*Generating Unlabeled Connected Cubic Planar Graphs Uniformly at Random**

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ABSTRACT: We present an expected polynomial time algorithm to generate an unlabeled connected cubic planar graph uniformly at random. We first consider *rooted* connected cubic planar graphs, i.e., we count connected cubic planar graphs up to isomorphisms that fix a certain directed edge. Based on decompositions along the connectivity structure, we derive recurrence formulas for the exact number of rooted cubic planar graphs. This leads to rooted 3-connected cubic planar graphs, which have a unique embedding on the sphere. Special care has to be taken for rooted graphs that have a *sense-reversing* automorphism. Therefore we introduce the concept of *colored networks*, which stand in bijective correspondence to rooted 3-connected cubic planar graphs with given symmetries. Colored networks can again be decomposed along the connectivity structure. For rooted 3-connected cubic planar graphs embedded in the plane, we switch to the dual and count rooted triangulations. Since all these numbers can be evaluated in polynomial time using dynamic programming, *rooted* connected cubic planar graphs can be generated uniformly at random in polynomial time by inverting the decomposition along the connectivity structure. To generate connected cubic planar graphs *without a root* uniformly at random, we apply rejection sampling and obtain an expected polynomial time algorithm. © 2008 Wiley Periodicals, Inc. Random Struct. Alg., 32, 157–180, 2008

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1. INTRODUCTION

The number of planar graphs (and planar maps, i.e. planar graphs embedded in the plane), and various subclasses thereof, has been investigated for a long time (see [1–14, 18–20, 25, 27–34, 37–42, 46, 52–55]). Significant progress has been achieved recently, but the picture is still far from complete. When the graphs are counted up to isomorphisms, i. e., we consider *unlabeled* planar graphs, the situation is even more complex.

However, for unlabeled planar graphs where every vertex has exactly three neighbors, i.e., for *cubic* planar graphs, several results have been obtained. Brinkmann and McKay [15] developed an algorithm that produces a list of nonisomorphic planar cubic graphs as well as other classes of planar graphs. Using generating functions Gao and Wormald [29] determined the asymptotic growth of the number of cubic planar maps. Whitney's Theorem asserts that every three-connected planar graph has a unique embedding on the sphere (see e.g., [21]). Hence, there is a one-to-one correspondence between three-connected cubic planar graphs and triangulations, i.e., graphs embedded in the plane where every face is a triangle. Equivalently, a triangulation is a *maximal* planar graph in the sense that we cannot add an edge without destroying planarity. Tutte gave an exact formula for the number of labeled triangulations [51]. Since almost all triangulations do not have a nontrivial automorphism, this also yields the asymptotic number of unlabeled triangulations [50].

In this article, we study unlabeled connected cubic planar graphs. We design an algorithm that generates unlabeled connected cubic planar graphs uniformly at random and has polynomial expected running time.

A well-known technique for random sampling is based on Markov chains. If a Markov chain on a set of combinatorial objects has the uniform stationary distribution, it can be used for *Monte Carlo* algorithms to sample random instances *approximately* uniformly at random. These algorithms are efficient if the Markov chain is *rapidly mixing*, that is, the number of steps until the Markov chain is "close" to the uniform distribution (which is called the *mixing time*) is polynomial [36]. There are such efficient algorithms for the generation of certain random structures, for instance, triangulations of convex polygons [43, 44]. However, it is not known whether the Markov chain in [18] for the uniform generation of planar graphs is rapidly mixing. This crucial question is open even for the analogous Markov chain for outerplanar graphs.

Another powerful strategy for random sampling is the so-called *recursive method*. Early references for this method are Nijenhuis and Wilf [45] and Wilf [56]. The approach was systematized by Flajolet, Zimmerman, and Van Cutsem in [26]. In this approach, we need a decomposition strategy for the objects that we want to sample. From the decomposition strategy, we obtain recursive counting formulas, and these formulas can be used to sample such objects for instance from the uniform distribution, that is, to generate them *uniformly at random*. The idea is that the sampling procedure is branching recursively into subroutines with the right probabilities that are evaluated with the help of the counting formulas. One advantage of the recursive method for random sampling is that it can be used to sample the objects *exactly* uniformly at random. Often, the counting formulas are interesting in their own right. The running time for the generation improves considerably if we allow a precomputation step for the values of the dynamic programming arrays.

In this article, we use the recursive method for sampling unlabeled connected cubic planar graphs uniformly at random, based on the decomposition of the graphs along the connectivity structure. Our approach here is motivated by Liskovets and Walsh [39], and in principle similar to the one for labeled planar graphs described in [8]; but for unlabeled structures we need several new techniques.

The main difficulties with unlabeled cubic planar graphs originate from several kinds of potential symmetries. One way to handle such symmetries in graphs is to specify a small (constant) number of vertices or edges of the graph, in order to break a certain degree of symmetry. This is called *rooting the graph*. In the present case, we first consider *rooted* unlabeled connected cubic planar graphs where the root is a distinguished oriented edge. The two end vertices of the root are called *poles*. We recursively *decompose* rooted unlabeled connected cubic planar graphs into smaller parts along the connectivity structure. The decomposition leads to a set of recursive counting formulas. In this way rooted connected cubic planar graphs are *counted* up to isomorphisms that fix the vertices of the root edge. Given all the necessary recursive counting formulas, we can then *generate* a rooted unlabeled connected cubic planar graph by the reverse operation of decomposition, with the correct probabilities given by the countings.

Clearly, generating a random rooted unlabeled connected cubic planar graph and then simply ignoring the root edge does not yield the uniform distribution, since unlabeled graphs generally correspond to different numbers of rooted graphs. But this imbalance can be compensated by *rejection sampling*, i.e., the generation algorithm is restarted with a probability that is inverse proportional to the size of the orbit of the root (which is linear). In this way we can generate unlabeled connected cubic planar graphs uniformly at random in expected polynomial time.

Next, we sketch how to decompose unlabeled rooted connected cubic planar graphs along the connectivity structure.

1.1. Sketch of the Decomposition

First, we classify rooted connected cubic planar graphs according to their connectivity types (Theorem 1). According to this classification, a rooted connected cubic planar graph is then decomposed into smaller graphs of different types. For this decomposition, we need two more types of auxiliary graphs as building blocks. The first type is *rooted 3-connected cubic planar graphs*, from which one type of graphs in Theorem 1 is built up. The second type is rooted connected cubic planar graphs with automorphisms that exchange the poles, which we call *pole-symmetric connected cubic planar graphs*. These are used as building blocks when the orientation of the root does not matter. To complete the decomposition we further decompose these two types of graphs, again following the connectivity structure.

To decompose rooted 3-connected cubic planar graphs, we use a classical theorem of Whitney (see e.g., [21]) saying that a rooted 3-connected planar graph can have either one or two embeddings in the plane where the root edge is embedded on the outer face. In the first case we say that the graph has a *sense-reversing automorphism*, or it is *symmetric*.

For the decomposition of symmetric 3-connected cubic planar graphs, we need a well-known concept of networks: A *network* is a graph equipped with an oriented edge (which may or may not be in the graph, and whose two end vertices are again called poles) such that if we insert a directed edge between the poles into the network, then the resulting graph is a rooted *2-connected* graph. We introduce a new concept, a *colored network*, which is a network where some vertices are colored so that they can store and recover the information on the symmetry due to a sense-reversing automorphism. We then derive a bijection between symmetric 3-connected cubic planar graphs and colored networks (Theorem 2), and a decomposition for colored networks.

Next, we study embedded 3-connected cubic planar graphs, i.e., rooted 3-connected cubic maps. For rooted 3-connected cubic maps we study the dual objects [50], namely, planar triangulations. We decompose a slightly more general class of objects, rooted *near-triangulations*, i.e., embedded graphs where every face except the outer face is a triangle. For these objects an exact formula was given by Brown [16]. In our case, we need to control the number of vertices at the outer face as well as the degree of one of the poles. Therefore, we present the decomposition of near-triangulations with this additional parameter.

Finally, we decompose pole-symmetric connected cubic planar graphs until we end up with pole-symmetric 3-connected graphs. Since a pole-symmetric 3-connected graph might additionally have a sense-reversing automorphism, we decompose also pole-symmetric 3-connected graphs with sense-reversing automorphism again using colored networks. Planar graphs with various symmetries also attracted much attention in graph drawing [22–24].

1.2. Plan of the Paper

Figure 1 shows the dependencies between the concepts in the decomposition and the counting formulas, and also gives an overview of the dependencies of the remaining sections in this article.

