Introduction to Random Graphs

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Combinatorics Workshop 2019, Incheon, Korea, 13-15 August 2019

I. Erdős-Rényi Random Graphs

II. Higher-Dimensional Analogues

III. Topological Aspects

Part I

Erdős-Rényi Random Graphs

(I) Threshold phenomena

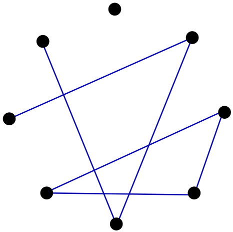
(II) Connectedness

(III) Largest component

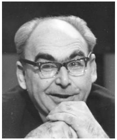
Random graph models

Let $G(n, m)$ denote a uniform random graph:

a graph taken uniformly at random from the set $\mathcal{G}(n, m)$ of all graphs on vertex set $[n] := \{1, \ldots, n\}$ with $m = m(n)$ edges



Paul Erdős (1913 – 1996)

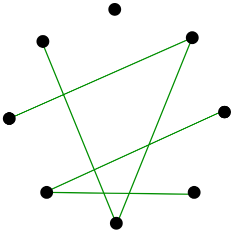


Alfréd Rényi (1921 – 1970)

Random graph models

Let $G(n, p)$ denote a binomial random graph:

a graph on vertex set $[\, n\,]$, in which each pair of vertices is joined by an edge with probability $p = p(n)$, independently



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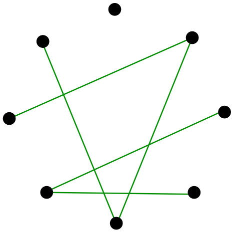
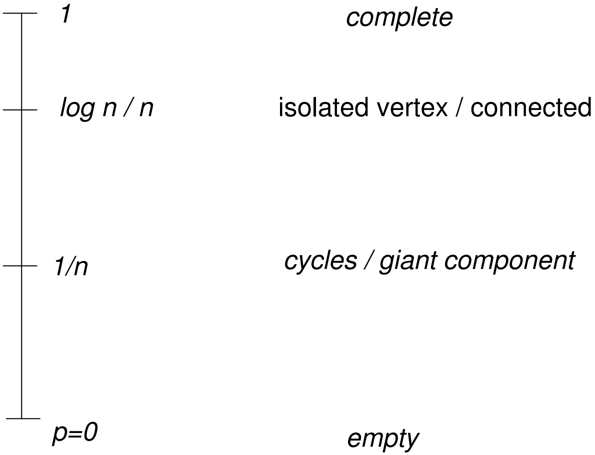


$\Longrightarrow$

$G(n, m)$ and $G(n, p)$ are 'essentially equivalent' when $m \sim \binom{n}{2}\, p$

Threshold phenomena in $G(n, p)$

Let $p = p(n) \in [0, 1]$



Thresholds in $G(n,p)$

Let $\mathcal{A}$ be a monotone increasing property

e.g.

– $G(n,p)$ contains no isolated vertex

– $G(n,p)$ is connected

Threshold

A function $p^* = p^*(n)$ is called a threshold for $\mathcal{A}$ if

$$\mathbb{P}[\; G(n,p) \,\text{satisfies}\, \mathcal{A} \;] \quad \xrightarrow{n\to\infty} \quad \begin{cases} 0 & \textit{if} \quad p \ll p^* \\ 1 & \textit{if} \quad p \gg p^* \end{cases}$$

Sharp thresholds in $G(n, p)$

Let $\mathcal{A}$ be a monotone increasing property

e.g.

– $G(n, p)$ contains no isolated vertex

– $G(n, p)$ is connected

Sharp threshold

A function $p^* = p^*(n)$ is called a sharp threshold for $\mathcal{A}$ if $\forall \varepsilon > 0$,

$$\mathbb{P}[\ G(n,p) \text{ satisfies } \mathcal{A}\ ] \xrightarrow{n\to\infty} \begin{cases} 0 & \textit{if} \quad p \ \leq \ (1-\varepsilon)p^* \\ 1 & \textit{if} \quad p \ \leq \ (1+\varepsilon)p^* \end{cases}$$

Sharp threshold for isolated vertices

A sharp threshold for property that $G(n,p)$ contains no isolated vertex is

$$p^* = \frac{\log n}{n}.$$

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Theorem

Let

$$p = \frac{\log n + c(n)}{n}$$

where $|c(n)| \to \infty$ arbitrarily slowly as $n \to \infty$. Then

$$\mathbb{P}\big[\, G(n, p) \text{ contains no isolated vertex } \big] \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } c(n) \to -\infty \\ 1 & \text{if } c(n) \to \infty \end{cases}$$

First moment method

Markov's inequality

Let X be a non-negative integer-valued random variable. Then for any $t > 0$

$$\mathbb{P}[X \geq t] \quad \leq \quad \frac{\mathbb{E}[X]}{t}$$

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For example, let $X = X(n) \ = \ \#$ isolated vertices in $G(n, p)$.

