Escaping unimodularity and irreducibility for Pisot numeration

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Overview

1. Fractals and Numeration.
2. Main differences when escaping unimodularity and irreducibility.
3. New developments and perspectives.

1. Fractals and Numeration

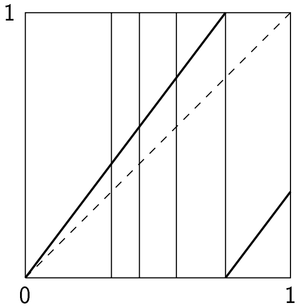

Figure : T_β for $\beta^3 = \beta + 1$.

Let $\beta > 1$ be a Pisot number. Define

$$\begin{aligned} T_\beta : [0,1) &\to [0,1) \\ x &\mapsto \beta x - \lfloor \beta x \rfloor \end{aligned}$$

Every real $x \in [0,1]$ has a β-*expansion*:

$$(x)_\beta = .d_1 d_2 d_3 \cdots$$

with $d_i \in \mathcal{A} = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$.

$([0,1), T_\beta)$ is conjugate to a either sofic or of finite type *subshift*, the admissibility depending on $(1)_\beta$.

Key words: unit, non-unit, irreducible, reducible...

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Example: β smallest Pisot number, $(1)_\beta = .10001$

$$\sigma_\beta : 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$$

$$M_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad f(X)g(X) = (X^3 - X - 1)(X^2 - X + 1)$$

Consider the number field $K = \mathbb{Q}(\beta)$ and the finite set of places $S = S_\infty \cup \{\mathfrak{p} : \mathfrak{p} \mid (\beta)\}$. The *representation space* is

$$K_\beta := K_\infty \times \prod_{\mathfrak{p}|(\beta)} K_\mathfrak{p} = \prod_{\mathfrak{p}\in S} K_\mathfrak{p}$$

where

- $K_\infty = K \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$.
- $K_\mathfrak{p}$ finite extension of $\mathbb{Q}_p$, for $\mathfrak{p} \mid (p)$.

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Cut out the first (expanding) place: K'_β. Here $\times\beta$ is a contraction!

Embed K into K_β, K'_β diagonally by δ, δ'.

The *x-tiles*

For $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$,

$$\mathcal{R}(x) = \overline{\bigcup_{k\geq 0} \delta'(\beta^k T_\beta^{-k}(x))} \;\in\; K'_\beta$$

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Cut and project scheme:

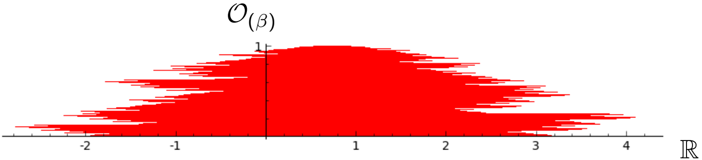
$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\pi_{\mathfrak{p}_1}} & K_\beta & \xrightarrow{\pi} & K'_\beta \\
\cup & & \cup & & \cup \\
\mathbb{Z}[\beta^{-1}] & \overset{1-1}{\longleftrightarrow} & \delta(\mathbb{Z}[\beta^{-1}]) & \overset{1-1}{\longleftrightarrow} & \delta'(\mathbb{Z}[\beta^{-1}])
\end{array}$$

- $\delta(\mathbb{Z}[\beta^{-1}])$ is a lattice in K_β.
- $\delta'(\mathbb{Z}[\beta^{-1}] \cap [0,1))$ is a Delone set in K'_β.

Rauzy fractals

- are compact with non-zero Haar measure.
- are the closure of their interior.
- have fractal boundary with zero Haar measure.
- are self-similar (IFS).
- provide a multiple tiling of K'_β.
- under some conditions provide a tiling.

Non-unit example: $\beta^2 = 2\beta + 2$, $K'_\beta = \mathbb{R} \times K_{(\beta)} \cong \mathbb{R} \times \mathbb{Q}_2^2$.



