Asymptotic properties of graphs on orientable surfaces

Mihyun Kang

Doctoral School Discrete Mathematics

Outline

I. Erdős–Rényi Random Graphs

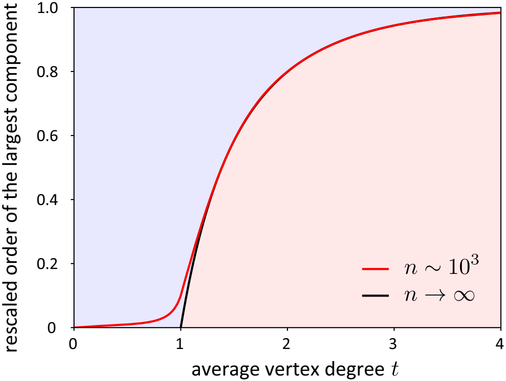
II. Random Graphs on Surfaces

III. Proof Ideas

Part I.

Erdős–Rényi Random Graph $G(n, m)$

chosen uniformly at random among all graphs with n vertices and m edges

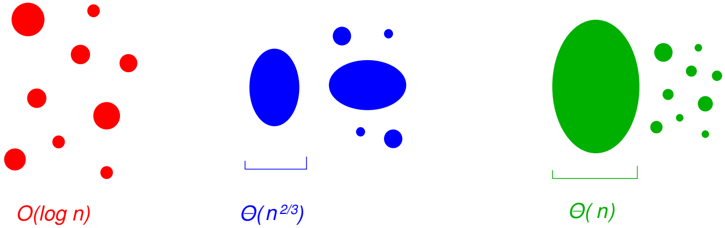


Emergence of "Giant" Component

$L(d) = \#$ vertices in the largest component in $G(n, m)$, where $m = d \cdot \frac{n}{2}$

Theorem [ERDŐS–RÉNYI 60]

- If $d < 1$, whp $L(d) = O(\log n)$
- If $d = 1$, whp $L(d) = \Theta(n^{2/3})$
- If $d > 1$, whp $L(d) = \Theta(n)$



Emergence of "Giant" Component

$L(d) = \#$ vertices in the largest component in $G(n, m)$, where $m = d \cdot \frac{n}{2}$

Theorem [ERDŐS–RÉNYI 60]

- If $d < 1$, whp $L(d) = O(\log n)$

  and the largest component is a tree and all other components are either trees or unicyclic components.

- If $d = 1$, whp $L(d) = \Theta(n^{2/3})$

Emergence of "Giant" Component

$L(d) \;=\; \#$ vertices in the largest component in $G(n,m)$, where $m = d \cdot \frac{n}{2}$

Theorem [Erdős–Rényi 60]

- If $d < 1$, whp $L(d) \;=\; O(\log n)$

  and the largest component is a tree and all other components are either trees or unicyclic components.

- If $d = 1$, whp $L(d) \;=\; \Theta(n^{2/3})$

- If $d > 1$, whp $L(d) \;=\; (1 + o(1))\,\rho n$

  where $\rho = \rho(d) \in (0,1)$ is the unique positive solution of $1 - \rho = e^{-d\,\rho}$,

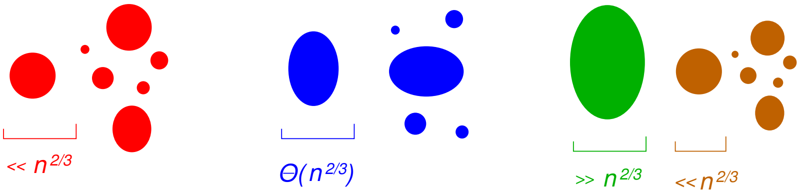
  and the largest component contains more than two cycles,
  while all but the largest component are trees or unicyclic components of order $O(\log n)$.

Critical Phenomenon

Let $d = 1 + \epsilon$ for $\epsilon = o(1)$.

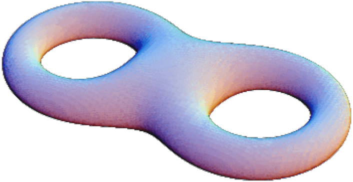
Theorem [Bollobás 84; Łuczak 90]

- If $\epsilon\, n^{1/3} \to -\infty$, whp $\quad L(1+\epsilon) = 2\epsilon^{-2} \log|\epsilon|^3 n \ll n^{2/3}$
- If $\epsilon\, n^{1/3} \to \lambda$, a constant, whp $\quad L(1+\epsilon) = \Theta(n^{2/3})$
- If $\epsilon\, n^{1/3} \to +\infty$, whp $\quad L(1+\epsilon) = (1+o(1))\, 2\,\epsilon\, n \gg n^{2/3}$



Part II.

