible polynomial $F(x, y) \in \mathbb{Q}[x, y]$ which divides $f(x) - g(y)$ and defines a plane curve of genus zero and with at most two points at infinity. This result can be applied, for example, to Diophantine equations of the shape $f(x) = g(y)$, where f and g depend on unknown parameters, that is, we have again more than two variables.

Another series of Tichy's papers is related to Diophantine problems of linear recurrence sequences. In [13] he provided together with C. Fuchs (under suitable and natural conditions) an explicit upper bound for the number of solutions (x, n) of the equation $G_n = x^q$, where G_n is a linear recurrence sequence and $q > 1$ a given integer. Two further papers with Fuchs and Pethő [11, 12] deal with recurrences

$$G_{n+d}(x) = A_{d-1}(x)G_{n+d-1}(x) + \cdots + A_0(x)G_n(x), \quad \text{for } n \geq 0,$$

of polynomials $G_n(x), A_j(x) \in \mathbf{K}[x]$, where $\mathbf{K}$ is a field of characteristic 0. They consider the Diophantine equation

$$G_n(x) = G_m(P(x))$$

and obtain an upper bound for the number of solutions (m, n) that depends only on d.

4. Actuarial Mathematics

In the early 80's, Robert Tichy got into contact with some problems from actuarial risk theory that deal with the stochastic modelling of the free reserve in a portfolio of insurance contracts. He quickly realized that rigorous analytical techniques could be employed to develop new solution procedures for ruin-related quantities and, in turn, these practical problems can also trigger theoretical research questions. Many equations in this field are of renewal type and Robert Tichy not only contributed criteria for existence and uniqueness of corresponding solutions, but also developed a rather versatile numerical solution method based on Quasi-Monte Carlo techniques (cf. [30]), in that way linking risk theory with uniform distribution theory. Later on, this method was extended to more general risk models in [4].

Exact solutions of partial integro-differential equations occurring in risk theory are rather scarce, mostly due to rather unpleasant boundary conditions. However, such exact expressions are extremely helpful since they allow to tune model parameters towards a given target for ruin-related quantities, such as ensuring a certain degree of solvency. In this connection, Robert Tichy introduced Laplace transform methods to the field of linear dividend barrier problems [31] and together with Siegl in [24] considerably enlarged the available set of exact solutions, developing a recursive solution algorithm (for which a rigorous convergence proof could then be given in Albrecher and Tichy [6]; further extensions of that method to arbitrary moments of expected dividend payments and discounted penalty functions were recently derived in [3]). Further objects of study in this context range from intricate recurrences (for the exact solution of finite-time ruin probabilities under interest force [5]) to certain types of differential equations with delayed arguments, that occur when implementing securitization strategies in the risk model [26, 27].

Among his most frequently cited papers in this field is [25], where additional stochastic factors in the dividend risk model were introduced and where it was shown that the toolkit of analytical solution procedures carries over in a transparent way, an approach that was subsequently taken up by several other research groups for further investigation.

Robert Tichy also contributed to the development of efficient Quasi-Monte Carlo methods for pricing products that bridge insurance activities with the financial market, such as catastrophe bonds [1, 2]. Such pricing problems sometimes lead to integrals with singular integrands, and the suitability of Quasi-Monte Carlo techniques in these situations was then investigated in [17].

Being himself a fully qualified actuary, Robert Tichy has a lot of insight in practical issues of the insurance industry, which is an ideal complement to his profound knowledge of mathematics. In this respect, he recently also played an instrumental role in installing a Master program in Financial and Actuarial Mathematics in Graz, which will offer future students an entry to this fascinating field of applied mathematics.

Gaudeat igitur, iuvenis dum est!

References

[1] Hansjörg Albrecher, Jürgen Hartinger, and Robert F. Tichy, *Multivariate approximation methods for the pricing of catastrophe-linked bonds*, Internat. Ser. Numer. Math. **145** (2003), 21–39.

[2] ———, *Quasi-Monte Carlo techniques for CAT bond pricing*, Monte Carlo Methods & Appl. (2004), no. 10, 197–212.

[3] Hansjörg Albrecher, Jürgen Hartinger, and Robert F. Tichy, *On the distribution of dividend payments and the discounted penalty function in a risk model with linear dividend barrier*, Scand. Actuar. J. (2005), no. 2, 103–126.

[4] Hansjörg Albrecher, Reinhold Kainhofer, and Robert F. Tichy, *Simulation methods in ruin models with non-linear dividend barriers*, Math. Comput. Simulation **62** (2003), 277–287.

[5] Hansjörg Albrecher, Jozef Teugels, and Robert F. Tichy, *On a gamma series expansion for the time-dependent probability of collective ruin*, Insurance: Mathematics and Economics **29** (2001), no. 3, 345–355.

[6] Hansjörg Albrecher and Robert F. Tichy, *On the convergence of a solution procedure for a risk model with gamma-distributed claims*, Schweiz. Aktuarver. Mitt. **2** (2000), 115–127.

