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Random Graphs: Theory and Applications from Nature to Society to the Brain

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1 Introduction

The theory of random graphs deals with asymptotic properties of graphs equipped with a certain probability distribution; for example, it studies how the component structure of a uniform random graph evolves as the number of edges increases. Since the foundation of the theory of random graphs by Erdős and Rényi five decades ago, various random graph models have been introduced and studied. Graph theory has meanwhile found its way into other sciences as a rich source of models describing fundamental aspects of a broad range of complex phenomena. This article is a gentle introduction to the theory of random graphs and its recent developments (with focus on the phase transition and critical phenomena, a favourite topic of the first author) and applications.

This is an extended version of the article entitled "Random Graphs: from Nature to Society to the Brain" [35] published in *Seoul Intelligencer*, a special issue of the *Mathematical Intelligencer*, on the occasion of International Congress of Mathematicians in Seoul in 2014.

2 Erdős-Rényi Random Graphs

2.1 The Beginning

Erdős and Rényi initiated the theory of random graphs in their article [27] entitled "On random graphs I" published in 1959, in which they addressed, among other things, the questions of the probability of a random graph being connected, and the probability that the largest component of a random graph covers almost all

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vertices. In a subsequent paper entitled "On the evolution of random graphs" published in 1960 [28] , Erdős and Rényi discovered that a random graph undergoes a drastic change in the number of vertices of the largest component when the number of edges is around half the number of vertices, i.e. there is a *phase transition* in the evolution of a random graph in view of the *emergence of the giant component*.

Before we discuss the phase transition phenomenon in more details, we introduce some basic definitions. A graph $G = (V,E)$ is a pair consisting of a set V of vertices (or nodes) and a set E of edges (or lines), which are 2-element subsets of V. The number of edges incident to a vertex is called the *degree* of a vertex. A sequence of vertices $(v_1,\ldots,v_k)$ in which each two consecutive vertices form an edge is called a *path* from v_1 to v_k if $v_1 \neq v_k$, and it is called a *cycle* if $v_1 = v_k$. We say that a vertex v is *reachable* from another vertex w if there exists a path from v to w. Reachability is an equivalence relation, and equivalence classes are called *components* of G. The number of vertices in a component is called the *order* of the component. If a graph has only a single component, then it is called *connected*. A connected graph with no cycle is called a *tree*, and a graph without cycles is called a *forest*. A connected graph with exactly one cycle is called *unicyclic*.

2.2 Erdős-Rényi Random Graph Models

There are three random graph models hidden under the name of the Erdős-Rényi random graphs:

(1) The uniform random graph $G(n,m)$ is a graph chosen uniformly at random from the set of all graphs with vertex set $[n] := \{1,\cdots,n\}$ and m edges, for an integer $0 \leq m \leq \binom{n}{2}$.
(2) The binomial random graph $G(n,p)$ is a graph with vertex set $[n]$ in which each pair of vertices is joined by an edge independently with probability p, for a real number $0 \leq p \leq 1$.
(3) The Erdős-Rényi process $\{G_n(m) : m = 0,\ldots,\binom{n}{2}\}$ begins with a graph $G_n(0)$ with n isolated vertices and no edges, and in each step $1 \leq m \leq \binom{n}{2}$ a new random edge is added to an evolving graph $G_n(m-1)$ to obtain a new graph $G_n(m)$. The graph $G_n(m)$ created by the Erdős-Rényi process is distributed like the uniform random graph $G(n,m)$.

The three models are essentially equivalent when the parameters are appropriately selected, i.e., $m = p\binom{n}{2}$. The uniform random graph and binomial random graph were studied earlier, among others by Gilbert [29]. A quarter of a century later Bollobás [13] observed that the uniform random graph $G(n,m)$ can be considered as a graph $G_n(m)$ created by the Erdős-Rényi process.

When discussing properties of the Erdős-Rényi random graphs we shall parametrise $m = t\,n/2$ or $p = t/(n-1)$, so t denotes the expected degree of a random

vertex, and shall be concerned with properties that hold *with high probability* (in short *whp*), meaning with probability tending to one as the number n of vertices tends to ∞.

2.3 Emergence of the Giant Component

One of the most important discoveries by Erdős and Rényi [28] deals with the appearance of the phase transition in the order of the largest component: in short, the order of the largest component in the Erdős-Rényi random graph changes from logarithmic to linear order when the expected degree passes through one (for example, from 0.99 to 1.01) as more edges are added.

To be more precise, we consider the Erdős-Rényi random graph with expected degree $t > 0$. If the expected degree t is smaller than the critical value one, *whp* the Erdős-Rényi random graph consists of trees and unicyclic components and the largest component is a tree of order $O(\log n)$; such components are called "small". On the other hand, if the expected degree t is larger than one, *whp* there is a unique largest component of linear order (called "the giant component"), while all but the giant component are trees or unicyclic components of order $O(\log n)$; in other words, all but the giant component are "small". If the expected degree t is equal to the critical value one, *whp* the order of the largest component is $\Theta(n^{2/3})$.

The reason why the expected degree one is so crucial in view of the emergence of the giant component in $G(n,p)$ was nicely explained by Karp [40] who used the following component-exposure process based on the breadth-first search. Given a vertex v, we first expose the neighbours (say children) of v. And then we expose the neighbours of each of the neighbours of v, one after another. And we continue until there are no more vertices left in the component of v. Roughly speaking, when $k = o(n)$ vertices are exposed so far, the number of children of each vertex is a binomial random variable with parameters $n-k$ and p, and thus its expectation is $(n-k)p \sim t$. Since the binomial distribution $\mathrm{Bi}(n-k,p)$ converges to the Poisson distribution $\mathrm{Po}(t)$ with mean $t = p(n-1)$ when $n \to \infty$ and t is a fixed constant, we can approximate the component-exposure process by the Galton-Watson branching process with Poisson offspring distribution $\mathrm{Po}(t)$; it starts with a unisexual organism which generates a random number of children according to $\mathrm{Po}(t)$, and each of its children generates a random number of children independently according to $\mathrm{Po}(t)$, and so on. The classical branching process theory says that if $t < 1$, then with probability one the branching process dies out; this corresponds to small components in $G(n,p)$. On the other hand, if $t > 1$, with positive probability the process continues forever. The survival probability is given as the unique positive solution $\rho = \rho(t) \in (0,1)$ of the equation

$$1 - \rho = e^{-t\rho}. \tag{1}$$

This survival probability corresponds to the probability that a random vertex in

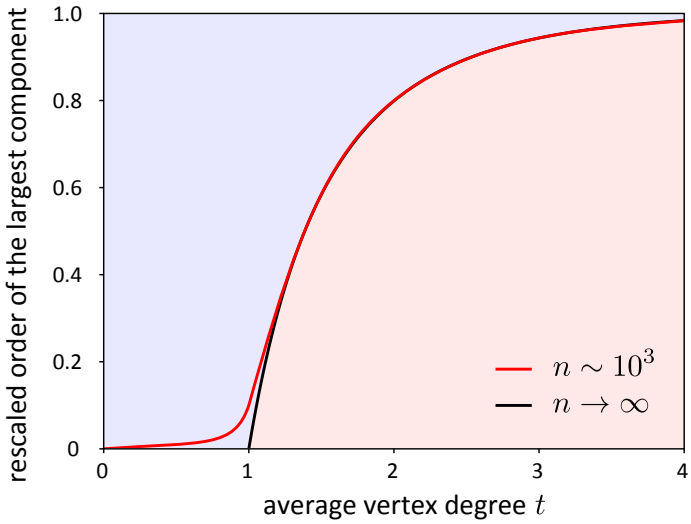


Figure 1: Phase transition in the Erdős-Rényi random graph.

