Random Cubic Planar Graphs

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ABSTRACT: We show that the number of labeled cubic planar graphs on n vertices with n even is asymptotically $\alpha n^{-7/2}\rho^{-n}n!$, where $\rho^{-1} \doteq 3.13259$ and α are analytic constants. We show also that the chromatic number of a random cubic planar graph that is chosen uniformly at random among all the labeled cubic planar graphs on n vertices is three with probability tending to $e^{-\rho^4/4!} \doteq 0.999568$ and four with probability tending to $1 - e^{-\rho^4/4!}$ as $n \to \infty$ with n even. The proof given combines generating function techniques with probabilistic arguments. © 2006 Wiley Periodicals, Inc. Random Struct. Alg., 30, 78–94, 2007

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1. INTRODUCTION

Random planar *maps* are well-studied objects in combinatorics [2, 4, 34, 35]. In contrast, random planar graphs did not receive much attention until recently. For planar graphs, we do not distinguish between different embeddings of the same graph. We are interested in the asymptotic number of labeled planar graphs (or subclasses of labeled planar graphs) and the properties of a graph chosen uniformly at random from the set of all labeled planar graphs on n vertices for large n. To study properties of random planar structures mainly three

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approaches have been applied: the first is the probabilistic method [7, 8, 15, 22–24, 30, 32]; the second is based on connectivity decomposition and generating functions [1, 3, 5, 7, 9–12, 25, 29]; the third is the matrix integral method [6, 13, 14, 16, 17, 27].

In this paper we combine the first two approaches to determine the asymptotic number of labeled *cubic* planar graphs (i.e., labeled planar graphs where every vertex has degree three) and to study typical properties of a *random* cubic planar graph (i.e., a graph that is chosen uniformly at random among all the labeled cubic planar graphs), such as connectedness, components, containment of a triangle, and the chromatic number. Note that cubic planar *maps* were enumerated only recently by Gao and Wormald [21].

We first apply well-known connectivity decomposition techniques [28], which specialize nicely in the case of cubic graphs. From that, we derive a system of *algebraic* equations that describe the exponential generating function for the number of labeled *connected* cubic planar graphs. Using the singularity analysis method discussed in [19] we then derive the asymptotic number of labeled connected cubic planar graphs. From the relation between labeled connected graphs and labeled graphs we also derive the asymptotic number of labeled cubic planar graphs.

Using the asymptotic numbers obtained and probabilistic arguments we investigate the asymptotic probability of connectedness of a random cubic planar graph, the limiting distribution of the number of components isomorphic to a given graph (for example K_4) in a random cubic planar graph, and the asymptotic probability of the containment of a triangle in a random cubic planar graph. Having these, we determine the chromatic number of a random cubic planar graph.

Based on the connectivity decomposition exact counting formulas and a deterministic polynomial time sampling procedure then follow from general principles [19, 20, 31].

2. ROOTED CUBIC GRAPHS

To count labeled cubic planar (simple) graphs, we introduce "labeled rooted cubic planar graphs." We will present a decomposition scheme for such graphs, which can then be used to count (unrooted) labeled cubic planar (simple) graphs.

From now on, except in part of Section 5, we will consider only labeled graphs and thus leave out the term "labeled" unless explicitly stated otherwise.

A *rooted cubic graph* $G = (V, E, st)$ consists of a connected cubic multigraph $G = (V, E)$ and an ordered pair of adjacent vertices s and t such that the *underlying graph* G^- obtained by deleting an edge between s and t is simple. Thus, in G, if s and t are distinct there may be either one or two edges between them, and if $s = t$ there is a loop at this vertex, and otherwise there are no loops or parallel edges. The oriented edge st is called the *root* of G, and s and t are the *poles*. Thus, G^- is obtained from G by deleting the root edge. Note that a rooted cubic graph must have at least 4 vertices: we may not have a "triple edge."

The following lemma is easily checked. Note that $G\backslash\{s, t\}$ denotes the graph G less the vertices s and t.

Lemma 1. *A rooted cubic graph $G = (V, E, st)$ has exactly one of the following types (Figure 1).*

- *b: the root is a self-loop.*
- *d: G^- is disconnected.*
- *s: G^- is connected but there is a cut edge in G^- that separates s and t.*

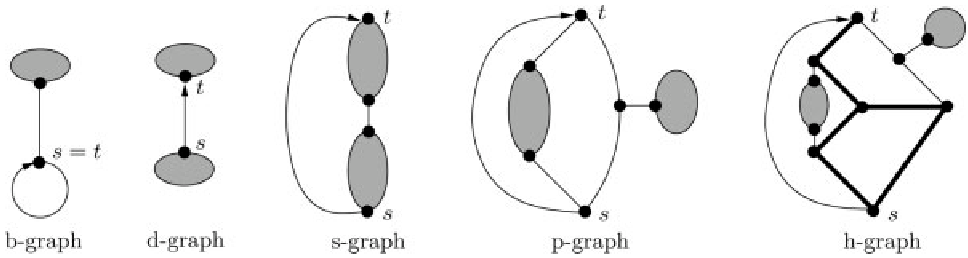

Fig. 1. The five types of rooted cubic graphs in Lemma 1.

- *p: G^- is connected, there is no cut edge in G^- separating s and t, and either st is an edge of G^- or $G\backslash\{s,t\}$ is disconnected.*
- *h: G^- is connected, there is no cut-edge in G^- separating s and t, G is simple and $G\backslash\{s,t\}$ is connected.*

We will make use of a *replacement* operation for rooted cubic graphs. We are often interested in rooted cubic graphs that are not d-graphs, i.e., b-, s-, p-, or h-graphs: let us call these c-graphs. Let $G = (V_G, E_G, s_G t_G)$ be a rooted cubic graph, let $u_G v_G$ be obtained by orienting an edge in G^-, and let $H = (V_H, E_H, s_H t_H)$ be a c-graph. The rooted cubic graph G' obtained from G by the *replacement* of $u_G v_G$ by H has vertex set the disjoint union of V_G and V_H, edge set the disjoint union of $E_G\backslash\{u_G v_G\}$ and $E_H\backslash\{s_H t_H\}$ together with the edges $u_G s_H$ and $v_G t_H$, and the same root as G. When we perform a replacement by H we always insist that H is a c-graph. The following result may be compared with network decomposition results of Trakhtenbrot [33, 36].

