

# Annealed bounds for the return probability of the simple random walk on critical percolation clusters

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**Abstract:** Critical Bernoulli percolation on a unimodular transitive graph and on the 2-dim. euclidean lattice has almost surely finite connected components. Estimating the expected return probability of the simple random walk is difficult, due to the heavy tails of the cluster-size distribution[1]. Annealed upper and lower bounds are presented and compared to the conventional technique: instead of only estimating the spectral gap, the whole spectrum of the graph Laplacian is involved. This leads to an improvement of the upper bound [2]. In the case of regular trees, this is good enough to distinguish the decay of the expected return probability on finite clusters from that on the incipient infinite cluster, as it is predicted by the Alexander-Orbach conjecture[3,4]. The proof involves using the property of cartesian products of finite graphs with cycles of a specific length to be Hamiltonian[5].

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1. W.Kirsch, P.Müller: ‘Spectral properties of the Laplacian on bond-percolative graphs’: *Math. Z.* 252, 899-916, 2006
2. F.Sobieczky: ‘Bounds for the return probability of the delayed random walk on finite percolation clusters in the critical case’, arXiv:0812.0117v2
3. M. T. Barlow, T. Kumagai: ‘Random walk on the incipient infinite cluster on trees’, *Illinois Journ. Math.*, 50, 33-65, 2006
4. G. Kozma, A. Nachmias: ‘The Alexander-Orbach conjecture holds in high dimensions’, arXiv:0806.1442v1
5. V. Batagelj, T. Pisanski: ‘Hamiltonian cycles in the cartesian product of a tree and a cycle’, *Discr. Math.*, 38, 311-312, 1982

**Keywords:** Interlacing, random walk, critical percolation, return-probability, IDS

# Introduction of the Model, Problem-Definition

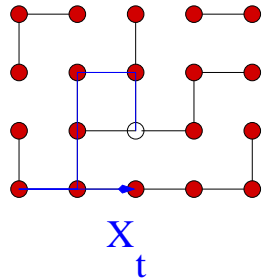
Transitive Graph  $G$   
(e.g.  $\langle \mathbb{Z}^2, \text{N.N.} \rangle$ )

Percolation

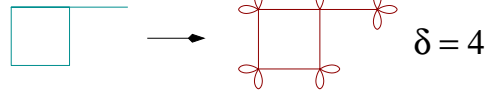
Random Walk  
on random  $C_0$

Expected Return-

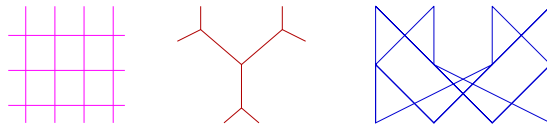
Probability:  $\bar{P}_t(o) = \mathbb{E}[P_t[X_t = X_0]]$



Simple  $\leftrightarrow$  Delayed Random Walk



Unimodularity & spectral Properties



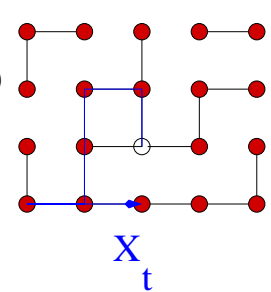
# Presentation of the results

Transitive Graph  $G$   
(e.g.  $\langle \mathbb{Z}^2, \text{N.N.} \rangle$ )

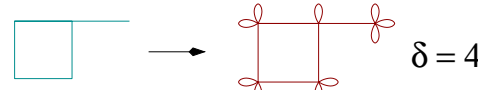
**Percolation**

**Random Walk**  
on random  $C_0$

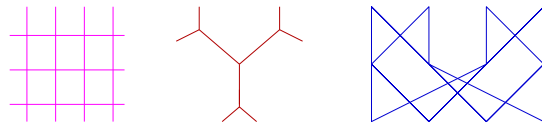
Expected Return-  
Probability:  $\bar{P}_t(o) = \mathbb{E}[P_t[X_t = X_0]]$



Simple  $\leftrightarrow$  Delayed Random Walk



Unimodularity & spectral Properties



For Bernoulli-Perc. at Criticality:

**Expected Return-Probability**

$$c t^{-\xi} \leq \bar{P}_t(o) \leq \mathbb{E}|C_o|^{-1} \leq c' t^{-\eta}$$

**Integrated Density of States**

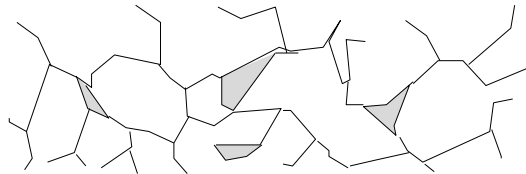
$$b E^\xi \leq N(E) - N(0) \leq b' E^\rho$$

with  $\xi, \varepsilon, \zeta, \rho, c', b, b' > 0$  known.

Consequences for the IIC

**Incipient Infinite Cluster**

$$\mathbb{E}_{\text{IIC}} [P^o[X = o]] \leq \frac{c''}{\sqrt{t}} \quad (c'' > 0)$$



# Illustration of methods

Transitive Graph  $G$   
(e.g.  $\langle \mathbb{Z}^2, \text{N.N.} \rangle$ )

**Percolation**

Random Walk  
on random  $C_0$

Expected Return-  
Probability:  $\bar{P}_t(o) = \mathbb{E}[P_t[X_t = X_0]]$

Simple  $\leftrightarrow$  Delayed Random Walk

$\delta = 4$

Unimodularity & spectral Properties

For Bernoulli-Perc. at Criticality:

Expected Return-Probability

$$c t^{-\xi} \leq \bar{P}_t(o) \leq \mathbb{E}[|C_0|^{-1}] \leq c' t^{-\eta}$$

Integrated Density of States

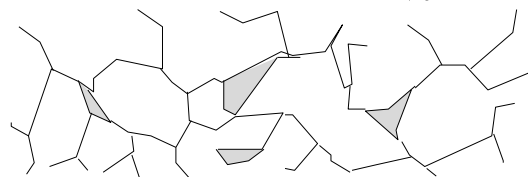
$$b E^\xi \leq N(E) - N(0) \leq b' E^\rho$$

with  $\xi, \varepsilon, \zeta, \rho, c', b, b' > 0$  known.

Consequences for the IIC

Incipient Infinite Cluster

$$\mathbb{E}_{\text{IIC}} [P^o[X = o]] \leq \frac{c''}{\sqrt{t}} \quad (c'' > 0)$$



Method A: Bulk-Spectrum

$$\hat{P} = \hat{D} \hat{A} \quad \{\beta_i\} = \sigma(\hat{P}) \quad 1 = \beta_1 > \beta_2 \dots$$

$C_0$   $P_{N'}$

Edge removal and insertion:  $S$

$R = \text{Rank}(S) \ll |C_0| + N' =: \bar{N}$

Interlacing:  $\beta_j \leq \bar{\beta}_{j-R}$

Method B: The Spectrum at the edge

Edge-Removal:

$\beta_j \leq \bar{\beta}_j$

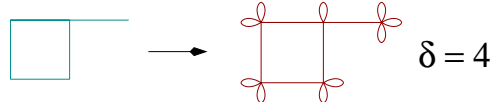
$\delta = 3$

$P_t(o) P_t(o) = P_{2t}(o)$

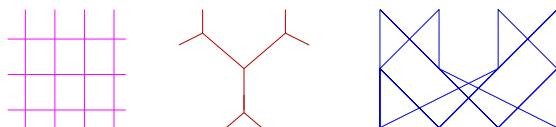


## Nearest Neighbour Random Walks on $H_o(\omega)$

Simple  $\leftrightarrow$  Delayed Random Walk



Unimodularity & spectral Properties



$X_n^\omega : \tilde{\Omega}^\omega \rightarrow \mathcal{C}_o(\omega)$  (discrete time  $n \in \mathbb{N}$ )

Choose among two initial distributions:

$\nu_o$  (Atom at  $o$ ), and  $\bar{\nu}$  (UNIF( $\mathcal{C}_o$ ))