$G(n,p)$ lies in the giant component. This approach can be made mathematically rigorous and yields that if $t > 1$, then *whp* the order of the giant component in $G(n,p)$ is $\rho n + o(n)$.

2.4 Critical Window of the Phase Transition

Returning to the result of Erdős and Rényi, the order of the largest component changes from logarithmic to sublinear and then to linear order; more precisely it changes from $O(\log n)$ to $\Theta(n^{2/3})$ and to $\Theta(n)$ depending on whether the expected degree t satisfies $t < 1$, $t = 0$, or $t > 1$. Erdős and Rényi described this phenomenon as a "double jump" and considered it to be one of the most striking facts concerning random graphs.

But it leads to several natural questions: *are there 'real' jumps?* In other words, is the phase transition in the Erdős-Rényi random graph discontinuous? Or is it continuous? If it is continuous, how "smooth" or how "sharp" is the phase transition? How big should the difference between the expected vertex degree and the critical value one be, so as to be able to distinguish the order of the largest component from the order of the second largest component?

Bollobás [13] showed that there is in fact no jump, but a *smooth phase transition with three different phases* when the number of edges is around half the number of vertices: the subcritical phase in which *whp* there are many small components of almost equal order; the critical phase in which *whp* there are a few large components of the same order up to constant factor; and the supercritical phase which is characterised by the fact that *whp* there is a unique largest component that is much

larger than the second largest component. The result of Bollobás was improved by Łuczak [44].

To state the results of Bollobás and Łuczak, we consider the binomial random graph $G(n,p)$ with $p = \frac{t}{n-1}$, where $t = 1 \pm \varepsilon$ for $\varepsilon = \varepsilon(n) > 0$ satisfying $\varepsilon \to 0$, so the expected degree converges to one. In order to capture how fast ε tends to zero, we let $\omega(n)$ be any function tending to infinity arbitrarily slowly with n and let $\varepsilon = \varepsilon(n)$ satisfy $\omega(n) \leq \varepsilon^3 n$. In other words, we let $\varepsilon \to 0$ and $\varepsilon^3 n \to \infty$. If $t = 1-\varepsilon$, then for any fixed positive integer i *whp* the order of the i-th largest component is asymptotically $2\varepsilon^{-2}\log(\varepsilon^3 n)$, which is substantially smaller than $n^{2/3}$. On the other hand, if $t = 1+\varepsilon$, *whp* the order of the largest component is asymptotically $2\varepsilon n$, which is substantially larger than $n^{2/3}$, and the order of the second largest component is substantially smaller than $n^{2/3}$.

When $\varepsilon^3 n = \Theta(1)$, Aldous [4] provided a precise description of the sequence of the orders (rescaled by $n^{2/3}$) of largest components, using multiplicative coalescent processes and a sequence of lengths of the excursions of a reflecting inhomogeneous Brownian motion.

2.5 Limit Theorems for the Giant Component

Can we say more about the distribution of the order of the giant component when the expected degree t is larger than one?

Let $I \subset (1,\infty)$ be a compact interval, let $p = p(n)$ be a sequence such that $t = p(n-1) \in I$ for all n and let $\rho = \rho(t) \in (0,1)$ be the unique positive solution of the equation (1). Let $L_1(t)$ denote the order of the giant component in $G(n,p)$.

The first limit theorem for the giant component is *a strong law of large numbers* which provides that for any $\delta_1, \delta_2 > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$1-\delta_2 \leq \mathbb{P}[\, \left|\frac{L_1(t)}{n} - \rho\right| \leq \delta_1 \,] \leq 1+\delta_2.$$

So, the typical value of $L_1(t)$ can be determined up to fluctuations of order $o(n)$.

A natural question is whether we can characterise the distribution of $L_1(t)$ more precisely. We define $\mu = \mu(\rho,n)$ and $\sigma = \sigma(t,\rho,n)$ as

$$\mu := \rho n \quad \textit{and} \quad \sigma := \sqrt{\frac{\rho(1-\rho)}{(1-t(1-\rho))^2}\, n}.$$

We can show that $\sigma^{-1}(L_1(t)-\mu)$ converges in distribution to $N(0,1)$, where $N(0,1)$ denotes the standard normal distribution. This provides a *central limit theorem* for $L_1(t)$: for any $a < b$ with $a,b \in \mathbb{R}$ and any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$\frac{1-\delta}{\sqrt{2\pi}}\int_a^b \exp\left(-\frac{t^2}{2}\right) dt \leq \mathbb{P}[\, a \leq \frac{L_1(t)-\mu}{\sigma} \leq b \,] \leq \frac{1+\delta}{\sqrt{2\pi}}\int_a^b \exp\left(-\frac{t^2}{2}\right) dt.$$

Thus one can estimate the value of $L_1(t)$ up to an error of $o(\sigma) = o(\sqrt{n})$.

Indeed, one can derive even a stronger result, a *local limit theorem* for $L_1(t)$: for any compact interval $J \subset \mathbb{R}$ and any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and any integer $k \in \mathbb{N}$ satisfying $\sigma^{-1}(k-\mu) \in J$, we have

$$\frac{1-\delta}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(k-\mu)^2}{2\sigma^2}\right) \leq \mathbb{P}[\, L_1(t) = k \,] \leq \frac{1+\delta}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(k-\mu)^2}{2\sigma^2}\right).$$

These results were established by Stepanov [64] and Pittel and Wormald [54] who used counting techniques, and were reproved by Behrisch, Coja-Oghlan and Kang [10] who applied Stein's method to the total number of vertices outside the giant component.