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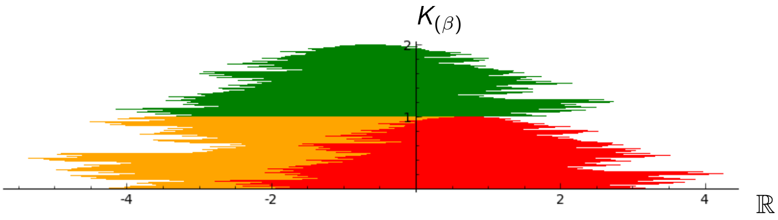

Figure : $\beta^{-1}\mathcal{R}(0)$ for $\beta^2 = 2\beta + 2$.

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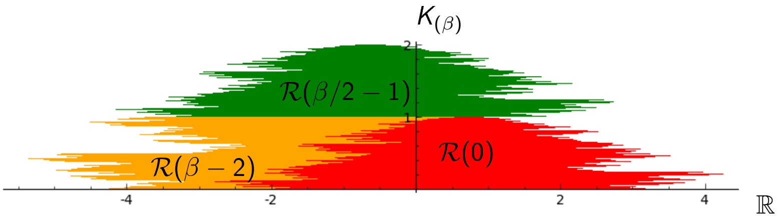

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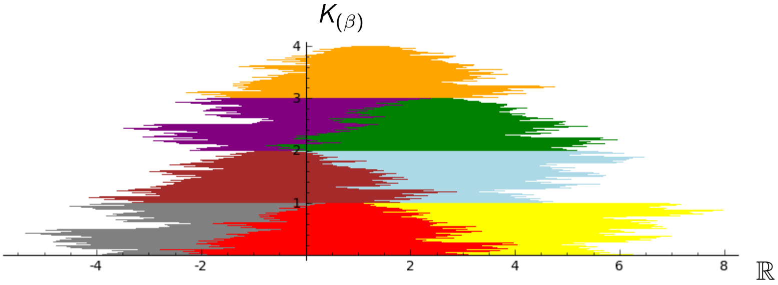
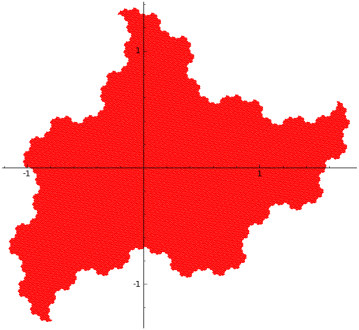


Figure : $\beta^{-2}\mathcal{R}(0)$ for $\beta^2 = 2\beta + 2$.

Reducible example: $\beta^3 = \beta + 1$, $K'_\beta = \mathbb{C}$.



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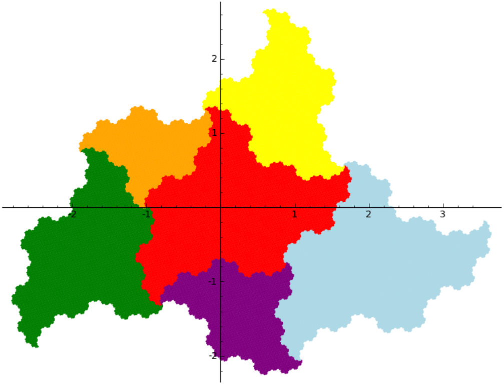

Figure : $\beta^{-5}\mathcal{R}(0)$ for $\beta^3 = \beta + 1$.

How to construct a natural extension for the circle-doubling map $(\mathbb{T}, T_2)$?

$$\mathcal{X} := \varprojlim(\mathbb{T}, T_2) = \{(x_i)_{i\geq 0} \in \mathbb{T}^{\mathbb{N}} : x_i = T_2 x_{i+1},\ \forall i\}$$
$$\mathcal{T}_2(x_0, x_1, x_2, \ldots) = (T_2 x_0, x_0, x_1, \ldots)$$

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We have

$$\varprojlim(\mathbb{T}, T_2) \cong (\mathbb{R} \times \mathbb{Q}_2)/\delta(\mathbb{Z}[\tfrac{1}{2}]) \cong \widehat{\mathbb{Z}[\tfrac{1}{2}]}$$

which is called the *dyadic solenoid*.