Note that a cubic planar graph has an even number of vertices n and $3n/2$ edges. Thus, we assume that the number of vertices in the counting formulas, except those for the triangulations and colored networks, is even.

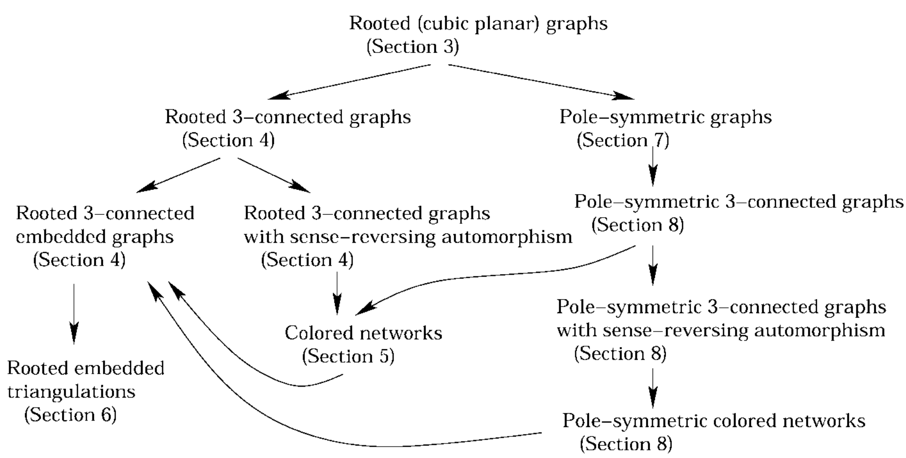

Fig. 1. Dependencies of the sections and concepts in this article.

2. PRELIMINARIES

In this section, we introduce concepts that we need for the decomposition of cubic planar graphs. First, we recall some classical graph-theoretical concepts. A graph is *simple* if it does not contain multiple-edges or self-loops. A graph is *planar* if it can be embedded in the plane without crossing edges, and it is *cubic* if every vertex has degree three. A graph G is called *k-connected* if $|G| > 2$ and $G - K$ is connected for every set $K \subset G$ with $|K| < k$. A k-point set K of a graph G is called a *k-cut* of G if $G - K$ is disconnected. A single point that forms a 1-cut is called a *cut-vertex*. A 2-cut is also called a *split pair*. Two adjacent cut-vertices form a *cut-edge*.

2.1. Rooted Graphs

Now we introduce some new concepts for convenient and exhaustive decomposition of cubic planar graphs. Let G be a simple graph with distinguished vertices s, t and let $e = st$ be a directed edge from s to t such that if we insert the edge e into G the resulting multigraph G^* is connected. We call the edge e the *root-edge* of G^* and call $G^* = (G; st)$ a *rooted* graph and G the *unrooted* graph of G^*. For technical reasons, we allow the situations that the root-edge of G^* might be a self-loop or a multi-edge, and we say that an unrooted graph G is planar, cubic, or k-connected if G^* has the corresponding property – but we will always use the term *unrooted* for G to distinguish from the standard/classical case. We say that a cut-edge *separates* two vertices u, v if u, v are contained in two different connected (in a classical sense) components after deleting the cut-edge (but not its ends) from G.

During the process of decomposing graphs we often make use of the following *replacement* operation: Let H^* and I^* be rooted graphs. Let uv be a non-root edge of H^* and st be the root-edge of I^* – thus we allow the case $s = t$ (by the definition of a root-edge), but $u \neq v$. *Replacing* an edge uv of H^* with I^* results in a new rooted graph $G^* := H^* \cup I \cup \{us, vt\}$ obtained from H^* and I (without the root edge st) by connecting edges us and vt, where the root-edge of G^* is the same for H^*. If H^* and I^* are cubic, the resulting graph G^* is again cubic. Then we have the following structure theorem for rooted connected cubic planar graphs, which may be compared with the characterization of networks by Trakhtenbrot [48, 54]:

Theorem 1. *Let $G^* = (G; st)$ be a rooted connected cubic planar graph and G be its unrooted graph. Then G^* is of precisely one of the following types:*

- *d: The root edge st is a cut-edge in G^*.*
- *s: The unrooted graph G has a cut-edge that separates s and t.*
- *p: The unrooted graph G has no cut-edge separating s and t. In addition, either s and t are adjacent by a non-cut-edge in G, or they are not adjacent in G but form a split pair of G.*
- *h: The rooted graph G^* is built from a uniquely determined 3-connected rooted cubic planar graph H^*, where the root is not a self-loop or a multi-edge in H^*, by replacing some non-root edges of H^* (in the above sense) with s-, p-, h-, or g-graphs.*
- *g: The root is a self-loop, i.e., $s = t$.*

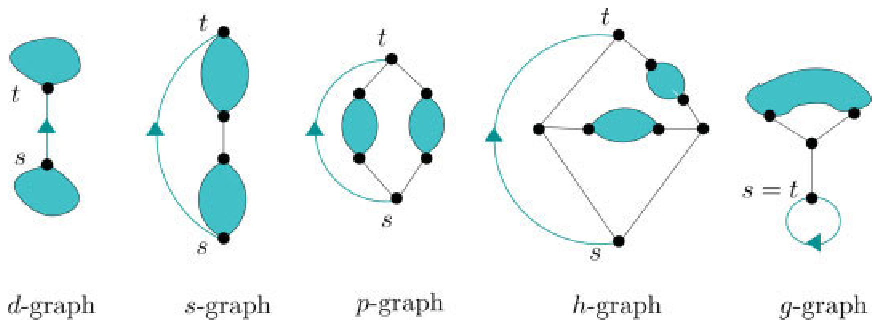

Fig. 2. Types of rooted connected cubic planar graphs. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

The types of rooted connected cubic planar graphs G^* according to Theorem 1 are called d- (for "disconnected"), s- (for "serial"), p- (for "parallel"), h-, or g-graphs, respectively (see Fig. 2).

Proof. The cases that G^* is a g- or d-graph are obviously disjoint from the other cases. If G^* is an s-graph it cannot be a p-graph, and vice versa. If G^* contains a split pair, one of the corresponding components does not contain a root vertex. We replace this component by an edge if we can preserve three-regularity. If we iterate this process, the graph will eventually be 3-connected, or an s- or p-graph. Since the replacement operations are confluent and always terminating, the resulting 3-connected cubic graph H is unique. ∎

From now on, we use the terms *graphs* and *rooted graphs* meaning (connected) cubic planar graphs and rooted (connected) cubic planar graphs, respectively, unless explicitly stated otherwise.

For technical reasons (indeed for exhaustive decomposition later on), we allow in Theorem 1 the cases that the root of a rooted graph might be a self-loop (for g-graphs) or a multi-edge (for s-, and p-graphs). However, what we really want to count is rooted *simple* graphs, which correspond to "standard" rooted graphs that have neither a multi-edge nor a self-loop, but a distinguished directed edge. It is easy to see that the only nonsimple rooted graphs among the above d-, s-, p-, h-, or g-graphs are the following: g-graphs; s-graphs that can be split into two g-graphs; p-graphs that can be split into a single edge and an s-, p-, h-, or g-graph. Thus by using only *simple* d-, s-, p-, or h-graphs, we will enumerate rooted *simple* graphs at the end of Section 3.

2.2. Networks

For dealing with at least 2-connected graphs, we sometimes make use of the following well-known concept of networks (see e.g., [1,8]). A *network* N is a graph with two distinguished vertices s, t, so-called *poles*, such that the corresponding rooted graph N^* (obtained from N by inserting a directed edge from s to t, where s is called the south-pole and t the north-pole) is 2-connected. The unique network with 2 vertices only is called the *trivial* network. Note that for a network N, the rooted graph N^* cannot be a d- or a g-graph. The networks corresponding to the remaining cases we call s-, p-, or h-networks, respectively. Every split pair $\{k_1, k_2\}$ in N induces a partition of the edge set, and each of these parts is again a network, where k_1k_2 are the poles. These networks are called *subnetworks* of N. If $k_1, k_2 \notin \{s, t\}$ the split pair $\{k_1, k_2\}$ induces two other networks N_1 and N_2 with the poles st and k_1k_2, respectively. Then N^* can be considered to be obtained by *replacing* an edge uv of N_1^* with N_2^*.

