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On complete convergence of triangular arrays of independent random variables

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Abstract

Given a triangular array $\mathbf{a} = \{a_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ of positive reals, we study the complete convergence property of $T_n = \sum_{k=1}^{k_n} a_{n,k} X_{n,k}$ for triangular arrays $\mathscr{X} = \{X_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ of independent random variables. In the Gaussian case we obtain a simple characterization of density type. Using Skorohod representation and Gaussian randomization, we then derive sufficient criteria for the case when $X_{n,k}$ are in L^p, and establish a link between the L^p-case and L^{2p}-case in terms of densities. We finally obtain a density type condition in the case of uniformly bounded random variables.

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1. Introduction and results

Throughout this paper, we let $\mathscr{X} = \{X_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ denote a triangular array of real centered independent random variables, and $\mathbf{a} = \{a_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ with $\{k_n, n \geqslant 1\}$ non-decreasing, a triangular array of positive reals. When the random variables are symmetric (resp. identically distributed), we will say that the triangular array $\mathscr{X}$ is symmetric (resp. iid). Set, for every $n \geqslant 1$,

$$T_n = \sum_{k=1}^{k_n} a_{n,k} X_{n,k}, \quad A_n = \sum_{k=1}^{k_n} a_{n,k}, \quad B_n^2 = \sum_{k=1}^{k_n} a_{n,k}^2, \quad C_n = A_n / B_n. \tag{1}$$

Let $(\Omega, \mathscr{A}, \mathbf{P})$ be the basic probability space on which $\mathscr{X}$ is defined. Note that $C_n \geqslant 1$. We investigate under what conditions the sequence T_n/A_n converges completely to 0: $T_n/A_n \stackrel{\text{c.c.}}{\to} 0$, which means, as is well-known,

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that for any $\varepsilon > 0$

$$\sum_n \mathbf{P}\{|T_n|/A_n > \varepsilon\} < \infty.$$

The study of this property originates from a well-known paper by Hsu and Robbins (1947) who proved in the case of a single iid sequence $\xi = \{\xi, \xi_n, n \geqslant 1\}$ with partial sums $S_n = \sum_{k=1}^n \xi_k$, $n = 1, 2, \ldots$ that $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^2 < \infty$ imply $S_n/n \xrightarrow{\text{c.c.}} 0$. Shortly afterward, Erdős (1949) proved the validity of the converse implication. Since then, the study of various possible generalizations of this result (subsequence case, the theorems of Baum and Katz (1965), extensions to triangular arrays of independent random variables, Banach space valued random variables) have received a lot of attention. See, for example, the works of Pruitt (1966), Rohatgi (1971), Fazekas (1985, 1992), Hu et al. (1989), Kuczmaszewska and Szynal (1988, 1990, 1994), Gut (1992), Li et al. (1992), Rao et al. (1993), Sung (1997), Adler et al. (1999), Hu et al. (1999), Ahmed et al. (2002). The purpose of the present paper is to present new necessary as well as sufficient criteria for the complete convergence of triangular arrays of independent random variables, and discuss their relations with known results in the literature.

We start our investigations with the Gaussian case, because of the classical Gaussian randomization procedure for sums of independent random variables, and also because this case is in general very informative. If $\mathscr{X}$ is Gaussian, the problem can be simply settled. Put

$$\mathscr{L}(\mathbf{a}) = \limsup_{x \to \infty} \frac{\log \sharp\{n : C_n \leqslant x\}}{x^2}.$$

Then we have the following characterization.

Theorem 1. *Assume that the $X_{n,k}$ are iid standard Gaussian variables. Then we have*

$$T_n/A_n \xrightarrow{\text{c.c.}} 0 \iff \mathscr{L}(\mathbf{a}) = 0.$$

In view of this complete result, it is natural to attack the general iid case using invariance principles. Applying Skorohod embedding for the row sums of the triangular array $\mathscr{X}$ leads, under natural conditions on the stopping times in the Skorohod representation, to a necessary and sufficient criterion for $T_n/A_n \xrightarrow{\text{c.c.}} 0$, see Proposition 10. This condition, in turn, leads to sufficient criteria under the existence of higher moments. In particular, we will prove:

Theorem 2. *Assume that $\mathbf{E}X_{n,k}^2 = 1$ and $X_{n,k} \in L^{2p}$ for some $p \geqslant 2$. Then the relation*

$$\sum_n \frac{(\sum_{k=1}^{k_n} a_{n,k}^4)^{p/2}}{(\sum_{k=1}^{k_n} a_{n,k}^2)^p} < \infty,$$

implies $T_n/A_n \xrightarrow{\text{c.c.}} 0$.

To compare this result with the Gaussian case, note that $\mathscr{L}(\mathbf{a}) = 0$ is equivalent to

$$\sum_n \exp\left(-\delta \frac{\left(\sum_{k=1}^{k_n} a_{n,k}\right)^2}{\sum_{k=1}^{k_n} a_{n,k}^2}\right) < \infty \quad \text{for all } \delta > 0.$$

In the case when $\mathscr{X}$ is also symmetric, the condition in Theorem 2 can be weakened.

Theorem 3. *Assume that $\mathscr{X}$ is symmetric, $\mathbf{E}X_{n,k}^2 = 1$ and $X_{n,k} \in L^{2p}$ for some $p \geqslant 2$. Then the relation*

$$\sum_n \frac{(\sum_{k=1}^{k_n} a_{n,k}^4)^{p/2}}{(\sum_{k=1}^{k_n} a_{n,k})^{2p} \log^p n} < \infty,$$

implies $T_n/A_n \xrightarrow{\text{c.c.}} 0$.

Recall that the array $\mathscr{X}$ is *stochastically bounded* by a random variable X if there is a constant D such that $\mathbf{P}\{|X_{n,k}| > x\} \leqslant D\mathbf{P}\{|DX| > x\}$ for all $x > 0$ and for all $n \geqslant 1$, $1 \leqslant k \leqslant k_n$. We will prove the following result.