Indeed, much stronger results hold. Stepanov [64] and Pittel and Wormald [54] derived a *local limit theorem for the joint distribution* of the number of vertices and edges in the giant component. Behrisch, Coja-Oghlan and Kang [10] reproved this result using the so-called two-round exposure and "smoothing technique" as well as Fourier analysis. Let $E(t)$ denote the number of edges in the giant component in $G(n,p)$. We define

$$\begin{aligned}
\mu_e &:= \frac{t(1-(1-\rho)^2)}{2} n, \\
\sigma_e &:= \sqrt{\left(\frac{t^2(1-\rho)^2(2-(2t-1)\rho(1-\rho))}{(1-t(1-\rho))^2} + \frac{t(1-(1-\rho)^2)}{2}\right) n}, \\
\sigma_j &:= \frac{t(1-\rho)(1-(1-\rho)^2+t\rho(1-\rho))}{(1-t(1-\rho))^2} n^2.
\end{aligned}$$

Then for any compact intervals $J, J_e \subset \mathbb{R}$ and for any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and any integers $k, \ell \in \mathbb{N}$ satisfying $\sigma^{-1}(k-\mu) \in J$ and $\sigma_e^{-1}(\ell - \mu_e) \in J_e$, we have

$$(1-\delta)\Phi(k,\ell) \leq \mathbb{P}[\, L_1(t) = k \text{ and } E(t) = \ell \,] \leq (1+\delta)\Phi(k,\ell),$$

where

$$\begin{aligned}
\Phi(k,\ell) := &\frac{1}{2\pi\sqrt{\sigma^2\sigma_e^2 - \sigma_j^2}} \\
&\cdot \exp\left(-\frac{\sigma^2\sigma_e^2}{2(\sigma^2\sigma_e^2 - \sigma_j^2)}\left(\frac{(k-\mu)^2}{\sigma^2} - \frac{2\sigma_j(k-\mu)(\ell-\mu_e)}{\sigma^2\sigma_e^2} + \frac{(\ell-\mu_e)^2}{\sigma_e^2}\right)\right).
\end{aligned}$$

As for local limit theorems in the (more sophisticated) *supercritical regime* when $t = 1+\varepsilon$ for $\varepsilon = \varepsilon(n) > 0$ satisfying $\varepsilon \to 0$ and $\varepsilon^3 n \to \infty$, Bollobás and Riordan [17]

gave a simple new proof of a strong law of large numbers for the order of the giant component, using the Galton-Watson branching process. Pittel and Wormald [54] established a central limit theorem for the order of the giant component all the way through the supercritical regime, by counting connected graphs. Their result was reproved recently by Bollobás and Riordan [18], using random walk and martingale arguments.

3 Generalisations of Erdős-Rényi Random Graphs

Since the seminal work of Erdős and Rényi [28], various random graph models have been extensively studied. Examples include random hypergraphs, random planar graphs, random graph processes, and inhomogeneous random graphs, each of which we briefly review below.

3.1 Random Hypergraphs

One of the most natural generalisations of the Erdős-Rényi random graph $G(n,p)$ is the random k-uniform hypergraph $H_k(n,p)$, which is a hypergraph with vertex set $[n]$ in which each of the $\binom{n}{k}$ possible edges (i.e. k-element subsets of $[n]$) is present independently with probability p.

We say that a vertex v in a hypergraph H is reachable from w (or v and w are vertex-connected) if there exists a sequence of edges $(e_1,\dots,e_\ell)$ such that $v \in e_1$, $w \in e_\ell$ and $e_i \cap e_{i+1} \neq \emptyset$ for all $1 \leq i \leq \ell-1$. Reachability is an equivalence relation, and the equivalence classes are called the components of H.

Phase transition phenomena were discovered also in random hypergraphs. The critical point of the emergence of the giant component in $H_k(n,p)$ was first determined by Schmidt-Pruzan and Shamir in [61]. To be more precise, let $p = t(k-1)^{-1}\binom{n-1}{k-1}^{-1}$ for $t > 0$. Schmidt-Pruzan and Shamir showed that if $t < 1$, then *whp* the number of vertices in the largest component is $O(\log n)$, but if $t > 1$, then *whp* there is a unique component containing a linear number of vertices. Indeed, more is known: *whp* the number of vertices in the giant component is $\rho n + o(n)$, where $\rho = \rho(k,t) \in (0,1)$ is the unique positive solution to the equation

$$1 - \rho = \exp(t((1-\rho)^{k-1} - 1)).$$

When $k = 2$, this corresponds to the result of the graph case.

Karoński and Łuczak [39] studied the phase transition in the early supercritical phase, when $t = 1 + o((\log n/n\log\log n)^{1/3})$, proving a local limit theorem for the number of vertices in the largest component. Behrisch, Coja-Oghlan and Kang [11] established central and local limit theorems for the number of vertices in the largest component when $t > 1+\varepsilon$ for an arbitrarily small but fixed $\varepsilon > 0$. In addition, they derived the local limit theorem for the joint distribution of the

number of vertices and the number of edges in the largest component of $H_k(n,p)$. As an application, Behrisch, Coja-Oghlan and Kang [11] obtained an asymptotic formula for the probability that $H_k(n,p)$ is connected. To this end, they applied new purely probabilistic approaches, such as two-round edge exposure, Stein's method, and Fourier analysis. Bollobás and Riordan [18] subsequently proved that the distribution of the number of vertices in the largest component tends to a normal distribution for $t = 1+\varepsilon$ whenever $\varepsilon = \varepsilon(n) > 0$ satisfies $\varepsilon^3 n \to \infty$, i.e. throughout the supercritical regime.

Such local limit theorems are closely related to the *asymptotic number of connected k-uniform hypergraphs* with n vertices and m edges. Using combinatorial enumeration, Karoński and Łuczak [39] derived the asymptotic number of connected k-uniform hypergraphs with given numbers of vertices and edges in the barely supercritical regime when $m - \frac{n}{k-1} \ll \frac{\log n}{\log\log n}$. Behrisch, Coja-Oghlan and Kang derived the corresponding result from the local limit theorem in the strictly supercritical regime when $m - \frac{n}{k-1} = \Theta(n)$, and Bollobás and Riordan [19] all the way through the supercritical regime $m - \frac{n}{k-1} = o(n)$. Sato and Wormald [60] also derived the asymptotic number of connected 3-uniform hypergraphs using cores and kernels when $n^{1/3}\log^2 n \ll m - \frac{n}{2} \ll n$.

3.2 Random Hypergraphs as Random Simplicial Complexes

We can view random hypergraphs as random simplicial complexes, for example random 3-uniform hypergraphs as random simplicial 2-complexes, by regarding 3-element subsets as trianglular 2-cells and 2-element subsets as 1-cells. Topological aspects of random simplicial complexes – such as collapsibility and vanishing of the top homology – were investigated, for example, in [5, 6, 43].