Theorem 1. *(a) Let H be a 3-connected simple rooted cubic graph, let F be a set of oriented edges of H^-, and for each $uv \in F$ let H_{uv} be a c-graph. Let G be obtained by replacing the edges $uv \in F$ by H_{uv}. Then G is an h-graph. Further, if H is planar and each H_{uv} is planar then so is G.*

(b) Let $G = (V, E, st)$ be an h-graph. Then there is a unique 3-connected rooted cubic graph H (called the core *of G) such that we can obtain G by replacing some oriented edges e of H^- by c-graphs H_e. Further, H is simple, and if G is planar then so is H and each H_e.*

Proof. (a) Note that H is an h-graph; and if G' is an h-graph and we replace an oriented edge by a c-graph then we obtain another h-graph (which is planar if both the initial and the replacing graph are). Thus, part (a) follows by induction on the number of edges replaced.

(b) The main step is to identify the core H. Let W be the set of vertices $v \in V\backslash\{s,t\}$ such that there is a set of three pairwise internally vertex-disjoint (or equivalently, edge-disjoint) paths between v and $\{s,t\}$. Then W is non-empty. For, let P_1 and P_2 be internally vertex-disjoint paths between s and t in G^-. There must be a path Q between an internal vertex of P_1 and an internal vertex of P_2 (since neither P_1 nor P_2 is just a single edge, and $G\backslash\{s,t\}$ is connected), and we can insist that Q be internally vertex-disjoint from P_1 and P_2. Now the terminal vertices of Q must both be in W.

Let H be the graph with vertex set $V_H = W \cup \{s,t\}$, where for distinct vertices u and v in V_H we join u and v in H if there is a $u - v$ path in G using no other vertices in V_H. Thus, in particular if vertices $u, v \in V_H$ are adjacent in G, then they are adjacent also in H.

It is easy to check that H is 3-connected and thus also is simple.

Let X be the set of vertices of G not in H. If $X = \emptyset$ then $G = H$ and we are done: suppose then that X is non-empty. Consider a component C of the subgraph of G induced by X. We claim that there are distinct vertices u and v in V_H that are adjacent in H but not in G, vertices x and y in C (possibly $x = y$), and edges ux and vy in G that are the only edges between C and V_H. Let H_{uv} be the rooted cubic graph obtained from C by adding the root edge xy. Now it is clear that we may obtain G by starting with H and replacing any edge uv of H not in G by the corresponding H_{uv}.

We have now seen that the rooted cubic graph H is simple and 3-connected, and we may obtain G by starting with H and replacing some edges e of H^- by c-graphs H_e. Finally, it is easy to see that H is unique. For if H' also has these properties, then we immediately see that $V_H = V_{H'}$, and it follows easily that the graphs are the same. ∎

We are interested here only in planar graphs. However, all results in Sections 2 and 3 can be formulated more generally for subclasses of connected cubic graphs that are closed under replacements.

3. DECOMPOSING ROOTED GRAPHS

In this section we decompose rooted graphs into b-, d-, s-, p-, and h-graphs. The decomposition can be formulated with algebraic equations for the corresponding exponential generating functions.

Exponential Generating Functions

Let b_n, d_n, s_n, p_n, h_n, and c_n be the number of b-, d-, s-, p-, h-, and c-graphs on n vertices, respectively. Thus, $c_n = b_n + s_n + p_n + h_n$. Let $B(x), D(x), S(x), P(x), H(x)$, and $C(x)$ be the corresponding exponential generating functions. For instance, $B(x)$ is defined by

$$B(x) := \sum_{n\geq 0} \frac{b_n}{n!} x^n.$$

Note that $b_n = d_n = s_n = p_n = h_n = c_n = 0$ for all odd n, due to cubicity, also for $n = 0$ by convention, and for $n = 2$. Thus, for instance, $B(x)$ is of the form $\sum_{n\geq 2} \frac{b_{2n}}{(2n)!} x^{2n}$.

b-Graphs

The structure of a b-graph is restricted by 3-regularity and the shaded area in Figure 2 together with an oriented edge between u and v is a d-, s-, p-, or h-graph. Therefore, $B(x) = x^2/2(D(x) + S(x) + P(x) + H(x))$, where the factor $1/2$ is due to the orientation of the edge between u and v. This can be rewritten as $B(x) = x^2(D(x) + C(x) - B(x))/2$.

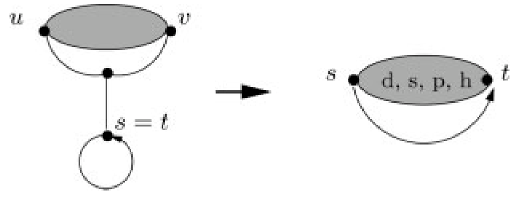

Fig. 2. Decomposing a b-graph.

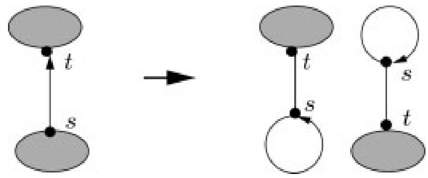

Fig. 3. Decomposing a d-graph.

d-Graphs

A d-graph can be decomposed uniquely into two b-graphs as shown in Figure 3. We therefore have $D(x) = B(x)^2/x^2$.

s-Graphs

For a given s-graph G, the graph G^- has a cut-edge that separates s and t and that is closest to s as in Figure 4. (Note that the cut edge could be a second copy of st.) We obtain $S(x) = (S(x) + P(x) + H(x) + B(x))(P(x) + H(x) + B(x)) = C(x)^2 - C(x)S(x)$.