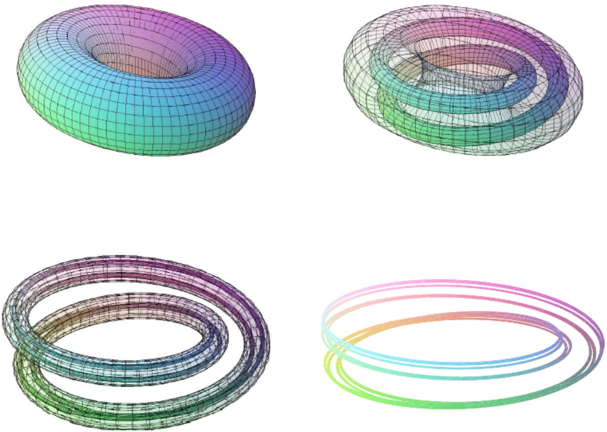

Figure : Visualizing the dyadic solenoid as a nested intersection of solid tori $\bigcap_{k\geq 0} f^k(S^1 \times D^2)$, with $f(t,z) = (T_2(t), z/4 + e^{2\pi i t}/2)$.

Let $V = \{v_1, \ldots, v_n\}$ with the $v_i \in \{T_\beta^k(1) : k \geq 0\} \cup \{0\}$ ordered increasingly. Define

$$\mathcal{X} = \bigcup_{i=1}^{m-1} [v_i, v_{i+1}) \times (\delta'(v_i) - \mathcal{R}(v_i)),$$

$$\mathcal{T}_\beta : \mathcal{X} \to \mathcal{X}, \quad (x, \mathbf{y}) \mapsto \big(T_\beta(x), \beta \cdot \mathbf{y} - \delta'(\lfloor \beta x \rfloor)\big)$$

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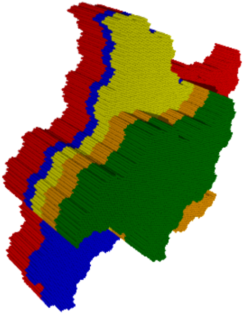
Theorem

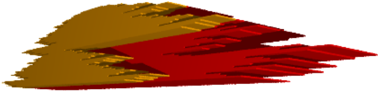
$(\mathcal{X}, \mathscr{B}, \mu, \mathcal{T}_\beta)$ is a natural extension of $([0,1), B, \mu \circ \pi^{-1}, T_\beta)$.

- $\overline{\mathcal{X}} = \overline{\bigcup_{x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)} \{x\} \times \big(\delta'(x) - \mathcal{R}(x)\big)}$.
- $K_\beta = \overline{\mathcal{X}} + \delta(\mathbb{Z}[\beta^{-1}])$.

Smallest Pisot number natural extension in $\mathbb{R} \times \mathbb{C}$:



Natural extension associated to $\beta^2 = 2\beta + 2$ in $\mathbb{R}^2 \times \mathbb{Q}_2^2$:



What we want is that these natural extensions are conjugate to toral/solenoidal automorphisms!

2. Main Differences

Framework: β **non-unit**.

Integral β-tiles

For $x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)$,

$$\mathcal{S}(x) = \{(z_\mathfrak{p})_{\mathfrak{p} \in S \setminus \{\mathfrak{p}_1\}} \in \mathcal{R}(x) : z_\mathfrak{p} = 0 \ \text{ for each } \ \mathfrak{p} \mid (\beta)\}$$

Properties:

1. $\mathcal{S}(x)$ form "slices" of $\mathcal{R}(0)$ and of $\mathcal{X}$.

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3. For $x \in \mathbb{Z}[\beta] \cap [0,1)$,

$$\mathcal{S}(x) = \operatorname*{Lim}_{k \to \infty} \delta'_{\infty}(\beta^k (T_{\beta}^{-k}(x) \cap \mathbb{Z}[\beta])) \in K'_{\infty}$$

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4. $\mathcal{S}(x) - \delta'_\infty(x)$ is close to $\mathcal{S}(y) - \delta'_\infty(y)$ if $|x-y|_\mathfrak{p}$ is small $\forall\ \mathfrak{p} \mid (\beta)$.

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5. $\mathcal{S}(x)$ are SRS tiles (Berthé, Siegel, Steiner et al. 2011).
6. $\{\mathcal{S}(x) : x \in \mathbb{Z}[\beta] \cap [0,1)\}$ forms a weak m-tiling of K'_∞.

Dynamically very important: if we have a periodic tiling we get the conjugation with a toral/solenoidal translation $\rightarrow$ *pure discrete spectrum*.

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Let $\widehat{V} = \{T_\beta^k(1) : k \geq 0\} \setminus \{0\}$.