Motivated partly by the study of random simplicial complexes, we consider higher order connectivity. The notion of higher order connectivity in hypergraphs is however ambiguous and in fact there are several possible definitions. As an example, we shall consider the one suggested by Bollobás and Riordan [18], i.e. the j-tuple connectivity: a j-element subset J_1 is said to be *reachable* from another j-element subset J_2 if there exists a sequence of edges $(E_1,\dots,E_\ell)$ such that $J_1 \subseteq E_1$, $J_2 \subseteq E_\ell$ and $|E_i \cap E_{i+1}| \geq j$ for each $i = 1,\dots,\ell-1$. The reachability is an equivalence relation on j-element subsets, and the equivalence classes are called j-tuple connected components, or *j-components* in short. The case $j = 1$ corresponds to the notion of vertex-connectedness.

In view of the emergence of a giant j-component for any $1 \leq j \leq k-1$, Cooley, Kang and Person [25] showed that $H_k(n,p)$ undergoes a phase transition at the threshold $p_{k,j} := \frac{1}{\binom{\binom{k}{j}-1}\cdot\binom{n-j}{k-j}}$. Let $p = t \cdot p_{k,j}$ for $t > 0$ and let $L_j(t)$ denote the number of j-element subsets contained in the largest j-component in $H_k(n,p)$. For an arbitrarily small but fixed $\varepsilon > 0$, Cooley, Kang and Person showed that

whp $L_j(t) = \Omega(\varepsilon n^j)$ and the giant component is unique if $t = 1+\varepsilon$, but $L_j(t) = O(\varepsilon^{-2}\log n)$ if $t = 1-\varepsilon$.

Cooley, Kang and Koch [24] took a closer look at the giant component in the supercritical regime and proved that when $t = 1+\varepsilon$ for $\varepsilon = \varepsilon(n) > 0$ satisfying $\varepsilon \to 0$ and $\varepsilon^3 n \to \infty$, *whp* $L_j(t) = (1+o(1))2\varepsilon \frac{1}{\binom{k}{j}-1}\binom{n}{j}$. For $k = 2, j = 1$, the threshold and the number of vertices in the giant component match those for the graph case: $p_{2,1} = \frac{1}{n-1}$ and *whp* $L_1(t) = (1+o(1))2\varepsilon n$.

These results open up many new questions. What can we say about $L_j(t)$ at the criticality when $t = 1$? What about the number of j-element subsets contained in the second largest j-component in the supercritical regime? What is the actual distribution of the number of j-element subsets contained in the giant j-component? What about central or local limit theorems for the giant j-component?

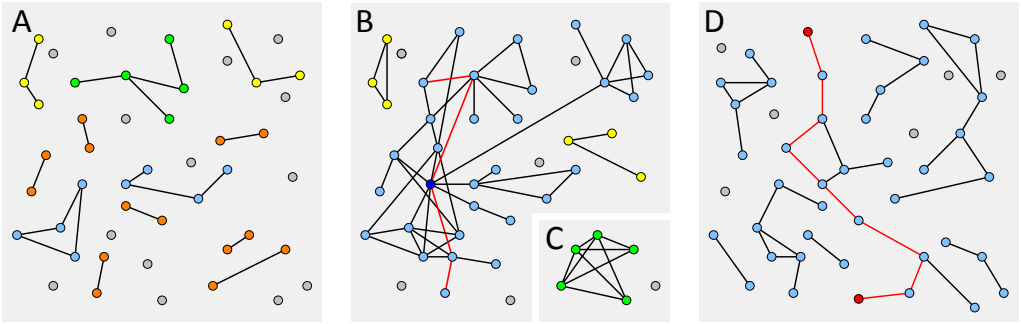
3.3 Random Planar Graphs

A graph is called *planar* if it can be embedded in the plane without crossing edges. One of the most well-known results about planar graphs is the Four Colour Theorem: it states roughly that given any separation of a plane into regions, say countries, the countries can be coloured using at most four colours so that no two countries sharing a common border have the same colour.

Random planar graphs have attained a considerable attention since McDiarmid, Steger and Welsh derived important asymptotic properties of random planar graphs [48] and Giménez and Noy determined the exact asymptotic number of labelled planar graphs [30].

Consider a uniform random planar graph $P(n,m)$ which is a graph chosen uniformly at random among all labelled planar graphs with n vertices and m edges. Łuczak and Kang [36] showed that there are, surprisingly, *two critical periods* in the evolution of a random planar graph. The first one takes place when the giant component is formed; this happens when $m = n/2 + O(n^{2/3})$, analogously to the uniform random graph $G(n,m)$. The second critical period of a random planar graph occurs when the giant component covers nearly all vertices; this happens when $m = n + O(n^{3/5})$.

As we have seen in Section 2.3 – after expressing the results in terms of $G(n,m)$ – the giant component in $G(n,m)$ suddenly emerges at $m = n/2 + O(n^{2/3})$. If $m = n/2 + s$ and $-n \ll s \ll -n^{2/3}$, then *whp* $G(n,m)$ consists of isolated trees and unicyclic components, so it is clearly planar, and the largest component is a tree of order $(1+o(1))\frac{n^2}{2s^2}\log\frac{|s|^3}{n^2}$. On the other hand, if $n^{2/3} \ll s \ll n$, then *whp* $G(n,m)$ contains exactly one complex component (called the *giant component*) of order $(4+o(1))s$, while the second largest component is *whp* a tree of order $(1+o(1))\frac{n^2}{2s^2}\log\frac{s^3}{n^2}$, a remarkable similarity between the subcritical graph and



When the number of edges is small, the random graph consists of many small components

As more edges are added, a **giant component** emerges (blue).

Planar graph can be drawn in plane so that its edges do not cross.

The **degree of a vertex** is the number of adjacent edges. The dark blue vertex in B has the largest degree 9.

The **order of a component** is the number of vertices it contains. The blue component in B has order 30, the coloured components in A have order 5 (green), 4 (blue), 3 (yellow), 2 (orange) and 1 (grey).

The graph **diameter** is the longest minimum distance between any two vertices. The largest component in A has a diameter three, in B four, in C two, and in D eight.

In a **"small-world"** network, the typical shortest distance between any two vertices in the large component is "short", i. e., the largest component has a small diameter (marked by a red line in B and D).

High **clustering** means that any neighbours of a vertex are also very likely to be connected, as in the green component in C.

Real-world networks often have a small **diameter** and exhibit high **clustering**.

Figure 2: Basic concepts in graph theory.

the supercritical graph after the removal of the giant component. Furthermore, if $s \gg n^{2/3}$, then *whp* $G(n,m)$ contains a topological copy of $K_{3,3}$ and thus is not planar.