Theorem 4. *Let $\mathscr{X}$ be a symmetric triangular array stochastically bounded by a square integrable random variable X. Assume that for any $\varepsilon > 0$:*

(a) $\sum_{1 \leqslant k \leqslant l < \infty} \mathbf{P}\{|X| \geqslant \varepsilon A_l / a_k\} < \infty$.
Further assume that for some integer $r \geqslant 2$ and any $\varepsilon > 0$,
(b) $\sum_{n \geqslant 1} \mathbf{P}\{|T_n| > \varepsilon A_n\}^r < \infty$.
Then
(c) $T_n / A_n \xrightarrow{\text{c.c.}} 0$.
Conversely, if the triangular array $\mathscr{X}$ is iid symmetric, then (c) *implies* (a).

The next result concerns the uniformly bounded case. We show that a condition similar to that assumed in the Gaussian case suffices for complete convergence. Put, for any positive integer n,

$$V_n^2 = \sum_{k=1}^{k_n} a_{n,k}^2 X_{n,k}^2.$$

Theorem 5. *Let $\mathscr{X}$ be a triangular array of real centered, uniformly bounded independent random variables. Assume that for any $\varepsilon > 0$*

$$\mathbf{E} \sup_{m \geqslant 1} \frac{\sharp\{n : m < A_n / V_n \leqslant m+1\}}{\exp\{\varepsilon m^2\}} < \infty.$$

Then $T_n / A_n \xrightarrow{\text{c.c.}} 0$.

Our final result establishes a link between the complete convergence of arrays in the L^p and L^{2p}-case. Remarkably, the link is provided by the density condition in the Gaussian case in Theorem 1. We need a preliminary definition.

Definition. Let $p \geqslant 2$. We say that $\mathbf{a}$ is p-regular if any triangular array $\mathscr{X}$ of real centered iid random variables with finite pth moments satisfies $T_n / A_n \overset{\text{c.c.}}{\to} 0$.

Let $\mathbf{a}$ be a triangular array of positive reals. Define

$\mathbf{a}^2 := \{a_{n,k}^2, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$.

Then we have:

Theorem 6. *Let $p \geqslant 2$ and assume that $\mathbf{a}^2$ is p-regular. Then $\mathbf{a}$ is $2p$-regular iff $\mathscr{L}(\mathbf{a}) = 0$.*

2. Proofs

Proof of Theorem 1. Before giving the proof, recall for the reader's convenience an elementary estimate for Gaussian random variables due to Komatsu–Pollak (see Mitrinović, 1970, p. 178).

Lemma 7. *The Mills's ratio $R(x) = \mathrm{e}^{x^2/2} \int_x^\infty \mathrm{e}^{-t^2/2}\,\mathrm{d}t$ satisfies*

$$\frac{2}{\sqrt{x^2+4}+x} \leqslant R(x) \leqslant \frac{2}{\sqrt{x^2+\frac{8}{\pi}}+x} \leqslant \sqrt{\frac{\pi}{2}} \quad \textit{for all } x > 0.$$

Note that

$$\mathbf{P}\{|T_n|/A_n > \varepsilon\} = \mathbf{P}\{|\mathscr{N}(0,1)| > \varepsilon C_n\} \asymp \frac{1}{1+\varepsilon C_n} \mathrm{e}^{-(\varepsilon C_n)^2/2} \quad \text{as } n \to \infty,$$

where the symbol $\asymp$ means that the ratio of the two sides is between positive constants. Thus it follows that $T_n/A_n \xrightarrow{\text{c.c.}} 0$ if and only if the series

$$\sum_n e^{-\delta C_n^2}$$

converges for any $\delta > 0$. And this is equivalent to $\mathscr{L}(\mathbf{a}) = 0$ (for a proof, see e.g. Weber, 1995, pp. 402–403). $\square$

Proof of Theorem 6. The proof relies upon several intermediate results. Let $\boldsymbol{\xi} = \{\xi_k, k \geqslant 1\}$ be a sequence of real centered independent square integrable random variables defined on the probability space $(\Omega, \mathscr{A}, \mathbf{P})$, and let $\mathbf{w} = \{w_k, k \geqslant 1\}$ be a sequence of positive reals. Put, for any positive integer m,

$$S_m = \sum_{k=1}^m w_k \xi_k, \quad W_m = \sum_{k=1}^m w_k, \quad M_m = \sum_{k=1}^m w_k^2.$$

Recall the Skorohod embedding scheme (see e.g. Breiman, 1968): there exists, after suitably enlarging the probability space, a linear Brownian motion $\mathscr{B} = \{B(t), 0 \leqslant t < \infty\}$ starting at 0, and a sequence $\tau_1, \tau_2, \ldots$ of independent non-negative random variables with $\mathbf{E}\tau_k = w_k^2 \mathbf{E}\xi_k^2$, $k \geqslant 1$ such that, with $\tau_0 = 0$ a.s.,

$$\left\{ B\left(\sum_{j=0}^k \tau_j\right) - B\left(\sum_{j=0}^{k-1} \tau_j\right), k \geqslant 1 \right\} \stackrel{\mathscr{D}}{=} \{w_k \xi_k, k \geqslant 1\}.$$

Put, for any real x,

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} \, du.$$

Lemma 8. *Let ε, h, δ be positive numbers with $\varepsilon > h > \sqrt{2\delta}$, and put*

$$\Delta_m = \Delta_m(\delta) = \mathbf{P}\left\{ \left| \sum_{j=0}^m \tau_j - M_m \right| \geqslant \delta M_m \right\}.$$

Then, for any positive integer m, we have

$$\Psi\left((\varepsilon + h)\frac{W_m}{\sqrt{M_m}}\right) - 4\Psi\left(\frac{hW_m}{\sqrt{2\delta M_m}}\right) - \Delta_m \leqslant \mathbf{P}\{|S_m| > \varepsilon W_m\} \leqslant \Psi\left((\varepsilon - h)\frac{W_m}{\sqrt{M_m}}\right) + 4\Psi\left(\frac{hW_m}{\sqrt{2\delta M_m}}\right) + \Delta_m.$$