p-Graphs

For a given p-graph, we distinguish whether s and t are adjacent in G^-. Both situations are depicted in Figure 5. We obtain $P(x) = x^2(S(x) + P(x) + H(x) + B(x)) + x^2/2(S(x) + P(x) + H(x) + B(x))^2 = x^2C(x) + x^2C(x)^2/2$, where the factor $1/2$ in the latter term is there because two c-graphs are not ordered.

h-Graphs

From Theorem 1 we know that an h-graph is built from a rooted three-connected cubic planar graph by replacing some edges, except the root edge, by b-, s-, p-, or h-graphs, i.e., c-graphs; see Figure 6. Let $m_{n,l}$ be the number of rooted 3-connected cubic planar graphs on n vertices and l edges and let $M(x,y) := \sum_{n,l\geq 0} \frac{m_{n,l}}{n!} x^n y^l$ be its exponential generating function. Clearly, $m_{n,l} = 0$ for odd n, $n = 0, 2$ or $l \neq 3n/2$ since a cubic planar graph on n vertices has $3n/2$ edges. Hence,

$$M(x,y) = \sum_{n\geq 2} \frac{m_{2n,3n}}{(2n)!} x^{2n} y^{3n},$$

which we will determine in Section 5 (see Eq. (5)).

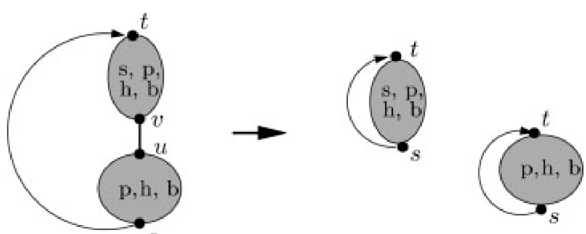

Fig. 4. Decomposing an s-graph.

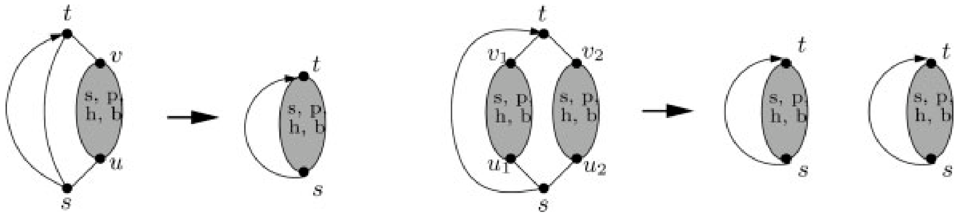

Fig. 5. Decomposing two types of a p-graph.

Note that the variable y in $M(x, y)$ marks the edges in rooted 3-connected cubic planar graphs. Thus, in order to derive the exponential generating function for h-graphs, we replace the variable y in $M(x, y)$ by $C(x) + 1$ (where the constant term 1 is there because an edge need not be replaced) and divide this by $C(x) + 1$, because we do not replace the root edge. Thus, we get

$$H(x) = \frac{M(x, (C(x) + 1))}{(C(x) + 1)}. \tag{1}$$

4. CUBIC PLANAR GRAPHS

For $k = 0, 1, 2, 3$ let $g_n^{(k)}$ be the number of k-vertex-connected cubic planar (simple) graphs on n vertices and $G^{(k)}(x)$ be the corresponding exponential generating functions. Note that $g_n^{(k)} = 0$ for odd n and also for $n = 0, 2$ except that we set $g_0^{(0)} = 1$ by convention.

If we select an arbitrary edge in a connected cubic planar (simple) graph and orient this edge, we obtain a rooted cubic graph $G = (V, E, st)$ that is neither a b-graph, nor an s- or p-graph where s and t are adjacent in the underlying graph G^-; see Figure 7. Note that the number of connected cubic planar (simple) graphs with one distinguished oriented edge is counted by $3x\frac{dG^{(1)}(x)}{dx}$, and the number of s- (resp. p-)graphs $G = (V, E, st)$ where s and t are adjacent in G^- as depicted in the middle (resp., right) picture in Figure 7 is counted by $B(x)^2$ (resp., $x^2C(x)$). Therefore, we get

$$3x\frac{dG^{(1)}(x)}{dx} = D(x) + S(x) + P(x) + H(x) - B(x)^2 - x^2C(x). \tag{2}$$

Finally, the exponential generating function for connected cubic planar graphs and that for not necessarily connected ones are related by the following well-known identity (see [26]).

$$G^{(0)}(x) = \exp(G^{(1)}(x)). \tag{3}$$

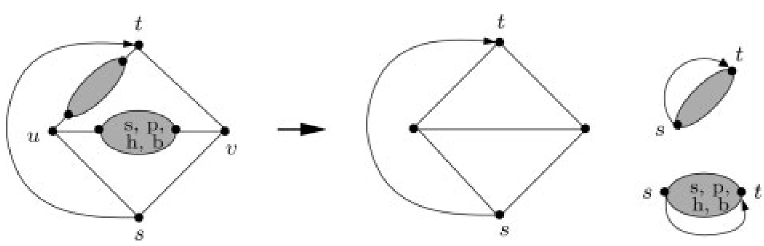

Fig. 6. Decomposing an h-graph.

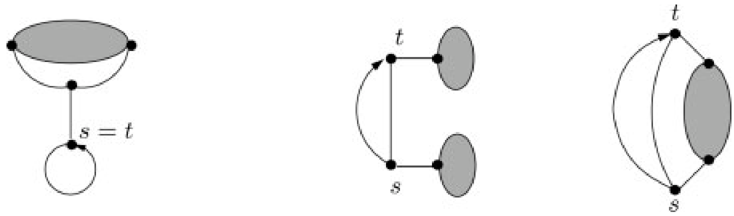

Fig. 7. Types of rooted cubic graphs that are not simple.

5. THREE-CONNECTED CUBIC PLANAR GRAPHS

The number of labeled rooted three-connected cubic planar graphs is closely related to that of *rooted triangulations*. A rooted triangulation is an edge-maximal plane graph with a distinguished directed edge on the outer face, called the *root edge*. Tutte [34] derived exact and asymptotic formulas for the number of such objects up to isomorphisms that preserve the outer face and the root edge. Since such objects do not have non-trivial automorphisms that fix the root edge, we can easily obtain the number of labeled objects from the number of unlabeled objects. Note that labeled rooted three-connected planar graphs with at least four vertices have exactly two non-equivalent embeddings in the plane. Using duality, we can compute the number of labeled rooted three-connected cubic planar graphs from the number of unlabeled rooted triangulations.