Another random structure relevant to the behaviour of a random planar graph $P(n,m)$ is a uniform random forest $F(n,m)$, i.e. a forest chosen uniformly at random among all labelled forests with n vertices and m edges. Łuczak and Pittel [46] found that although the giant component in $F(n,m)$ emerges at $m = n/2 + O(n^{2/3})$, the critical behaviour of $F(n,m)$ is somewhat different from that of $G(n,m)$ when $n^{2/3} \ll s \ll n$. Let $m = n/2 + s$. If $s \ll -n^{2/3}$, then the structures of both $F(n,m)$ and $G(n,m)$ are similar, e.g. the order of the largest tree in $F(n,m)$ is *whp* $(1+o(1))\frac{n^2}{2s^2}\log\frac{|s|^3}{n^2}$. However in the supercritical phase, when $n^{2/3} \ll s \ll n$, the giant tree of $F(n,m)$ is *whp* of order $(2+o(1))s$, which is roughly *half* the or-

der of the largest component of $G(n,m)$, while the second largest tree of $F(n,m)$ is of order $\Theta(n^{2/3})$, which is by far larger than the second largest component of $G(n,m)$.

Łuczak and Kang showed that as far as $m = n/2 + s$ with $n^{2/3} \ll s \ll n$ (i.e. the supercritical regime of the first critical period), the behaviour of $P(n,m)$ is similar to that of $F(n,m)$. Namely, *whp* the order of the largest complex component is $(2+o(1))s$, while the second largest component has $\Theta(n^{2/3})$ vertices. So, unlike in $G(n,m)$, there is no similarity between the subcritical planar graph and the supercritical planar graph after the removal of the giant component.

The second critical period in the evolution of $P(n,m)$ does not correspond to phase transitions in $F(n,m)$ and $G(n,m)$. To see why $P(n,m)$ features the second type of the critical behaviour, we note that the maximum number of edges in $F(n,m)$ is $n-1$, while that in $P(n,m)$ is $3n-6$, so the growth rate of the order of complex components in $P(n,m)$ must change at some point. Łuczak and Kang proved that this occurs when $m = n + O(n^{3/5})$. To be more precise, we let $m = n+t$, where $t = o(n)$. If $t \ll -n^{3/5}$ but $n/2+t \gg n^{2/3}$, the giant component of $P(n,m)$ contains *whp* $n-(2+o(1))|t|$ vertices, while for $n^{3/5} \ll t \ll n^{2/3}$ the giant component contains $n-(\alpha+o(1))(n/t)^{3/2}$ vertices for some computable constant $\alpha > 0$. The unfortunate condition $t \ll n^{2/3}$ is a result of the proof method and most likely can be replaced by $t \ll n$.

3.4 Random Graph Processes

A natural modification of the Erdős-Rényi process is the class of random processes based on the paradigm of the *power of multiple choices*, which are now known as Achlioptas processes. In each step of Achlioptas processes, *two* or more potential edges are chosen randomly, and according to a certain rule, one of them is chosen and added to the evolving graph.

One natural question is whether there is a simple rule that shifts the critical time for the emergence of a giant component. The product rule was suggested as the most likely to delay the critical time: the product rule selects between the two given potential edges the one that minimises the product of the sizes of the components of its endvertices. Shortly thereafter, Bohman and Frieze showed that a much simpler rule, now known as the Bohman-Frieze process, delays the critical time [12]: if the first edge would join two isolated vertices, it is added to the evolving graph; otherwise, the second edge is added. Their work showed that this simple rule delayed the appearance of the giant component, that is, the critical point of the phase transition in the Bohman-Frieze process is strictly larger than one. It initiated three major directions of research into Achlioptas processes.

The first direction is concerned with testing the power and limits of Achlioptas processes. How much can we accelerate or delay the phase transition? How

long can we delay the formation of a Hamiltonian cycle? These questions can be asked in the original context of an Achlioptas process or in the off-line case in which all pairs of edges are given first, then the choices are made. The results often generalise to the case in which ℓ edges are presented at each step instead of 2, for some fixed constant ℓ. Recent results concern mostly the delaying of the phase transition, the avoidance of small subgraphs, and the acceleration of the appearance of Hamiltonian cycles.

The second direction involves choosing one fixed Achlioptas rule and exploring the fine details of the evolution of the graph. For example, Spencer and Wormald [62], Janson and Spencer [34] examined the state of the Bohman-Frieze process after $m = tn/2$ rounds. Spencer and Wormald [62] showed there is a critical time $t_c \sim 1.176$ at which the Bohman-Frieze process undergoes a phase transition: in the subcritical region $t = t_c - \varepsilon$ (for a constant $\varepsilon > 0$) the largest component is of order $O(\log n)$, while in the supercritical region $t = t_c + \varepsilon$ there is a giant component of order $\Omega(n)$. Janson and Spencer [34] studied the barely supercritical phase with $t = t_c + \varepsilon$ for a constant small $\varepsilon > 0$ and showed that *whp* the largest component is of order $\Theta(\varepsilon n)$, and the order of the second largest component is $\Theta(\varepsilon^{-2} \log n)$. Kang, Perkins and Spencer [37] looked closer into the component size distribution near the criticality. In fact, the Bohman-Frieze process is shorthand for a much wider class of Achlioptas rules, so-called 'bounded-size rules' introduced by Spencer and Wormald [62]. In a bounded-size rule, the choice between the two edges can only depend on the sizes of the components that may be connected by these edges, and all components of order larger than K must be treated equally, for some fixed constant K.

The third direction is to understand the detailed behaviour of a general class of Achlioptas processes. The distribution of component sizes in bounded-size rules was determined [58, 59]. The interest in Achlioptas processes increased immensely, when Achlioptas, D'Souza and Spencer [1] conjectured, based on extensive simulations, that the product rule behaves quite differently from the Erdős-Rényi process. It exhibits the so-called *explosive percolation*, in other words, the order of the largest component "jumps" from sublinear to linear order within sublinear steps of the process. However, Riordan and Warnke proved that this is not the case [57, 58]: the phase transition for a large class of generalised Achlioptas processes including the product rule is continuous.

Despite the intensive study of Achlioptas processes, the detailed behaviour of the component size distribution of a large class of Achlioptas processes, including the product rule, is not known. What can be further said about the phase transition for the product rule? What is the critical time for the emergence of a giant component? How large is the giant component shortly after the critical point? What is the size of the critical window?

3.5 Inhomogeneous Random Graphs

Another well-studied natural generalisation of the Erdős-Rényi random graph are random graphs with a given degree sequence – an example of inhomogeneous random graphs. Łuczak [45] and Chung and Lu [23] studied the component structure of a random graph with a given degree sequence. The phase transition in random graphs with a given degree sequence has been much more extensively studied, among others by Molloy and Reed [50, 51], Newman, Strogatz, and Watts [52], Kang and Seierstad [38], and Riordan [56].