Proof. We observe that

$$\begin{aligned}
\mathbf{P}\{|S_m| > \varepsilon W_m\} &= \mathbf{P}\left\{ \left| B\left(\sum_{j=0}^m \tau_j\right) \right| > \varepsilon W_m \right\} \\
&\leqslant \mathbf{P}\{|B(M_m)| > (\varepsilon - h) W_m\} + \mathbf{P}\left\{ \left| B\left(\sum_{j=0}^m \tau_j\right) \right| > \varepsilon W_m, |B(M_m)| \leqslant (\varepsilon - h) W_m \right\} \\
&\leqslant \Psi\left(\frac{(\varepsilon - h) W_m}{\sqrt{M_m}}\right) + \mathbf{P}\left\{ \left| B\left(\sum_{j=0}^m \tau_j\right) - B(M_m) \right| \geqslant h W_m \right\} \\
&\leqslant \Psi\left(\frac{(\varepsilon - h) W_m}{\sqrt{M_m}}\right) + \mathbf{P}\left\{ \left| \sum_{j=0}^m \tau_j - M_m \right| \geqslant \delta M_m \right\} + \mathbf{P}\left\{ \sup_{|\theta - 1| \leqslant \delta} |B(\theta M_m) - B(M_m)| \geqslant h W_m \right\} \\
&= \Psi\left(\frac{(\varepsilon - h) W_m}{\sqrt{M_m}}\right) + \mathbf{P}\left\{ \left| \sum_{j=0}^m \tau_j - M_m \right| \geqslant \delta M_m \right\} + \mathbf{P}\left\{ \sup_{|\theta - 1| \leqslant \delta} |B(\theta) - B(1)| \geqslant h \frac{W_m}{\sqrt{M_m}} \right\}.
\end{aligned}$$

Conversely,

$$\begin{aligned}\Psi\left((\varepsilon+h)\frac{W_m}{\sqrt{M_m}}\right) &= \mathbf{P}\{|B(M_m)|>(\varepsilon+h)W_m\}\\
&\leqslant \mathbf{P}\left\{\left|B\left(\sum_{j=0}^m \tau_j\right)\right|>\varepsilon W_m\right\}+\mathbf{P}\left\{|B(M_m)|>(\varepsilon+h)W_m,\left|B\left(\sum_{j=0}^m \tau_j\right)\right|\leqslant \varepsilon W_m\right\}\\
&\leqslant \mathbf{P}\left\{\left|B\left(\sum_{j=0}^m \tau_j\right)\right|>\varepsilon W_m\right\}+\mathbf{P}\left\{\left|B\left(\sum_{j=0}^m \tau_j\right)-B(M_m)\right|\geqslant hW_m\right\}\\
&\leqslant \mathbf{P}\left\{\left|B\left(\sum_{j=0}^m \tau_j\right)\right|>\varepsilon W_m\right\}+\mathbf{P}\left\{\left|\sum_{j=0}^m \tau_j-M_m\right|\geqslant \delta M_m\right\}\\
&\quad+\mathbf{P}\left\{\sup_{|\theta-1|\leqslant\delta}|B(\theta M_m)-B(M_m)|\geqslant hW_m\right\}\\
&= \mathbf{P}\left\{\left|B\left(\sum_{j=0}^m \tau_j\right)\right|>\varepsilon W_m\right\}+\mathbf{P}\left\{\left|\sum_{j=0}^m \tau_j-M_m\right|\geqslant \delta M_m\right\}+\mathbf{P}\left\{\sup_{|\theta-1|\leqslant\delta}|B(\theta)-B(1)|\geqslant h\frac{W_m}{\sqrt{M_m}}\right\}.\end{aligned}$$

Since B has stationary increments, we get by using scale invariance, the symmetry of the law of B and Eq. (1.5.1) in Csörgő and Révész (1981, p. 43),

$$\begin{aligned}\mathbf{P}\left\{\sup_{|\theta-1|\leqslant\delta}|B(\theta)-B(1)|\geqslant h\frac{W_m}{\sqrt{M_m}}\right\} &= \mathbf{P}\left\{\sup_{u\in[0,2\delta]}|B(u)|\geqslant \frac{hW_m}{\sqrt{M_m}}\right\}\\
&= \mathbf{P}\left\{\sup_{0\leqslant u\leqslant 1}|B(u)|\geqslant \frac{hW_m}{\sqrt{2\delta M_m}}\right\}\\
&= \mathbf{P}\left\{\max\left(\sup_{0\leqslant u\leqslant 1}B(u),\ \sup_{0\leqslant u\leqslant 1}(-B(u))\right)\geqslant \frac{hW_m}{\sqrt{2\delta M_m}}\right\}\\
&\leqslant 2\mathbf{P}\left\{\sup_{0\leqslant u\leqslant 1}B(u)\geqslant \frac{hW_m}{\sqrt{2\delta M_m}}\right\}=4\Psi\left(\frac{hW_m}{\sqrt{2\delta M_m}}\right).\end{aligned}$$

Consequently,

$$\Psi\left((\varepsilon+h)\frac{W_m}{\sqrt{M_m}}\right)-4\Psi\left(\frac{hW_m}{\sqrt{2\delta M_m}}\right)-\Delta_m\leqslant \mathbf{P}\{|S_m|>\varepsilon W_m\}\leqslant \Psi\left((\varepsilon-h)\frac{W_m}{\sqrt{M_m}}\right)+4\Psi\left(\frac{hW_m}{\sqrt{2\delta M_m}}\right)+\Delta_m.$$

This completes the proof. □

We shall apply Lemma 8 to triangular arrays. Let again $\mathscr{X}=\{X_{n,k},1\leqslant k\leqslant k_n,n\geqslant 1\}$ be a triangular array of real centered independent random variables and $\mathbf{a}=\{a_{n,k},1\leqslant k\leqslant k_n,n\geqslant 1\}$ a triangular array of positive reals. By considering, if necessary, a larger probability space, we can always assume that there exists a sequence $\boldsymbol{\xi}^1,\boldsymbol{\xi}^2,\ldots$ such that for each positive integer n,

$$\boldsymbol{\xi}^n=\{\xi_{n,k},k\geqslant 1\},\quad \text{with } \xi_{n,k}=X_{n,k},\ 1\leqslant k\leqslant k_n,$$

and $\boldsymbol{\xi}^n$ is a sequence of independent random variables. Further the sequences $\boldsymbol{\xi}^1,\boldsymbol{\xi}^2,\ldots$ are mutually independent. By suitably enlarging the probability space, there exists for each integer n a linear Brownian motion $\mathscr{B}^n=\{B^n(t),0\leqslant t<\infty\}$ starting at 0 and a sequence $\tau_1^n,\tau_2^n,\ldots$ of independent non-negative random variables with $\mathbf{E}\tau_k^n=a_{n,k}^2\mathbf{E}\xi_{n,k}^2$, $k\geqslant 1$ such that, with $\tau_0^n=0$ a.s.,

$$\left\{B^n\left(\sum_{j=0}^k \tau_j^n\right)-B^n\left(\sum_{j=0}^{k-1}\tau_j^n\right),\ k\geqslant 1\right\}\stackrel{\mathscr{D}}{=}\{a_{n,k}\xi_{n,k},\ k\geqslant 1\}.$$