2.3. Symmetries

A rooted connected cubic planar graph $G^* = (G; st)$ might have two kinds of symmetries. It might have a *sense-reversing automorphism* φ, i. e., $\varphi \neq \mathrm{id}$, but $\varphi(s) = s$ and $\varphi(t) = t$, in which case we call G^* *symmetric*; or it might have a *pole-exchanging automorphism* ψ, i.e., $\psi(s) = t$ and $\psi(t) = s$, in which case we call G^* *pole-symmetric*. For the decomposition of G^*, we use the *maximal* sense-reversing automorphism φ in the sense that it has the largest number of vertices u, v, and, w where u is fixed by φ, but v and w are exchanged by φ, i.e., $\varphi(u) = u$, $\varphi(v) = w$ and $\varphi(w) = v$. Note that such φ exists. Similarly we use the *maximal* pole-exchanging automorphism.

For later decomposition it is useful to color the edges and vertices of G^* according to the symmetry. If G^* is symmetric, we let φ be its maximal sense-reversing automorphism. We color the edge uv and the vertices u, v, *φ-blue* if its end vertices u, v, are fixed by φ, that is, $\varphi(u) = u, \varphi(v) = v$, and *$\varphi$-red* if u, v are exchanged by φ, that is, $\varphi(u) = v, \varphi(v) = u$. Otherwise edges are not colored. Edges and their φ-images are called *φ-corresponding edges*. If G^* is pole-symmetric, we let ψ be its maximal pole-exchanging automorphism. We color the edge uv and the vertices u, v, *ψ-blue* if u, v are fixed by ψ, and *ψ-red* if u, v are exchanged by ψ. Edges and their ψ-images are called *ψ-corresponding edges*.

If G^* is both symmetric and pole-symmetric, then we first color the edges with *ψ-blue* and *ψ-red*, and next color the rest edges with φ-blue and φ-red as earlier. There are two cases where we should make this coloring more clear: The first case is that there is one (indeed only one if any) edge uv such that u, v are fixed by ψ, but in addition are exchanged by φ. In this case, we color uv by *ψ-blue*, not by φ-red. The second case is that there is one (and only one if any) edge uv such that u, v are exchanged by ψ, but are fixed by φ. In this case, we color uv by *ψ-red*, not by φ-blue. In other words, the ψ-colors always come first.

Whenever a graph is symmetric or pole-symmetric, we always equip it with the colors of its edges and vertices attained as earlier using maximal automorphisms. We will sometimes use only *blue* or *red* colors instead of having the name of the maximal automorphism in front of them, when only one type of the two symmetries is involved.

3. COUNTING ROOTED CUBIC GRAPHS

In this section, we present a decomposition of rooted cubic graphs, and derive recurrence formulas for counting them. Let $r(n)$ be the number of rooted cubic graphs on n vertices. According to Theorem 1 we have $r(n) = d(n) + s(n) + p(n) + h(n) + g(n)$, for $n \geq 0$, where the functions $d(n), s(n), p(n), h(n), g(n)$ count the number of d-, s-, p-, h- and g-graphs on n vertices, respectively. It will be convenient to use the numbers $c(n) = r(n) - d(n), n \geq 1$, $c(0) = 1$, and call the corresponding rooted graphs *c-graphs* (c for "connected"). Since there is no rooted cubic graph with odd number of vertices these functions take values zero for odd n.

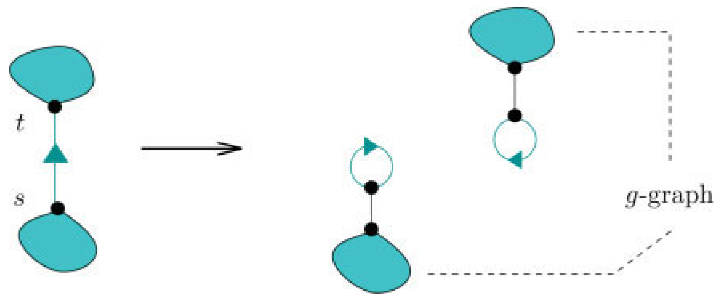

Fig. 3. Decomposition of a d-graph. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

3.1. Counting *d*-Graphs

Since a d-graph without the root edge is disconnected, we can decompose a d-graph G^* uniquely into two g-graphs (see Fig. 3): To obtain the first g-graph, we shrink the connected component containing t in G into the vertex t, connect s and t, and think of a self-loop at t as a root-edge. To obtain the second g-graph, we do the same for the connected component containing s. For $0 \leq n \leq 9$, we have $d(n) = 0$, and for $n \geq 10$,

$$d(n) = \sum_{i=6}^{n-4} g(i)g(n+2-i).$$

3.2. Counting *s*-Graphs

Note that every s-graph G^* has a unique cut-edge uv separating s, t such that u is closest to the vertex s (in this article, *closest* is meant with respect to the length of a shortest connecting path). It could be the case that $u = s$, or $v = t$. An s-graph can be split into a p-, h-, or g-graph rooted at su, and an s-, p-, h-, or g-graph (i.e., a c-graph) rooted at vt (see Fig. 4).

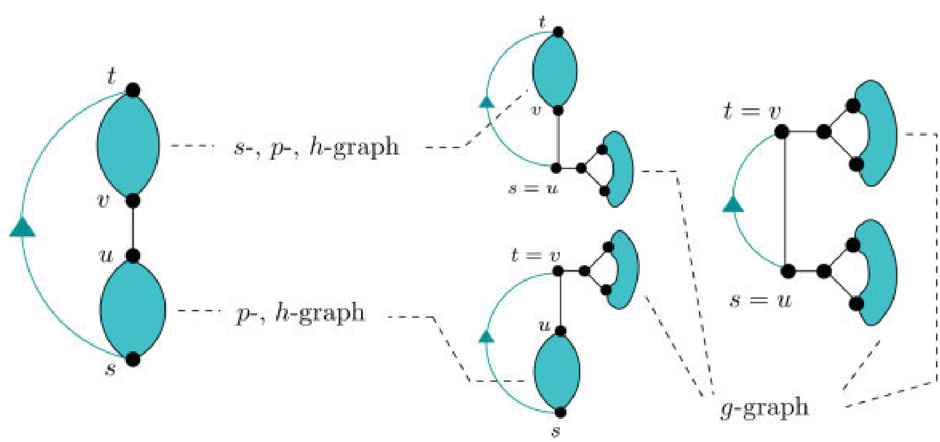

Fig. 4. Decomposition of an s-graph. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

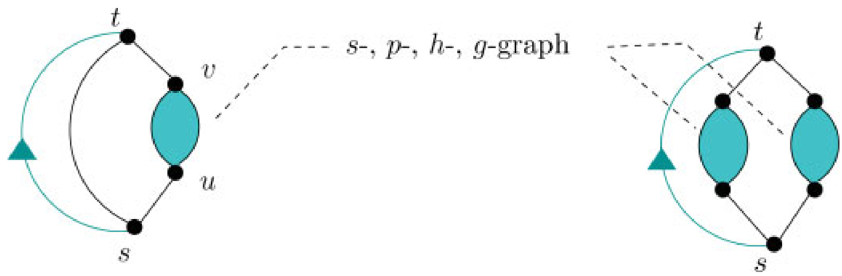

Fig. 5. Decomposition of a p-graph. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

For $0 \leq n \leq 7$, we have $s(n) = 0$, and for $n \geq 8$,

$$s(n) = \sum_{i=4}^{n-4} (p(i) + h(i) + g(i))c(n-i).$$

3.3. Counting p-Graphs

A p-graph G^* could contain a nonroot edge between s and t. If this is the case (see Fig. 5, left part), then due to the three-regularity of G^*, there are unique nonpole neighbors u and v for s and t, respectively, and $\{u, v\}$ induces a smaller graph rooted at uv that is a c-graph. These graphs are counted by $c(n-2)$. If the poles are not adjacent (see Fig. 5, right part), a p-graph can be split into two c-graphs, of size i and $n-i-2$. These subgraphs are counted by $c(i)$ and $c(n-i-2)$. We must not count twice the case where the two subgraphs are isomorphic. We obtain the following formula: For $0 \leq n \leq 3$, $p(n) = 0$, and for $n \geq 4$,

$$p(n) = c(n-2) + \sum_{i=4}^{(n-4)/2} c(i)c(n-i-2) + \binom{c((n-2)/2)+1}{2}.$$

3.4. Counting g-Graphs

A g-graph G^* has a unique neighbor u of the pole $s = t$, which in turn is adjacent to two other distinct vertices w_1 and w_2 in G^* (see Fig. 6). Note that the orientation of the root of the remainder does not matter. Thus, we obtain the same g-graph in two ways, unless there is an automorphism mapping w_1 to w_2 and w_2 to w_1. The number of rooted graphs that have such a *pole-exchanging* automorphism will be counted by $\tilde{r}(n)$ in Section 7. Therefore, for $0 \leq n \leq 5$, $g(n) = 0$, and for $n \geq 6$,

$$g(n) = (r(n-2) - g(n-2) + \tilde{r}(n-2))/2.$$

3.5. Counting h-Graphs

Let G^* be an h-graph. Theorem 1 asserts that there is a unique 3-connected rooted cubic graph H^*, such that we can derive G^* from H^* by replacing edges in H^* with rooted c-graphs