Let t_n be the number of *unlabeled* rooted triangulations on $n+2$ vertices. From the formulas Tutte computed for unlabeled rooted triangulations on $n+3$ vertices, it follows that the ordinary generating function $T(z)$ for t_n, i.e., $T(z) = \sum_{n\geq 1} t_n z^n$, satisfies the following.

$$\begin{aligned} T(z) &= u(1-2u) \\ z &= u(1-u)^3. \end{aligned} \tag{4}$$

The first terms of $T(z)$ are $z + z^2 + 3z^3 + 13z^4 + 68z^5 + 399z^6 + \cdots$. Further, $T(z)$ has a dominant singularity at $\xi = 27/256$ and the asymptotic growth of t_n is $\alpha_4 n^{-5/2}\xi^{-n}n!$, where α_4 is a constant. Let $\tilde{T}(x,y)$ be the corresponding ordinary generating function, but where x marks the number of faces and y marks the number of edges. By Euler's formula, a triangulation on $n+2$ vertices has $2n$ faces and $3n$ edges. Therefore, $\tilde{T}(x,y) := \sum_{n\geq 1} t_n x^{2n} y^{3n}$ can be computed by $\tilde{T}(x,y) = T(x^2y^3)$.

We now determine the exponential generating function $M(x,y)$ for the number of *labeled* rooted 3-connected cubic planar graphs, which was needed in the decomposition of h-graphs in Section 3. Note that the number of labeled rooted 3-connected cubic planar maps on $2n$ vertices (and hence with $3n$ edges) is twice the number of labeled rooted 3-connected cubic planar graphs on $2n$ vertices (and hence with $3n$ edges). Since the dual of a rooted 3-connected cubic *map* on $2n$ vertices is a rooted triangulation on $n+2$ vertices, we have $2m_{2n,3n} = (2n)!\, t_n$ for $n \geq 2$. We therefore obtain

$$M(x,y) = \sum_{n\geq 2} \frac{m_{2n,3n}}{(2n)!} x^{2n} y^{3n} = \frac{1}{2}\big(\tilde{T}(x,y) - x^2y^3\big) = \frac{1}{2}\big(T(x^2y^3) - x^2y^3\big). \tag{5}$$

Thus, $M(x,y) = (x^4y^6 + 3x^6y^9 + 13x^8y^{12} + 68x^{10}y^{15} + 399x^{12}y^{18} + \cdots)/2$. Furthermore, the dominant singularity of $M(x) = M(x,1) = 1/2(T(x^2) - x^2)$ is the square-root of the dominant singularity of $T(z)$ and the asymptotic growth of m_n with n even is $\alpha_3 n^{-5/2}\theta^{-n} n!$, where $\theta = 3\sqrt{3}/16$ and α_3 is a constant.

6. SINGULARITY ANALYSIS

We summarize the equations derived so far.

$$B(x) = x^2(D(x) + C(x) - B(x))/2 \tag{6}$$
$$C(x) = S(x) + P(x) + H(x) + B(x) \tag{7}$$
$$D(x) = B(x)^2/x^2 \tag{8}$$
$$S(x) = C(x)^2 - C(x)S(x) \tag{9}$$
$$P(x) = x^2C(x) + x^2C(x)^2/2. \tag{10}$$

We can also describe the substitution in Eq. (1) for $H(x)$ algebraically, using Eqs. (4) and (5).

$$2(C(x) + 1)H(x) = u(1 - 2u) - u(1 - u)^3 \tag{11}$$
$$x^2(C(x) + 1)^3 = u(1 - u)^3. \tag{12}$$

Using algorithms for computing resultants and factorizations (these are standard procedures in e.g., Maple or Mathematica), we obtain a single algebraic equation $Q(C(x), x) = 0$ from Eqs. (6)–(12) that describes the generating function $C(x)$ uniquely, given sufficiently many initial terms of c_n. This is in principle also possible for all other generating functions involved in the above equations; however, the computations turn out to be more tedious, whereas the computations to compute the algebraic equation for $C(x)$ are manageable.

From this equation, following the discussion in Section VII.4 in [19], one can obtain the two dominant singularities ρ and $-\rho$ of $C(x)$, where ρ is an analytic constant and the first digits are $\rho \doteq 0.319224$. One can also compute the expansion at the dominant singularity ρ. Changing the variables $Y = C(x) - C(\rho)$ and $X = x - \rho$ in $Q(C(x), x) = 0$, one can symbolically verify that the equation $Q(C(x), x) = 0$ can be written in the form

$$(aY + bX)^2 = pY^3 + qXY^2 + rX^2Y + sX^3 + \text{ higher order terms},$$

where a, b, p, q, r, s are constants that are given analytically. This implies the following expansion of $C(x)$ near the dominant singularity ρ.

$$C(x) = C(\rho) + b\rho/a\ (1 - x/\rho) + \beta_1(1 - x/\rho)^{3/2} + O((1 - x/\rho)^2),$$

where $\beta_1 := \rho^{3/2}/a\ \sqrt{p(b/a)^3 - q(b/a)^2 + r(b/a) - s}$ is a positive constant. For large n, the coefficient c_n^+ of x^n on the right-hand side satisfies

$$c_n^+ \sim \beta_2\, n^{-5/2}\, \rho^{-n}\, n!,$$

where $\beta_2 = \beta_1/\Gamma(3/2) = 2\beta_1/\sqrt{\pi}$. Similarly we get the expansion at the dominant singularity $-\rho$

$$C(x) = C(\rho) + b\rho/a\ (1 + x/\rho) + \beta_1(1 + x/\rho)^{3/2} + O((1 + x/\rho)^2),$$

and for large n, the coefficient c_n^- of x^n on the right-hand side satisfies

$$c_n^- \sim \beta_2\, n^{-5/2}\, (-\rho)^{-n}\, n!.$$

Following Theorem VI.8 [19], the asymptotic number c_n is then the summation of these two contributions c_n^+ and c_n^-, and thus for large even n

$$c_n \sim 2\beta_2\, n^{-5/2}\, \rho^{-n}\, n!,$$

whereas $c_n = 0$ for odd n.