In fact, in each step, it would be enough to let k run between 1 and k_n. By applying Lemma 8 with the choice $\boldsymbol{\xi}=\boldsymbol{\xi}^n$, $m=k_n$, we now easily deduce the following corollary.

Corollary 9. *Let ε, h, δ be positive reals with $\varepsilon > h > \sqrt{2\delta}$. Then, with notation* (1), *for $n = 1, 2, \ldots$*

$$\Psi((\varepsilon+h)C_n) - 4\Psi\left(\frac{h}{\sqrt{2\delta}} C_n\right) - \Delta_n(\delta) \leqslant \mathbf{P}\{|T_n| > \varepsilon A_n\} \leqslant \Psi((\varepsilon-h)C_n) + 4\Psi\left(\frac{h}{\sqrt{2\delta}} C_n\right) + \Delta_n(\delta),$$

where

$$\Delta_n(\delta) = \mathbf{P}\left\{\left|\sum_{j=0}^{k_n} \tau_j^n - B_n^2\right| \geqslant \delta B_n^2\right\}.$$

This result will allow us to establish the following statement.

Proposition 10. *Assume that $\mathscr{X}$ and* **a** *satisfy*

$$\sum_m \Delta_m(\delta) < \infty \quad \textit{for all } \delta > 0. \tag{2}$$

Then

$$T_n/A_n \xrightarrow{\text{c.c.}} 0 \iff \mathscr{L}(\mathbf{a}) = 0.$$

This proposition can be viewed as an extension of Theorem 1, since in the Gaussian case $\tau_j^n \stackrel{\text{a.s.}}{=} a_{n,j}^2$.

Proof. The key lies in the comparison between $\Psi((\varepsilon+h)C_n)$ and $\Psi(\frac{h}{\sqrt{2\delta}}C_n)$, which is achieved by using Lemma 7. The implication $\mathscr{L}(\mathbf{a}) = 0 \Rightarrow T_n/A_n \xrightarrow{\text{c.c.}} 0$ is easy. Indeed, if $\mathscr{L}(\mathbf{a}) = 0$, then for any $\rho > 0$ the series $\sum_n \mathrm{e}^{-\rho C_n^2}$ converges, or equivalently,

$$\sum_n \Psi(\rho C_n) < \infty \quad \text{for all } \rho > 0. \tag{3}$$

Let $\varepsilon > 0$, and choose h, δ in Corollary 9 such that $h = \varepsilon/2 > \sqrt{2\delta}$. By Corollary 9 and the assumption made, the series $\sum_n \mathbf{P}\{|T_n| > \varepsilon A_n\}$ converges provided

$$\sum_n \Psi((\varepsilon - h)C_n) < \infty, \quad \sum_n \Psi\left(\frac{h}{\sqrt{2\delta}} C_n\right) < \infty.$$

And this holds true if $\sum_n \Psi((\varepsilon/2)C_n) < \infty$, which is satisfied by assumption. Hence the first part of Proposition 10 is proved.

Conversely, if $T_n/A_n \xrightarrow{\text{c.c.}} 0$, then the series $\sum_n \mathbf{P}\{|T_n| > \varepsilon A_n\}$ converges for any $\varepsilon > 0$. We shall prove that (3) holds true. We distinguish two cases.

Case 1: $\liminf_{n\to\infty} C_n = \infty$. Let $\rho > 0$ be fixed, we choose ε, h, δ such as $\varepsilon = h = \rho/2$, $\delta = 1/8$, so that $h/\sqrt{2\delta} = 2\rho$. Then $\Psi((\varepsilon+h)C_n) = \Psi(\rho C_n)$ and $\Psi((h/\sqrt{2\delta})C_n) = \Psi(2\rho C_n)$. By Lemma 7,

$$\Psi(\rho C_n) \asymp \frac{1}{1+\rho C_n} \mathrm{e}^{-(\rho C_n)^2/2}, \quad \Psi(2\rho C_n) \asymp \frac{1}{1+2\rho C_n} \mathrm{e}^{-2(\rho C_n)^2},$$

so that, for any $\rho < \rho_1 < \rho_2 < 2\rho$, if n is sufficiently large

$$\Psi(\rho C_n) \geqslant \mathrm{e}^{-(\rho_1 C_n)^2/2}, \quad 4\Psi(2\rho C_n) \leqslant \mathrm{e}^{-(\rho_2 C_n)^2/2}.$$

Therefore,

$$\Psi(\rho C_n) - 4\Psi(2\rho C_n) \geqslant \mathrm{e}^{-(\rho_1 C_n)^2/2}(1 - \mathrm{e}^{-(\rho_2^2 - \rho_1^2)(C_n)^2/2}) \geqslant (1/2)\mathrm{e}^{-(\rho_1 C_n)^2/2}$$

for n sufficiently large. In view of Corollary 9, and assumption (2) this implies that the series $\sum_n \mathrm{e}^{-(\rho_1 C_n)^2/2}$ converges. This being true for any $\rho > 0$ and any $\rho_1 > \rho$, it follows that (3) is satisfied, as claimed.