[7] István Berkes, Walter Philipp, and Robert F. Tichy, *Empirical processes in probabilistic number theory: the LIL for the discrepancy of* $(n_k\omega) \bmod 1$, Illinois J. Math. **50** (2006), no. 1-4, 107–145 (electronic).

[8] Yuri F. Bilu and Robert F. Tichy, *The Diophantine equation* $f(x) = g(y)$, Acta Arith. **95** (2000), no. 3, 261–288.

[9] Michael Drmota and Robert F. Tichy, *Sequences, discrepancies and applications*, Lecture Notes in Mathematics, vol. 1651, Springer-Verlag, Berlin, 1997.

[10] Philippe Flajolet, Peter Grabner, Peter Kirschenhofer, Helmut Prodinger, and Robert F. Tichy, *Mellin transforms and asymptotics: digital sums*, Theoret. Comput. Sci. **123** (1994), no. 2, 291–314.

[11] Clemens Fuchs, Attila Pethő, and Robert F. Tichy, *On the Diophantine equation* $G_n(x) = G_m(P(x))$, Monatsh. Math. **137** (2002), no. 3, 173–196.

[12] ———, *On the Diophantine equation* $G_n(x) = G_m(P(x))$*: higher-order recurrences*, Trans. Amer. Math. Soc. **355** (2003), no. 11, 4657–4681 (electronic).

[13] Clemens Fuchs and Robert F. Tichy, *Perfect powers in linear recurring sequences*, Acta Arith. **107** (2003), no. 1, 9–25.

[14] Peter J. Grabner, Peter Kirschenhofer, and Robert F. Tichy, *Combinatorial and arithmetical properties of linear numeration systems*, Combinatorica **22** (2002), no. 2, 245–267, Special issue: Paul Erdős and his mathematics.

[15] Peter J. Grabner, Pierre Liardet, and Robert F. Tichy, *Odometers and systems of numeration*, Acta Arith. **70** (1995), no. 2, 103–123.

[16] ———, *Spectral disjointness of dynamical systems related to some arithmetic functions*, Publ. Math. Debrecen **66** (2005), no. 1-2, 213–243.

[17] Jürgen Hartinger, Reinhold Kainhofer, and Robert F. Tichy, *Quasi-Monte Carlo algorithms for unbounded, weighted integration problems*, J. Complexity **20** (2004), no. 5, 654–668.

[18] Clemens Heuberger, Attila Pethő, and Robert F. Tichy, *Complete solution of parametrized Thue equations*, Proceedings of the 13th Czech and Slovak International Conference on Number Theory (Ostravice, 1997), vol. 6, 1998, pp. 93–114.

[19] ———, *Thomas' family of Thue equations over imaginary quadratic fields*, J. Symbolic Comput. **34** (2002), no. 5, 437–449.

[20] Clemens Heuberger and Robert F. Tichy, *Effective solution of families of Thue equations containing several parameters*, Acta Arith. **91** (1999), no. 2, 147–163.

[21] Gerhard Larcher and Robert F. Tichy, *Some number-theoretical properties of generalized sum-of-digit functions*, Acta Arith. **52** (1989), no. 2, 183–196.

[22] Viktor Losert, Werner-Georg Nowak, and Robert F. Tichy, *On the asymptotic distribution of the powers of* $s \times s$*-matrices*, Compositio Math. **45** (1982), no. 2, 273–291.

[23] Harald Niederreiter and Robert F. Tichy, *Solution of a problem of Knuth on complete uniform distribution of sequences*, Mathematika **32** (1985), no. 1, 26–32.

[24] Thomas Siegl and Robert F. Tichy, *Lösungsmethoden eines Risikomodells bei exponentiell fallender Schadensverteilung*, Schweiz. Aktuarver.Mitt. (1996), no. 1, 85–118.

[25] ———, *A process with stochastic claim frequency and a linear dividend barrier*, Insurance Math. Econom. **24** (1999), no. 1-2, 51–65, 1st IME Conference (Amsterdam, 1997).

[26] ———, *Ruin theory with risk proportional to the free reserve and securitization*, Insurance Math. Econom. **26** (2000), no. 1, 59–73.

[27] ———, *A model in ruin theory using derivative securities*, Mitt. Schweiz. Aktuarvereinigung (2002), no. 1, 13–30.

[28] Jörg M. Thuswaldner and Robert F. Tichy, *An Erdős-Kac theorem for systems of q-additive functions*, Indag. Math. (N.S.) **11** (2000), no. 2, 283–291.

[29] ———, *Waring's problem with digital restrictions*, Israel J. Math. **149** (2005), 317–344, Probability in mathematics.

[30] Robert F. Tichy, *Über eine zahlentheoretische Methode zur numerischen Integration und zur Behandlung von Integralgleichungen*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **193** (1984), no. 4-7, 329–358.

[31] ———, *Bemerkung zu einem versicherungsmathematischen Modell*, Mitt. Verein. Schweiz. Versicherungsmath. (1987), no. 2, 237–241.

[32] ———, *Ein metrischer Satz über vollständig gleichverteilte Folgen*, Acta Arith. **48** (1987), no. 2, 197–207.

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