Random Graphs on Surfaces



Graphs on a Surface

Embeddability on $\mathbb{S}_g$

Let $\mathbb{S}_g$ be the orientable surface of genus $g > 0$.

Graphs on a Surface

Embeddability on $\mathbb{S}_g$

Let $\mathbb{S}_g$ be the orientable surface of genus $g > 0$.

- A connected graph is embeddable on $\mathbb{S}_g$, if it has an embedding on $\mathbb{S}_g$ such that every face is homeomorphic to a disc.

Graphs on a Surface

Embeddability on $\mathbb{S}_g$

Let $\mathbb{S}_g$ be the orientable surface of genus $g > 0$.

- A connected graph is embeddable on $\mathbb{S}_g$, if it has an embedding on $\mathbb{S}_g$ such that every face is homeomorphic to a disc.
- A graph is embeddable on $\mathbb{S}_g$, if each of its connected components is embeddable on a surface $\mathbb{S}_{g_i}$ and $\sum_i g_i \leq g$.

Graphs on a Surface

Embeddability on $\mathbb{S}_g$

Let $\mathbb{S}_g$ be the orientable surface of genus $g > 0$.

- A connected graph is embeddable on $\mathbb{S}_g$, if it has an embedding on $\mathbb{S}_g$ such that every face is homeomorphic to a disc.
- A graph is embeddable on $\mathbb{S}_g$, if each of its connected components is embeddable on a surface $\mathbb{S}_{g_i}$ and $\sum_i g_i \leq g$.
- A connected graph embedded on $\mathbb{S}_g$ is called a map on $\mathbb{S}_g$.

Graphs on a Surface

Embeddability on $\mathbb{S}_g$

Let $\mathbb{S}_g$ be the orientable surface of genus $g > 0$.

- A connected graph is embeddable on $\mathbb{S}_g$, if it has an embedding on $\mathbb{S}_g$ such that every face is homeomorphic to a disc.
- A graph is embeddable on $\mathbb{S}_g$, if each of its connected components is embeddable on a surface $\mathbb{S}_{g_i}$ and $\sum_i g_i \leq g$.
- A connected graph embedded on $\mathbb{S}_g$ is called a map on $\mathbb{S}_g$.

Facewidth

- The facewidth of a map is the minimum number of intersections that a non-contractible circle has with the map.
- The facewidth of a graph is the maximal facewidth of all its embeddings.

Random Planar Graphs

Let $P(n, m)$ be a uniform random planar graph with n vertices and m edges and $L(d)$ denote the number of vertices in the largest component in $P(n, d \cdot \frac{n}{2})$. Let $d = 1 + \epsilon$ for $\epsilon = o(1)$.

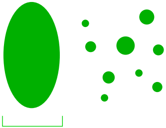
Random Planar Graphs

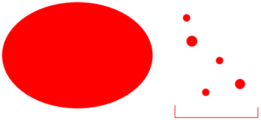
Let $P(n, m)$ be a uniform random planar graph with n vertices and m edges and $L(d)$ denote the number of vertices in the largest component in $P(n, d \cdot \frac{n}{2})$. Let $d = 1 + \epsilon$ for $\epsilon = o(1)$.

Theorem [K.– Łuczak 12]

- If $\epsilon n^{1/3} \to -\infty$, whp $L(1+\epsilon) = 2\epsilon^{-2} \log|\epsilon|^3 n \ll n^{2/3}$

  If $\epsilon n^{1/3} \to \infty$, whp $L(1+\epsilon) = (1+o(1))\,\epsilon\, n \gg n^{2/3}$





Random Planar Graphs

Let $P(n, m)$ be a uniform random planar graph with n vertices and m edges and $L(d)$ denote the number of vertices in the largest component in $P(n, d \cdot \frac{n}{2})$. Let $d = 1 + \epsilon$ for $\epsilon = o(1)$.

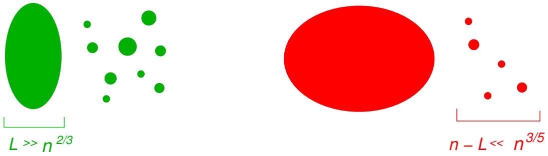
Theorem [K.– ŁUCZAK 12]

- If $\epsilon n^{1/3} \to -\infty$, whp $\quad L(1+\epsilon) = 2\epsilon^{-2} \log |\epsilon|^3 n \ll n^{2/3}$

  If $\epsilon n^{1/3} \to \infty$, whp $\quad L(1+\epsilon) = (1+o(1))\,\epsilon\, n \gg n^{2/3}$

- If $\epsilon n^{2/5} \to -\infty$, whp $\quad n - L(2+\epsilon) = |\epsilon\, n| \gg n^{3/5}$

  If $\epsilon n^{2/5} \to \infty$, whp $\quad n - L(2+\epsilon) = \Theta(\epsilon^{-3/2}) \ll n^{3/5}$



Random Graphs with Genus $g > 0$

Let $S_g(n, m)$ be a uniform random graph on a surface of genus $g > 0$ with n vertices and m edges and $L(d)$ denote the number of vertices in the largest component in $S_g(n, d \cdot \frac{n}{2})$. Let $d = 1 + \epsilon$ for $\epsilon = o(1)$.