Case 2: $\liminf_{n\to\infty} C_n < \infty$. In this case there exist a sequence of indices $\{n_j, j \geqslant 1\}$ and a real t such that $\lim_{j\to\infty} C_{n_j} = t$. Choose $\rho > 0$ such that $\Psi(\rho t) > 4\Psi(2\rho t)$, and let again ε, h, δ such as $\varepsilon = h = \rho/2$, $\delta = 1/8$. Applying Corollary 9 for $n = n_j$, $j = 1, 2, \ldots$ gives

$$\Psi(\rho C_{n_j}) - 4\Psi(2\rho C_{n_j}) \leqslant \mathbf{P}\{|T_{n_j}| > \varepsilon A_{n_j}\} + \Delta_{n_j}(\delta).$$

Letting now j tend to infinity implies

$$0 < \Psi(\rho t) - 4\Psi(2\rho t) \leqslant \liminf_{j\to\infty}(\mathbf{P}\{|T_{n_j}| > \varepsilon A_{n_j}\} + \Delta_{n_j}(\delta)),$$

which contradicts the fact that both series $\sum_n \mathbf{P}\{|T_n| > \varepsilon A_n\}$, $\sum_n \Delta_n(\delta)$ converge. The proof is now complete. $\square$

We can now pass to the proof of Theorem 6. Let $p \geqslant 2$ and let $\mathbf{a} = \{a_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be a triangular array of positive reals such that $\mathbf{b} = \mathbf{a}^2$ is p-regular. Let $\mathscr{X} = \{X_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ be a triangular array of real centered iid random variables with finite $2p$-th moment. We shall make use of the fact (Fisher (1992), Theorem 2.1) that for each n, we can assume that $\{\tau_k^n, 1 \leqslant k \leqslant k_n\} \stackrel{\mathscr{D}}{=} \{a_{n,k}^2 \theta_k^n, 1 \leqslant k \leqslant k_n\}$, and $\{\theta_k^n, 1 \leqslant k \leqslant k_n\}$ is an iid sequence with finite pth moments. As $\mathbf{b}$ is p-regular, (2) is satisfied. Using Proposition 10, we get the desired conclusion. $\square$

Remark. Although the characterization given in Theorem 6 is simple, it is rather abstract. Usually condition (2) is as difficult to check as the fact that $\mathbf{a}$ is $2p$-regular. Thus the interest in a statement like Theorem 6 is the link established between p-regularity and $2p$-regularity, via the arrays $\mathbf{a}$ and $\mathbf{b}$.

It is possible to check directly condition (2), by imposing conditions on the weights, which, however, appear to be stronger than the condition $\mathscr{L}(\mathbf{a}) = 0$. To see this, we shall use some arguments from Weber (2006). In order to avoid unnecessarily heavy notation, we simply return to the setting considered in Lemma 8, and will bound the quantity

$$\Delta_m = \Delta_m(\delta) = \mathbf{P}\left\{\left|\sum_{j=0}^{m} \tau_j - M_m\right| \geqslant \delta M_m\right\}.$$

Using inequality (1.2) in Davis (1976) we see that if $\mathbf{E}|\xi_i|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$, the sequence of stopping times τ_i satisfies

$$\mathbf{E}\tau_i^{1+\varepsilon/2} \leqslant C w_i^{2+\varepsilon} \mathbf{E}|\xi_i|^{2+\varepsilon}, \tag{4}$$

where the constant C depends on ε only. Let $p \geqslant 2$. Assume that for any positive integer j, $\xi_i \in L^{2p}$, and moreover

$$Q_p(\xi) := \sup_{j \geqslant 1} \|\xi_i\|_p < \infty.$$

Put for any positive integer l,

$$x_l = \tau_l - \mathbf{E}\tau_l = \tau_l - w_l^2.$$

Then using (4) with $2(p-1) = \varepsilon$ gives

$$\mathbf{E}|x_l|^p \leqslant 2^p(\mathbf{E}|\tau_l|^p + w_l^{2p}) \leqslant C'_p(1 + Q_p^p(\xi)) w_l^{2p},$$

where C'_p depends on p only, and may vary in the next lines. Further note that in the case $\xi_l \in L^4$, $l \geqslant 1$ we have

$$0 \leqslant \mathbf{E}x_l^2 = \mathbf{E}\tau_l^2 - (\mathbf{E}\tau_l)^2 \leqslant \mathbf{E}\tau_l^2 \leqslant C'_2 w_l^4 \mathbf{E}|\xi_l|^4.$$

Apply now Rosenthal's inequality (see e.g. Petrov, 1995, p. 59). In view of centering and independence of the x_l's, we get

$$\begin{aligned}\mathbf{E}\left|\sum_{l=1}^{m}(\tau_l - w_l^2)\right|^p &\leqslant C'_p\left(\sum_{l=1}^{m} w_l^{2p} + \left(\sum_{l=1}^{m} \mathbf{E}x_l^2\right)^{p/2}\right) \\ &\leqslant C'_p(1 + Q_p^p(\xi))\left(\sum_{l=1}^{m} w_l^{2p} + \left(\sum_{l=1}^{m} w_l^4\right)^{p/2}\right) \leqslant C'_p(1 + Q_p^p(\xi))\left(\sum_{l=1}^{m} w_l^4\right)^{p/2}.\end{aligned}$$

Consequently, by using Chebyshev's inequality,

$$\Delta_m(\delta) = \mathbf{P}\left\{\left|\sum_{j=0}^{m} \tau_j - M_m\right| \geqslant \delta M_m\right\} \leqslant C_p'(1 + Q_p^p(\xi))\left(\frac{\sum_{l=1}^m w_l^4}{(\delta M_m)^2}\right)^{p/2}.$$

We thus see that condition (2) holds provided

$$\sum\nolimits_m \left(\frac{[\sum_{l=1}^m w_l^4]^{1/2}}{M_m}\right)^p < \infty.$$

For triangular arrays, this means that

$$\sum_n \left(\frac{[\sum_{k=1}^{k_n} a_{n,k}^4]^{1/2}}{\sum_{k=1}^{k_n} a_{n,k}^2}\right)^p < \infty,$$

establishing Theorem 2. As we noted earlier, $\mathscr{L}(\mathbf{a}) = 0$ is equivalent to

$$\sum_n \exp\left(-\delta \frac{\left(\sum_{k=1}^{k_n} a_{n,k}\right)^2}{\sum_{k=1}^{k_n} a_{n,k}^2}\right) < \infty \quad \text{for all } \delta > 0.$$

