On the Connectivity of Random Graphs from Addable Classes

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Abstract

A class $\mathcal{A}$ of graphs is called *weakly addable* (or *bridge-addable*) if for any $G \in \mathcal{A}$ and any two distinct components C_1 and C_2 in G, any graph that can be obtained by adding an edge between C_1 and C_2 is also in $\mathcal{A}$. McDiarmid, Steger and Welsh conjectured in [6] that a graph chosen uniformly at random among all graphs with n vertices in a weakly addable $\mathcal{A}$ is connected with probability at least $e^{-1/2+o(1)}$, as $n \to \infty$. In this paper we show that the conjecture is true under a stronger assumption. A class $\mathcal{G}$ of graphs is called *bridge-alterable*, if for any $G \in \mathcal{G}$ and any bridge e in G, $G \in \mathcal{G}$ if and only if $G - e \in \mathcal{G}$. We prove that a graph chosen uniformly at random among all graphs with n vertices in a bridge-alterable $\mathcal{G}$ is connected with probability at least $e^{-1/2+o(1)}$, as $n \to \infty$.

The main tool in our analysis is a tight enumeration result that addresses the number of ways in which a given forest can be complemented to a forest with fewer components.
Keywords: Random Graphs, Addable Graph Classes, Connectivity, Tree Enumeration

1 Introduction

A class $\mathcal{A}$ of graphs is called *weakly addable* or *bridge-addable* if for any $G \in \mathcal{A}$ and any two vertices u and v in distinct components of G, the graph that is obtained by adding the edge $\{u, v\}$ to G is also in $\mathcal{A}$. The notion of weakly addable classes was introduced by McDiarmid, Steger and Welsh [6] as a general model for studying properties of classes of graphs. Indeed, many "natural" classes, like all H-free and all H-minor-free graphs, where H is 2-edge-connected, are easily seen to be weakly addable. The main objects of study in this work is a random graph that is sampled uniformly at random from the set $\mathcal{A}_n$ of all graphs with n vertices contained in a given weakly addable class $\mathcal{A}$, and we consider their asymptotic probability of being connected. Asymptotics are always as $n \to \infty$.

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One particular example of a weakly addable class is the class of forests. A result of Rényi [8] shows that the number of forests with n vertices is $e^{1/2+o(1)}n^{n-2}$. Together with Cayley's [3] famous formula n^{n-2} for the number of trees with n vertices, this implies that the asymptotic probability that a random forest is connected is $e^{-1/2+o(1)}$. McDiarmid, Steger and Welsh [6] conjectured that the class of forests is the "least connected" weakly addable class, in the sense that the asymptotic probability that a random graph from any weakly addable class is connected is at least $e^{-1/2+o(1)}$. Moreover, they derived a lower bound of e^{-1} for this probability. Balister, Bollobás and Gerke improved this bound to $e^{-0.7983+o(1)}$ in [2].

Addario-Berry, McDiarmid and Reed observed in [1] that all "natural" weakly addable classes, like H-free or H-minor-free graphs, also satisfy the property of being *monotone*: a class $\mathcal{A}$ is called monotone, if for any $G \in \mathcal{A}$, each graph that is obtained by deleting edges from G is also in $\mathcal{A}$. In an earlier version of [1], it was shown that the asymptotic probability that a random graph from a monotone weakly addable class is connected is at least $e^{-0.54076+o(1)}$.

Following the notation from [1], we say that a class of graphs $\mathcal{G}$ is *bridge-alterable* if for any $G \in \mathcal{G}$ and any bridge e in G, $G \in \mathcal{G}$ if and only if $G - e \in \mathcal{G}$. Note that any monotone weakly addable class is bridge-alterable, and that any bridge-alterable class is weakly addable. Our main result provides a lower bound for the asymptotic probability that a random graph from a bridge-alterable class is connected.

Theorem 1.1. *Let $\mathcal{G}$ be a non-empty bridge-alterable class and $\mathcal{G}_n$ the set of all graphs with n vertices in $\mathcal{G}$. Let G_n denote a graph chosen uniformly at random among all graphs in $\mathcal{G}_n$. Then*

$$\Pr[\mathsf{G}_n \textit{ is connected}] \geq e^{-1/2+o(1)}.$$

The same result was obtained independently by Addario-Berry, McDiarmid and Reed in [1].