Consider a sequence $\mathcal{D} = \{d_0(n), d_1(n), \ldots\}$, such that $d_i(n) = 0$ for $i \geq n$, and $\sum_{i\geq 0} d_i(n) = n$. The value $d_i(n)$ denotes the number of vertices of degree i in a graph of order n. Consider a uniform random graph $G_n(\mathcal{D})$ with $\mathcal{D}$ as a degree sequence. Under a mild smoothness condition on $\mathcal{D}$, $\lambda_i(n) := d_i(n)/n$ converges to a constant: we define $\lambda_i^* = \lim_{n\to\infty} \lambda_i(n)$ and $Q(\mathcal{D}) = \sum_{i\geq 1} i(i-2)\lambda_i^*$. Molloy and Reed [50, 51] showed that if $Q(\mathcal{D}) < 0$, then *whp* all components in $G_n(\mathcal{D})$ are of order $\Theta(\log n)$, while if $Q(\mathcal{D}) > 0$, then *whp* $G_n(\mathcal{D})$ contains a unique giant component of order $\Theta(n)$. Kang and Seierstad [38] studied the critical phase when $\sum_{i\geq 1} i(i-2)\lambda_i(n)$ converges to 0. To capture how fast the quantity $\sum_{i\geq 1} i(i-2)\lambda_i(n)$ converges to 0, we let τ_n be the largest zero of the function $Q_n(x) := \sum_{i\geq 1} i(i-2)\lambda_i(n)x^i$, i.e. $Q_n(\tau_n) = 0$. Kang and Seierstad determined the order of the largest component in a weakly supercritical regime with a logarithmic gap, i.e. when $(1-\tau_n)n^{1/3} \gg \log n$. Here the parameter $1-\tau_n$ plays the same role for $G_n(\mathcal{D})$ as $t-1$ does for $G(n,p)$ when $p = t/(n-1)$. More recently, Riordan [56] determined the exact width of the critical window and the limiting distribution of the asymptotic order of the largest component when the maximum degree is bounded.

During the last few decades, it was observed that real-world networks – arising in the fields of economy, physics, and social sciences – belong to the class of the so-called *small-world networks* which are characterised by *high clustering* (meaning that vertices are highly but locally connected among each other) and *a small diameter* (meaning that short paths link globally all vertices of the network, so all vertices are linked through relatively few number of steps). Furthermore, the number of connections (i.e. the degree of a vertex) in many real-world networks has a *power law distribution*. This property, among others, motivated the preferential attachment model introduced and studied, for example, by Barabási and Albert [2, 7].

To model and analyse real-world networks, numerous random graphs are introduced as their stochastic models, and many of them are special cases of *inhomogeneous random graphs* introduced by Bollobás, Janson, and Riordan [15], where vertices come in different types, and the probability of realising an edge depends on the types of its terminal vertices. In particular, the edges appear independently and the number of edges is linear in the number of vertices. Among other things,

Bollobás, Janson, and Riordan [15] determined the critical point of the phase transition and the order of the giant component after the transition, by relating their model to multi-type branching processes.

4 Related Areas and Applications

Random graphs have been extensively studied since their introduction, and became one of the central themes of contemporary mathematics, partly because they are closely related to various random discrete structures such as random surfaces, random maps, random matrices, random satisfiability problems, Ising and Potts models, and percolation, and partly because they are useful for modelling, analysis, and solving of structural and algorithmic problems arising in mathematics, theoretical computer science, natural sciences, social sciences, and life sciences [2]. The intense study of random discrete structures, in particular the study of their phase transition phenomena, has brought together different fields of research, such as discrete mathematics, probability theory, theoretical computer science, and statistical physics.

4.1 Phase Transition

The phase transition deals with a sudden change in the properties of a large structure caused by altering a critical parameter. It is observed in mathematics and natural sciences in many different contexts. We have seen the phase transitions in various random graph models through Sections 2.3–3.5.

The phase transitions that everybody is most familiar with are those of water: from ice (solid) to water (liquid), and from water to vapour (gas). There are two critical temperatures: zero degree Celsius, the freezing point of water, and one hundred degree Celsius, the boiling point of water. At low temperatures, in the solid state, the atoms and molecules interact strongly with their neighbours and are densely packed, typically in a regular pattern. At intermediate temperatures, the interactions are weakened, resulting in a constantly changing short-range order. And at high temperatures, the molecules barely interact and display a sparse, highly dynamic, and a rather random pattern – see Figure 3. The intriguing fact is that these temperature-induced changes do not occur continuously but exhibit two sharp jumps – phase transitions – at the melting and boiling temperatures.

A well-known example that exhibits phase transition is the percolation. In physics, materials science, and geography, the theory of percolation deals with questions related to the passage of fluid or gas through porous or disordered media. It can be applied to a wide range of seemingly unrelated phenomena, for example, the change of the earth's surface caused by weathering or erosion, the spreading of forest fire, the functioning of a cigarette filter or a coffee percolator.

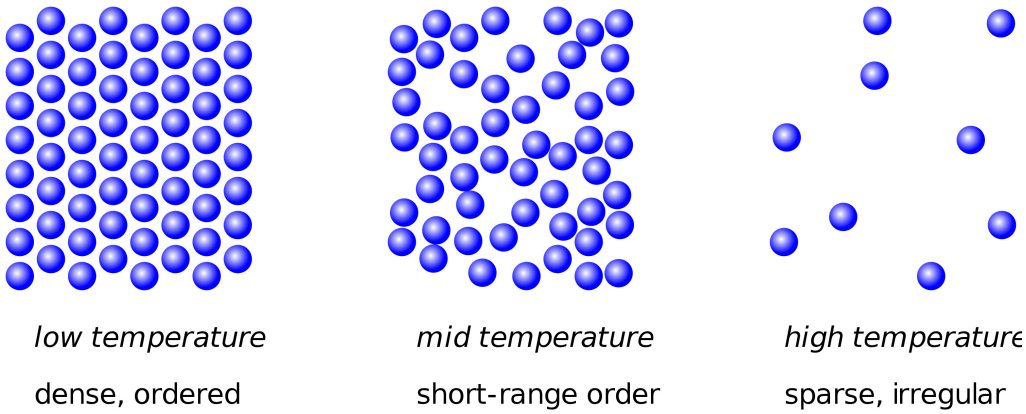


Figure 3: Organisation of atoms or molecules at different temperatures.