Proof of Theorem 3. Since $\mathscr{X}$ is symmetric, it has the same law as $\mathscr{X} = \{\varepsilon_{n,k} X_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$, where $\boldsymbol{\varepsilon} = \{\varepsilon_{n,k}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ is a Rademacher sequence defined on a joint probability space $(\Omega_\varepsilon, \mathscr{A}_\varepsilon, \mathbf{P}_\varepsilon)$ (with corresponding expectation symbol $\mathbf{E}_\varepsilon$). Put

$$Y_n = \sum_{k=1}^{k_n} a_{n,k} \varepsilon_{n,k} X_{n,k}, \quad Q_n = \frac{\sum_{k=1}^{k_n} a_{n,k}^2 X_{n,k}^2}{B_n^2}.$$

Let $\{\Omega_n, n \geqslant 1\}$ be a sequence of positive reals. Write

$$\mathbf{P}\left\{\frac{|T_n|}{A_n} > \varepsilon\right\} = \mathbf{E}\mathbf{P}_\varepsilon\left\{\frac{|Y_n|}{A_n} > \varepsilon\right\} \leqslant \mathbf{P}\{Q_n > \Omega_n\} + \mathbf{E}\mathbf{1}_{\{Q_n \leqslant \Omega_n\}} \mathbf{P}_\varepsilon\left\{\frac{|Y_n|}{A_n} > \varepsilon\right\}.$$

Further, there exists an absolute constant C such that

$$\mathbf{P}_\varepsilon\left\{\frac{|Y_n|}{A_n} > \varepsilon\right\} \leqslant \exp\left\{-C \frac{\varepsilon^2 A_n^2}{\sum_{k=1}^{k_n} a_{n,k}^2 X_{n,k}^2}\right\} = \exp\left\{-C \frac{\varepsilon^2 A_n^2}{Q_n B_n^2}\right\} = \exp\left\{-C \frac{\varepsilon^2 C_n^2}{Q_n}\right\}.$$

We deduce that

$$\mathbf{P}\left\{\frac{|T_n|}{A_n} > \varepsilon\right\} \leqslant \mathbf{P}\{Q_n > \Omega_n\} + \exp\left\{-C \frac{\varepsilon^2 C_n^2}{\Omega_n}\right\}.$$

It follows that if

$$\text{(a)} \quad \sum_{n=1}^{\infty} \mathbf{P}\{Q_n > \Omega_n\} < \infty, \quad \text{(b)} \quad \sum_{n=1}^{\infty} \exp\left\{-C \frac{\varepsilon^2 C_n^2}{\Omega_n}\right\} < \infty$$

then $T_n/A_n \xrightarrow{\text{c.c.}} 0$. Choosing in particular (with $L > 1$)

$$\Omega_n = C_n^2/(L \log n),$$

shows that $T_n/A_n \xrightarrow{\text{c.c.}} 0$, provided that

$$\sum_{n=1}^{\infty} \mathbf{P}\{Q_n > \lambda C_n^2/\log n\} < \infty$$

for any $\lambda > 0$. To connect the last sum with the sum in Theorem 3, we use Rosenthal's inequality. Recall that we assumed for $1 \leqslant k \leqslant k_n, n \geqslant 1$ that $\mathbf{E}X_{n,k}^2 = 1$, and for some $p \geqslant 2$, $X_{n,k} \in L^{2p}$. Put

$$Y_{n,k} = a_{n,k}^2(X_{n,k}^2 - 1), \quad 1 \leqslant k \leqslant k_n, \ n \geqslant 1,$$

then for sufficiently large n we have

$$\begin{aligned}\mathbf{P}\{Q_n > \lambda C_n^2/\log n\} &= \mathbf{P}\left\{\frac{\sum_{k=1}^{k_n} a_{n,k}^2 X_{n,k}^2}{B_n^2} > \lambda \frac{A_n^2}{B_n^2 \log n}\right\} = \mathbf{P}\left\{\sum_{k=1}^{k_n} a_{n,k}^2 X_{n,k}^2 > \lambda A_n^2 \log n\right\} \\ &\leqslant \mathbf{P}\left\{\sum_{k=1}^{k_n} a_{n,k}^2 (X_{n,k}^2 - 1) > \frac{\lambda}{2} A_n^2 \log n\right\} \leqslant \left(\frac{\mathbf{E}|\sum_{k=1}^{k_n} Y_{n,k}|^p}{(\frac{\lambda}{2} A_n^2 \log n)^p}\right).\end{aligned}$$

Now, by Rosenthal's inequality

$$\mathbf{E}\left|\sum_{k=1}^{k_n} Y_{n,k}\right|^p \leqslant \left(C_0 \frac{p}{\log p}\right)^p \left\{\left|\mathbf{E}\left(\sum_{k=1}^{k_n} Y_{n,k}\right)^2\right|^{p/2} + \sum_{k=1}^{k_n} \mathbf{E}|Y_{n,k}|^p\right\},$$

where C_0 is an absolute constant. But

$$\mathbf{E}\left|\sum_{k=1}^{k_n} Y_{n,k}\right|^2 = \sum_{k=1}^{k_n} a_{n,k}^4 \mathbf{E}(X_{n,k}^2 - 1)^2 \leqslant C\|X\|_4^4 \sum_{k=1}^{k_n} a_{n,k}^4,$$

so that

$$\begin{aligned}\mathbf{E}\left|\sum_{k=1}^{k_n} Y_{n,k}\right|^p &\leqslant \left(C_0 \frac{p}{\log p}\right)^p \left\{\left(C\|X\|_4^4 \sum_{k=1}^{k_n} a_{n,k}^4\right)^{p/2} + \|X\|_{2p}^{2p} \sum_{k=1}^{k_n} a_{n,k}^{2p}\right\} \\ &\leqslant C_p \max(\|X\|_4^4, \|X\|_{2p}^{2p}) \left(\sum_{k=1}^{k_n} a_{n,k}^4\right)^{p/2}.\end{aligned}$$