Since the generating functions for $B(x), D(x), S(x), P(x), H(x)$ are related with $C(x)$ by algebraic equations, they all have the same dominant singularities ρ and $-\rho$. The singular expansion of $G^{(1)}(x)$ can be obtained from Eq. (2) through a term-by-term integration, and thus we obtain the singular expansions at ρ and $-\rho$,

$$G^{(1)}(x) = G^{(1)}(\rho) + c(1 - x/\rho)^2 + \beta_3(1 - x/\rho)^{5/2} + O((1 - x/\rho)^3),$$
$$G^{(1)}(x) = G^{(1)}(\rho) + c(1 + x/\rho)^2 + \beta_3(1 + x/\rho)^{5/2} + O((1 + x/\rho)^3),$$

where c and β_3 are analytically given constants. Thus, for an analytically given constant α_1 and for large even n we get

$$g_n^{(1)} \sim \alpha_1\, n^{-7/2}\, \rho^n\, n!,$$

whereas $g_n^{(1)} = 0$ for odd n.

Because of Eq. (3), the generating functions $G^{(0)}(x)$ and $G^{(1)}(x)$ have the same dominant singularities ρ and $-\rho$, and indeed we may see that $g_n^{(1)}/g_n^{(0)} \to e^{-\lambda}$ where $\lambda = G^{(1)}(\rho)$. Based on the above decomposition it is also easy to derive equations for the exponential generating function $G^{(2)}(x)$ for the number of biconnected cubic planar graphs, which has a slightly larger radius of convergence η (whose first digits are 0.319521).

We finally obtain the following.

Theorem 2. *The asymptotic number of cubic planar graphs, connected cubic planar graphs, 2-connected cubic planar graphs, and 3-connected cubic planar graphs is given by the following. For large even n*

$$g_n^{(0)} \sim \alpha_0\, n^{-7/2}\, \rho^{-n}\, n!$$
$$g_n^{(1)} \sim \alpha_1\, n^{-7/2}\, \rho^{-n}\, n!$$
$$g_n^{(2)} \sim \alpha_2\, n^{-7/2}\, \eta^{-n}\, n!$$
$$g_n^{(3)} \sim \alpha_3\, n^{-7/2}\, \theta^{-n}\, n!.$$

All constants are analytically given. Also $\alpha_1/\alpha_0 = e^{-\lambda}$ where $\lambda = G^{(1)}(\rho)$. The first digits of ρ^{-1}, η^{-1}, and θ^{-1} are 3.132595 , 3.129684, *and* 3.079201, *respectively.*

Table 1 shows the exact numbers $g_n^{(0)}$, $g_n^{(1)}$, $g_n^{(2)}$, and $g_n^{(3)}$ of cubic planar graphs, connected cubic planar graphs, 2-connected cubic planar graphs, and 3-connected cubic planar graphs, up to $n = 20$.

TABLE 1. The Exact Number of Cubic Planar Graphs on n Vertices Depending on the Connectivity

n	$g_n^{(0)}$	$g_n^{(1)}$	$g_n^{(2)}$	$g_n^{(3)}$
4	1	1	1	1
6	60	60	60	60
8	13475	13440	13440	10920
10	5826240	5813640	5700240	4112640
12	4124741775	4116420000	3996669600	2654467200
14	4379810575140	4371563196000	4217639025600	2625727104000
16	6541927990422825	6530471307360000	6272314592544000	3697449275520000
18	13108477865022540000	13087079865123264000	12526155233399808000	7034785952882688000
20	33981214383613597525425	33929276115192441984000	32381500604547878784000	17394357294393311232000

7. RANDOM CUBIC PLANAR GRAPHS

In this section, we use Theorem 2 to investigate the connectedness, components, and the chromatic number of a random cubic planar graph. Throughout the section, for $k = 0, 1, 2, 3$ let $G_n^{(k)}$ denote a random graph chosen uniformly at random among all the k-vertex-connected cubic planar graphs on vertices $1, \ldots, n$ for even n.

7.1. Connectedness

Theorem 3. *Let $\lambda = G^{(1)}(\rho)$. As $n \to \infty$ with n even,* $\Pr(G_n^{(0)}$ *is connected*$) \to e^{-\lambda}$*, whereas each of* $\Pr(G_n^{(0)}$ *is 2-connected*$)$, $\Pr(G_n^{(1)}$ *is 2-connected*$)$ *and* $\Pr(G_n^{(2)}$ *is 3-connected*$)$ *tends to 0.*

Proof. From Theorem 2, we see that as $n \to \infty$ with n even

$$\Pr\left(G_n^{(0)} \text{ is connected}\right) = g_n^{(1)}/g_n^{(0)} \to \alpha_1/\alpha_0 = e^{-\lambda}.$$

Also,

$$\Pr\left(G_n^{(0)} \text{ is 2-connected}\right) = g_n^{(2)}/g_n^{(0)} \sim \alpha_2/\alpha_0(\eta/\rho)^{-n} \to 0,$$

with a similar proof in the other cases. ∎

Using the numbers in Table 1 we compute the probability that $G_n^{(0)}$ is connected, up to $n = 20$, in Table 2.

7.2. Components of $G_n^{(0)}$

In order to discuss coloring later (Theorem 6) we need to find the limiting probability that $G_n^{(0)}$ has a component isomorphic to K_4. Here we consider a more general problem.