Note that the bound in the above theorem is tight, since the class of forests is weakly addable and monotone. We shall prove the theorem in the next section. Here we give a main idea of our proof, and highlight differences from the earlier approaches [1, 2, 6]. First of all, let us discuss why the probability that a random forest is connected approaches $e^{-1/2}$. Recall Cayley's formula, which says that the number of trees with n vertices is n^{n-2}. Moreover, it is well-known that the number of forests with $k \geq 2$ components and n vertices is

$$(1+o(1))\frac{1}{2^{k-1}(k-1)!}n^{n-2}, \quad \text{as } n \to \infty. \tag{1.1}$$

This formula remains valid even if k is allowed to go very slowly with n to infinity (say $1 \leq k \leq \sqrt{\log n}$). Probably the first reference to this enumeration result is the paper [8] by Rényi, and it has been rediscovered many times with many different approaches in the literature. In addition to that, it can be shown that the number of forests with more than $k(n)$ components, where $k(n)$ is any function going to infinity, is $o(n^{n-2})$. By putting these facts together we see that the asymptotic number of forests with n vertices is

$$(1+o(1))\sum_{k=1}^{\sqrt{\log n}}\frac{1}{2^{k-1}(k-1)!}n^{n-2} + o(n^{n-2}) = e^{1/2+o(1)}\,n^{n-2}.$$

It is an immediate consequence of this formula that the probability that a random forest is connected is asymptotically $e^{-1/2+o(1)}$. An alternative way of proving this (and much more) is to use generating functions, see e.g. [4].

All the previous proofs, and also our proof, follow a similar line of reasoning in order to prove lower bounds for the asymptotic probability that a random graph from a weakly addable class is connected. In particular, it is shown that there exists a constant $\beta > 0$ such the number of graphs with $2 \leq k \leq o(\log n)$ components and n vertices is at most $(1 + o(1))\frac{\beta^{k-1}}{(k-1)!} \cdot X$, where X is the number of connected graphs with n vertices from the considered class, and that the number of graphs with "too many" components is $o(X)$. More specifically, $\beta \approx 0.79$ in [2] and $\beta \approx 0.54$ in an early version of [1]. In this work and in [1], the best possible value of β (up to $(1+o(1))$ factor), namely $\beta = 1/2$, is obtained. We show this by *explicitly* computing bounds for the number of graphs with k components, instead of relating this number to X.

2 The Proof of Theorem 1.1

From this section on, we assume that a non-empty bridge-alterable class $\mathcal{G}$ is given. For any positive integer n, we denote by $\mathcal{G}_n$ the set of all graphs with n vertices contained in $\mathcal{G}$ and by $[n]$ the set $\{1, \ldots, n\}$.

Let us begin with reviewing some basic facts from [1]. Given a graph $G \in \mathcal{G}_n$, we let $b(G)$ be the graph that is obtained from G by removing all bridges, and let $[G]$ be the set of all graphs $G' \in \mathcal{G}_n$ such that $b(G') = b(G)$. As already argued in [1], we observe that $\mathcal{G}_n$ is the disjoint union of equivalence classes $[G_1], \ldots, [G_m]$, for some positive integer m (depending on $\mathcal{G}_n$) and some graphs $G_i \in \mathcal{G}_n$, for $1 \leq i \leq m$. To see this, note simply that if $G \in \mathcal{G}_n$, then $[G] \subseteq \mathcal{G}_n$, as $\mathcal{G}$ is closed under deleting and adding bridges.

A new key observation leading to Theorem 1.1 is the following.

Lemma 2.1. *Let $G \in \mathcal{G}_n$ and let G be a graph chosen uniformly at random from $[G]$. Then*

$$\Pr[\mathsf{G} \textit{ is connected}] \geq e^{-1/2+o(1)}.$$

The key lemma says that the desired bound in Theorem 1.1 holds for a graph chosen uniformly at random from each equivalence class. Since $\mathcal{G}_n$ is the disjoint union of (finite) equivalence classes, Theorem 1.1 follows immediately. In the next section we prove Lemma 2.1.

3 Proof of Lemma 2.1

Let $G \in \mathcal{G}_n$. Since $[G] = [b(G)]$, throughout this section we will assume, without loss of generality, that G is bridgeless, i.e., $G = b(G)$. Our first step towards the proof of Lemma 2.1 is the following result about the number of connected graphs in $[G]$. We include a proof of it, although similar statements are probably known.