Choose among two transition kernels:

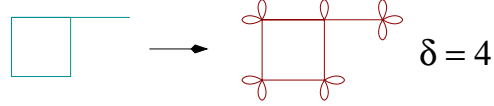
1.  $P_{kl} = \begin{cases} 1/\deg_\omega(k) & k \sim l \text{ in } H_o(\omega), \\ 0 & \text{otherwise.} \end{cases}$
2.  $\hat{P}_{kl} = \begin{cases} 1/\delta & k \sim l \text{ in } H_o(\omega), \\ 1 - \deg_\omega(w)/\delta & k = l \\ 0 & \text{otherwise.} \end{cases}$

(1. Simple Random Walk; 2. Delayed Random Walk)

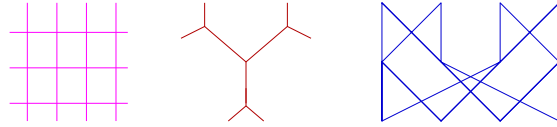


## Unimodularity: The Mass-Transport-Principle

Simple  $\leftrightarrow$  Delayed Random Walk



Unimodularity & spectral Properties



MTP: For diagonally invariant functions  $f : V \times V \rightarrow \mathbb{R}$  :

$$\sum_{v \in V} f(v, w) = \sum_{v \in V} f(w, v).$$

Then:  $\mathbb{E}_\mu [\mathbb{P}_o[X_t = o]] = \mathbb{E}_\mu [\mathbb{P}_-[X_t = X_0]].$

Proof:

$$\sum_{v \in V} \mathbb{E}_\mu \left[ \mathbb{P}_o[X_t = o] \frac{\chi_{\{v \in \mathcal{C}_o\}}}{|\mathcal{C}_o|} \right] = \sum_{v \in V} \mathbb{E}_\mu \left[ \mathbb{P}_v[X_t = v] \frac{\chi_{\{o \in \mathcal{C}_v\}}}{|\mathcal{C}_v|} \right] = \sum_{v \in V} \mathbb{E}_\mu \left[ \mathbb{P}_v[X_t = v] \frac{\chi_{\{v \in \mathcal{C}_o\}}}{|\mathcal{C}_o|} \right],$$

which equals the right-hand-side, since  $v \in \mathcal{C}_o \Leftrightarrow o \in \mathcal{C}_v$ . □

Note:  $P_t(o) = \mathbb{E} \left[ \frac{1}{|\mathcal{C}_o|} \text{Tr} \exp(-t(1 - \hat{P})) \right].$

## Annealed Return Probability on finite Clusters

**Theorem 1:** Let  $\widehat{X}_t$  be the continuous time delayed random walk on the  $\mu$ -a.s. finite critical percolation cluster of a unimodular transitive graph. If  $a, \nu > 0$ , such that  $\mathbb{P}[|\mathcal{C}_o| \geq m] \geq a m^{-\nu}$ , and  $b > \frac{1}{2}$  such that  $\mathbb{E}_\mu[|\mathcal{C}_o|^{2b-1}] < \infty$ , then the annealed return probability  $P_t(o) = \mathbb{E}_\mu \mathbb{P}_o[X_t = o]$  satisfies

$$a \frac{e^{-\delta}}{[t]^{1+\nu}} \leq P_t(o) - \mathbb{E}_\mu [|\mathcal{C}_o|^{-1}] \leq c(\delta) \frac{\mathbb{E}_\mu[|\mathcal{C}_o|^{2b-1}]}{t^b}.$$

For  $G = \langle \mathbb{Z}^2, N.N \rangle$ , and  $\alpha > 0$  s.t.  $\mathbb{E}[|\mathcal{C}_o|^\alpha] < \infty$ ,

$$\frac{e^{-4}}{2} [t]^{-3/2} \leq P_t(o) - \mathbb{E}_\mu [|\mathcal{C}_o|^{-1}] \leq c_\mu t^{-\frac{1}{2}(1+\alpha)}.$$