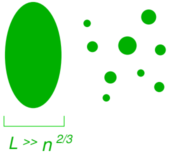
Theorem [K.– Mosshammer–Sprüssel 15+]

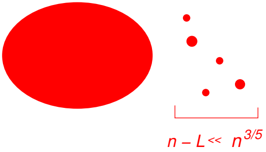
- If $\epsilon n^{1/3} \to -\infty$, whp $\quad L(1+\epsilon) = 2\epsilon^{-2} \log|\epsilon|^3 n \ll n^{2/3}$

  If $\epsilon n^{1/3} \to \infty$, whp $\quad L(1+\epsilon) = (1+o(1))\,\epsilon n \gg n^{2/3}$

- If $\epsilon n^{2/5} \to -\infty$, whp $\quad n - L(2+\epsilon) = |\epsilon n| \gg n^{3/5}$

  If $\epsilon n^{2/5} \to \infty$, whp $\quad n - L(2+\epsilon) = \Theta(\epsilon^{-3/2}) \ll n^{3/5}$





Part III.

Proof Ideas

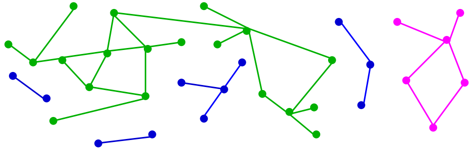
- Internal structure: Core-Kernel
- Constructive decomposition along connectivity
- Singularity analysis of generating functions
- Probabilistic analysis

Internal Structure

complex com.

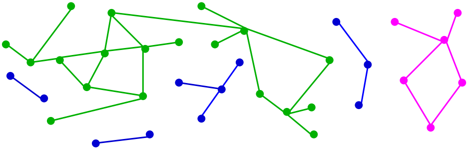
unicyc. com.

trees

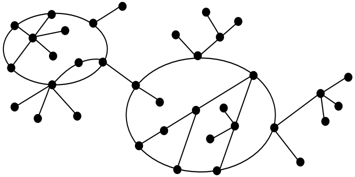


Internal Structure

complex com.
unicyc. com.
trees

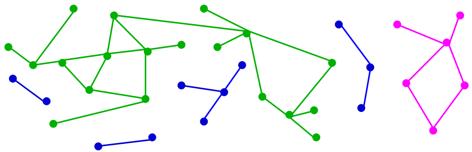


⇒ Core of G: maximal subgraph of G with minimum degree two

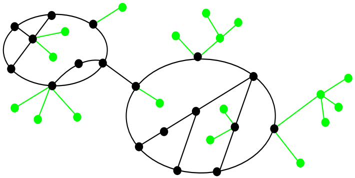


Internal Structure

complex com.
unicyc. com.
trees



⇒ Core of G: maximal subgraph of G with minimum degree two

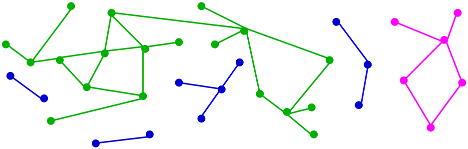


Internal Structure

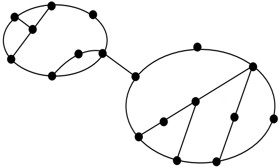
complex com.

unicyc. com.

trees



⇒ Core of G: maximal subgraph of G with minimum degree two

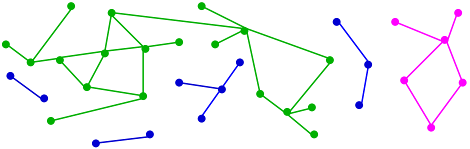


Internal Structure

complex com.