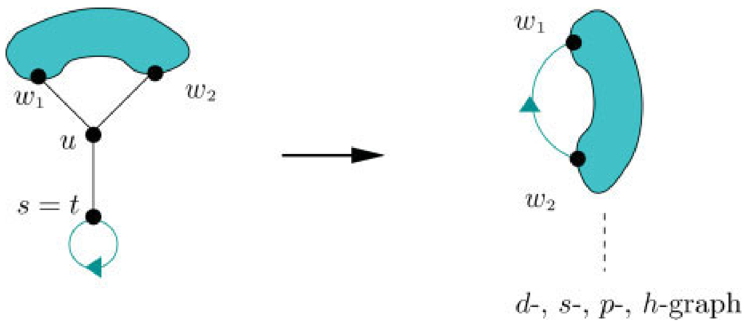

Fig. 6. Decomposition of a g-graph. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

(see Fig. 7). We call H^* the *core* of G^* and denote $H^* = \text{core}(G^*)$. Note that core(G^*) might have a sense-reversing automorphism.

We describe how to construct an h-graph G^* from symmetric and from asymmetric 3-connected networks. We first *uniquely* order the edges of the core of G^*. For that, we define two sequences, for each of the two possible embeddings of the core in the plane that have the root at the outer face. To define the sequence for one of these embeddings, we do a depth-first search traversal of the core, beginning with the root edge and visiting the neighbors of a vertex in clockwise order with respect to the embedding. We label the vertices with numbers according to the order of their occurrence in the traversal. We can now denote edges by pairs of such vertex labels. The sequence we are interested in is the sequence of edges represented in that way, in the order as we traversed them in the depth-first search.

If the core is asymmetric, the two sequences are distinct. Thus we can now order the edges of the core uniquely according to the lexicographically smaller sequence. If the graph has a symmetric core, clearly both edge sequences are the same. In this case, we first specify a face incident to the root edge so that core(G^*) is (uniquely) embedded in the plane in such a way that this selected face becomes the outer face. Then we can uniquely order the edges of core(G^*) in the following way. Let φ be a *maximal sense-reversing automorphism* of core(G^*). We start with the (φ-)*blue* edges according to the traversal. Then we list the (φ-)*red* edges. We continue with the uncolored edges, and order them according to the traversal. Edges and their (φ-)corresponding edges, are consecutive but

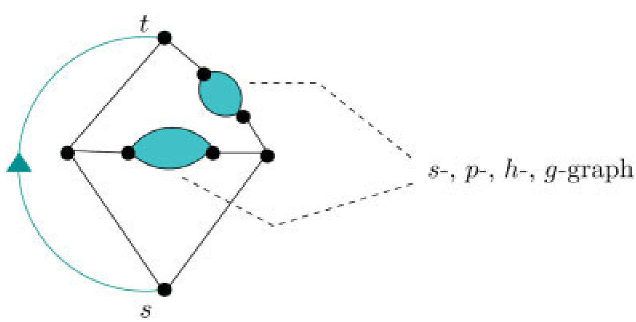

Fig. 7. Decomposition of an h-graph. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

arbitrarily ordered, and each pair of corresponding edges attain unique order according to the traversal.

Let $b_{b,r,l}(n)$ denote the number of symmetric h-graphs B^* on n vertices, where core(B^*) has b blue and r red edges, and the first l edges of core(B^*) are edges of B^*. To generate such a symmetric h-graph B^*, we have to start from a symmetric core. We look at the $(l+1)$-th edge uv of core(B^*). It is either blue (when $l+1 \leq b$), red (when $b < l+1 \leq b+r$), or uncolored (when $l+1 > b+r$). If it is blue we can replace it by an edge or an arbitrary s-, p-, h-, or g-graph (i.e., c-graph). If it is red we can replace it by an edge or a graph with a pole-exchanging automorphism (counted by $\tilde{c}(n)$, again see Section 7). If it is not colored, we should tell if l is even or odd. If l is odd, then uv is the corresponding edge of lth edge and thus it should also be an edge. If l is even, then we can replace uv and its corresponding uncolored edges in pairs (i.e., the $l+1$-st and $l+2$-nd core edges) by the same copy of an edge or a c-graph. When l equals the number of edges of B^*, i.e., $l = 3n/2 - 1$, we have a 3-connected symmetric network with b blue and r red edges, which will be counted in Section 5. Thus for $0 \leq n \leq 3$, $b_{b,r,l}(n) = 0$, and for $n \geq 4$ and $b, r \geq 0$ satisfying $1 \leq b+r \leq n, 3n/2 - 1 - b - r = 0 (\mathrm{mod}\ 2)$ we have

$$b_{b,r,l}(n) = \begin{cases} b_{b,r,l+1}(n) + \sum_{i\geq 4} c(i) b_{b,r,l+1}(n-i), & \text{when } l+1 \leq b \\ b_{b,r,l+1}(n) + \sum_{i\geq 4} \tilde{c}(i) b_{b,r,l+1}(n-i), & \text{when } b < l+1 \leq b+r \\ b_{b,r,l+1}(n), & \text{when } l+1 > b+r, l = 1 (\mathrm{mod}\ 2) \\ b_{b,r,l+2}(n) + \sum_{i\geq 4} c(i) b_{b,r,l+2}(n-2i), & \text{when } l+1 > b+r, l = 0 (\mathrm{mod}\ 2). \end{cases}$$

Let $a_l(n)$ denote the number of asymmetric h-graphs A^* on n vertices, where the first l edges of core(A^*) are edges of A^*. To generate such an asymmetric h-graph A^*, we could first take an h-graph which is already asymmetric (counted by a_{l+1}) and might replace the $l+1$-st core edge by a c-graph or not. Or we could take a symmetric h-graph whose core has b blue and r red edges, for any $b, r \geq 0$ satisfying $b+r \geq 1$, and the $l+1$-st edge is red, and replace this edge by a c-graph that has no automorphism exchanging the poles, which will be counted by $c(i) - \tilde{c}(i)$. Finally, we could take such a symmetric h-graph, where the $l+1$-st edge is not fixed by the automorphism. If l is odd, then the $l+1$-st edge can be replaced by any c-graph. If l is even, then we can substitute two different c-graphs for the corresponding $l+1$-st and $l+2$-nd core edges. Again when l equals the number of edges of A^*, we have to count the number of 3-connected asymmetric networks; for that we refer to Section 5. Hence for $0 \leq n \leq 5$, $a_l(n) = 0$, and for $n \geq 6$,

$$\begin{aligned} a_l(n) = a_{l+1}(n) + \sum_{i\geq 4} c(i) a_{l+1}(n-i) + \sum_{b+r\geq 1, b<l+1\leq b+r, i\geq 4} (c(i) - \tilde{c}(i)) b_{b,r,l+1}(n-i) \\ + \begin{cases} \sum_{1\leq b+r<l+1} \sum_{i\geq 4} c(i) b_{b,r,l+1}(n-i)), & \\ & \text{when } l = 1 (\mathrm{mod}\ 2) \\ \sum_{1\leq b+r<l+1} \left(\sum_{i,j\geq 4} c(i)c(j) b_{b,r,l+2}(n-i-j) - \sum_{i\geq 4} c(i) b_{b,r,l+2}(n-2i) \right) /2, & \\ & \text{when } l = 0 (\mathrm{mod}\ 2). \end{cases} \end{aligned}$$

With these numbers we can compute $h(n) = a_0(n) + \sum_{b,r} b_{b,r,0}(n)$, where the summation is over all pairs of $b, r \geq 0$ satisfying $1 \leq b+r \leq n$, $3n/2 - 1 - b - r = 0 (\mathrm{mod}\ 2)$.