IF

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$$= \quad \mathbb{P}[X \geq 1] \quad \leq \quad \mathbb{E}[X] \quad \xrightarrow{n \to \infty} \quad 0$$

Second moment method

Chebyshev's inequality

Let X be a random variable with $\mathbb{E}[X] > 0$. Then

$$\mathbb{P}[X = 0] \quad \leq \quad \mathbb{P}\big[\, |X - \mathbb{E}[X]| \geq \mathbb{E}[X] \,\big] \quad \leq \quad \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2}$$

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$$\mathbb{P}\big[G(n,p) \text{ contains no isolated vertex}\big]$$
$$= \mathbb{P}[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2} \xrightarrow{n \to \infty} 0$$

Variation of second moment method

Let $X = X_1 + X_2 + \ldots$ be a sum of indicator random variables with $\mathbb{E}[X] > 0$. Then

$$\mathbb{P}\left[X = 0\right] \quad \leq \quad \frac{\mathrm{Var}\left[X\right]}{\mathbb{E}\left[X\right]^2} \quad \leq \quad \frac{1}{\mathbb{E}\left[X\right]} + \frac{\sum_{i \neq j} \mathrm{Cov}\left[X_i, X_j\right]}{\mathbb{E}\left[X\right]^2},$$

where $\mathrm{Cov}\left[X_i, X_j\right] \; = \; \mathbb{E}\left[X_i X_j\right] - \mathbb{E}\left[X_i\right]\mathbb{E}\left[X_j\right]$.

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where $\mathrm{Cov}[X_i, X_j] = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i]\mathbb{E}[X_j]$.

IF $\quad \mathbb{E}[X] \xrightarrow{n \to \infty} \infty \quad$ and $\quad \dfrac{\sum_{i \neq j} \mathrm{Cov}[X_i, X_j]}{\mathbb{E}[X]^2} \xrightarrow{n \to \infty} 0,$

THEN

$\mathbb{P}[G(n,p)$ contains no isolated vertex$]$

$$= \quad \mathbb{P}[X=0] \quad \leq \quad \frac{1}{\mathbb{E}[X]} + \frac{\sum_{i \neq j} \mathrm{Cov}[X_i, X_j]}{\mathbb{E}[X]^2} \quad \xrightarrow{n \to \infty} \quad 0$$

Sharp threshold for isolated vertices

Theorem

Let
$$p = \frac{\log n + c(n)}{n}$$
where $|c(n)| \to \infty$ arbitrarily slowly as $n \to \infty$. Then

$$\mathbb{P}\big[\, G(n,p) \text{ contains no isolated vertex } \big] \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } c(n) \to -\infty \\ 1 & \text{if } c(n) \to \infty \end{cases}$$

Proof ideas

Note that the function $F : [0,1] \to [0,1]$ defined by

$$F(p) := \mathbb{P}[\ G(n,p) \text{ contains NO isolated vertex }]$$

is monotone increasing in p. To prove the statement, we may assume without loss of generality that $|c(n)| \ll \log n$.

Proof ideas - contd

For each $v \in [n]$, let $X_v = \begin{cases} 1 & \text{if } v \text{ is isolated in } G(n,p) \\ 0 & \text{otherwise.} \end{cases}$

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Set $X = \sum_{v \in [n]} X_v$. Then

$$\mathbb{E}[X] = \sum_{v \in [n]} \mathbb{E}[X_v]$$

Proof ideas - contd

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$$\mathbb{E}[X] = \sum_{v \in [n]} \mathbb{E}[X_v]$$
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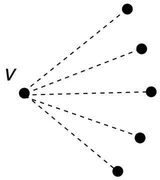

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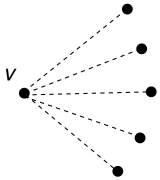

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{v \in [n]} \mathbb{E}[X_v] \\
&= n\,(1-p)^{n-1} \\
&= \exp\big(\log n - pn + p + O(p^2 n)\big),
\end{aligned}
$$

using $1 - x = \exp(-x + O(x^2))$ for $x = o(1)$.

Proof ideas - contd

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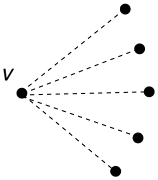

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using $\quad 1 - x = \exp(-x + O(x^2)) \quad$ for $\quad x = o(1)$.

Taking $p = \frac{\log n + c(n)}{n}$ with $|c(n)| \ll \log n$, we have

$$\mathbb{E}[X] = (1 + o(1)) \exp\big(-c(n)\big).$$

Proof ideas - contd

Recall X denotes the number of isolated vertices in $G(n, p)$ and

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Case (1): assume that $c(n) \to \infty$.

Using first moment method, we have

$$\mathbb{P}[X \geq 1] \leq \mathbb{E}[X] = (1 + o(1)) \exp\big(-c(n)\big) \to 0,$$

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Therefore, if $p = \frac{\log n + c(n)}{n}$ with $c(n) \to \infty$,

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Proof ideas - contd

Case (2): assume that $c(n) \to -\infty$.