Lemma 3.1. *Let $G \in \mathcal{G}_n$ be a bridgeless graph with t components of sizes $n_1, \ldots, n_t$. Then the number of connected graphs in $[G]$ equals $\left(\prod_{i=1}^{t} n_i\right) \cdot n^{t-2}$.*

Proof. We denote the t components of G by $C_1, \ldots, C_t$ and assume that C_i has n_i vertices, for each $1 \leq i \leq t$. We first identify each component C_i in G with integer ("vertex") i from $[t]$. Every connected graph H in $[G]$ can then be mapped to a tree T on $[t]$ as follows: for each pair $i, j \in [t]$ with $i \neq j$, we insert an edge in T between vertices i and j if and only if there is an edge (indeed a bridge) in H connecting C_i and C_j. This mapping guarantees that each tree T on $[t]$ gives rise to $\prod_{i=1}^{t} n_i^{d_i}$ connected graphs that are in $[G]$, where d_i is the degree of vertex i in T.

From the Prüfer code representation of trees (see the original paper [7] or the book [5] for a modern treatment), it follows that the number of trees with degree sequence $d_1, \dots, d_t$ is $\binom{t-2}{d_1-1,\dots,d_t-1}$. Thus, the number of connected graphs in $[G]$ is equal to

$$\sum_{\substack{d_1+\dots+d_t=2(t-1) \\ \forall 1\le i\le t\,:\, d_i\ge 1}} \binom{t-2}{d_1-1,\dots,d_t-1} \prod_{i=1}^{t} n_i^{d_i} = \sum_{\substack{d_1+\dots+d_t=t-2 \\ \forall 1\le i\le t\,:\, d_i\ge 0}} \binom{t-2}{d_1,\dots,d_t} \prod_{i=1}^{t} n_i^{d_i+1}.$$

By applying the binomial theorem we infer that this equals $\left(\prod_{i=1}^{t} n_i\right) \cdot n^{t-2}$, as claimed. ∎

Note that Lemma 3.1 provides an enumeration result that addresses the number of ways in which any given forest can be completed to a tree. In particular, if we take $n_1 = n_2 = \dots = n_n = 1$, then we recover Cayley's formula. On the other hand, by fixing any forest F, then Lemma 3.1 enumerates all trees that contain F. Our main contribution is the proof of the following claim, which concerns only *partial* completions, i.e., the considered graphs have $k \ge 2$ components.

Lemma 3.2. *Let $G \in \mathcal{G}_n$ be a bridgeless graph with t components of sizes $n_1, \dots, n_t$. Then, the number of graphs with $k \ge 2$ components in $[G]$ is at most*

$$c_{n,k} \cdot \frac{1}{2^{k-1}(k-1)!} \left(\prod_{i=1}^{t} n_i\right) \cdot n^{t-2}, \text{ where } c_{n,k} = e^{5(2^{k-1}-1)/n}.$$

Note that the bound in the lemma above is asymptotically tight for the class of forests with $k \ge 2$ components, even for k that grows sufficiently slowly with n. Indeed, for such k we have that $c_{n,k} = 1 + o(1)$ and therefore the provided upper bound coincides with (1.1). However, it is doubtful that this bound is best possible. Moreover, the bound is very weak if $k = \Omega(\log n)$. For large k we will use the following statement, which was observed in a similar (though not so explicit) form in [2, 1]. We include a proof for completeness.

Lemma 3.3. *Let $G \in \mathcal{G}_n$ be a bridgeless graph with t components of sizes $n_1, \dots, n_t$. Then, the number of graphs with $k \ge 2$ components in $[G]$ is at most*

$$\frac{1}{(k-1)!} \left(\prod_{i=1}^{t} n_i\right) \cdot n^{t-2}.$$

Proof. Let $\mathcal{F}_k$ be the set of graphs with k components in $[G]$. For any $k \ge 1$ we construct an auxiliary bipartite graph $B_k = (L \cup R, E)$ with vertex set $L \cup R$ and edge set E, where L contains one vertex for each graph in $\mathcal{F}_k$, R contains one vertex for each graph in $\mathcal{F}_{k+1}$ and E contains an edge $\{\ell, r\}$ with $\ell \in L$ and $r \in R$ if and only if the graph corresponding to r can be obtained from the graph corresponding to ℓ by deleting a bridge. Let us count the number of edges in B_k in two ways. First, the number of ways to remove a bridge from any graph in $\mathcal{F}_k$ is $t - k$. Thus