3.6. Counting Simple Graphs

We can also compute the number of rooted *simple* cubic planar graphs. Note that a g-graph together with the root edge is not simple, since the root edge creates a self-loop. Every d-, or h-graph together with the root edge is simple. But some s-graphs and some p-graphs together with the root edge are not simple: An s-graph that can be split into two g-graphs, together with the root edge, is not simple, because the root edge forms a multi-edge between the poles (see Fig. 4, right most part). A p-graph that can be split into a single edge and a c-graph is not simple, because the root edge again forms a multi-edge (see Fig. 5, left part). Thus the number of rooted simple cubic planar graphs on n vertices is

$$d(n) + h(n) + s(n) - \sum_i g(i)g(n-i) + p(n) - c(n-2).$$

4. ROOTED 3-CONNECTED GRAPHS

In this section, we study (edge)-rooted 3-connected graphs. If we additionally specify a face incident to the root edge (there are one or two ways to do so), then there is a unique embedding in the plane where this face is the outer face. We call the resulting graph a *face-rooted* 3-connected graph, which is also called a *3-connected map*. Then we can instead make use of the dual objects, i.e., *face-rooted triangulations*.

If a 3-connected cubic planar map has n vertices in total and k vertices on the outer face, the face-rooted triangulation has $\frac{n}{2} + 2$ vertices in total (due to Euler's formula) and its s-pole has degree k. Let $t(n)$ be the number of face-rooted triangulations on n vertices. Let $a(n)$ and $b(n)$ be the numbers of asymmetric and symmetric (edge)-rooted 3-connected cubic planar graphs, respectively. There are either one or two ways to obtain a face-rooted graph from a (edge)-rooted one: If the graph is *symmetric*, then there is only one corresponding face-rooted graph. Otherwise there are two. Thus, we get $t(n/2+2) = 2a(n) + b(n)$, and thus $h_{3n/2-1}(n) = a(n) + b(n) = (t(n/2+2) + b(n))/2$.

Since it is not easy to deal with the asymmetric case explicitly, we provide a decomposition, counting formulas, and a generation procedure for rooted symmetric 3-connected cubic planar graphs (Section 4, 5) and for rooted 3-connected cubic planar graphs embedded in the plane, i.e., rooted 3-connected cubic maps (Section 6).

Having these, we can also count the asymmetric case: The number of asymmetric 3-connected cubic planar graphs are the half of the number of rooted 3-connected cubic maps $t(n/2+2)$ minus the number of symmetric 3-connected cubic planar graphs $b(n)$, i.e., $a_{3n/2-1}(n) = (t(n/2+2) - b(n))/2$. Furthermore, we can *generate* asymmetric 3-connected cubic planar graphs, using *rejection sampling*: First we generate a face-rooted 3-connected graph. Then we check whether it is symmetric or not. This can be done in linear time using known techniques for planar graph isomorphism [35]. If the graph is symmetric, we reject it and restart the procedure. The procedure will terminate after a constant number of rounds in expectation, since there are more asymmetric than symmetric graphs. For 3-connected graphs there are even asymptotically more asymmetric than symmetric graphs and one round will suffice in most cases, which we know from the result of Tutte [50].

In the rest of this section we will focus on the symmetric rooted 3-connected graphs. Note that there is a unique symmetric rooted 3-connected graph on four vertices, i.e., K_4, and thus $b(4) = 1$.

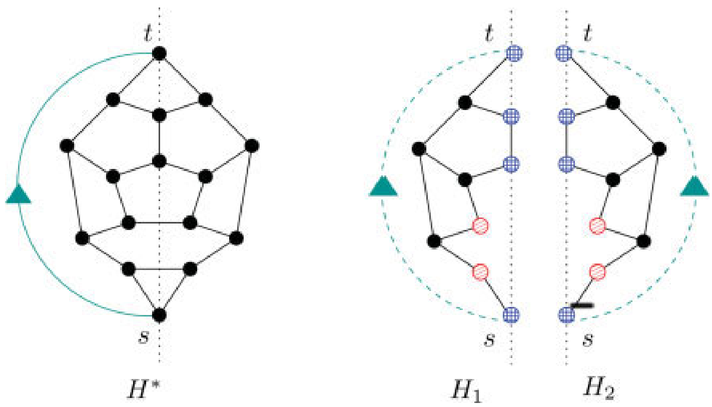

Fig. 8. Decomposition of a symmetric rooted 3-connected graph. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

Let H^* be a symmetric rooted 3-connected graph on at least six vertices, and φ be its maximal sense-reversing automorphism. Recall that each vertex is colored either red, blue, or uncolored, induced by φ (see Section 2), and that the poles are blue. Note that due to the 3-regularity of H^*, the number of blue vertices must be even, but the number of red vertices can be even or odd. We can think of H^* as being embedded in the plane in such a way that φ corresponds to a reflection, the blue vertices are aligned on the reflection axis, and the red vertices are connected by an edge crossing this axis perpendicularly (see Fig. 8, left part). Our arguments do not rely on such a representation, however.

If we remove from H^* the blue vertices and the edges between red vertices (i. e., we cut the unrooted graph H along the symmetry axis), then the resulting graph has exactly two connected components. The graphs induced by these components and the blue vertices are isomorphic and will be called H_1 and H_2 (see Fig. 8, right part). Recall that a network N is a graph with two distinguished vertices, such that the rooted graph N^* obtained by inserting a directed edge between the poles is 2-connected. We claim that H_1 is a network and hence H_1^* is 2-connected: Suppose there was a cut-vertex v in H_1^*. Then this cut-vertex together with the cut-vertex $\varphi(v)$ in H_2^* forms a 2-cut in H^*, contradicting the 3-connectivity of H^*.

Now we extract some more properties of the networks H_1 and H_2 and define the corresponding *colored networks* in such a way that we can *recursively decompose* them, and that we can establish a *bijection* between symmetric rooted 3-connected graphs and colored networks.

Definition 1. *A colored network is a network N in which some vertices are colored red and blue such that:*

(P1) *The colored vertices have degree* 2 *in* N^*, *and all other vertices have degree* 3.
(P2) N^* *has an embedding in the plane in such a way that all colored vertices and the poles lie on the outer face.*
(P3) *N and all subnetworks of N contain at least one colored vertex.*
(P4) *No nontrivial subnetwork of N has two blue poles.*

The bijection to symmetric rooted 3-connected graphs is as follows.

Theorem 2. *For $n \geq 6, b, r, 1 \leq b + r \leq n$ with n, b even, there is a bijection between the following two sets of objects:*

i. colored networks with b blue vertices, r red vertices, and a total of $(n+b)/2$ vertices that have no blue cut-vertex and whose both poles are blue, and
ii. face-rooted 3-connected graphs on n vertices that have a nontrivial automorphism that fixes b vertices and exchanges r pairs of adjacent vertices.

Proof. Given a symmetric rooted 3-connected graph H^*, we first check that both H_1 and H_2, constructed as described earlier, are networks that have blue poles and no blue cut-vertex, and satisfy properties (P1) − (P4): First H_1^* and H_2^* are 2-connected, since if there was a nonblue cut-vertex in H_1^*, we would also have a corresponding cut-vertex in H_2^*, and together they would form a 2-cut in H^*. If H_1 had a blue cut-vertex, then this vertex together with a pole of H_1 would form a 2-cut in H^*. These contradict the assumption that H^* is 3-connected. (P1) and (P2) are immediate from the definition of H_1 and H_2. (P3): Every subnetwork contains a colored vertex, since otherwise its poles would be a 2-cut in H^*. (P4): No subnetwork has two blue pole vertices, since these blue pole vertices would be a 2-cut in H^*.