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For $v \neq w$,

$$\begin{aligned} \mathrm{Cov}[X_v, X_w] &= \mathbb{E}[X_v X_w] - \mathbb{E}[X_v]\mathbb{E}[X_w] \\ &= (1-p)^{2n-3} - (1-p)^{2n-2} \\ &= p(1-p)^{2n-3} \end{aligned}$$

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and therefore

$$\frac{\sum_{v \neq w} \mathrm{Cov}[X_v, X_w]}{\mathbb{E}[X]^2} = \frac{n(n-1)p(1-p)^{2n-3}}{n^2(1-p)^{2n-2}} \to 0$$

Proof ideas - contd

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Using second moment method, we have

$$\mathbb{P}[X = 0] \leq \frac{1}{\mathbb{E}[X]} + \frac{\sum_{v \neq w} \mathrm{Cov}[X_v, X_w]}{\mathbb{E}[X]^2} \to 0.$$

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Therefore, if $p = \frac{\log n + c(n)}{n}$ with $c(n) \to -\infty$,

$$\mathbb{P}\left[G(n,p) \text{ contains no isolated vertex}\right] = \mathbb{P}[X = 0] \to 0.$$

Sharp threshold for isolated vertices

Theorem

Let
$$p = \frac{\log n + c(n)}{n}$$
where $|c(n)| \to \infty$ arbitrarily slowly as $n \to \infty$. Then

$$\mathbb{P}\big[\, G(n,p) \text{ contains no isolated vertex } \big] \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } c(n) \to -\infty \\ 1 & \text{if } c(n) \to \infty \end{cases}$$

Isolated vertices in critical window

Theorem

Let $$p = \frac{\log n + c(n)}{n}, \quad \text{where} \quad c(n) \to c \in \mathbb{R}.$$

Let $X = X(n)$ be # isolated vertices in $G(n,p)$. Then

$$X \xrightarrow{D} \mathrm{Po}(e^{-c}).$$

It means, for every $\ell = 0, 1, 2, \ldots$

$$\lim_{n\to\infty} \mathbb{P}[X = \ell] = e^{-e^{-c}}\, e^{-c\,\ell}/\ell!$$

In particular,

$$\mathbb{P}\big[\,G(n,p) \text{ contains no isolated vertex}\,\big] = \mathbb{P}[X = 0] \to e^{-e^{-c}}$$

Isolated vertices in $G(n,p)$

Theorem

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Minimal obstruction for connectedness

$$\mathbb{P}\big[\ G(n,p)\ \text{ is connected}\ \big]$$
$$= \mathbb{P}\big[G(n,p) \text{ contains no isolated vertex}\ \big] + o(1)$$

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if

$$p = \frac{\log n + c(n)}{n}.$$

$\Longrightarrow$ higher-dimensional analogue

With high probability ...

whp = with probability tending to one as $n \to \infty$

Given a property $\mathcal{A}$, we say

whp $G(n,p)$ satisfies $\mathcal{A}$ if $\mathbb{P}[G(n,p) \text{ satisfies } \mathcal{A}] \to 1$

Emergence of giant component

Let $d = (n-1)\,p$ be a constant.

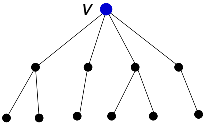
$|L_1| = \#$ vertices in largest component in $G(n,p)$.

Theorem [Erdős–Rényi 60]

- If $d < 1$, whp $\quad |L_1| = O(\log n)$
- If $d > 1$, whp $\quad |L_1| = \Theta(n)$

BFS tree and GW tree

(1) Breadth-First Search tree

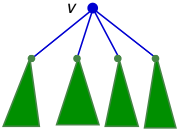


Construct spanning tree T_v
of component C_v that contains vertex v

(2) $\#$ *neighbours of v* $\sim Bi(n-1,p) \approx Po(d)$

(3) Coupling BFS tree with Galton-Watson tree

with offspring distribution $Po(d)$



ℓ subtrees

$$\rho := \mathbb{P}\,(\text{GW tree is infinite})$$

$$1-\rho = \sum_\ell \mathbb{P}\,(Po(d)=\ell)\,(1-\rho)^\ell$$

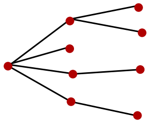
$$= \exp(-d\,\rho)$$

Galton-Watson tree

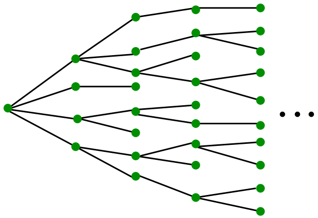
Theorem

Let ρ be a solution of $1 - \rho = \exp(-d\,\rho)$.

- If $d < 1$, then $\rho \ = \ 0$.
- If $d > 1$, then $\rho \ \in \ (0, 1)$.



'small' component in $G(n, p)$



'giant' component in $G(n, p)$

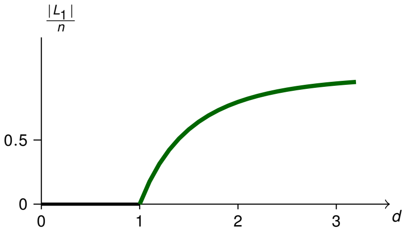
Largest component

Assume $d = (n-1)\,p$ is a constant and $1 - \rho = \exp(-d\,\rho)$

$|L_1| = \#$ vertices in largest component in $G(n,p)$

Theorem [Erdős–Rényi 60; Karp 91]

- If $d < 1$, whp $|L_1| = O(\log n)$
- If $d > 1$, whp $|L_1| = (1 + o(1))\,\rho\,n$



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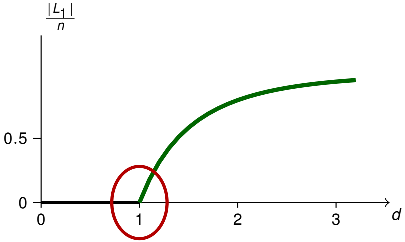