Therefore,

$$\mathbf{P}\{Q_n > \lambda C_n^2/\log n\} \leqslant C_p \max(\|X\|_4^4, \|X\|_{2p}^{2p}) \left(\frac{(\sum_{k=1}^{k_n} a_{n,k}^4)^{p/2}}{(\frac{\lambda}{2} A_n^2 \log n)^p}\right).$$

This completes the proof of Theorem 3. $\square$

Proof of Theorem 4. Let $Y_1, \ldots, Y_n$ be independent symmetric random variables, $S_n = Y_1 + \cdots + Y_n$. One part of the Hoffmann-Jørgensen (1974) inequality states that

$$\mathbf{P}\{|S_n| > 3^p t\} \leqslant C_p \mathbf{P}\left\{\max_{1 \leqslant k \leqslant n} |X_k| > t\right\} + C_p\{\mathbf{P}(|S_n| > t)\}^{2^p} \tag{5}$$

for any integer $p \geqslant 1$, where C_p is a constant depending on p. By (5) we have

$$\mathbf{P}\{|T_n| > 3^p \varepsilon A_n\} \leqslant DC_p \sum_{k=1}^{k_n} \mathbf{P}\{|Da_k X| > \varepsilon A_n\} + C_p(\mathbf{P}\{|T_n| > \varepsilon A_n\})^{2^p}. \tag{6}$$

Choosing p large enough and summing (6) for $n = 1, 2, \ldots$ we get

$$\sum_{n=1}^{\infty} \mathbf{P}\{|T_n| > 3^p \varepsilon A_n\} \leqslant DC_p \sum_{\substack{1 \leqslant k \leqslant k_n \\ n \geqslant 1}} \mathbf{P}\{|X| > \varepsilon A_n/Da_k\} + C_p \sum_{n=1}^{\infty} (\mathbf{P}\{|T_n| > \varepsilon A_n\})^{2^p}.$$

Assumptions (a) and (b) therefore imply (c). Conversely if (c) is true, then

$$\begin{aligned}
\mathbf{P}\{|T_n|>\varepsilon A_n\} &\geqslant \frac{1}{2}\mathbf{P}\left\{\max_{1\leqslant k\leqslant k_n}|a_kX_k|\geqslant \varepsilon A_n\right\}=\frac{1}{2}\left[1-\mathbf{P}\left\{\max_{1\leqslant k\leqslant k_n}|a_kX_k|<\varepsilon A_n\right\}\right]\\
&=\frac{1}{2}\left[1-\prod_{k=1}^{k_n}(1-\mathbf{P}\{|a_kX_k|\geqslant\varepsilon A_n\})\right]\geqslant\frac{1}{2}\left[1-\prod_{k=1}^{k_n}\mathrm{e}^{-\mathbf{P}\{|a_kX_k|\geqslant\varepsilon A_n\}}\right]\\
&=\frac{1}{2}\left[1-\mathrm{e}^{-\sum_{k=1}^{k_n}\mathbf{P}\{|a_kX_k|\geqslant\varepsilon A_n\}}\right]:=\frac{1}{2}[1-\mathrm{e}^{-\lambda_n}].
\end{aligned}$$

From this estimate and (c) follows that λ_n tends to 0, and then the chain of estimates can be continued as

$$\tfrac{1}{2}[1-\mathrm{e}^{-\lambda_n}]=\tfrac{1}{2}[\lambda_n+\mathcal{O}(\lambda_n^2)]\geqslant\tfrac{1}{4}\lambda_n,$$

for any integer n sufficiently large. Therefore, for n large

$$\mathbf{P}\{|T_n|>\varepsilon A_n\}\geqslant\tfrac{1}{4}\lambda_n.$$

And consequently (c) implies $\sum_n\lambda_n<\infty$, which is exactly (a). □

Proof of Theorem 5. The proof is based on a convexity argument enabling us to use the Gaussian randomization technique. First of all, there is no loss of generality in assuming that for any $n\geqslant 1$ and $1\leqslant k\leqslant k_n$ we have

$$|X_{n,k}|\leqslant 1 \quad \text{a.s.}$$

Let $\mathscr{X}'$ be an independent copy of $\mathscr{X}$ defined on a joint probability space $(\Omega',\mathscr{A}',\mathbf{P}')$ with corresponding expectation symbol $\mathbf{E}'$. Write $T_n'=\sum_{k=1}^{k_n}a_{n,k}X_{n,k}'$. Let $\boldsymbol{\varepsilon}=\{\varepsilon_{n,k},1\leqslant k\leqslant k_n,n\geqslant 1\}$ be a triangular array of independent Rademacher random variables defined on a joint probability space $(\Omega_\varepsilon,\mathscr{A}_\varepsilon,\mathbf{P}_\varepsilon)$, with corresponding expectation symbol $\mathbf{E}_\varepsilon$. Similarly, let $\mathbf{g}=\{g_{n,k},1\leqslant k\leqslant k_n,n\geqslant 1\}$ be a triangular array of independent $\mathscr{N}(0,1)$ distributed random variables defined on a joint probability space $(\Omega_g,\mathscr{A}_g,\mathbf{P}_g)$, with corresponding expectation symbol $\mathbf{E}_g$. Let A be any real number and consider the convex non-decreasing function $\varphi_A(x)=(x-A)^+$. If X is any random variable, then for any positive real a, $a\mathbf{P}\{X>A+a\}\leqslant\mathbf{E}\varphi_A(X)$. Applying this for $A=A_n\varepsilon=a$ and $X=T_n$ and then using Jensen's inequality lead to

$$\begin{aligned}
(\varepsilon A_n)\mathbf{P}\{T_n>2\varepsilon A_n\}&\leqslant\mathbf{E}\varphi_{\varepsilon A_n}(T_n)=\mathbf{E}\varphi_{\varepsilon A_n}(T_n-\mathbf{E}'T_n')\\
&\leqslant\mathbf{E}\mathbf{E}'\varphi_{\varepsilon A_n}(T_n-T_n')=\mathbf{E}\mathbf{E}_\varepsilon\varphi_{\varepsilon A_n}\left(\sum_{k=1}^{k_n}a_{n,k}\varepsilon_{n,k}X_{n,k}\right)\\
&=\mathbf{E}\mathbf{E}_\varepsilon\varphi_{\varepsilon A_n}\left(\frac{\sum_{k=1}^{k_n}a_{n,k}\varepsilon_{n,k}(\mathbf{E}_g|g_{n,k}|)X_{n,k}}{(2/\pi)^{1/2}}\right)\\
&\leqslant\mathbf{E}\mathbf{E}_\varepsilon\mathbf{E}_g\varphi_{\varepsilon A_n}\left(\frac{\sum_{k=1}^{k_n}a_{n,k}\varepsilon_{n,k}|g_{n,k}|X_{n,k}}{(2/\pi)^{1/2}}\right)\\
&=\mathbf{E}\mathbf{E}_g\varphi_{\varepsilon A_n}\left(\frac{\sum_{k=1}^{k_n}a_{n,k}g_{n,k}X_{n,k}}{(2/\pi)^{1/2}}\right).
\end{aligned}\tag{7}$$