Conversely, we have to (re-)construct for every colored network H_1 with blue poles, but without a blue cut-vertex, the corresponding symmetric rooted 3-connected graph H^*. First we make an isomorphic copy H_2 of H_1. Then we identify corresponding blue vertices in H_1 and H_2, and connect by edges corresponding red vertices in H_1 and H_2. Finally, we add the root edge between the poles. The constructed graph H^* is clearly a symmetric, planar (by (P2)), and cubic (by (P1)) network. Suppose H^* was not 3-connected. Then there is a split pair $\{k_1, k_2\}$ in H^* that determines two subnetworks N_1 and N_2. We distinguish four cases:

1. Both of k_1, k_2 are blue. – This is impossible because then H_1 or H_2 would contain a subnetwork with two blue poles k_1, k_2, contradicting (P4).
2. Exactly one of k_1, k_2 is blue. – Then wlog. k_2 is blue, and k_1 is in $H_1 \setminus H_2$. Let N_1' and N_2' be those (nonempty) parts of N_1 and N_2 that also lie in H_1. By (P3) there are colored vertices $v_1 \in N_1'$ and $v_2 \in N_2'$. Since H_2^* is 2-connected, there is a path from v_1 to v_2 passing through H_2^* and avoiding k_2 (and k_1), which contradicts the assumption that k_1, k_2 is a split pair in H^*.
3. None of k_1, k_2 is blue, and either both lie in H_1 or both lie in H_2. – Suppose wlog. both vertices lie in H_1. Then k_1 and k_2 define a nontrivial subnetwork in H_1. But since every such subnetwork contains a colored vertex, this contradicts that k_1, k_2 is a 2-cut in H^*.
4. Again none of k_1, k_2 is blue, but this time k_1 is in $H_1 \setminus H_2$, and k_2 is in $H_2 \setminus H_1$. – It cannot be that H_1 contains vertices from both N_1 and N_2 because of 2-connectivity; the same for H_2. Thus H_1 either equals N_1 or N_2, which is impossible by (P3), since a colored vertex in H_1 again contradicts that k_1, k_2 is a 2-cut in H^*. ∎

Because we only consider *cubic* networks, a colored network with blue poles s and t (see Fig. 9, middle part) has a unique colored subnetwork with poles u, v where u is adjacent to s and v is adjacent to t, and both u and v are not blue (see Fig. 9, right-most part). On the other hand, given a colored network N whose both poles u, v are not blue, we can obtain a unique colored network from N by adding new blue poles s and t and new edges su and tv. Given a colored network whose both poles are not blue and that have no blue cut-vertex, we

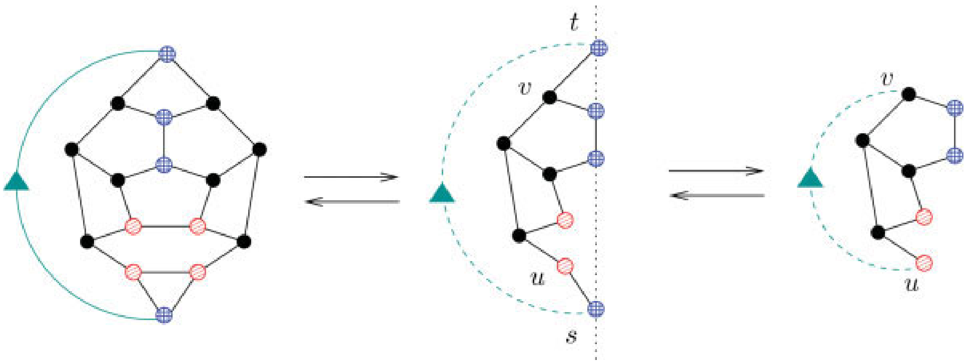

Fig. 9. Rooted symmetric 3-connected graph and colored networks. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

can thus construct its unique corresponding rooted symmetric 3-connected graph by first constructing a colored network with blue poles as above, and then making its isomorphic copy, identifying corresponding blue vertices, connecting corresponding red vertices, and adding the root edge (see Fig. 9, left-most part), and vice versa. Thus we obtained the following one-to-one correspondence.

Corollary 1. *For $n \geq 6, b, r, 1 \leq b + r \leq n$ with n, b even, there is a bijection between the following two sets of objects:*

i. *Colored networks with b blue vertices, r red vertices, and $(n + b)/2 - 2$ vertices in total, that have no blue cut-vertex and whose both poles are not blue, and*
ii. *Face-rooted 3-connected graphs on n vertices that have a nontrivial automorphism that fixes b vertices and exchanges r pairs of adjacent vertices.*

Using Corollary 1, we reduce the problem of decomposing and counting symmetric rooted 3-connected graphs to the problem of decomposing and counting colored networks that have no blue cut-vertex and whose both poles s, t are not blue, which will be done in Section 5.

5. COLORED NETWORKS

Let $n'_{b,r}(n)$ denote the number of colored networks with b blue vertices, r red vertices, and n the total number of vertices, that have no blue cut-vertex and whose both poles s, t are not blue. From Corollary 1, we have $b(n) = \sum_{b+r\geq 1} n'_{b,r}((n + b)/2 - 2)$ for $n \geq 6$.

When we in addition require that the first colored vertex has distance k to the s-pole (which we will use in Theorem 3 in Section 8), we use $n'_{b,r,k}(n)$ instead. To recursively decompose such colored networks we need auxiliary objects, colored s-networks such that both poles are not blue, but that *might* have a blue cut-vertex. The exact formulas require rather tedious case studies and thus we only sketch how to decompose colored networks.

5.1. Colored *s*-Networks

The cut-edge uv in a colored s-network which is closest to s induces a split colored p- or h-network with non-blue poles s, u, and a remaining part with nonblue poles v, t, which is again an arbitrary colored network that has no blue cut-vertex.

5.2. Colored *p*-Networks

The colored vertices of a colored p-network must all lie in one of its two parts, by property (P2). Since every subnetwork has to contain at least one colored vertex (P3), the other part of the p-network must be a single edge. Moreover the poles s, t are not colored. One (but not both) of the unique neighbors of s and t in this subnetwork might be blue (P4).

5.3. Colored *h*-Networks

The core of a colored h-network G^* has a unique embedding in the plane where the root edge and the core edges that are replaced by colored networks lie on the outer face. Hence, the edges of core(G^*) (on the outer face, in particular) have a unique ordering. Along the edges of core(G^*) on the outer face, we split off a subnetwork N^* from G^*. The remaining graph H^* after splitting off N^* from G^* is again an h-network. However, it might be the case that all colored vertices lie in N^* and if in addition the replaced edge is the k-th edge of core(G^*), then H^* is a 3-connected network with at least k vertices on the outer face. In this case the dual of H^* is a rooted triangulation, one of whose poles has degree at least k, which will be studied in Section 6.

6. CUBIC PLANAR MAPS AND TRIANGULATIONS

The dual of a face-rooted 3-connected cubic planar graph, i.e., a 3-connected cubic planar map, is a face-rooted triangulation. The root-face (incident to the root-edge) becomes the s-pole and the other face incident to the root-edge becomes the t-pole in the dual. In our drawings, the root-face incident to the root will always be the outer face.

To derive a recursion, we generalize the notion of a triangulation, as Tutte did [49]: We consider 3-connected planar maps where all the faces except the outer face are triangles, i.e., we do not require that the outer face is a triangle. Then we distinguish between external and internal vertices and edges, where the external vertices and edges are defined to be the vertices and edges on the outer face. We call such objects *near-triangulations*. By 3-connectivity, in a near-triangulation there is no internal edge connecting two external vertices.

If a 3-connected cubic planar map has n vertices in total and k vertices on the outer face, then the face-rooted triangulation has $\frac{n}{2} + 2$ vertices in total and its s-pole has degree k. To count the number of such triangulations, we use the function $t_{k,l}(n)$ which denotes the number of rooted near-triangulations with n vertices, where the s-pole has degree k and there are l vertices on the outer face.

If the unique internal vertex adjacent to the two poles has no internal edge connecting it to an external vertex except the poles, then we remove the pole edge and move the t-pole to the unique internal vertex adjacent to the two former poles. This results in a near triangulation

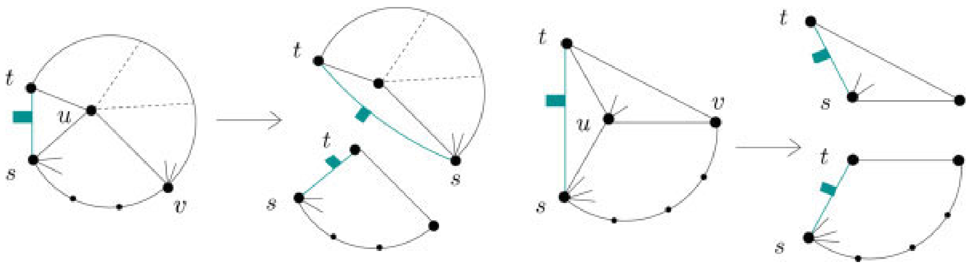

Fig. 10. Decomposition of a rooted near-triangulation. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

with $l+1$ vertices on the outer face such that the s-pole has degree $k-1$. Otherwise, we remove the edge between the poles and decompose such triangulations along the edge, say uv, connecting to the first such external vertex, say v, according to a traversal of the outer face starting from the s-pole ending at the t-pole. Then one of the two split triangulations has the new t-pole at the vertex u, and the other one has the new s-pole at v (see Fig. 10, left part), but it has the new s-pole at u when the number of edges on the outer face is 3 (see Fig. 10, right part). All these cases can be computed inductively using the value of $t_{k,l}(n)$ for lexicographically smaller arguments.