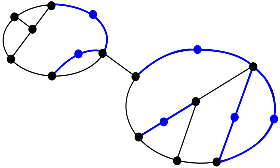
unicyc. com.

trees

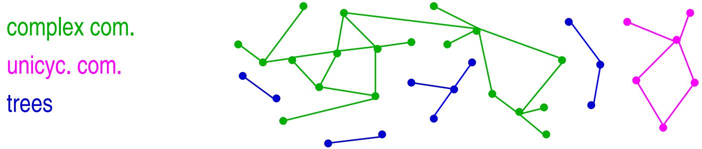


⇒ Core of G: maximal subgraph of G with minimum degree two

⇒ Kernel of G: obtained from core of G by replacing each path by an edge

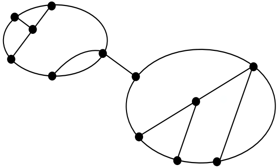


Internal Structure

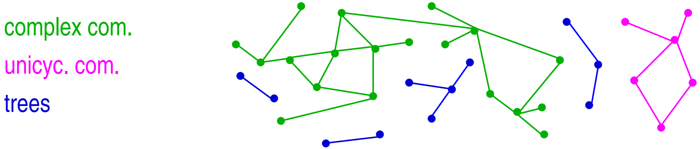


⇒ Core of G: maximal subgraph of G with minimum degree two

⇒ Kernel of G: obtained from core of G by replacing each path by an edge



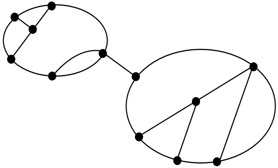
Internal Structure



⇒ Core of G: maximal subgraph of G with minimum degree two

⇒ Kernel of G: obtained from core of G by replacing each path by an edge

⇒ G is embeddable on $\mathbb{S}_g$ if and only if kernel of G is embeddable on $\mathbb{S}_g$



Typical Kernel on $\mathbb{S}_g$ with $g > 0$

- Cubic weighted multigraphs

Typical Kernel on $\mathbb{S}_g$ with $g > 0$

- ▷ Cubic weighted multigraphs
- ▷ Constructive decomposition: 1-conn. $\Longrightarrow$ 2-conn.

Theorem [ROBERTSON–VITRAY 90]

- Every 1-connected graph C embeddable on $\mathbb{S}_g$ with facewidth $k \geq 2$ has a unique 2-conn. component embeddable on $\mathbb{S}_g$ with facewidth k, while all other 2-connected components are planar.

Typical Kernel on $\mathbb{S}_g$ with $g > 0$

- ▷ Cubic weighted multigraphs
- ▷ Constructive decomposition: 1-conn. $\Longrightarrow$ 2-conn. $\Longrightarrow$ 3-conn.

Theorem [Robertson–Vitray 90]

- Every 1-connected graph C embeddable on $\mathbb{S}_g$ with facewidth $k \geq 2$ has a unique 2-conn. component embeddable on $\mathbb{S}_g$ with facewidth k, while all other 2-connected components are planar.
- Every 2-connected graph B embeddable on $\mathbb{S}_g$ with facewidth $k \geq 3$ has a unique 3-conn. component embeddable on $\mathbb{S}_g$ with facewidth k, while all other 3-conn. components are planar.

Typical Kernel on $\mathbb{S}_g$ with $g > 0$

- ▷ Cubic weighted multigraphs
- ▷ Constructive decomposition: 1-conn. $\Longrightarrow$ 2-conn. $\Longrightarrow$ 3-conn.

Theorem [ROBERTSON–VITRAY 90]

- Every 1-connected graph C embeddable on $\mathbb{S}_g$ with facewidth $k \geq 2$ has a unique 2-conn. component embeddable on $\mathbb{S}_g$ with facewidth k, while all other 2-connected components are planar.
- Every 2-connected graph B embeddable on $\mathbb{S}_g$ with facewidth $k \geq 3$ has a unique 3-conn. component embeddable on $\mathbb{S}_g$ with facewidth k, while all other 3-conn. components are planar.
- Every 3-connected graph embeddable on $\mathbb{S}_g$ has a unique embedding on $\mathbb{S}_g$ if its facewidth is at least $2g + 3$

Typical Kernel on $\mathbb{S}_g$ with $g > 0$

- ▷ Cubic weighted multigraphs
- ▷ Constructive decomposition: 1-conn. $\Longrightarrow$ 2-conn. $\Longrightarrow$ 3-conn.
- ▷ Dual of 3-connected cubic maps on $\mathbb{S}_g$ are triangulations on $\mathbb{S}_g$

  in which separating double edges, separating loops, and separating pair of loops are forbidden

Theorem [Robertson–Vitray 90]

- Every 1-connected graph C embeddable on $\mathbb{S}_g$ with facewidth $k \geq 2$ has a unique 2-conn. component embeddable on $\mathbb{S}_g$ with facewidth k, while all other 2-connected components are planar.
- Every 2-connected graph B embeddable on $\mathbb{S}_g$ with facewidth $k \geq 3$ has a unique 3-conn. component embeddable on $\mathbb{S}_g$ with facewidth k, while all other 3-conn. components are planar.
- Every 3-connected graph embeddable on $\mathbb{S}_g$ has a unique embedding on $\mathbb{S}_g$ if its facewidth is at least $2g + 3$