Lemma 2. *Let H be a given connected cubic planar graph, and let $\lambda_H = \rho^{v_H}/\mathrm{Aut}(H)$, where ρ is as in Theorem 2, v_H denotes the number of vertices in H (and hence it is even), and* $\mathrm{Aut}(H)$ *denotes the size of its automorphism group. Let the random variable $X_H = X_H(n)$ be the number of components of $G_n^{(0)}$ isomorphic to H for even n. Then X_H has asymptotically the Poisson distribution* $\mathrm{Po}(\lambda_H)$ *with mean λ_H; that is, for $k = 0, 1, 2, \ldots$*

$$\Pr(X_H(n) = k) \to e^{-\lambda_H}\frac{\lambda_H^k}{k!} \quad \text{as } n \to \infty.$$

In particular, the probability that $G_n^{(0)}$ has at least one component isomorphic to H tends to $1 - e^{-\lambda_H}$ as $n \to \infty$ with n even.

This result can be proved along the lines of the proof of Theorem 5.6 of [30]; see also [3]. Indeed, we may obtain the following generalization.

TABLE 2. The Probability that a Random Cubic Planar Graph is Connected

n	4	6	8	10	12	14	16	18	20
$g_n^{(1)}/g_n^{(0)}$	1	1	0.997403	0.997837	0.997982	0.998117	0.998249	0.998368	0.998472

Lemma 3. *Let $H_1, \dots, H_m$ be given pairwise non-isomorphic connected cubic planar graphs; and as before let $\lambda_{H_i} = \rho^{v_{H_i}}/\mathrm{Aut}(H_i)$ and let the random variable $X_{H_i} = X_{H_i}(n)$ be the number of components of $G_n^{(0)}$ isomorphic to H_i, where n is even. Then $X_{H_1}, \dots, X_{H_m}$ are asymptotically jointly distributed like independent random variables* $\mathrm{Po}(\lambda_{H_1}), \dots, \mathrm{Po}(\lambda_{H_m})$, *and so the total number of components isomorphic to some H_i is asymptotically* $\mathrm{Po}\left(\sum_i \lambda_{H_i}\right)$.

Let us observe here that if $H_1, H_2, \dots$ is an enumeration of all the pairwise non-isomorphic connected cubic planar graphs, then $\sum_i \lambda_{H_i} = G^{(1)}(\rho)$. For

$$G^{(1)}(\rho) = \sum_n g_n^{(1)} \cdot \frac{1}{n!}\rho^n = \sum_n \sum_{i: v_{H_i}=n} \frac{n!}{\mathrm{Aut}(H_i)} \cdot \frac{1}{n!}\rho^n = \sum_i \frac{\rho^{v_{H_i}}}{\mathrm{Aut}(H_i)} = \sum_i \lambda_{H_i}. \quad (13)$$

Next we want to show that $G_n^{(0)}$ usually has a giant component.

Lemma 4. *For any $\varepsilon > 0$ there exists t such that the probability is less than ε that each component in $G_n^{(0)}$ has order at most $n - t$.*

Proof. Let $\mathcal{C}(n)$ denote the set of labeled cubic planar (simple) graphs on the vertices $1, \dots, n$ and so $|\mathcal{C}(n)| = g_n^{(0)}$. By Theorem 2, there are constants $\alpha > 0$ and $\beta > 1$ such that

$$g_n^{(0)} \sim \alpha n^{-\beta} \rho^{-n} n!$$

as $n \to \infty$ with n even. Thus, there is an n_0 such that for all even $n \geq n_0$

$$\frac{1}{2}\alpha n^{-\beta}\rho^{-n} n! \leq g_n^{(0)} \leq 2\alpha n^{-\beta}\rho^{-n} n!.$$

Let t be a positive integer at least n_0 sufficiently large that

$$8\alpha \cdot 2^{\beta} \cdot \frac{(t-1)^{-(\beta-1)}}{\beta - 1} < \varepsilon.$$

The reason for this choice will of course emerge shortly. Let $\mathcal{D}(n)$ be the set of graphs $G \in \mathcal{C}(n)$ such that each component has order at most $n - t$. Then for even $n \geq 3t$,

$$\begin{aligned}
|\mathcal{D}(n)| &\leq \sum_{j=t}^{n/2} \binom{n}{j} g_j^{(0)} g_{n-j}^{(0)} \\
&\leq 4\alpha^2 \rho^{-n} n! \sum_{j=t}^{n/2} j^{-\beta}(n-j)^{-\beta} \\
&\leq 4\alpha^2 \rho^{-n} n! \left(\frac{n}{2}\right)^{-\beta} \sum_{j=t}^{n/2} j^{-\beta} \\
&\leq 8\alpha g_n^{(0)} \, 2^{\beta} \sum_{j=t}^{n/2} j^{-\beta}.
\end{aligned}$$

But

$$\sum_{j=t}^{n/2} j^{-\beta} \leq \int_{t-1}^{n/2} x^{-\beta} dx < \frac{(t-1)^{-(\beta-1)}}{\beta - 1}.$$

Thus, our choice of t yields $|\mathcal{D}(n)|/g_n^{(0)} < \varepsilon$ as required. ∎

Theorem 4. *The number of components of $G_n^{(0)}$ is asymptotically* $1 + \mathrm{Po}(\lambda)$, *where* $\lambda = G^{(1)}(\rho)$.

Observe that this theorem shows again (as in Theorem 4) that the probability that $G_n^{(0)}$ is connected tends to $e^{-\lambda}$ as $n \to \infty$.

Proof. We may use Lemmas 3 and 4, together with (15), and follow the lines of the proof of Theorem 5.5 of [30]. ∎

7.3. Triangles and Other Subgraphs

In order to discuss coloring later we also need to know about triangles, in particular the unsurprising result that $G_n^{(k)}$ usually contains at least one triangle. In fact, far more is true.

Lemma 5. *Let $Y_n^{(k)}$ be the number of triangles in $G_n^{(k)}$. Then there exists $\delta > 0$ such that for even n*

$$\Pr\left(Y_n^{(k)} \geq \delta n\right) = 1 - e^{-\Omega(n)}.$$

We shall avoid using round-down $\lfloor x \rfloor$ and round-up $\lceil x \rceil$ in order to keep our formulas readable.