Largest component – contd

Assume $d = (n-1)\,p \to 1$ and $1-\rho = \exp(-d\,\rho)$

$|L_1| = \#$ vertices in largest component in $G(n,p)$

Let $\varepsilon = \varepsilon(n)$ satisfy $\varepsilon > 0,\ \varepsilon \to 0,\ \varepsilon^3 n \to \infty$

Theorem [Bollobás 84; Łuczak 90; Bollobás–Riordan 12]

- If $d = 1 - \varepsilon$, whp $\quad |L_1| = (1+o(1))\,2\varepsilon^{-2}\log(\varepsilon^3 n)$
- If $d = 1 + \varepsilon$, whp $\quad |L_1| = (1+o(1))\,2\,\varepsilon\,n$

$\Longrightarrow$ higher-dimensional analogue

Part II

Higher-Dimensional Analogues

(I) Random hypergraphs

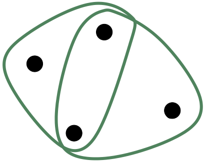
(II) Random simplicial complexes

Random hypergraphs

Let $H_k(n,p)$ denote a random binomial k-uniform hypergraph on vertex set $[n] := \{1, 2, \ldots, n\}$,

in which each k-(element sub)set of vertex set $[n]$ is

a hyperedge with probability p, independently



Note $H_2(n,p) = G(n,p)$

In the section (I) we assume $k \geq 2,\ 1 \leq j \leq k-1$.

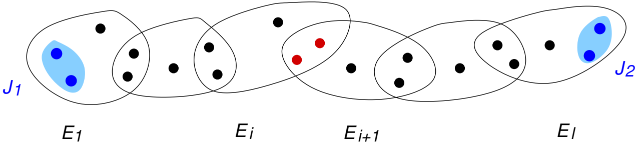
High-order components

- Given two j-(element sub)sets J_1, J_2, we say

  J_1 is reachable from J_2

  if $\exists$ sequence $E_1, \ldots, E_\ell$ of hyperedges such that

  $J_1 \subseteq E_1$, $J_2 \subseteq E_\ell$, and $|E_i \cap E_{i+1}| \geq j, \quad i \in [\ell - 1]$.



- Reachability is an equivalence relation on j-sets, and equivalence classes are called j-(tuple)component.
- If H consists of a single j-component, it is j-connected.

Sharp threshold for j-connectedness

Theorem [Cooley–K.–Koch 16]

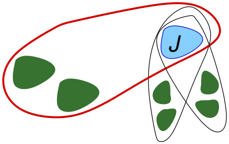
Let
$$p = \frac{j \log n + c(n)}{\binom{n-j}{k-j}}.$$

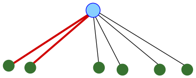
Then
$$\mathbb{P}\left[H_k(n,p) \text{ is } j\text{-connected} \right] \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } c(n) \to -\infty \\ e^{-\frac{e^{-c}}{j!}} & \text{if } c(n) \to c \in \mathbb{R} \\ 1 & \text{if } c(n) \to \infty \end{cases}$$

an isolated j-set is a minimal obstruction for j-connectedness

Heuristics for threshold for giant component

Component exploration & Breadth-First Search tree





- Begin with a j-set J
- Discover all hyperedges that contain that j-set J

  $\exists$ $\binom{n-j}{k-j}$ such hyperedges containing J, each with prob. p
- For each hyperedge E containing J,

  discover $\left(\binom{k}{j}-1\right)$ new j-sets in E

$$\mathbb{E}\left[\,\#\ j\text{-sets discovered from } J\,\right] \;=\; \left(\binom{k}{j}-1\right)\binom{n-j}{k-j}\,p \;=:\; d$$

Largest j-component

Assume $d \;=\; \left(\binom{k}{j}-1\right)\binom{n-j}{k-j}\,p \;\to\; 1.$

Let $|L_j| \;=\; \#$ j-sets in largest j-component in $H_k(n,p)$

Let $\varepsilon = \varepsilon(n)$ satisfy $\;\varepsilon > 0,\; \varepsilon \to 0,\; \varepsilon^3\, n^j \to \infty,\; \ldots$

Theorem [Cooley–K.–Koch 18; Cooley–Fang–Del Giudice–K. 19]

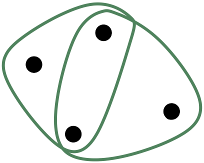
- If $d \;=\; 1-\varepsilon,$ whp $\quad |L_j| \;=\; (1+o(1))\,\frac{2\left(\binom{k}{j}-1\right)}{\varepsilon^2}\log\left(\varepsilon^3\binom{n}{j}\right)$
- If $d \;=\; 1+\varepsilon,$ whp $\quad |L_j| \;=\; (1+o(1))\,\frac{2\varepsilon}{\binom{k}{j}-1}\binom{n}{j}$

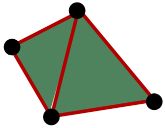
Random simplicial complexes

Random k-dimensional simplicial complex $\mathcal{G}_p$ arising from $H_{k+1}(n,p)$ by taking its downward-closure, i.e.