Generating Functions

Theorem [Fang–K.– Mosshammer–Sprüssel 15]

$$G_g(y) = \sum_{\sum g_i \leq g} \frac{1}{k!} \prod_{i=1}^{k} C_{g_i}(y)$$

$$C_g(y) \sim C_g^{\text{fw}\geq 2}(y) \sim B_g^{\text{fw}\geq 2}\left(\frac{y}{1 - Q_0(y)}\right)$$

$$B_g(y) \sim B_g^{\text{fw}\geq 3}(y) \sim T_g^{\text{fw}\geq 3}(y(1 + N_0(y)))$$

$$T_g^{\text{fw}\geq 3}(y) \sim T_g^{\text{fw}\geq 2g+3}(y) \sim S_g(y)$$

$G_g(y)$: cubic multigraphs on $\mathbb{S}_g$

$C_g(y)$: 1-connected cubic multigraphs on $\mathbb{S}_g$
$Q_0(y)$: 1-connected cubic multigraphs on $\mathbb{S}_0$ rooted at a loop

$B_g(y)$: 2-connected cubic multigraphs on $\mathbb{S}_g$
$N_0(y)$: 2-connected cubic multigraphs on $\mathbb{S}_0$ rooted at an edge

$T_g(y)$: 3-connected cubic graphs on $\mathbb{S}_g$
$S_g(y)$: simple triangulations on $\mathbb{S}_g$

Cubic Multigraphs on $\mathbb{S}_g$

Theorem [Fang–K.– Mosshammer–Sprüssel 15]

Let $K_g(n)$ denote the number of cubic weighted multigraphs embeddable on $\mathbb{S}_g$ with n vertices, for $n \in \mathbb{N}$ even and $g \geq 0$. Then

$$K_g(n) \sim c_g \, n^{5/2(g-1)-1} \, \gamma^n \, n!$$

where c_g is a constant depending only on genus g and $\gamma = \frac{79^{3/4}}{54^{1/2}} \approx 3.6$.

Cubic Multigraphs on $\mathbb{S}_g$

Theorem [Fang–K.– Mosshammer–Sprüssel 15]

Let $K_g(n)$ denote the number of cubic weighted multigraphs embeddable on $\mathbb{S}_g$ with n vertices, for $n \in \mathbb{N}$ even and $g \geq 0$. Then

$$K_g(n) \sim c_g \, n^{5/2(g-1)-1} \, \gamma^n \, n!$$

where c_g is a constant depending only on genus g and $\gamma = \frac{79^{3/4}}{54^{1/2}} \approx 3.6$.

Theorem [K.– Mosshammer–Sprüssel 15+]

Let $\mathcal{G}$ be a graph chosen uniformly at random among all cubic weighted multigraphs embeddable on $\mathbb{S}_g$ with n vertices, for $n \in \mathbb{N}$ even and $g \geq 1$. Then with probability tending to one as $n \to \infty$

Cubic Multigraphs on $\mathbb{S}_g$

Theorem [Fang–K.– Mosshammer–Sprüssel 15]

Let $K_g(n)$ denote the number of cubic weighted multigraphs embeddable on $\mathbb{S}_g$ with n vertices, for $n \in \mathbb{N}$ even and $g \geq 0$. Then

$$K_g(n) \sim c_g \, n^{5/2(g-1)-1} \, \gamma^n \, n!$$

where c_g is a constant depending only on genus g and $\gamma = \frac{79^{3/4}}{54^{1/2}} \approx 3.6$.

Theorem [K.– Mosshammer–Sprüssel 15+]

Let $\mathcal{G}$ be a graph chosen uniformly at random among all cubic weighted multigraphs embeddable on $\mathbb{S}_g$ with n vertices, for $n \in \mathbb{N}$ even and $g \geq 1$. Then with probability tending to one as $n \to \infty$

- $\mathcal{G}$ is not embeddable on $\mathbb{S}_{g-1}$;

Cubic Multigraphs on $\mathbb{S}_g$

Theorem [Fang–K.– Mosshammer–Sprüssel 15]

Let $K_g(n)$ denote the number of cubic weighted multigraphs embeddable on $\mathbb{S}_g$ with n vertices, for $n \in \mathbb{N}$ even and $g \geq 0$. Then

$$K_g(n) \sim c_g \, n^{5/2(g-1)-1} \, \gamma^n \, n!$$

where c_g is a constant depending only on genus g and $\gamma = \frac{79^{3/4}}{54^{1/2}} \approx 3.6$.