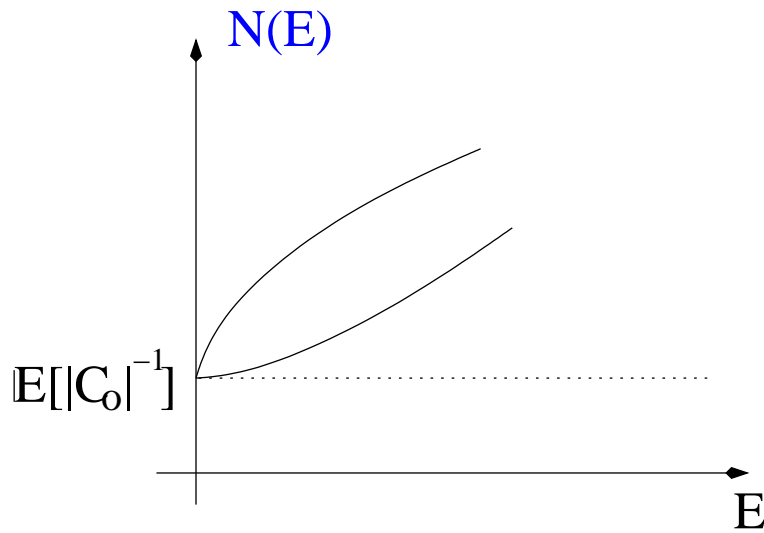
Moreover, if  $G$  is the homogeneous tree of degree three, then  $\nu = 3/4 - \epsilon$ , for any  $\epsilon > 0$ .

M. T. Barlow, T. Kumagai: 'Random walk on the incipient infinite cluster on trees', Illinois Journ. Math., 50, (2006), pp. 33-65

## Integrated Density of states on finite Clusters

**Theorem 2:** For the integrated density of states  $E \mapsto N_N(E)$  of the Neumann Laplacian corresponding to critical 2D-Bernoulli bond percolation there is a constant  $\alpha > 0$ , such that for  $\nu := \frac{1}{2}(1 + \alpha)$ , and for all positive  $E < 0.0002$  and  $c_\mu = (8 + \sqrt{3\pi})\mathbb{E}_\mu[|\mathcal{C}_o|^\alpha]$

$$\frac{1}{1000} \frac{E^{3/2}}{(\log 1/E)^{3/2}} \leq N_N(E) - N_N(0) \leq c_\mu \left(\frac{e}{\nu}\right)^\nu E^{\frac{1}{2}(1+\alpha)}.$$

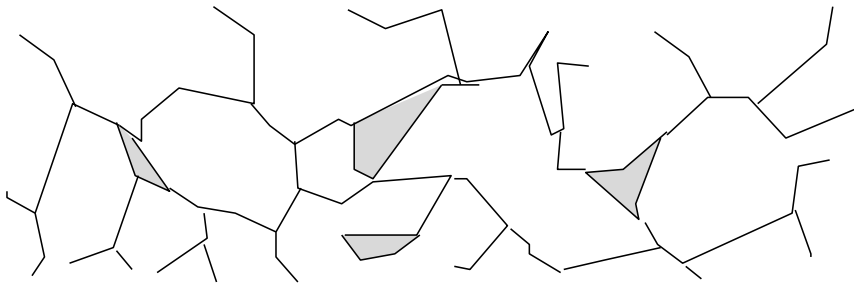


### Theorem 3: On the infinite incipient cluster

## Consequences for the IIC

### Incipient Infinite Cluster

$$\mathbb{E}_{IIC} [\mathbb{P}^o [ X = o ] ] \leq \frac{c''}{\sqrt{t}} \quad (\text{some } c'' > 0)$$



$$\mathbb{E}[\mathbb{P}_o[X_t = o] | o \leftrightarrow \partial B_r(o)] \rightarrow \mathbb{E}_{IIC}[\mathbb{P}_o[X_t = o]], \text{ as } r \rightarrow \infty.$$