Proof. Let us consider $Y_n^{(0)}$: the other cases are very similar. Let $\delta > 0$ be sufficiently small that

$$\frac{\rho^2(1-4\delta)}{4e\delta} > 2.$$

By Theorem 2 there exist constants $\alpha > 0$, $\beta > 1$, and $n_0 \geq 2/\delta$ such that for all even $n \geq n_0$

$$\frac{1}{2}\alpha n^{-\beta}\rho^{-n} n! \leq g_n^{(0)} \leq 2\alpha n^{-\beta}\rho^{-n} n!. \tag{14}$$

Assume for a contradiction that for some even $n \geq n_0$

$$\Pr\left(Y_n^{(0)} \leq \delta n\right) \geq e^{-\delta n}. \tag{15}$$

Consider the following construction of cubic planar graphs on vertices $1, \ldots, n+2\delta n$:

- pick an ordered list of $2\delta n$ *special* vertices, say $s_1, s_2, \ldots, s_{2\delta n}$; there are $\frac{(n+2\delta n)!}{n!}$ choices;
- take a cubic planar graph G on the remaining n vertices with at most δn triangles; by (14) and (15) there are at least $e^{-\delta n} g_n^{(0)} \geq e^{-\delta n}\frac{1}{2}\alpha n^{-\beta}\rho^{-n} n!$ choices;
- pick a set of δn vertices in G that form an independent set and list them in increasing order, say $v_1, v_2, \ldots, v_{\delta n}$; the number of choices is at least

$$\frac{n(n-4)\cdots(n-4\delta n+4)}{(\delta n)!} \geq \frac{n^{\delta n}(1-4\delta)^{\delta n}}{(\delta n)!} \geq \left(\frac{1-4\delta}{\delta}\right)^{\delta n};$$

- construct a cubic graph G' in such a way that for each v_i we select its two largest neighbors, say m and l, and insert s_{2i-1} on the edge (v_i, m) and s_{2i} on (v_i, l) together with an edge (s_{2i-1}, s_{2i}); see Figure 8.

For a given set of δn triangles in G', there is at most one construction as above yielding G' with these as the new triangles (see Figure 8 and note that we can identify v_i in the triangle

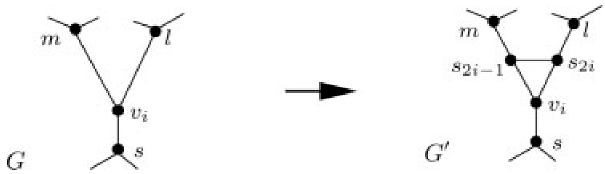

Fig. 8. Creating a new triangle.

as the vertex adjacent to s). But G' has at most $2\delta n$ triangles. Hence, the same graph G' is constructed at most $\binom{2\delta n}{\delta n} \le 2^{2\delta n}$ times. But of course $g^{(0)}_{n+2\delta n}$ is at least the number of graphs constructed in this way. Thus,

$$\begin{aligned} g^{(0)}_{n+2\delta n} &\ge \frac{(n+2\delta n)!}{n!} \cdot e^{-\delta n} \frac{1}{2} \alpha n^{-\beta} \rho^{-n} n! \cdot \left(\frac{1-4\delta}{\delta}\right)^{\delta n} \cdot 2^{-2\delta n} \\ &> \frac{1}{2}\alpha (n+2\delta n)!(n+2\delta n)^{-\beta} \rho^{-n-2\delta n} \rho^{2\delta n} e^{-\delta n} \left(\frac{1-4\delta}{\delta}\right)^{\delta n} 4^{-\delta n} \\ &\ge \frac{1}{4} g^{(0)}_{n+2\delta n} \left(\frac{\rho^2(1-4\delta)}{4e\delta}\right)^{\delta n} \\ &> g^{(0)}_{n+2\delta n}, \end{aligned}$$

a contradiction. ∎

It is triangles that we need to know about for coloring, but we could also ask about appearances of other subgraphs. Here is one such result, which may be proved along the lines of the proof of Theorem 4.1 in [30].

Theorem 5. *Let H be a fixed connected planar graph with one vertex of degree 1 and each other vertex of degree 3. Let k be 0 or 1. Then there exists $\delta > 0$ such that for even n*

$$\Pr\left(G^{(k)}_n \text{ contains } < \delta n \text{ copies of } H\right) = e^{-\Omega(n)}.$$

Note that each copy of H contributes at least one cut-edge to the graph, and each such edge is counted at most twice, so we see that $G^{(0)}_n$ and $G^{(1)}_n$ are very far from being 2-edge-connected; see Theorem 3 above.

7.4. Coloring

Finally we can give a full story about the chromatic number $\chi(G^{(k)}_n)$.

Theorem 6. *Let $\nu = \rho^4/4! \doteq 0.000432$, where ρ is as in Theorem 2. Then as $n \to \infty$ with n even*

$$\begin{aligned} \Pr\left(\chi(G^{(0)}_n) = 3\right) &\to e^{-\nu} \doteq 0.999568, \\ \Pr\left(\chi(G^{(0)}_n) = 4\right) &\to 1 - e^{-\nu}. \end{aligned}$$

For $k = 1, 2, 3$ we have $\Pr(\chi(G^{(k)}_n) = 3) \to 1$ as $n \to \infty$.

Proof. By Brook's Theorem (see, e.g., [18]), for a cubic graph G with at least one triangle, $\chi(G) = 3$ unless there is a component K_4, in which case $\chi(G) = 4$. Thus, the theorem follows from Lemmas 2 and 5. ■

8. CONCLUDING REMARKS

Using the decomposition in Section 3 we can derive recursive counting formulas that count the exact number of cubic planar graphs. The decomposition and the counting formulas also yield a deterministic polynomial time sampling procedure—this is known as the *recursive method* for sampling [20, 31]. The sampling procedure was implemented in [29], where several other empirical properties of a random cubic planar graph are discussed, e.g., the number of cut-edges and the diameter. The present decomposition, exact and asymptotic enumeration, and the sampling procedure can also be adapted to multi-graphs, i.e., graphs that might contain double edges and loops.

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REFERENCES

[1] R. Arratia, A. Barbour, and S. Tavaré, Logarithmic combinatorial structures: A probabilistic approach, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2003.