- Irreducible case: all directions of the domain exchange can be identified, by choosing a natural anti-diagonal lattice $\delta'(L)$, where

$$L = \langle \widehat{V} - \widehat{V} \rangle_{\mathbb{Z}}$$

- **Reducible case**: $\#\widehat{V} > d = \deg(\beta)$ and $\delta'(L) \subset K'_\beta$ may not be a lattice.

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This is the case for the family $\beta^3 - t\beta^2 - (t+1)\beta - 1$, $t \in \mathbb{N}$: $L = \mathbb{Z}[\beta]$ dense in K'_β.

But not for $\beta^3 = t\beta^2 - \beta + 1$, $t \geq 2$: L has rank 2.

Let $Z' = K'_\infty \times \prod_{\mathfrak{p}|(\beta)} \overline{\mathbb{Z}[\beta]}$ be the *stripe space*.

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Proposition

If (QM) holds:

1. $\mathcal{R}(0) + \delta'(L)$ is a multiple tiling of Z'.
2. $\mathcal{R}(0) + \delta'(L)$ is a tiling of Z' iff $\{\mathcal{R}(x) : x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)\}$ is a tiling of K'_β.

Figure : Periodic tiling for $\beta^2 = 2\beta + 2$.

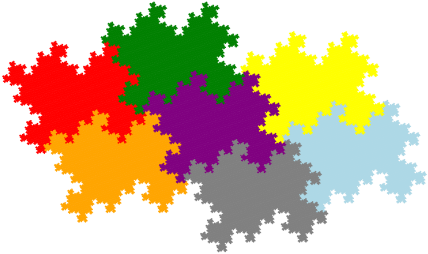

Figure : Periodic tiling for $\beta^3 = 2\beta^2 - \beta + 1$.

Property (W):

$$\forall\, y \in P,\ \exists\, z \in \mathbb{Z}[\beta^{-1}] \cap [0,\varepsilon),\ k \geq 0 :\ T_\beta^k(y+z) = T_\beta^k(z) = 0$$

Theorem (M., Steiner 201?)

The following are equivalent:

- (W) holds.
- $\overline{\mathcal{X}} + \delta(\mathbb{Z}[\beta^{-1}])$ is a tiling of K_β.
- $\{\mathcal{R}(x) :\ x \in \mathbb{Z}[\beta^{-1}] \cap [0,1)\}$ is an aperiodic tiling of K'_β.
- $\{\mathcal{S}(x) : x \in \mathbb{Z}[\beta] \cap [0,1)\}$ is a weak tiling of K'_∞.

If (QM) holds, the following is also equivalent to the ones above:

- $\mathcal{R}(0) + \delta'(L)$ is a periodic tiling of Z'.

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If (QM) holds, the following is also equivalent to the ones above:

- $\mathcal{R}(0) + \delta'(L)$ is a periodic tiling of Z'.

Remark: Integral β-tiles provide an easy proof that for quadratic Pisot numbers the statements above hold!

Consider $\beta^3 = \beta + 1$.

(Ei and Ito 2005) + (Ei, Ito and Rao 2006)

- There is no periodic translation set for the smallest Pisot β.
- They construct an "ad hoc" lattice and fundamental domain in order to have a periodic tiling.

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- There is no periodic translation set for the smallest Pisot β.
- They construct an "ad hoc" lattice and fundamental domain in order to have a periodic tiling.

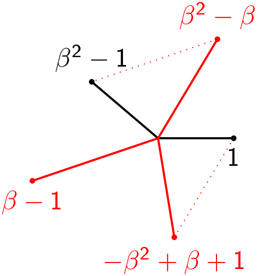
We saw $L = \mathbb{Z}[\beta]$ dense in K'_β.

But actually we have natural periodic translation sets:

$$\Lambda = \langle v_i - v_j : i, j = a_1, \ldots, a_d \rangle_{\mathbb{Z}}$$

have rank $d - 1$.

Take $\Lambda = \langle \beta^2 - 2\beta + 1, -\beta^2 + 2 \rangle_{\mathbb{Z}} = \langle v_3 - v_2, v_4 - v_2 \rangle_{\mathbb{Z}}$.