- 0-simplices are singletons of $[\,n\,]$
- k-simplices are hyperedges of $H_{k+1}(n,p)$
- $\forall i \in [\,k-1\,]$, i-simplices are $(i+1)$-(element sub)sets that are contained in hyperedges of $H_{k+1}(n,p)$

e.g. $k=2$





Cohomology groups

Let X be a k-dimensional simplicial complex. For $0 \leq j \leq k-1$

- $C^j(X)$ denotes the set of $\{0,1\}$-functions on j-simplices
- coboundary operator $\delta^j \colon C^j(X) \to C^{j+1}(X)$, $h \to \delta^j h$, is defined such that for each $(j+1)$-simplex σ

$$\left[\delta^j h\right](\sigma) \quad := \sum_{j\text{-simplex } \tau \subset \sigma} h(\tau) \pmod 2$$

e.g. $\left[\delta^0 f\right](uv) := f(u) + f(v) \pmod 2$

$\left[\delta^1 g\right](uvw) := g(uv) + g(vw) + g(wu) \pmod 2$

- j-th cohomology group of X with coefficients in $\mathbb{F}_2$ is the quotient group

$$H^j(X;\ \mathbb{F}_2) \quad := \quad \frac{\text{Ker } (\delta^j)}{\text{Im } (\delta^{j-1})}$$

Cohomology groups – contd

$$H^j(X;\ \mathbb{F}_2)\ := \frac{\mathrm{Ker}\ (\delta^j)}{\mathrm{Im}\ (\delta^{j-1})}\ \neq 0 \iff \ \exists\ h \in \mathrm{Ker}\ (\delta^j) \setminus \mathrm{Im}\ (\delta^{j-1})$$

e.g. $\{0, 1\}$-function *h* on *j*-simplices that assigns

- **even** number of 1's on *j*-simplices

  that are contained in each $(j+1)$-simplex

- **odd** number of 1's on a set *J* of *j*-simplices s.t.

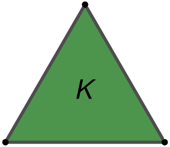
  every $(j-1)$-simplex is contained in even $\#$ *j*-simplices in *J*

$\Longrightarrow$

h is an obstacle for vanishing of cohomology group

Minimal obstruction $M = (K, C, J)$ for $k = 2$, $j = 1$

K = 2-simplex (i.e. hyperedge) in $\mathcal{G}_p$

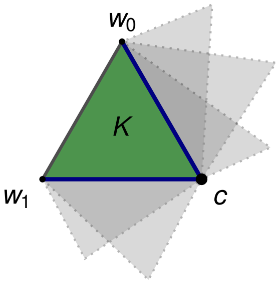


Minimal obstruction $M = (K, C, J)$ for $k = 2$, $j = 1$

K = 2-simplex (i.e. hyperedge) in $\mathcal{G}_p$

C = 0-simplex in K such that for each $w \in K \setminus C$,

1-simplex $C \cup \{w\}$ is contained in no other 2-simplex



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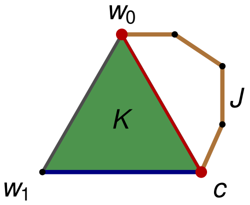
J = set of 1-simplices (i.e. a cycle) such that

- every 0-simplex is contained in even $\#$ 1-simplices in J
- it contains exactly one $C \cup \{w_0\}$, $w_0 \in K \setminus C$



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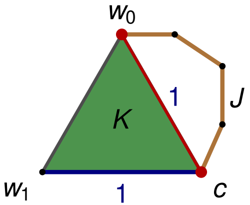


Minimal obstruction $M_j = (K, C, J)$ for $k \geq 2$, $1 \leq j \leq k-1$

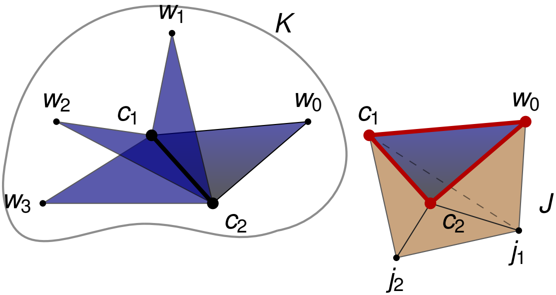
K = k-simplex (i.e. hyperedge) in $\mathcal{G}_p$

C = $(j-1)$-simplex in K such that for each $w \in K \setminus C$,

j-simplex $C \cup \{w\}$ is contained in no other k-simplex

J = set of j-simplices (i.e. a j-cycle) such that

- every $(j-1)$-simplex is contained in even $\#$ j-simplices in J
- it contains exactly one $C \cup \{w_0\}$, $w_0 \in K \setminus C$



Vanishing of cohomology groups in $\mathcal{G}_p$

Theorem [Cooley–del Giudice–K.–Sprüssel 19]

Let $k \geq 2$, $1 \leq j \leq k-1$, and

$$p = \frac{(j+1)\log n + \log\log n + c(n)}{(k-j+1)\binom{n}{k-j}}.$$

Then

$$\mathbb{P}\big[\, H^i(\mathcal{G}_p;\ \mathbb{F}_2) = 0,\ \ \forall\, i \in [\,j\,]\,\big]$$

$$\xrightarrow{n\to\infty} \quad \begin{cases} 0 & \text{if } c(n) \to -\infty \\ e^{-\frac{(j+1)e^{-c}}{(k-j+1)^2\, j!}} & \text{if } c(n) \to c \in \mathbb{R} \\ 1 & \text{if } c(n) \to \infty \end{cases}$$

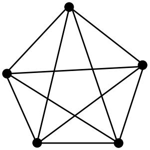
Part III

Topological Aspects

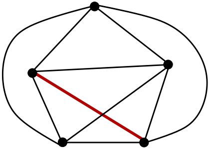
Guiding questions/themes

(1) What is a typical genus of Erdős-Rényi random graph?