[1] H. Kesten: ‘Subdiffusive behaviour of random walk on a random cluster’, Annales de l’Institut Henri Poincaré , Probabilités et Statistiques 22,(1986), 425-487

[2] G. Kozma, A. Nachmias: ‘The Alexander-Orbach conjecture holds in high dimensions’, arXiv:0806.1442v1 (2008)

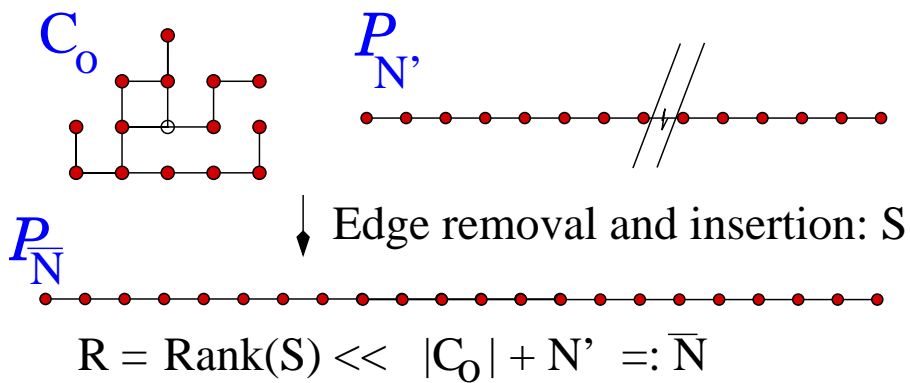
With  $H_r^o := \{\omega \in \Omega \mid o \leftrightarrow \partial B_r(o)\}$ , and [2] ( $\mathbb{P}[H_r^o] \sim r^{-1}$ ), t.e.  $c'' > 0$ , s.t.

$$\frac{1}{c''} \mathbb{E}_{IIC}[\mathbb{P}_-[X_t = X_o] | H_{2r}^o] \leq \mathbb{E}[\mathbb{P}_o[X_t = o]] \leq c'' \mathbb{E}_{IIC}[\mathbb{P}_-[X_t = X_o] | H_{r/2}^o].$$

Interlacing for the eigenvalues in the middle of  $\sigma(\hat{P})$

### Method A: Bulk-Spectrum

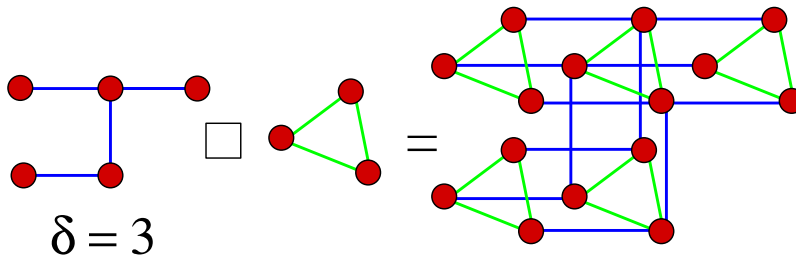
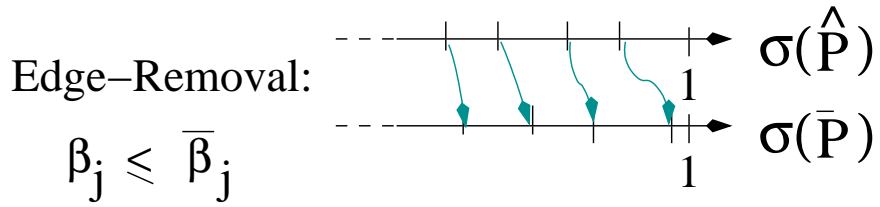
$$\hat{P} = \hat{D}^{-1} \hat{A} \quad \{ \beta_i \} = \sigma(\hat{P}) \quad 1 = \beta_1 > \beta_2 \dots$$



Interlacing:  $\beta_j \leq \bar{\beta}_{j-R}$

Cartesian Products with cycles of length  $\delta$

Method B: The Spectrum at the edge



$$P_t(o) \oplus P_t(o) = P_{2t}(o)$$