$$|E| = (t-k)|L| = (t-k)|\mathcal{F}_k| \le (n-k)|\mathcal{F}_k|.$$

On the other hand, let G' be a graph in $\mathcal{F}_{k+1}$, whose components have sizes $m_1, \dots, m_{k+1}$. Then, the number of ways to add a bridge to G' equals

$$\sum_{1\le i<j\le k+1} m_i m_j \ge \binom{k}{2} + k(n-k) \ge k(n-k).$$

Thus $k(n-k)|\mathcal{F}_{k+1}| \le |E| \le (n-k)|\mathcal{F}_k|$, from which we infer that $|\mathcal{F}_{k+1}| \le \frac{|\mathcal{F}_k|}{k}$. The claim then follows by applying Lemma 3.1 and by induction over k. ∎

Lemma 2.1 is an immediate consequence of Lemmas 3.2 and 3.3, because the number of graphs in $[G]$, divided by $(\prod_{i=1}^{t} n_i) \cdot n^{t-2}$ (i.e. the number of connected graphs in $[G]$), is at most

$$1 + \sum_{k=2}^{\min\{t,\sqrt{\log n}\}} c_{n,k} \frac{1}{2^{k-1}(k-1)!} + \sum_{k \ge \sqrt{\log n}} \frac{1}{(k-1)!} \overset{(c_{n,k}=1+o(1))}{\le} e^{1/2+o(1)} + o(1).$$

The rest of the paper is devoted to the proof of Lemma 3.2.

4 Proof of Lemma 3.2

Let $G \in \mathcal{G}_n$ be a bridgeless graph with components $C_1, \dots, C_t$ with $|C_i| = n_i$ for $1 \le i \le t$. For each $k \ge 1$ and each subset set $I \subseteq [t]$ we will denote by $F_k(I)$ the number of graphs with k components in $[\cup_{i \in I} C_i]$, where $\cup_{i\in I} C_i$ is a subgraph of G consisting of components C_i for $i \in I$, and we will abbreviate $s(I) = \sum_{i \in I} n_i$.

We shall treat the cases $k = 2$ and $k \ge 3$ separately. We prove the case $k \ge 3$ by induction, provided that the case $k = 2$ is true. As an induction hypothesis, we assume that the statement is true for any $k' \le k$ and any bridgeless graph with n' vertices for $n' \le n$. We first observe that $H_k([t]) = (k-1)F_k([t])$ counts the number of graphs with k components in $[G]$, such that one of the components that do not contain C_t is distinguished. We will provide an alternative way of enumerating those graphs counted by $H_k([t])$. To this end, we let $H_k^s([t])$ denote the number of graphs where the distinguished component has at most $n/2$ vertices and $H_k^\ell([t])$ the number of graphs where the distinguished component has more than $n/2$ vertices. Thus, $H_k([t]) = H_k^s([t]) + H_k^\ell([t])$.

We now construct all graphs counted by $H_k^s([t])$ by first fixing any $I \subseteq [t-1]$ such that $s(I) \le n/2$, then by picking any connected graph in $[\cup_{i\in I} C_i]$ (as the distinguished component), and finally by constructing any graph in $[\cup_{i \in [t]\setminus I} C_i]$ with $k-1$ components. From the induction hypothesis and from the fact that $c_{s([t]\setminus I),k-1} \le c_{n/2,k-1}$ (because $s([t] \setminus I) > n/2$), we obtain that $F_{k-1}([t] \setminus I) \le \frac{c_{n/2,k-1}}{2^{k-2}(k-2)!} \left(\prod_{i\in[t]\setminus I} n_i\right) s([t]\setminus I)^{t-|I|-2}$. From Lemma 3.1 we have $F_1(I) \le \left(\prod_{i \in I} n_i\right) s(I)^{|I|-2}$. Putting these together, we obtain that

$$\begin{aligned} H_k^s([t]) &= \sum_{I \subseteq [t-1], s(I) \le n/2} F_1(I)\, F_{k-1}([t] \setminus I) \\ &\le \frac{c_{n/2,k-1}}{2^{k-2}(k-2)!} \sum_{I \subseteq [t-1], s(I) \le n/2} \left(\prod_{i \in I} n_i\right) s(I)^{|I|-2} \cdot \left(\prod_{i \in [t]\setminus I} n_i\right) s([t]\setminus I)^{t-|I|-2}. \end{aligned}$$