* genus of a graph G is minimum number of handles that must be attached to a sphere in order to embed G without any crossing edges



K_5



genus of K_5 = 1

Guiding questions/themes

(2) How does a topological constraint influence component structure of a random graph?

- planarity
- upper bound on genus

Throughout the talk

- Let $\mathcal{G}(n, m)$ denote the set of all graphs on vertex set $[\,n\,] := \{1, \ldots, n\}$ with $m = m(n)$ edges
- Let $G(n, m)$ denote a graph taken uniformly at random from $\mathcal{G}(n, m)$
- Let $|L_1|$ denote $\#$ vertices in largest component

Planarity of $G(n,m)$

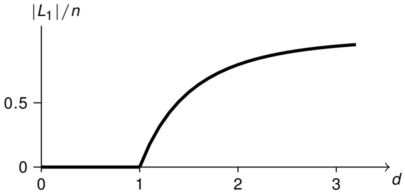
Theorem

- If $d = \frac{2m}{n} < 1$, whp $|L_1| = O(\log n)$,

  and $G(n,m)$ is planar

- If $d = \frac{2m}{n} > 1$, whp $|L_1| = (1 + o(1))\,\rho\, n$,

  where $1 - \rho = \exp(-d\,\rho)$, and $G(n,m)$ is not planar



Random planar graphs

- Let $\mathcal{P}(n, m)$ denote the set of all graphs on vertex set $[\,n\,]$ with $m = m(n)$ edges that are embeddable on the sphere without crossing edges

- Let $P(n, m)$ denote a graph taken uniformly at random from $\mathcal{P}(n, m)$

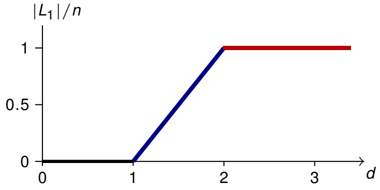
- For $1 \;\le\; m \;<\; \frac{n}{2}$,

$$\mathbb{P}[\; G(n, m) \text{ is planar }] \quad = \quad \frac{|\mathcal{P}(n, m)|}{|\mathcal{G}(n, m)|} \quad \xrightarrow{n\to\infty} \quad 1$$

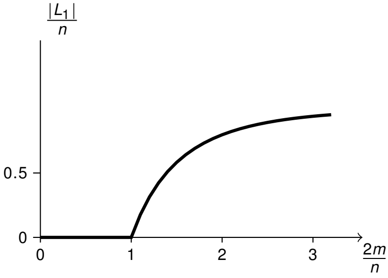
Random planar graph $P(n, m)$

Theorem [K.–Łuczak 2012; Giménez–Noy 2009]

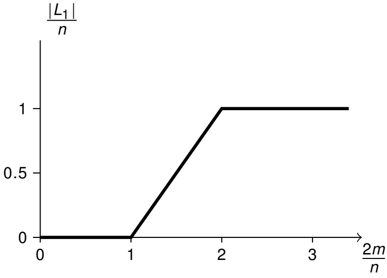
- If $\frac{2m}{n} < 1$, then whp $|L_1| = O(\log n)$.
- If $\frac{2m}{n} \to d \in (1, 2)$, then whp
$$|L_1| = (1 + o(1))\,(d-1)n.$$
- If $\frac{2m}{n} \to d \in [2, 6]$, then whp
$$|L_1| = (1 + o(1))\,n.$$



Phase transitions and critical phases

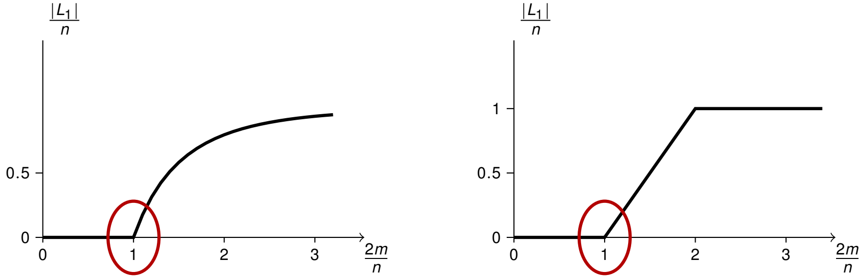


Uniform random graph $G(n, m)$



Random planar graph $P(n, m)$

Phase transitions and critical phases



Uniform random graph $G(n, m)$

Random planar graph $P(n, m)$

Weakly supercritical random graphs

Let $m = \frac{n}{2} + s$ for $s > 0$, $n^{2/3} \ll s \ll n$.

Uniform random graph $G(n, m)$ [Bollobás 84; Łuczak 90]

whp $$|L_1| = (4 + o(1))\, s$$

Random planar graph $P(n, m)$ [K.–Łuczak 2012]

whp $$|L_1| = (2 + o(1))\, s$$

Random graphs on a surface

- Let $\mathcal{S}_g(n, m)$ denote the set of all graphs on vertex set $[\,n\,]$ with m edges and with genus $\leq g$

  Note $$\mathcal{P}(n, m) \;=\; \mathcal{S}_0(n, m)$$

- Let $S_g(n, m)$ denote a graph taken uniformly at random from $\mathcal{S}_g(n, m)$

Random graphs on a surface

From which $g = g(n)$, are $S_g(n,m)$ and $G(n,m)$

not distinguishable under viewpoint of whp-properties?