Initially, $t_{2,3}(3) = 1$ and $t_{k,l}(n) = 0$ if $k = 2$ and $l > 3$ or $n > 3$, or if $l + k - 2 > n$. Otherwise

$$t_{k,l}(n) = t_{k-1,l+1}(n) + \sum_{k'+1,i\geq 3} t_{k-1,l}(i)t_{k',3}(n-i+2) + \sum_{k',l',i\geq 3} t_{k-1,l-l'+2}(i)t_{k',l'}(n-i+2).$$

The number $t(n)$ of face-rooted triangulations on n vertices is then $t(n) = \sum_{k\geq 2} t_{k,3}(n)$.

7. POLE-SYMMETRIC GRAPHS

In this section, we consider the decomposition of rooted graphs with a pole-symmetry and derive recurrence formulas to count them. We needed them in Section 3 for decomposition.

Let $\tilde{r}(n)$ be the number of pole-symmetric rooted graphs on n vertices, where the poles are distinct. According to Theorem 1 we have $\tilde{r}(n) = \tilde{d}(n) + \tilde{s}(n) + \tilde{p}(n) + \tilde{h}(n)$, where the functions $\tilde{d}(n), \tilde{s}(n), \tilde{p}(n), \tilde{h}(n)$ count the number of d-, s-, p-, and h-graphs with a pole symmetry on n vertices, respectively. Note that in $\tilde{r}(n)$ we did not count the rooted graphs where the poles are not distinct. By definition, a g-graph is pole-symmetric, $\tilde{g}(n) = g(n)$. Finally, let $\tilde{c}(n) = \tilde{s}(n) + \tilde{p}(n) + \tilde{h}(n) + \tilde{g}(n)$, and call the corresponding graphs pole-symmetric c-graphs.

If a d-graph G^* has a pole symmetry, then (its unrooted graph) G must have two identical components. This component together with the old root edge st and the self-loop ss, forms a g-graph rooted at the self-loop ss (see Fig. 3). Thus there is a bijection between g-graphs on $n/2+1$ vertices and pole-symmetric d-graphs on n vertices: For $0 \leq n \leq 9$, $\tilde{d}(n) = 0$, and for $n \geq 10$, $\tilde{d}(n) = g(n/2+1)$.

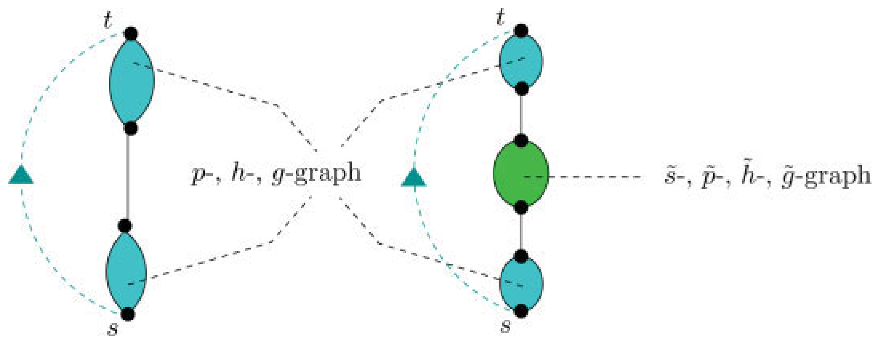

Fig. 11. Pole symmetric s-graphs. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

7.1. Pole-Symmetric *s*-Graphs

Here, we split off a p-, h- or g-graph at both poles simultaneously. The remaining graph is either a pole-symmetric c-graph, or an edge (see Fig. 11). For $0 \le n \le 7$, $\tilde{s}(n) = 0$. For $n \ge 8$,

$$\tilde{s}(n) = \sum_{i=4}^{(n-4)/2} (c(i) - s(i))\tilde{c}(n-2i) + p(n/2) + h(n/2).$$

7.2. Pole-Symmetric *p*-Graphs

The root edge st induces two components. One of them might be a single edge. Otherwise the two components, rooted at the corresponding neighbors of s and t, are pole-symmetric c-graphs. The two components are not ordered and we should not count such a pair of c-graphs twice, when they are isomorphic. Thus for $0 \le n \le 5$, $\tilde{p}(n) = 0$, and for $n \ge 6$,

$$\begin{aligned}\tilde{p}(n) &= \tilde{c}(n-2) + \frac{1}{2}\sum_{i=4}^{(n-4)/2} \tilde{c}(i)\tilde{c}(n-i-2) + \frac{1}{2}\binom{\tilde{c}((n-2)/2)}{2} + \tilde{c}((n-2)/2)\\ &= \tilde{c}(n-2) + \frac{1}{2}\sum_{i=4}^{(n-4)/2} \tilde{c}(i)\tilde{c}(n-i-2) + \binom{\tilde{c}((n-2)/2)+1}{2}.\end{aligned}$$

7.3. Pole-Symmetric *h*-Graphs

Let G^* be a pole-symmetric h-graph rooted at st. Since G^* is built up from its core(G^*) (Theorem 1), we will decompose G^* recursively into core(G^*) and c-graphs.

To recursively decompose G^*, we uniquely order the edges of core(G^*) using the coloring of core(G^*) attained by its symmetry type. Note that core(G^*) must be pole-symmetric, and in addition it might be symmetric. Let ψ be the maximal pole-symmetric automorphism of core(G^*) and φ be the maximal sense-reversing automorphism of core(G^*) if in addition core(G^*) is symmetric.

In the case when core(G^*) does not have a sense-reversing automorphism, we order the edges of core(G^*) in such a way that ψ-blue edges come first, which are followed by ψ-red edges, and next the uncolored ψ-corresponding edges are consecutively ordered (a vertex closer to s attains a smaller order). In the case when core(G^*) has a sense-reversing automorphism, we order the edges of core(G^*) in such a way that we start with ψ-blue edges followed by φ-blue edges, and next we list ψ-red edges and then φ-red edges, where each of ψ-and φ-corresponding pairs are consecutive, but ordered arbitrarily. Finally we list each pair of φ-corresponding edges, followed by its ψ-corresponding pair, consecutively in groups of four, in such a way that between the two ψ-corresponding edges a vertex closer to s attains a smaller order and the two φ-corresponding edges are ordered arbitrarily.

Having the unique ordering of the edges of core(G^*), we can obtain a pole-symmetric h-graph G^* from core(G^*) by replacing edges of core(G^*) according to this order. In other words, we can decompose the pole-symmetric h-graphs G^*, until we obtain core(G^*). In addition, recursive counting formulas can be directly derived from the decomposition. We will further decompose core(G^*), a rooted 3-connected pole-symmetric graph, in Section 8.

8. POLE-SYMMETRIC 3-CONNECTED GRAPHS

In this section, we consider rooted 3-connected pole-symmetric graphs. We again want to use colored networks. A (edge)-rooted 3-connected pole-symmetric graph G^* has either one embedding in the plane (if it has a sense-reversing automorphism), or two embedding in the plane (if it has not). In other words, by specifying an face incident to the edge-root, we can make either two face-rooted graphs or one face-rooted graph corresponding to G^*. If we can deal with over-all face-rooted 3-connected pole-symmetric graphs, and face-rooted 3-connected pole-symmetric graphs with sense-reversing automorphisms, then we know about those graphs without sense-reversing automorphisms. Especially for counting, let $\tilde{t}(n)$ be the number of face-rooted 3-connected pole-symmetric graphs and let $\tilde{h}_{3n/2-1}(n)$, $\tilde{a}(n)$, and $\tilde{b}(n)$ be the number of (edge)-rooted 3-connected pole-symmetric graphs, those without symmetry, and those with symmetry, respectively. Then $\tilde{t}(n) = 2\tilde{a}(n) + \tilde{b}(n)$ and $\tilde{h}_{3n/2-1}(n) = \tilde{a}(n) + \tilde{b}(n) = (\tilde{t}(n) + \tilde{b}(n))/2$.

We will thus first study face-rooted 3-connected pole-symmetric graphs using again colored networks. This time the colors correspond to the vertices and edges that are fixed by the pole-exchanging automorphism.