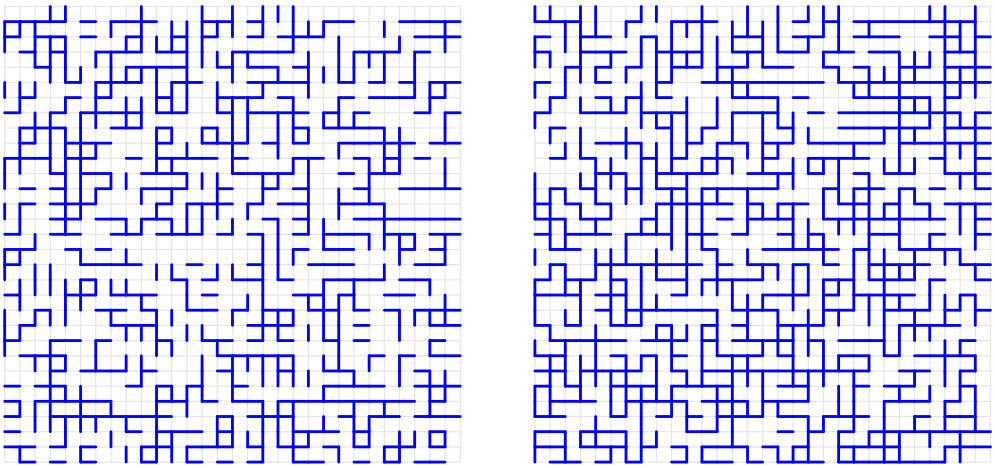
Assume that a liquid is poured on top of some porous material. Will the liquid be able to make its way from the top to the bottom? This physical question can be modelled mathematically by *bond percolation:* each bond (or edge or connection) between each two neighbours in a square lattice may be open (allowing the liquid through) with probability p, or closed with probability $1-p$, independently of each other. The bond percolation on the complete graph K_n (i.e. the graph with n vertices and $\binom{n}{2}$ edges) is just the binomial random graph $G(n,p)$. Another useful model is *site percolation*, where each site is occupied with probability p or empty with probability $1-p$.

The mathematical reformulation of the question from above is: for a given p, what is the probability that an open path exists from the top to the bottom? Interestingly, it turns out that for an infinite lattice there is a critical value p_c such that for p smaller than p_c the probability that such a path exists is zero, while for p larger than p_c the probability is one. In some cases p_c can be calculated explicitly. For example, for the bond percolation on the square lattice in two dimensions $\mathbb{Z}^2$, $p_c = 1/2$, a fact which was an open question for more than 20 years and was finally resolved by Kesten in the early 1980s [41].

Unfortunately, the exact calculation of p_c for most infinite lattice graphs is not known. For more detailed mathematical discussions on percolation, see the books with the same title "Percolation" by Grimmett [31] and by Bollobás and Riordan [16].

4.2 Social Sciences

Analysis of human interactions and communication by means of the theory of deterministic and random graphs is a useful tool in social sciences. One of the oldest and best-known examples is the 'six degree of separation' phenomenon described



$p = 0.45$ $p = 0.55$

Figure 4: Bond percolation on two-dimensional square lattice for two values of the probability p that a bond between two sites exists. Left: p below the percolation threshold $p_c = 0.5$; there is no path connecting two opposing sides of the square. Right: p above the percolation threshold; a path connecting the top with the bottom (and the left with the right) sides of the square exists.

decades ago [49]: any two people can be connected by a chain of acquaintances on average six persons long. Similarly to the Erdős-Rényi random graph, the graph of human acquaintanceship has a small diameter. However, unlike the Erdős-Rényi random graph, it has a high *clustering coefficient*, making it a 'small-world network'. The chances that two people who have a common friend also know each other are much higher than in a random graph. Many human interaction networks have this structure, for example, the scientific co-authorship network, known to mathematicians by the concept of the Erdős number.

The expansion of the internet into our daily lives greatly benefited social sciences: the patterns of human activity on the internet, such as navigation between the web pages or communication within virtual communities, is a rich source of data. In a recent example, the analysis of online-gaming patterns in internet identified clusters of players that coincided with the boundaries of traditional Chinese cultural regions [67]. While this coincidence is not surprising in itself, the fact that the player groups can be identified and analysed without using the geographical information is useful. The behaviour in the internet can be used to find culturally distinct groups, monitor their emergence and evolution, interactions, geographical distribution, etc. The network structure and dynamics of these groups can then help to predict their future development and influence on the society far beyond the limits of the internet.

The pattern of human mobility is relevant in many different contexts, for example, it has obvious consequences for the spread of infectious diseases. Modelling the human travel as diffusion on the Erdős-Rényi random graph does not take into account the geographical restrictions, which, unlike the case of communication over the internet, certainly play a strong role here. On the other hand, random walk on a regular lattice neglects the effect of long jumps enabled by air travel. In an interesting experiment reflecting the human mobility, the circulation of bank notes was monitored by volunteers using tracking websites. The analysis of the data revealed a power-law decay of distribution of travelling distances [20]. The presence of long-distance jumps means that the disease spreading across this network is a fast, super-diffusive process. Interestingly, the spread of plague in Europe in 14th century did not follow this trend, suggesting the absence of the long-range links and therefore a different structure of human contacts at the time [47].

The structure of the network determines the efficiency of spreading of information or diseases over it [26]. Will the spreading be more efficient on a well-connected network (a graph with a small diameter), where no two individuals are too far from each other, or is high clustering a more relevant network property? An experiment in which an internet community with a controlled network structure was created showed that a particular human behaviour spreads faster and further on a highly clustered rather than random network [22]. Reinforcement by multiple neighbours proved to be important for adopting the behaviour by an individual, and therefore for its spreading. A similar scenario may apply for spreading of a disease which requires multiple contacts, as opposed to a highly contagious disease that is transmitted on a single encounter. Another recent work identifies models in which some fraction of the network will not be reached [69]; the size of this fraction again depends on the network structure. All these findings have implications for designing strategies for distributing information or for vaccination [65]. At the time of writing this article, a severe Ebola epidemic broke out in West Africa. The variable network structures of the social contacts in different communities across the affected region is being used to explain and predict the differences between the involved countries in the spread and further development of the disease [42].

4.3 Man-made Networks

Large complex structures found in nature are often formed by individual uncorrelated interactions between a large number of their constituents; therefore it is not surprising that random graphs provide good models for many of them. Man-made structures, on the other hand, may be thought of as a result of rational design optimised for a given purpose; one would therefore expect a regular structure without much room for randomness (electrical circuits, cell phone network). Many creations of man that can be viewed as a network, however, exhibit sufficient 'randomness', often because their structure developed over time (like the world wide

web), or because they copy a pre-existing network (e.g. airport network, electricity network) or are subject to random or other constraints (e.g. geography). Random graphs with well-chosen characteristics can thus serve as useful models for the evolution and the function of many complex artificial structures.

New insights can be gained by studying the effects of interactions between two or more networks. If the interaction is such that the removal of a vertex in one network can eliminate vertices in the other network, the chain of subsequent vertex removals can lead to a large-scale fragmentation of both networks. A well known example of such a collapse of coupled networks is an electrical blackout in Italy in 2003, when a shutdown of power stations caused failure of internet network controlling the power grid [21]. Studies of interdependent networks can help to design networks resistant to this type of failure [55].

Although scale-free networks, such as internet, are robust against random removal of a vertex, they are highly sensitive against an attack: a targeted removal of a vertex with a high degree [3]. This knowledge can be used to design the ways to control the spread of viruses across the internet, or to identify the weak and sensitive points of the network that require extra protection against attacks. Understanding the properties of the underlying network structure can help in practical design of a real network with a balance between its effectiveness, redundancy, robustness and cost.