IF whp genus of $G(n,m)$ is T,

THEN $\forall\, g \geq T$, we have that

$$(1) \quad \frac{|\mathcal{S}_g(n,m)|}{|\mathcal{G}(n,m)|} \geq \frac{|\mathcal{S}_T(n,m)|}{|\mathcal{G}(n,m)|} \xrightarrow{n\to\infty} 1$$

(2) for every property $\mathcal{A}$,

whp $G(n,m)$ satisfies $\mathcal{A}$ iff whp $S_g(n,m)$ satisfies $\mathcal{A}$

Genus of weakly supercritical $G(n, m)$

Let $m = \frac{n}{2} + s$ for $s > 0$, $n^{2/3} \ll s \ll n$.

Let g denote the genus of $G(n, m)$.

Theorem [Dowden–K.–Krivelevich 2019]

whp
$$g = (1 + o(1)) \frac{8s^3}{3n^2}.$$

Largest component in weakly supercritical $S_g(n, m)$

Let $m = \frac{n}{2} + s$ for $s > 0$, $n^{2/3} \ll s \ll n$ and let $T = \frac{8s^3}{3n^2}$.

Let $|L_1| = \#$ vertices in largest component in $S_g(n, m)$.

Theorem [DOWDEN–K.–MOSSHAMMER–SPRÜSSEL 2019+]

whp

- $|L_1| = (4 + o(1))\, s$ *if* $g \geq (1 + o(1))T$

- $|L_1| = (2 + o(1))\, s$ *if* $g = o(T)$

Largest component in weakly supercritical $S_g(n,m)$

Let $m = \frac{n}{2} + s$ for $s > 0$, $n^{2/3} \ll s \ll n$ and let $T = \frac{8s^3}{3n^2}$.

Let $|L_1| = \#$ vertices in largest component in $S_g(n,m)$.

Theorem [Dowden–K.–Mosshammer–Sprüssel 2019+]

whp

- $|L_1| = (4 + o(1))\, s$ *if* $g \geq (1 + o(1))T$
- $|L_1| = (f(c) + o(1))\, s$ *if* $g = (c + o(1))T$ *for* $c \in (0,1)$
- $|L_1| = (2 + o(1))\, s$ *if* $g = o(T)$

where $f(c) \to 2$ as $c \to 0$ and $f(c) \to 4$ as $c \to 1$.

Genus of supercritical $G(n, m)$

Let $\frac{2m}{n} \to d > 1$ and g denote the genus of $G(n, m)$.

Theorem [Dowden–K.–Krivelevich 2019]

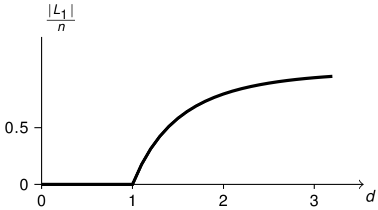
whp
$$g = \Theta(n)$$

Largest component L_1 in supercritical $S_g(n, m)$

Assume $\frac{2m}{n} \to d > 1$ and $g \gg n$

Theorem [DOWDEN–K.–MOSSHAMMER–SPRÜSSEL 2019+]

whp
$$|L_1| = (1 + o(1))\, \rho\, n,$$
where $1 - \rho = \exp(-d\, \rho)$.



Largest component L_1 in supercritical $S_g(n, m)$

Assume $\frac{2m}{n} \to d > 1$ and $g \ll n$.

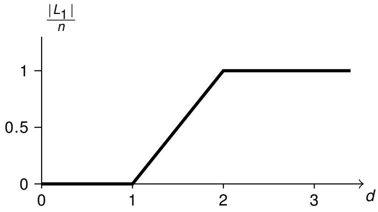
Theorem [Dowden–K.–Mosshammer–Sprüssel 2019+]

- If $d \in (1, 2)$, then whp

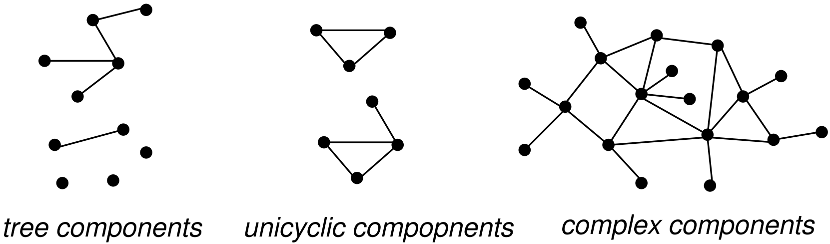
$$|L_1| = (1 + o(1))\,(d - 1)n.$$

- If $d \in [2, 6]$, then whp

$$|L_1| = (1 + o(1))\,n.$$



Component structure of $S_g(n, m)$



Enumeration of $|\mathcal{S}_g(n,m)|$

$$|\mathcal{S}_g(n,m)| \;=\; \#\text{ graphs on } [n] \text{ with } m \text{ edges and genus} \le g$$

$$= \;\sum\nolimits_{k,\ell} \binom{n}{k} C_g(k,k+\ell)\, U(n-k, m-k-\ell)$$

where

$$C_g(k,k+\ell) \;=\; \#\text{ complex part on } [k] \text{ with } k+\ell \text{ edges}$$