Theorem [K.– Mosshammer–Sprüssel 15+]

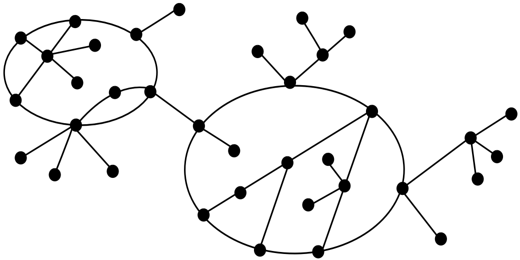
Let $\mathcal{G}$ be a graph chosen uniformly at random among all cubic weighted multigraphs embeddable on $\mathbb{S}_g$ with n vertices, for $n \in \mathbb{N}$ even and $g \geq 1$. Then with probability tending to one as $n \to \infty$

- $\mathcal{G}$ is not embeddable on $\mathbb{S}_{g-1}$;
- the largest component in $\mathcal{G}$ is a unique non-planar component and is of order $n - O(1)$.

Complex Graphs on $\mathbb{S}_g$

⇒ Core of G: maximal subgraph of G with minimum degree two

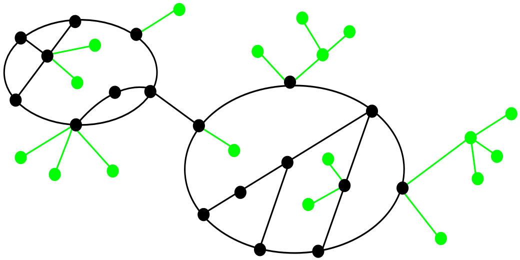
⇒ Kernel of G: graph obtained from the core of G by replacing each path



Complex Graphs on $\mathbb{S}_g$

⇒ Core of G: maximal subgraph of G with minimum degree two

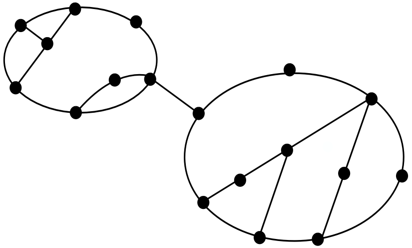
⇒ Kernel of G: graph obtained from the core of G by replacing each path



Complex Graphs on $\mathbb{S}_g$

⇒ Core of G: maximal subgraph of G with minimum degree two

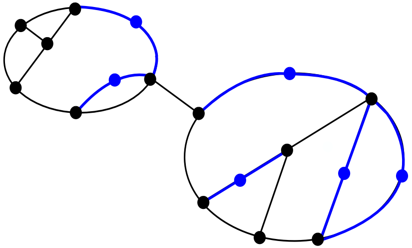
⇒ Kernel of G: graph obtained from the core of G by replacing each path



Complex Graphs on $\mathbb{S}_g$

⇒ Core of G: maximal subgraph of G with minimum degree two

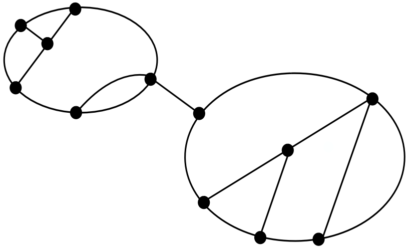
⇒ Kernel of G: graph obtained from the core of G by replacing each path



Complex Graphs on $\mathbb{S}_g$

⇒ Core of G: maximal subgraph of G with minimum degree two

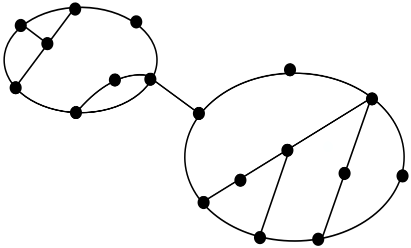
⇒ Kernel of G: graph obtained from the core of G by replacing each path



Complex Graphs on $\mathbb{S}_g$

⇒ Core of G: maximal subgraph of G with minimum degree two

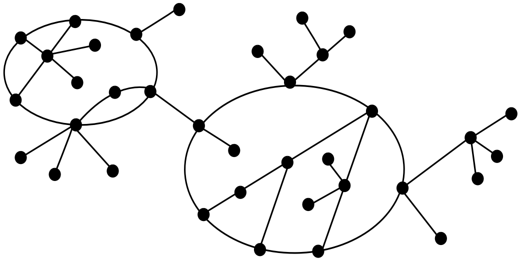
⇒ Kernel of G: graph obtained from the core of G by replacing each path