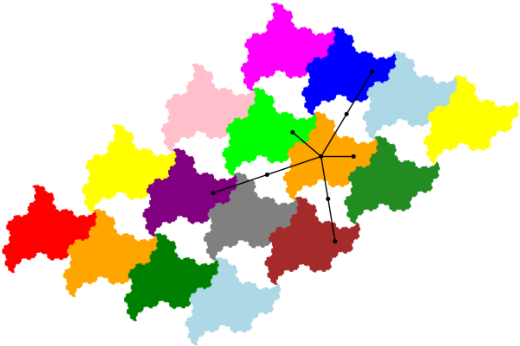


Figure : On the left-hand side red directions are identified mod Λ, while black directions are two times any of the red ones mod Λ.

Let $\mathcal{R} := \mathcal{R}(0)$ and denote by $\mathcal{R}_i$ its subtiles.

$$E : \mathcal{R} \to \mathcal{R}, \quad E(\mathcal{R}_i) = \mathcal{R}_i + 2\delta'(v_2), \qquad \text{for } i \in \mathcal{A}$$

coincides μ-a.e. with the **first return** to $\mathcal{R}$ under the transitive toral translation $\tau : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$, $\tau(x + \Lambda) = x + \delta'(v_2) + \Lambda$. This is equivalent to saying that $\widetilde{\mathcal{R}} + \Lambda$ is a tiling of $\mathbb{C}$.

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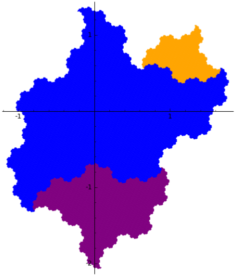

Figure : The extended domain $\widetilde{\mathcal{R}}$.

$$\begin{tikzcd}
X_\sigma \arrow[r] \arrow[d, "S"'] & \mathcal{R} \arrow[r] \arrow[d, "E"'] & \mathbb{C}/\Lambda \arrow[d, "\tau"'] \\
X_\sigma \arrow[r] & \mathcal{R} \arrow[r] & \mathbb{C}/\Lambda
\end{tikzcd}$$

$$\Lambda = \langle \beta - 2, \beta^2 - \beta - 1 \rangle_{\mathbb{Z}}.$$

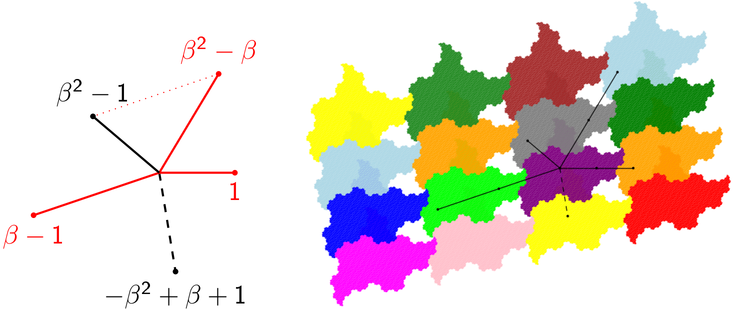

Figure : Red directions are identified mod Λ. The black dotted direction coincides with one translation vector.

3. Recent developments

Homological Pisot (Barge, Bruin, Jones, Sadun 2012)

Let σ be a Pisot substitution with $\deg(\beta) = d$. If the dimension of the first rational Čech cohomology of the tiling space is d, then we say that σ is *homological*.

Homological Pisot (Barge, Bruin, Jones, Sadun 2012)

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Perspectives:

- Can we translate the "homological" to Rauzy fractals?
- Can we find a homological non-unit irreducible Pisot substitution with coincidence rank greater than 1?