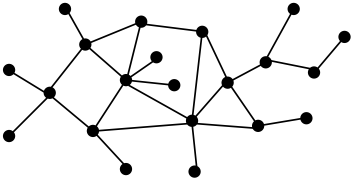
$U(n-k, m-k-\ell) \;=\; \#$ graphs consisting of trees

or unicyclic components

on $[n-k]$ with $m-k-\ell$ edges

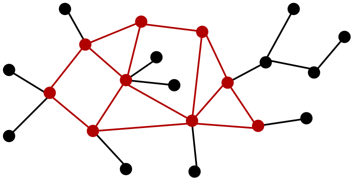
Core-Kernel approach

- Complex part G



Core-Kernel approach

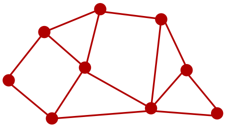
- Complex part G



2-Core $=$ max. subgraph of G with min. degree ≥ 2

Core-Kernel approach

- Complex part G



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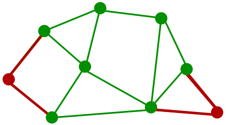
Core-Kernel approach

- Complex part G

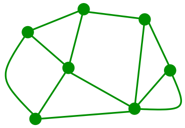


2-Core $=$ max. subgraph of G with min. degree ≥ 2

Kernel $=$ replace each path in 2-core by an edge

Core-Kernel approach

- Complex part G



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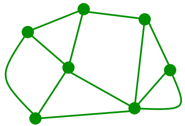
Core-Kernel approach

- Complex part G



$$\text{2-Core} \;=\; \text{max. subgraph of } G \text{ with min. degree} \geq 2$$

$$\text{Kernel} \;=\; \text{replace each path in 2-core by an edge}$$

- g is genus of G iff g is genus of kernel of G

Combinatorial Laplace's method

In order to to analyse a sum of the form

$$S(n) \;=\; \sum_{i \in I_n} Q(i)\, R(n-i)$$

Combinatorial Laplace's method

In order to to analyse a sum of the form

$$S(n) \;=\; \sum_{i \in I_n} Q(i)\,R(n-i) \;=\; \sum_{i \in I_n} \exp\Big(\log(Q(i)\,R(n-i))\Big),$$

let $A_n(i) = \log(Q(i)\,R(n-i))$

Combinatorial Laplace's method

In order to to analyse a sum of the form

$$S(n) \;=\; \sum_{i\in I_n} Q(i)\,R(n-i) \;=\; \sum_{i\in I_n} \exp\Big(\log(Q(i)\,R(n-i))\Big),$$

let $A_n(i) = \log(Q(i)\,R(n-i))$ and assume $A_n'(r) = 0,\; A_n''(r) < 0$:

$$\begin{aligned}
S(n) \;&=\; \sum_{i\in I_n} \exp\left(A_n(i)\right) \;=\; \sum_{i\in I_n} \exp\left(A_n(r) + \frac{A_n''(r)}{2}(i-r)^2 + \cdots\right) \\
&\sim\; \exp\left(A_n(r)\right) \sum_{i=r+O\left(\sqrt{1/|A_n''(r)|}\right)} \exp\left(-\frac{|A_n''(r)|}{2}(i-r)^2\right) \\
&\sim\; \exp\left(A_n(r)\right)\;\sqrt{2\pi/|A_n''(r)|}
\end{aligned}$$

Combinatorial Laplace's method

In order to to analyse a sum of the form

$$S(n) \;=\; \sum_{i \in I_n} Q(i)\, R(n-i) \;=\; \sum_{i \in I_n} \exp\Big(\log(Q(i)\, R(n-i))\Big),$$

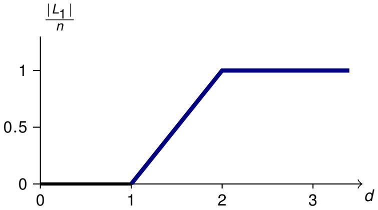
let $A_n(i) = \log(Q(i)\, R(n-i))$ and assume $A_n'(r) = 0,\; A_n''(r) < 0$:

$$\begin{aligned}
S(n) \;&=\; \sum_{i \in I_n} \exp\left(A_n(i)\right) \;=\; \sum_{i \in I_n} \exp\left(A_n(r) + \frac{A_n''(r)}{2}(i-r)^2 + \cdots\right) \\
&\sim\; \exp\left(A_n(r)\right) \sum_{i = r + O\left(\sqrt{1/|A_n''(r)|}\right)} \exp\left(-\frac{|A_n''(r)|}{2}(i-r)^2\right) \\
&\sim\; \exp\left(A_n(r)\right)\; \sqrt{2\pi/|A_n''(r)|}
\end{aligned}$$

This is an ideal scenario, but ...

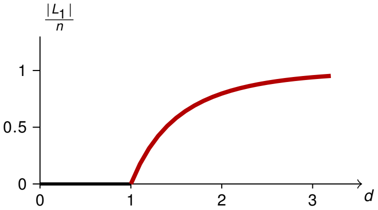
Summary & open problem

Largest component L_1 in $S_g(n,m)$ with $d = \frac{2m}{n} > 1$.



$S_g(n,m)$ for $g \ll n$

analogous to $P(n,m)$



$S_g(n,m)$ for $g \gg n$

analogous to $G(n,m)$

$\Longrightarrow$ behaviour of $|L_1|$ when $g = \Theta(n)$?