Complex Graphs on $\mathbb{S}_g$

⇒ Core of G: maximal subgraph of G with minimum degree two

⇒ Kernel of G: graph obtained from the core of G by replacing each path



Complex Graphs on $\mathbb{S}_g$

$C_g(k, k+\ell) = \#$ complex graphs on $\mathbb{S}_g$ with k vertices and $k+\ell$ edges

Complex Graphs on $\mathbb{S}_g$

Construct complex graph G embeddable on $\mathbb{S}_g$ by

- choosing the kernel of G from the set of possible candidates

$C_g(k, k+\ell)$ = # complex graphs on $\mathbb{S}_g$ with k vertices and $k+\ell$ edges

$$= \sum_{i,d} K_g(2\ell - d)$$

Complex Graphs on $\mathbb{S}_g$

Construct complex graph G embeddable on $\mathbb{S}_g$ by

- choosing the kernel of G from the set of possible candidates
- putting on its edges vertices of degree two to obtain the core of G

$C_g(k, k+\ell) = \#$ complex graphs on $\mathbb{S}_g$ with k vertices and $k+\ell$ edges

$$= \sum_{i,d} K_g(2\ell - d) \frac{(k)_i}{(2\ell - d)!} \binom{i - a\,\ell - 1}{3\ell - d - 1}$$

Complex Graphs on $\mathbb{S}_g$

Construct complex graph G embeddable on $\mathbb{S}_g$ by

- choosing the kernel of G from the set of possible candidates
- putting on its edges vertices of degree two to obtain the core of G
- adding a forest rooted at vertices of the core of G

$C_g(k, k+\ell) = \#$ complex graphs on $\mathbb{S}_g$ with k vertices and $k+\ell$ edges

$$= \sum_{i,d} K_g(2\ell - d) \frac{(k)_i}{(2\ell - d)!} \binom{i - a\,\ell - 1}{3\ell - d - 1} i\, k^{k-i-1}$$

Complex Graphs on $\mathbb{S}_g$

Construct complex graph G embeddable on $\mathbb{S}_g$ by

- choosing the kernel of G from the set of possible candidates
- putting on its edges vertices of degree two to obtain the core of G
- adding a forest rooted at vertices of the core of G

$C_g(k, k+\ell) = \#$ complex graphs on $\mathbb{S}_g$ with k vertices and $k+\ell$ edges

$$= \sum_{i,d} K_g(2\ell - d) \frac{(k)_i}{(2\ell - d)!} \binom{i - a\,\ell - 1}{3\ell - d - 1} i\, k^{k-i-1}$$

in which main contribution comes from the terms

$$i = (1 + O(\sqrt{\ell/k}) + O(1/\sqrt{\ell}))\sqrt{3k\ell} \quad \textit{and} \quad d = \Theta(\sqrt{\ell^3/k})$$

Graphs on $\mathbb{S}_g$

$S_g(n, m) = \#$ graphs on $\mathbb{S}_g$ with n vertices and m edges

Graphs on $\mathbb{S}_g$

$C_g(k, k+\ell)$ = # complex graphs on $\mathbb{S}_g$ with k vertices and $k+\ell$ edges

$U(n-k, m-k-\ell)$ = # graphs without complex components with $n-k$ vertices and $m-k-\ell$ edges

$S_g(n, m)$ = # graphs on $\mathbb{S}_g$ with n vertices and m edges

$$= \sum_{k,\ell} \binom{n}{k} C_g(k, k+\ell)\, U(n-k, m-k-\ell)$$

Graphs on $\mathbb{S}_g$

$C_g(k, k+\ell)$ = # complex graphs on $\mathbb{S}_g$ with k vertices and $k+\ell$ edges

$U(n-k, m-k-\ell)$ = # graphs without complex components with $n-k$ vertices and $m-k-\ell$ edges

$S_g(n, m)$ = # graphs on $\mathbb{S}_g$ with n vertices and m edges

$$= \sum_{k,\ell} \binom{n}{k} C_g(k, k+\ell)\, U(n-k, m-k-\ell)$$

- When $m = (1+\varepsilon)\frac{n}{2},\ n^{-1/3} \ll \varepsilon \ll 1$, the main contribution comes from

$$k = (1+o(1))\,\varepsilon\, n \quad \textit{and} \quad \ell = (\alpha + o(1))\,\varepsilon\, n^{1/3}.$$

Graphs on $\mathbb{S}_g$

$C_g(k, k+\ell) = \#$ complex graphs on $\mathbb{S}_g$ with k vertices and $k+\ell$ edges

$U(n-k, m-k-\ell) = \#$ graphs without complex components with $n-k$ vertices and $m-k-\ell$ edges