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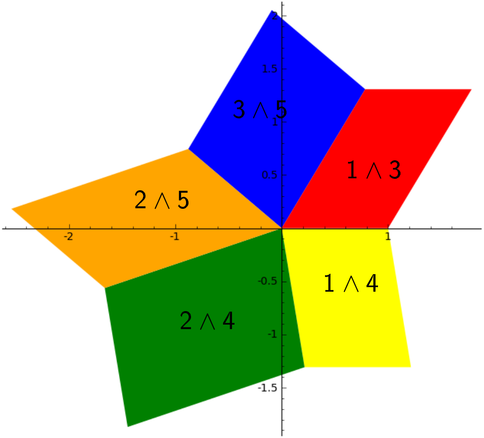
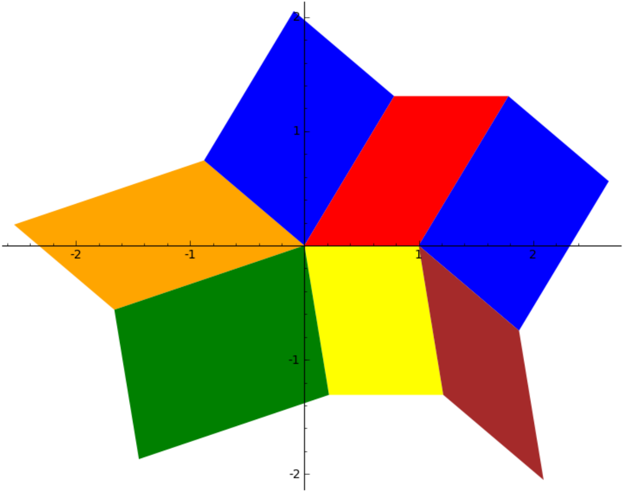
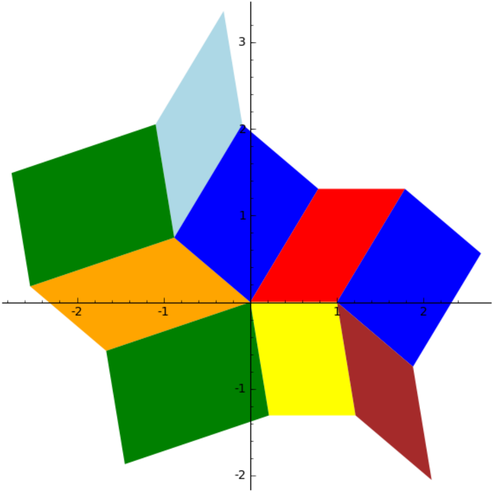
$$E^2(\sigma)(x, i \wedge j)^* = \sum_{\substack{\sigma(k)=p_1 i s_1 \\ \sigma(\ell)=p_2 j s_2}} (M_\sigma^{-1}(x + P(s_1) + P(s_2)), k \wedge \ell)^*$$

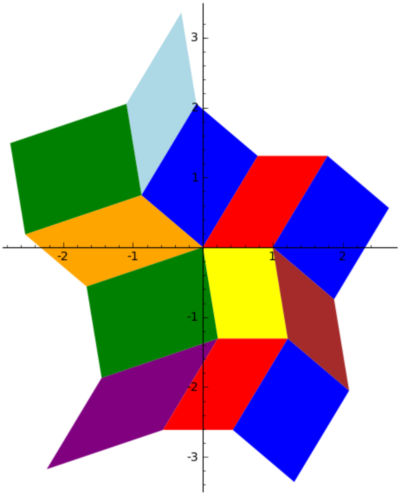
$$\mathcal{U} = (0, 1 \wedge 3)^* \cup (0, 1 \wedge 4)^* \cup (0, 2 \wedge 4)^* \cup (0, 2 \wedge 5)^* \cup (0, 3 \wedge 5)^*.$$

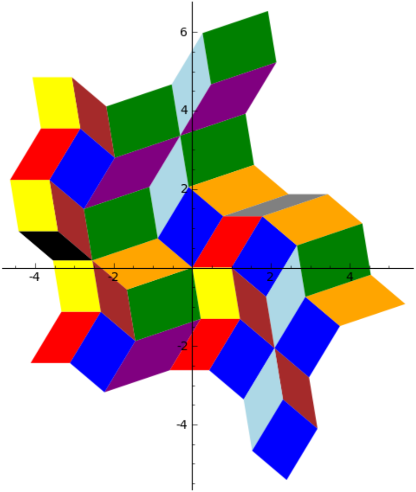


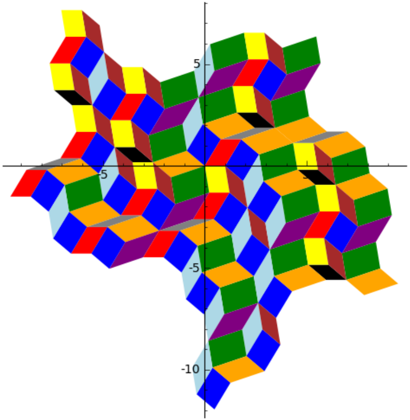
$\mathcal{U} \subset E^2(\mathcal{U})$!