In the last equality we used the fact that $\{\varepsilon_{n,k}|g_{n,k}|,1\leqslant k\leqslant k_n,n\geqslant 1\}\stackrel{\mathscr{D}}{=}\{g_{n,k},1\leqslant k\leqslant k_n,n\geqslant 1\}$. Applying it now to $A=A_n\varepsilon=a$ and $X=-T_n$, and arguing similarly also gives

$$(\varepsilon A_n)\mathbf{P}\{-T_n>2\varepsilon A_n\}\leqslant\mathbf{E}\mathbf{E}_g\varphi_{\varepsilon A_n}\left(\frac{\sum_{k=1}^{k_n}a_{n,k}g_{n,k}X_{n,k}}{(2/\pi)^{1/2}}\right).\tag{8}$$

As $\mathbf{P}\{|T_n|>2\varepsilon A_n\}\leqslant\mathbf{P}\{T_n>2\varepsilon A_n\}+\mathbf{P}\{-T_n>2\varepsilon A_n\}$, we obtain from (7) and (8)

$$(\varepsilon A_n)\mathbf{P}\{|T_n|>2\varepsilon A_n\}\leqslant 2\mathbf{E}\mathbf{E}_g\varphi_{\varepsilon A_n}\left(\frac{\sum_{k=1}^{k_n}a_{n,k}g_{n,k}X_{n,k}}{(2/\pi)^{1/2}}\right).$$

But,

$$\begin{aligned}\mathbf{E}_g\varphi_{\varepsilon A_n}\left(\frac{\sum_{k=1}^{k_n}a_{n,k}g_{n,k}X_{n,k}}{(2/\pi)^{1/2}}\right) &= \int_{\varepsilon A_n}^{\infty}\mathbf{P}\left\{\mathcal{N}(0,1)>\frac{(2/\pi)^{1/2}u}{V_n}\right\}\mathrm{d}u\\ &= \frac{V_n}{\sqrt{2/\pi}}\int_{(\sqrt{2/\pi}\varepsilon A_n)/V_n}^{\infty}\mathbf{P}\{\mathcal{N}(0,1)>v\}\,\mathrm{d}v\\ &= \frac{V_n}{\sqrt{2/\pi}}\int_{(\sqrt{2/\pi}\varepsilon A_n)/V_n}^{\infty}\int_v^{\infty}\mathrm{e}^{-w^2/2}\frac{\mathrm{d}w}{\sqrt{2\pi}}\,\mathrm{d}v\\ &= \frac{V_n}{2}\int_{(\sqrt{2/\pi}\varepsilon A_n)/T_n}^{\infty}R(v)\mathrm{e}^{-v^2/2}\,\mathrm{d}v\leqslant\sqrt{\frac{\pi}{2}}\frac{V_n}{2}R\left(\frac{\sqrt{2/\pi}\varepsilon A_n}{V_n}\right)\mathrm{e}^{-\varepsilon^2A_n^2/\pi V_n^2}\\ &\leqslant\frac{\pi V_n}{4}\mathrm{e}^{-\varepsilon^2A_n^2/\pi V_n^2}.\end{aligned}$$

Therefore,

$$\mathbf{P}\{|T_n|>2\varepsilon A_n\}\leqslant\mathbf{E}\frac{\pi V_n}{4\varepsilon A_n}\mathrm{e}^{-\varepsilon^2A_n^2/\pi V_n^2}.$$

We now make use of the boundedness assumption on the sequence $\mathscr{X}$. The above inequality becomes in this case

$$\mathbf{P}\{|T_n|>2\varepsilon A_n\}\leqslant\frac{\pi}{4\varepsilon}\mathbf{E}\mathrm{e}^{-\varepsilon^2A_n^2/\pi V_n^2},$$

since $V_n^2=\sum_{k=1}^{k_n}a_{n,k}^2X_{n,k}^2\leqslant\sum_{k=1}^{k_n}a_{n,k}^2\leqslant A_n^2$ a.s. Put for $m=1,2,\dots$:

$$J_m=\{n: m\leqslant A_n/V_n<m+1\}.$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty}\mathbf{P}\{|T_n|>2\varepsilon A_n\}&\leqslant\frac{\pi}{4\varepsilon}\sum_{m=1}^{\infty}\mathbf{E}\sum_{n\in J_m}\mathrm{e}^{-\varepsilon^2A_n^2/\pi V_n^2}\leqslant\frac{\pi}{4\varepsilon}\sum_{m=1}^{\infty}\mathbf{E}[\sharp\{J_m\}\mathrm{e}^{-\varepsilon^2m^2/2\pi}]\mathrm{e}^{\varepsilon^2m^2/2\pi}-(\varepsilon^2m^2/\pi)\\ &\leqslant\frac{\pi}{4\varepsilon}\mathbf{E}\sup_{m\geqslant1}[\sharp\{J_m\}\mathrm{e}^{-\varepsilon^2m^2/2\pi}]\left[\sum_{m=1}^{\infty}\mathrm{e}^{-\varepsilon^2m^2/2\pi}\right]\\ &\leqslant C_\varepsilon\mathbf{E}\sup_{m\geqslant1}\frac{\sharp\{n: m<A_n/V_n\leqslant m+1\}}{\exp\{\varepsilon^2m^2/2\pi\}}.\end{aligned}$$

This completes the proof of Theorem 5. $\square$

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