Now we will find an appropriate bound for $H_k^\ell([t])$. The graphs counted by $H_k^\ell([t])$ can be constructed by first fixing any $I \subseteq [t-1]$ such that $s(I) > n/2$, then by constructing any graph with $k-1$ components from $[\cup_{i\in I} C_i]$ with one (and necessarily) unique component that contains more than $n/2$ vertices, and by finally constructing any connected graph in

$[\cup_{i\in[t]\setminus I} C_i]$. Thus, again by applying the induction hypothesis and Lemma 3.1 we infer that

$$\begin{aligned} H_k^\ell([t]) &\le \sum_{I\subseteq[t-1],s(I)>n/2} F_{k-1}(I)\, F_1([t]\setminus I) \\ &\le \frac{c_{n/2,k-1}}{2^{k-2}(k-2)!} \sum_{I\subseteq[t-1],s(I)>n/2} \left(\prod_{i\in I} n_i\right) s(I)^{|I|-2}\cdot \left(\prod_{i\in[t]\setminus I} n_i\right) s([t]\setminus I)^{t-|I|-2}. \end{aligned}$$

Since $(k-1)F_k([t]) = H_k^s([t]) + H_k^\ell([t])$, we get the bound

$$(k-1)F_k([t]) \le \frac{c_{n/2,k-1}}{2^{k-2}(k-2)!} \sum_{I\subseteq[t-1]} \left(\prod_{i\in I} n_i\right) s(I)^{|I|-2}\cdot \left(\prod_{i\in[t]\setminus I} n_i\right) s([t]\setminus I)^{t-|I|-2}.$$

However, due to Lemma 3.1, the last sum is equal to $\sum_{I\subseteq[t-1]} F_1(I)F_1([t]\setminus I)$. But this counts the number of graphs with two components in $[G]$, i.e. $F_2([t])$, because the first term $F_1(I)$ counts the number of connected graphs in $[G]$ that do not contain C_t (as $I\subseteq[t-1]$) and the second term $F_1([t]\setminus I)$ counts the number of connected graphs in $[G]$ that contain C_t. By our assumption for $k=2$, we have $F_2([t]) \le \frac{1}{2}\, e^{5/n}\left(\prod_{i=1}^t n_i\right)\cdot n^{t-2}$. The proof is completed by observing that $c_{n/2,k-1}\cdot e^{5/n} = c_{n,k}$.

It remains to treat the case $k=2$. The crucial idea is to prove the stronger claim (which comes from magic!):

$$F_2([t]) \le \left(\prod_{i=1}^t n_i\right) n^{t-2}\cdot\left(\frac{1}{n}\left(\sum_{i=1}^t \frac{1}{n_i}\right) - \frac{f(t)}{2n^2}\right), \tag{4.1}$$

where $f(t) = t^2-5t+6$. To see that this is sufficient, let us write $\alpha = \frac{t}{n}$. Then $1/n \le \alpha \le 1$ and

$$\frac{1}{n}\left(\sum_{i=1}^t \frac{1}{n_i}\right) \le \frac{1}{n}\left(\sum_{i=1}^t 1\right) = \alpha,$$

so, the last multiplicative term in the right-hand side of (4.1) is at most

$$\alpha - \frac{f(t)}{2n^2} = \alpha - \frac{(\alpha n)^2 - 5\alpha n + 6}{2n^2} \le \alpha - \frac{\alpha^2}{2} + \frac{5}{2n} \le \frac{1}{2}\left(1+\frac{5}{n}\right) \le \frac{1}{2}\, e^{5/n}.$$

This proves the claim for the case $k=2$.