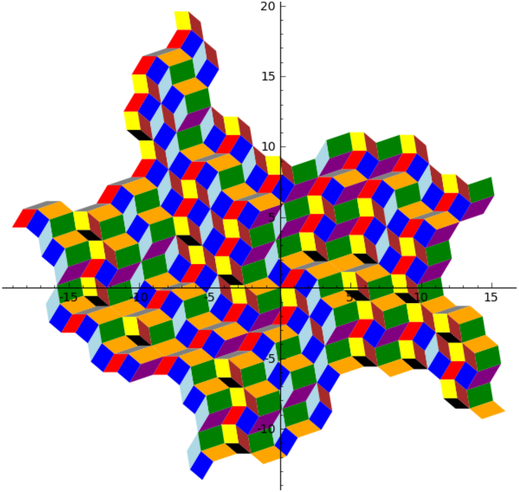


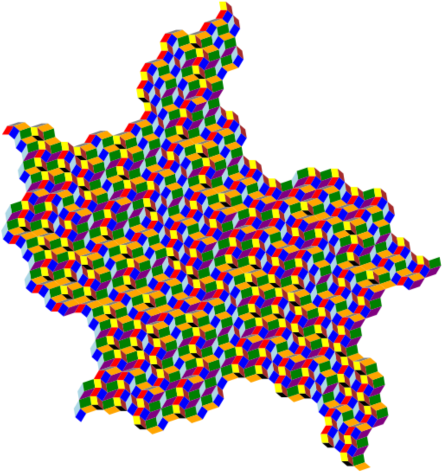

3
2
1
-1
-2
-2
-1
1
2


3
2
1
-2
-1
1
2
-1
-2
-3

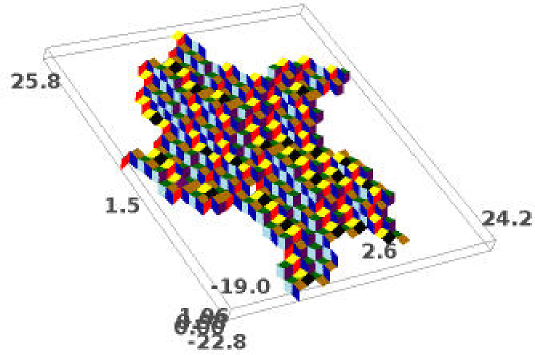

6
4
2
-2
-4
-4
-2
2
4

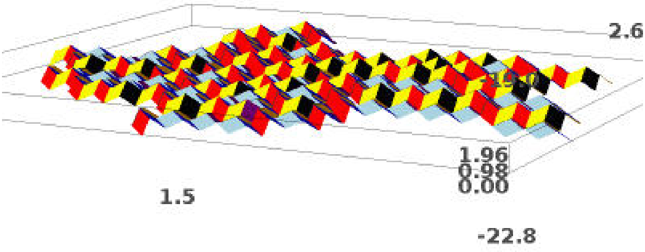

5
-5
-5
-10


20
15
10
5
-5
-10
-15
-10
-5
5
10
15

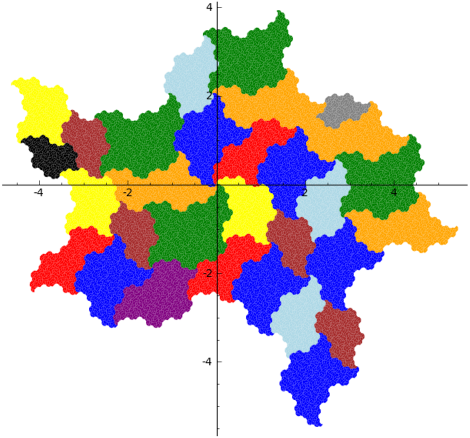


Stepped surfaces




2.6
-19.0
1.96
0.98
0.00
1.5
-22.8

Renormalizing the 10 pieces we get a self-replicating tiling:



Interpretation: every wedge can be thought as a union of two Hokkaido subtiles, and cutting in a suitable way we get the Hokkaido self-replicating tiling.

Merci

Vielen Dank

Have fun at FAN!