Given a face-rooted 3-connected pole-symmetric graphs H^* rooted at s, t, with the pole-exchanging automorphism ψ, fixing $\tilde{b}$ vertices and exchanging $\tilde{r}$ pairs of adjacent vertices, that have exactly $k \geq 3$ vertices on the outer face. Note first that both neighbors x, y of s and t can not be colored blue, i.e., the inner and outer faces of H^* can not be both triangles. If then, the other neighbors of x, y form a 2-cut of H^*, a contradiction.

If $k = 3$, then it implies that the unique neighbor u of s and t on the outer face must be blue. Then take u and its unique neighbor v (which must be blue) as new south- and north-poles, respectively. And we take as the outer face a face containing u and s, make the original root edge st undirected. Then the resulting graph is a face-rooted 3-connected *symmetric* graphs, where ψ is interpreted as a sense-reversing automorphism. Thus we can apply Theorem 2 immediately. This case is counted by $n'_{\tilde{b},\tilde{r}}((n + \tilde{b})/2 - 2)$ for $n \geq 6$, $1 \leq \tilde{b} + \tilde{r} \leq n$.

If $k \geq 4$ and the inner face of H^* is a triangle, i.e., there is a unique neighbor u of s and t on the *inner* face. Then it must be blue, and thus we can do the same as above. This case is also counted by $n'_{\tilde{b},\tilde{r}}((n + \tilde{b})/2 - 2)$ for $n \geq 6, 1 \leq \tilde{b} + \tilde{r} \leq n$.

Now we assume that $k \geq 4$ and that the inner face of H^* is not a triangle. Consider the pair uv of the two other neighbors of s. Then clearly both u and v are not blue. Remove s and t together with their incident edges. Remove all vertices that are fixed – the ψ*-blue* vertices – and edges that are exchanged by the pole-exchanging automorphism – the ψ*-red* vertices. Call the two graphs induced by the vertices in the two resulting connected components together with the blue vertices H_1 and H_2 – notice that the pair uv is uniquely ordered because of the face-rooting, say u is on the outer face, and thus we take them as poles of H_1. Thus H_1 is a colored network H_1 on $(n + \tilde{b})/2 - 1$ vertices. Moreover the $\lceil k/2 \rceil - 2$-th vertex of H_1 from u must be colored (either red or blue). Conversely, given a colored network we can construct the corresponding face-rooted pole-symmetric 3-connected network by reversing the above procedure. Thus we have the following bijection:

Theorem 3. *For $n \geq 6, k \geq 4, 1 \leq \tilde{b} + \tilde{r} \leq n$, there is a bijection between the following two sets of objects:*

> *Colored networks with $\tilde{b}$ blue, $\tilde{r}$ red, and $(n + \tilde{b})/2 - 1$ total vertices, such that the south-pole or north-pole are not blue. Moreover between s and the first (with respect to s) colored vertex there are $\lceil k/2 \rceil - 2$ edges on the outer face.*
> *Face-rooted 3-connected graphs on n vertices with a pole-exchanging automorphism fixing $\tilde{b}$ vertices and exchanging $\tilde{r}$ pairs of adjacent vertices. Moreover there are k vertices on the outer face and the inner face is not a triangle.*

Let $\tilde{t}_{\tilde{b},\tilde{r}}(n)$ be the number of face-rooted 3-connected graphs on n vertices with a pole-exchanging automorphism and $\tilde{b}$ blue vertices and $\tilde{r}$ pairs of red vertices, and $\tilde{t}_{\tilde{b},\tilde{r},k}(n)$ be those such that there are k vertices on the outer face and the inner face is not a triangle, respectively. Then we get $\tilde{t}_{\tilde{b},\tilde{r}}(n) = 2n'_{\tilde{b},\tilde{r}}((n + \tilde{b})/2 - 2) + \sum_{k\geq 4} \tilde{t}_{\tilde{b},\tilde{r},k}(n)$ and

$$\tilde{t}_{\tilde{b},\tilde{r},k}(n) = n'_{\tilde{b},\tilde{r},\lceil k/2\rceil - 2}((n + \tilde{b})/2 - 1).$$

Next we have to compute the number of pole-symmetric 3-connected graphs with a sense-reversing automorphism. Again we use colored networks, but impose the additional constraint that the colored network has a pole-exchanging automorphism. Along the lines of Theorem 2 we have a bijection between these pole-symmetric colored networks and pole-symmetric networks with a sense-reversing automorphism.

Corollary 2. *For $n \geq 6, 1 \leq b + r \leq n$ with n, b even, there is a bijection between the following two sets of objects:*

i. *pole-symmetric colored networks with b blue vertices, r red vertices, and $(n + b)/2$ vertices in total that have blue poles and no blue cut-vertex, and*
ii. *face-rooted pole-symmetric* 3*-connected graphs on n vertices that have a nontrivial sense-reversing automorphism that fixes b vertices and exchanges r pairs of adjacent vertices.*

In addition to Corollary 2, there is a one-to-one correspondence between pole-symmetric colored networks without a blue cut-vertex whose poles are not blue that have b blue vertices, r red vertices, and $(n + b)/2 - 2$ vertices in total, and rooted pole-symmetric 3-connected graphs on n vertices that have a nontrivial sense-reversing automorphism that fixes b vertices and exchanges r pairs of adjacent vertices, as in Corollary 1. The decomposition of pole-symmetric colored networks is a straightforward combination of the ideas in Section 5 and 7.

9. CONCLUSION

We presented a decomposition strategy for unlabeled connected cubic planar graphs along the connectivity structure. In order to count these objects we need a *unique* decomposition, and thus we used the well-known concept of rooting by an edge. For 3-connected rooted cubic planar graphs, however, we also had to control whether or not there is a sense-reversing automorphism, which in turn required to control networks that have a pole-exchanging automorphism, and finally networks with both a sense-reversing and a pole-exchanging automorphism. To count these objects we introduced the concept of *colored* networks, and proved several bijections. The decomposition together with the counting formulas immediately gives a polynomial time generation procedure for rooted unlabeled connected cubic planar graphs. Using rejection-sampling we obtain our result:

Theorem 4. *There is an algorithm that generates an unlabeled connected cubic planar graph on n vertices uniformly at random, and whose expected running time is in $\tilde{O}(n^{10})$. If we allow for a preprocessing, the algorithm can generate such an object in $O(n^3)$.*

Proof. The algorithm first generates a rooted unlabeled cubic planar graph G with n vertices, using the recursive decomposition along the connectivity structure (Section 2–Section 6), and the values of the counting formulas that can be computed efficiently using dynamic programming. Note that the representation size of all the values in this paper is linear, since the logarithm of the number of *unlabeled* planar graphs is linear.

The longest computation time is due to the five-dimensional tables for colored h-networks in Section 5, where the summation runs over at most three parameters, and where we have to perform multiplications with large numbers.

Assuming an $O(n \log n \log \log n)$ multiplication algorithm (see e.g. [17]), the number of computation steps needed to fill the five-dimensional table is within $\tilde{O}(n^9)$, where $\tilde{O}(\cdot)$ denotes growth up to logarithmic factors.

If we do not charge for the costs of a precomputation step, the actual generation of a rooted cubic planar graph can be done in quadratic time: The decomposition tree is of linear size; computing the probabilistic decisions at each branch also takes linear time, if we assume that we have access to the values in the table and the partial sums of the formulas.

To obtain *unrooted* cubic planar graphs, the algorithm computes the size o of the orbit of the root in the automorphism group of the graph G, which can be done in linear time using accordingly adapted graph isomorphism algorithms for planar graphs [35], and outputs the graph G with probability $1/o$. Since the size o of the orbit of the root is at most linear, the expected number of restarts is also linear. Thus the overall expected running time is in $\tilde{O}(n^{10})$, and in $O(n^3)$ with precomputation. ∎

It is easy to see that we can count and generate random cubic planar *multi-graphs* by only changing the initial values. Another by-product is the enumeration and uniform generation of rooted near-triangulations with a given number of vertices on the outer face, and a given number of vertices on the outer face of the dual. For the special case of rooted triangulations a linear-time generation procedure was known [47]. The present article also contains a series of new enumeration results and generation procedures of rooted cubic graphs with various sorts of symmetries and degrees of connectivity.

Many techniques presented in this paper also work for planar graphs. However, it is more difficult to count not-necessarily-cubic g-graphs, since there is no canonical way to decompose them into smaller rooted graphs. Another complication would be that we cannot decompose colored networks via networks where both poles are non-blue. The recurrences derived in this paper could serve as a starting point to compute the generating functions and the asymptotic behavior of the counting functions for the objects considered here.

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