4.4 Life Sciences

The recent boom in life sciences has generated huge amounts of data: genomes of whole organisms have been sequenced, proteins and patterns of their interactions have been identified, metabolic networks relating the biochemical reactions have been mapped. Detailed analysis of gene and protein interaction networks is expected to help us to understand the properties of the network that are determined by its large-scale structure rather than by the details of individual interactions, for example, how a combination of mutations of different proteins increases the risk of a disease, how the topology of a protein network affects its robustness against random mutations, or how the network may have evolved over time [8].

Biological networks are often scale-free, with a broad vertex degree distribution. Such networks are robust against removal of a randomly chosen vertex [3], a property important for resistance against random elimination of one unit, for example, mutation making a protein nonfunctional.

In graphs representing biological networks, some subgraphs (called motifs) can be identified as sub-units fulfilling a well-defined function. Analysis of a biological network in terms of subgraphs can therefore reveal so far unknown sub-units and thus help our understanding of the functionality of a large network [32].

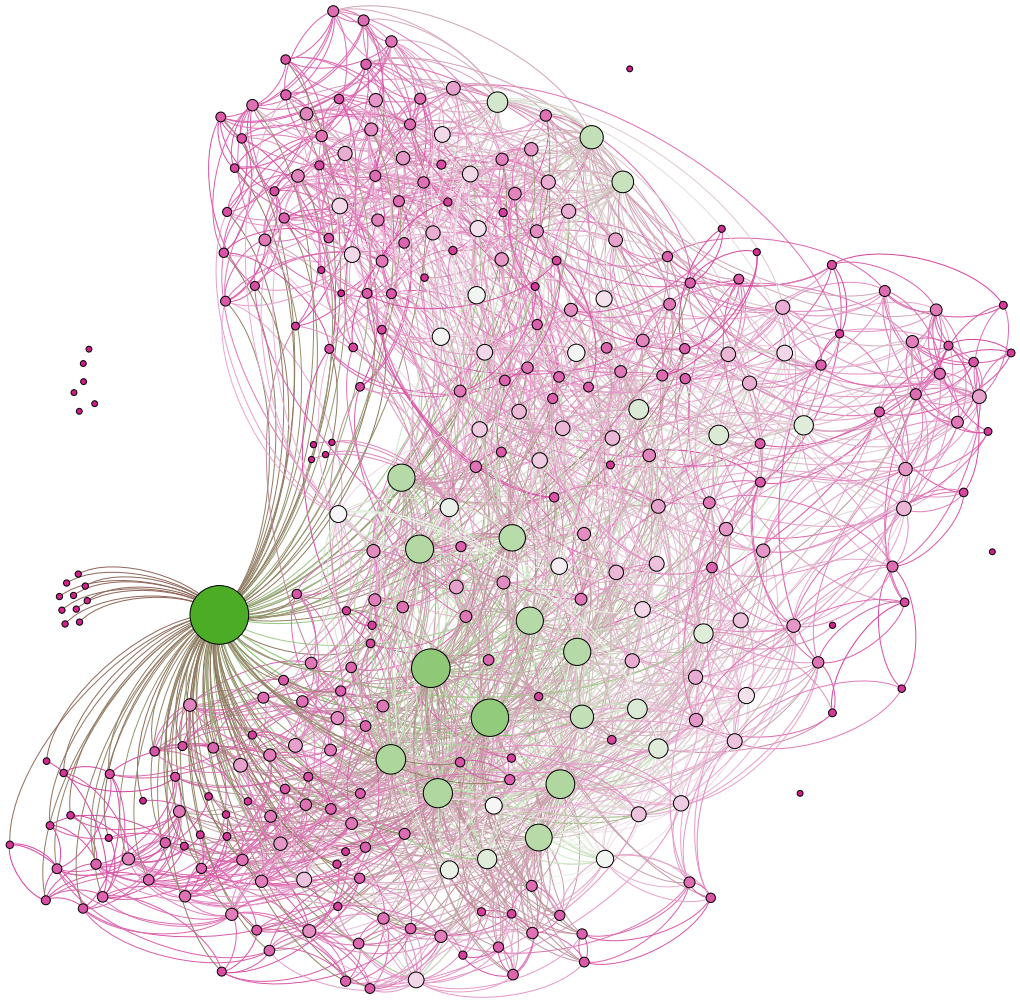


Figure 5: The graph depicting the connections between the neurons of *C. elegans*. The size and the colour of the vertices correspond to their degree. The data originate from ref. [68].

4.5 The Brain

The theory of deterministic or random graphs can help us describe, and in the future hopefully also understand, perhaps the most complex structure found in nature: the brain [53]. The network of interconnected neurons in the brain has been modelled as a random graph with different properties (e.g. scale-free network). While the 'brain' of a simple worm *C. elegans*, a widely studied model organism, consists of 302 neurons connected by around 7300 synapses, and has been mapped in detail [66], the human brain contains $\sim 10^{11}$ neurons, with the number of synapses hard to estimate (10^{14}–10^{15}?), let alone map. The way the neurons are

connected and interact with each other largely determines the brain functionality. For example, there is evidence that the functional connectivity structure within the brain changes in patients with Alzheimer's disease: the activity networks exhibit lower clustering, being closer to a random network than those of a healthy brain [63]. Experimental studies of spontaneous brain activity – the analysis of electric spikes produced by neurons, and of the size and duration of avalanches of this neuronal activity – often produce results that can be described by power-law distributions (scale-free). This has led to a hypothesis that brain, viewed as a network of neurons, may be operating at a critical state, a state close to a phase transition. This is relevant, as various network properties related to the communication efficiency, dynamic range of response, etc., are optimised at the critical point [9].

5 Conclusion

This article aims to give a flavour of how the field of random graphs has evolved over the last 50 years from the first definition of a random graph to a rich mathematical theory with applications across many scientific disciplines. It should, however, be mentioned that this article is not an exhaustive survey on the theory of random graphs, but rather a brief collection of special topics and results that are of particular interest to the authors.

From the theory point of view, a special focus was put on the fascinating phenomenon of phase transition in terms of the emergence of a giant component in a random graph. There are, however, also other important and interesting properties of random graphs, such as small subgraphs, long cycles, diameter, cliques, independent sets, and the chromatic number, to name a few. For a comprehensive account of these topics we refer the readers to two excellent books on random graphs by Bollobás [14] and by Janson, Łuczak, and Ruciński [33].

From the viewpoint of application, the theory of random graphs has proven to be appropriate for the description and analysis of complex structures arising everywhere from the nature to the society, even of the brain. On the other hand, diverse applications continue to motivate and inform the study of random graphs.

The expansion of random graph theory and its applications shows us again how much abstract mathematical ideas can teach us about the "real world".

Acknowledgements

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