We will prove (4.1) by induction on t. Note that for $t=2$ the statement is true, as $f(2)=0$ and

$$F_2([2]) = 1 = n_1 n_2 n^{2-2}\cdot\left(\frac{1}{n}\left(\frac{1}{n_1}+\frac{1}{n_2}\right) - \frac{f(2)}{2n^2}\right).$$

For $t\ge 3$, we observe that in each graph counted by $F_2([t])$, there are exactly $t-2$ edges (indeed bridges) that are inserted between $C_1,\dots,C_t$, because $F_2([t])$ counts the number of graphs with two components in $[G]$. Now let us denote by $F_2'([t])$ the number of graphs with two components in $[G]$, in which exactly one of those $t-2$ edges connecting some C_i and some C_j for $i\ne j$ is distinguished. So, $F_2'([t]) = (t-2)F_2([t])$. On the other hand, we can construct graphs counted by $F_2'([t])$, by first selecting two components C_i and C_j with $1\le i<j\le t$, and then fixing one of the possible $n_i n_j$ edges between C_i and C_j, and inserting that chosen

edge between C_i and C_j. In this way we obtain a new set of components, by removing C_i and C_j from the set $\{C_1, \dots, C_t\}$, and by including a new component of size $n_i + n_j$. By the induction hypothesis of (4.1), the number of graphs with two components that can be obtained from this new component set by inserting additional bridges is at most

$$N_{i,j} = \left((n_i + n_j) \prod_{\ell=1\dots t, \ell \neq i,j} n_\ell \right) n^{t-3} \left(\frac{1}{n} \left(-\frac{1}{n_i} - \frac{1}{n_j} + \frac{1}{n_i + n_j} + \sum_{\ell=1}^{t} \frac{1}{n_\ell} \right) - \frac{f(t-1)}{2n^2} \right).$$

Thus,

$$\begin{aligned} F_2'([t]) &\le \sum_{1 \le i < j \le t} n_i n_j \cdot N_{i,j} \\ &\le \left(\prod_{\ell=1}^{t} n_\ell \right) n^{t-2} \sum_{1 \le i < j \le t} (n_i + n_j) \left(\frac{1}{n^2} \left(-\frac{1}{n_i} - \frac{1}{n_j} + \frac{1}{n_i + n_j} + \sum_{\ell=1}^{t} \frac{1}{n_\ell} \right) - \frac{f(t-1)}{2n^3} \right). \end{aligned} \tag{4.2}$$

Note that

$$\begin{aligned} \frac{1}{n^2} \sum_{1 \le i < j \le t} (n_i + n_j) \left(-\frac{1}{n_i} - \frac{1}{n_j} + \frac{1}{n_i + n_j} \right) &= -\frac{1}{n^2} \sum_{1 \le i < j \le t} \left(1 + \frac{n_j}{n_i} + \frac{n_i}{n_j} \right) \\ = -\frac{\binom{t}{2}}{n^2} - \frac{1}{n^2} \sum_{\ell=1}^{t} \frac{n - n_\ell}{n_\ell} &= -\frac{\binom{t}{2} - t}{n^2} - \frac{1}{n} \left(\sum_{\ell=1}^{t} \frac{1}{n_\ell} \right). \end{aligned} \tag{4.3}$$

Since $\sum_{1 \le i < j \le t} (n_i + n_j) = (t-1)n$, we have

$$\frac{1}{n^2} \sum_{1 \le i < j \le t} (n_i + n_j) \left(\sum_{\ell=1}^{t} \frac{1}{n_\ell} \right) = \frac{t-1}{n} \left(\sum_{\ell=1}^{t} \frac{1}{n_\ell} \right), \tag{4.4}$$

and the error term involving f equals

$$-\sum_{1 \le i < j \le t} (n_i + n_j) \frac{f(t-1)}{2n^3} = -(t-1) \frac{f(t-1)}{2n^2}. \tag{4.5}$$

By plugging (4.3)–(4.5) into (4.2) we obtain that

$$F_2'([t]) \le \left(\prod_{\ell=1}^{t} n_\ell \right) n^{t-2} \left(\frac{t-2}{n} \left(\sum_{\ell=1}^{t} \frac{1}{n_\ell} \right) - \frac{\binom{t}{2} - t}{n^2} - (t-1) \frac{f(t-1)}{2n^2} \right).$$

Recall that $F_2'([t]) = (t-2) F_2([t])$. The proof of (4.1) completes with the observation

$$\frac{1}{t-2} \left(\binom{t}{2} - t + (t-1) \frac{f(t-1)}{2} \right) = \frac{f(t)}{2}.$$

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