[2] C. Banderier, P. Flajolet, G. Schaeffer, and M. Soria, Random maps, coalescing saddles, singularity analysis, and Airy phenomena, Random Struct Algor 19 (2001), 194–246.

[3] J. P. Bell, E. A. Bender, P. J. Cameron, and L. B. Richmond, Asymptotics for the probability of connectedness and the distribution of number of components, Electron J Comb 7 (2000), R33.

[4] E. Bender and L. Richmond, A survey of the asymptotic behaviour of maps, J Combinator Theory 40 (1986), 297–329.

[5] E. A. Bender, Z. Gao, and N. C. Wormald, The number of 2-connected labelled planar graph, Electron J Combinator 9 (2002).

[6] D. Bessis, C. Itzykson, and J. B. Zuber, Quantum field theory techniques in graphical enumeration, Adv Appl Math 1 (1980), 109–157.

[7] M. Bodirsky, O. Giménez, M. Kang, and M. Noy, "On the number of series-parallel and outerplanar graphs," Proceedings of European Conference on Combinatorics, Graph Theory, and Applications (EuroComb 2005), DMTCS Proceedings Series Volume AE, 2005, pp. 383–388.

[8] M. Bodirsky, C. Gröpl, D. Johannsen, and M. Kang, "A direct decomposition of 3-connected planar graphs," 17th Annual International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC05), 2005.

[9] M. Bodirsky, C. Gröpl, and M. Kang, "Generating labeled planar graphs uniformly at random," Proceedings of the Thirteenth International Colloquium on Automata, Languages and Programming (ICALP'03), Springer LNCS 2719, 2003, pp. 1095–1107.

[10] M. Bodirsky, C. Gröpl, and M. Kang, “Sampling unlabeled biconnected planar graphs,” Proceedings of the Sixteenth Annual International Symposium on Algorithms and Computation (ISAAC05), Springer LNCS 3827, 2005, pp. 593–603.

[11] M. Bodirsky and M. Kang, Generating random outerplanar graphs, Combinator Probab Comput 15 (2006), 333–343.

[12] N. Bonichon, C. Gavoille, and N. Hanusse, “Canonical decomposition of outerplanar maps and application to enumeration,” Coding and Generation, the Twentynineth International Workshop, Graph-Theoretic Concepts in Computer Science (WG03), Springer LNCS 2880, 2003, pp. 81–92.

[13] J. Bouttier, P. Di Francesco, and E. Guitter, Census of planar maps: From the one-matrix model solution to a combinatorial proof, Nuclear Phys B 645 (2002), 477–499.

[14] J. Bouttier, P. Di Francesco, and E. Guitter, Combinatorics of hard particles on planar graphs, Nuclear Phys B 655 (2003), 313–341.

[15] A. Denise, M. Vasconcellos, and D. Welsh, The random planar graph, Congress Numerant 113 (1996), 61–79.

[16] P. Di Francesco, 2D quantum gravity, matrix models and graph combinatorics, `http://arxiv.org/abs/math-ph/0406013`, 2004.

[17] P. Di Francesco, “Matrix model combinatorics: Applications to folding and coloring,” Random Matrix Models and Their Applications, Math. Sci. Res. Inst. Publ., 40, Cambridge Univ. Press, Cambridge, 2001, pp. 111–170.

[18] R. Diestel, Graph Theory, Springer-Verlag, New York, 1997.

[19] P. Flajolet and R. Sedgewick, Analytic Combinatorics, in preparation, October 23, 2006.

[20] P. Flajolet, P. Zimmerman, and B. V. Cutsem, A calculus for the random generation of labelled combinatorial structures, Theoret Comput Sci 132 (1994), 1–35.

[21] Z. Gao and N. C. Wormald, Enumeration of rooted cubic planar maps, Ann Combinator 6 (2002), 313–325.

[22] S. Gerke and C. McDiarmid, On the number of edges in random planar graphs, Combinator Probab Comput 13 (2004), 165–183.

[23] S. Gerke, C. McDiarmid, A. Steger, and A. Weißl, Random planar graphs with given average degree, preprint, 2005.

[24] S. Gerke, C. McDiarmid, A. Steger, and A. Weißl, “Random planar graphs with n nodes and a fixed number of edges,” Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA05), 2005, pp. 999–1007.

[25] O. Giménez and M. Noy, Asymptotic enumeration and limit laws of planar graphs, `http://arxiv.org/abs/math.CO/0501269`, 2005.

[26] F. Harary and E. Palmer, Graphical Enumeration. Academic Press, New York, 1973.

[27] M. Kang and M. Loebl, The enumberation of planar graphs via Wick's theorem, `http://arxiv.org/abs/math.CO/0605218`, 2006.

[28] V. A. Liskovets and T. Walsh, “Ten steps to counting planar graphs,” Proc. of Eighteenth Southeastern International Conference on Combinatoris, Graph Theory, and Computing, Congr. Numer., 1987, pp. 269–277.

[29] M. Löffler, Uniform sampling and enumeration of planar structures, Diploma Thesis, Humboldt-Universität zu Berlin, 2005.

[30] C. McDiarmid, A. Steger, and D. Welsh, Random planar graphs, J Combinator Theory B 93 (2005), 187–205.

[31] A. Nijenhuis and H. Wilf, Combinatorial Algorithms, Academic Press, 1979.

[32] D. Osthus, H. J. Prömel, and A. Taraz, On random planar graphs, the number of planar graphs and their triangulations, J Combinator Theory B 88 (2003), 119–13.

[33] B. A. Trakhtenbrot, Towards a theory of non-repeating contact schemes, Trudi Mat Inst Akad Nauk SSSR 51 (1958), 226–269. [In Russian].

[34] W. Tutte, A census of planar triangulations, Canad J Math 14 (1962), 21–38.

[35] W. Tutte, A census of planar maps, Canad J Math 15 (1963), 249–271.

[36] T. Walsh, Counting unlabelled three-connected and homeomorphically irreducible two-connected graphs, J Combinator Theory 32 (1982), 12–32.