$S_g(n, m) = \#$ graphs on $\mathbb{S}_g$ with n vertices and m edges

$$= \sum_{k,\ell} \binom{n}{k} C_g(k, k+\ell)\, U(n-k, m-k-\ell)$$

- When $m = (1+\varepsilon)\frac{n}{2},\ n^{-1/3} \ll \varepsilon \ll 1$, the main contribution comes from

$$k = (1+o(1))\,\varepsilon\, n \quad \textit{and} \quad \ell = (\alpha + o(1))\,\varepsilon\, n^{1/3}.$$

- When $m = (2+\varepsilon)\frac{n}{2},\ n^{-2/5} \ll \varepsilon \ll 1$, the main contribution comes from

$$k = n - (\beta + o(1))\,\varepsilon^{-3/2} \quad \textit{and} \quad \ell = t + (2\beta + o(1))\,\varepsilon^{-3/2}.$$

Two Critical Periods

Let $S_g(n, m)$ be a graph chosen uniformly at random among all graphs embeddable on $\mathbb{S}_g$ of genus $g \geq 1$ with n vertices and m edges.

Let $\mathcal{C}_1$ denote the largest component in $S_g(n, m)$.

Two Critical Periods

Let $S_g(n,m)$ be a graph chosen uniformly at random among all graphs embeddable on $\mathbb{S}_g$ of genus $g \geq 1$ with n vertices and m edges.

Let $\mathcal{C}_1$ denote the largest component in $S_g(n,m)$.

Theorem [K.– MOSSHAMMER–SPRÜSSEL 15+]

- When $m = (1+\varepsilon)\frac{n}{2}$, $n^{-1/3} \ll \varepsilon \ll 1$,

  $\mathcal{C}_1$ is not embeddable on $\mathbb{S}_{g-1}$ and the unique non-planar component.

  Furthermore, whp

$$|\mathcal{C}_1| = (1+o(1))\,\varepsilon\, n, \gg n^{2/3}$$

Two Critical Periods

Let $S_g(n,m)$ be a graph chosen uniformly at random among all graphs embeddable on $\mathbb{S}_g$ of genus $g \geq 1$ with n vertices and m edges.

Let $\mathcal{C}_1$ denote the largest component in $S_g(n,m)$.

Theorem [K.– Mosshammer–Sprüssel 15+]

- When $m = (1+\varepsilon)\frac{n}{2},\ n^{-1/3} \ll \varepsilon \ll 1$,

  $\mathcal{C}_1$ is not embeddable on $\mathbb{S}_{g-1}$ and the unique non-planar component.

  Furthermore, whp

  $$|\mathcal{C}_1| = (1+o(1))\,\varepsilon\, n, \gg\ n^{2/3}$$

- When $m = (2+\varepsilon)\frac{n}{2},\ n^{-2/5} \ll \varepsilon \ll 1$, whp

  $$n - |\mathcal{C}_1| = \Theta(\varepsilon^{-3/2})\ \ll\ n^{3/5}$$

Overview

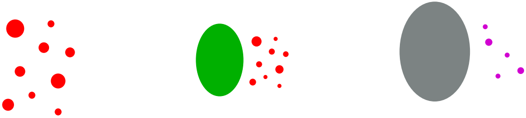
$G(n, m)$ Erdős–Rényi random graph [Erdős–Rényi 60]

- Phase transition and critical phenomenon
- Emergence of giant component when # edges $m = \frac{n}{2} + O(n^{3/2})$

Overview

$G(n,m)$ Erdős–Rényi random graph [Erdős–Rényi 60]

- Phase transition and critical phenomenon
- Emergence of giant component when # edges $m = \frac{n}{2} + O(n^{3/2})$



$S_g(n,m)$ with genus $g \geq 0$ [K.– Łuczak 12; K.– Mosshammer–Sprüssel 15+]

- Two critical periods
- Emergence of giant component when # edges $m = \frac{n}{2} + O(n^{2/3})$
- # vertices outside giant component when # edges $m = n + O(n^{3/5})$

Outlook

Let $S_g(n, m)$ be a graph chosen uniformly at random among all graphs embeddable on $\mathbb{S}_g$ of genus $g = g(n)$ with n vertices and m edges.

Outlook

Let $S_g(n, m)$ be a graph chosen uniformly at random among all graphs embeddable on $\mathbb{S}_g$ of genus $g = g(n)$ with n vertices and m edges.

Conjecture

- There exist $\alpha = \alpha(n), \beta = \beta(n)$ satisfying $\alpha(n) \leq \beta(n)$ such that
  - if $g \ll \alpha$,

    $S_g(n, m)$ exhibits the second critical phase

    analogous to $P(n, m)$, but with different critical exponent
  - if $g \gg \beta$,

    $S_g(n, m)$ does not exhibit the second critical phase

    analogous to $G(n, m)$