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A LAW OF THE ITERATED LOGARITHM FOR ARITHMETIC FUNCTIONS

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Abstract. Let $X, X_1, X_2, \dots$ be a sequence of centered iid random variables. Let $f(n)$ be a strongly additive arithmetic function such that $\sum_{p<n} \frac{f^2(p)}{p} \to \infty$ and put $A_n = \sum_{p<n} \frac{f(p)}{p}$. If $\mathbf{E}X^2 < \infty$ and f satisfies a Lindeberg-type condition, we prove the following law of the iterated logarithm:

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^N f(n)X_n}{A_N\sqrt{2N\log\log N}} \overset{a.s.}{=} \|X\|_2.$$

We also prove the validity of the corresponding weighted strong law of large numbers in L^1.

1. Main results

Let $X, X_1, X_2, \dots$ be i.i.d. random variables with $\mathbf{E}X = 0$, $\mathbf{E}X^2 < \infty$. In this paper we study the weighted LIL for (X_n), i.e. the relation

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^N d_n X_n}{\sqrt{2D_N^2 \log\log D_N^2}} \overset{a.s.}{=} \|X\|_2, \tag{1.1}$$

where $D_N^2 = \sum_{k=1}^N d_k^2$. By the Hartman-Wintner law of the iterated logarithm, relation (1.1) holds if $d_n = 1$ for all $n \geq 1$, and assuming slightly stronger moment conditions for X, (1.1) will actually hold for a large class of weight sequences (d_n). For example, a Skorohod embedding argument similar to the one used in Fisher [4] and Weber [11] shows that if $\mathbf{E}X^2 \log_+ |X| < \infty$ and

$$D_n \gg n, \quad d_n = O(D_n n^{-\gamma}) \tag{1.2}$$

for some $\gamma > 0$, then (1.1) holds. Condition (1.2) covers, with $\gamma = 1/2$, all sequences (d_n) regularly varying with a positive exponent, and because of the arbitrary value of γ, it leaves a lot of room for irregular sequences (d_n) as well. If we assume only $\mathbf{E}X^2 < \infty$, the situation changes radically. In this case condition (1.2) does not suffice for the LIL (1.1), and in addition to the order of magnitude of d_n and D_n, the distribution of the sequence $\{d_n/D_n,\ n \geq 1\}$ becomes crucial for the validity of (1.1). As a consequence, (1.1) fails for many irregular sequences (d_n), and in fact,

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proving (1.1) for an explicitly given (irregular) sequence (d_n) is generally a delicate problem. The situation is similar for the weighted strong law of large numbers

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N d_n X_n}{\sum_{n=1}^N d_n} \overset{a.s.}{=} \|X\|_1 \tag{1.3}$$

under $\mathbf{E}|X| < \infty$. Assuming slightly higher moment assumptions such as $\mathbf{E}|X|\log_+|X| < \infty$, one can give simple sufficient conditions for (1.3), similar to (1.2), but assuming only $X \in L_1$, the validity of (1.3) again becomes a delicate problem. In view of the special role irregular sequences and their distributions play in the theory, it is natural to ask if typical irregular sequences in number theory, such as additive and multiplicative arithmetic functions, can be used as coefficients in the LIL (1.1) and SLLN (1.3). The purpose of this paper is to study the case of additive functions and to prove a positive answer under natural conditions on f.

Let $f(n)$, $n = 1, 2, \ldots$, be a strongly additive arithmetic function, namely a function f satisfying

$$\begin{aligned} f(mn) &= f(m) + f(n), \qquad (m,n) = 1, \\ f(p^\alpha) &= f(p), \qquad p \text{ a prime}, \quad \alpha = 2, 3, \ldots. \end{aligned} \tag{1.4}$$

It follows that $f(n) = \sum_p f(p)\chi(p|n)$, so that f is completely determined by its values taken over the primes. (Here, and in the sequel, $\sum_p$ denotes the summation along the primes, χ denotes the indicator function and $m|n$ means that the integer m is a divisor of the integer n.) A typical example is $\omega(n)$, the number of different prime factors of n. Put

$$A_n = \sum_{p<n} \frac{f(p)}{p}, \qquad B_n = \sum_{p<n} \frac{f^2(p)}{p}. \tag{1.5}$$

By the classical central limit theorem of Erdős and Kac [3], if $|f(p)| = O(1)$ and $B_n \to \infty$, then we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N : f(n) \le A_N + xB_N^{1/2}\} = (2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2} du. \tag{1.6}$$

The same conclusion holds for unbounded $f(p)$, provided

$$\lim_{n\to\infty} \frac{1}{B_n} \sum_{\{p<n,\, |f(p)| \ge \varepsilon B_n^{1/2}\}} \frac{f^2(p)}{p} = 0 \qquad \text{for any } \varepsilon > 0. \tag{1.7}$$

(See Kubilius [7], Shapiro [9].) Condition (1.7) is the analogue of the Lindeberg condition of classical probability theory and, as Kubilius [7] showed, it is also necessary for (1.6) under mild regularity conditions on the sequence B_n. Condition (1.7) is satisfied if, for example,

$$f(p) = o(B_p^{1/2}), \qquad B_p \to \infty. \tag{1.8}$$

The last relation is stronger than (1.7), but it is still sharp: Halberstam [5] proved that replacing the o by O in (1.8), the central limit theorem (1.6) becomes generally false. For additional limit theorems related to (1.6), see Kubilius [7]; for an alternative approach via the theory of mixing random variables, see Philipp [8]. The purpose of this paper is to show that under condition (1.8) the sums $\sum_{n\le N} f(n)X_n$ satisfy the LIL for any centered i.i.d. sequence X_n with finite variances and the strong law of large numbers for any i.i.d. sequence X_n with finite

mean. This result establishes a connection between two different types of probabilistic behavior of arithmetic functions, namely the "density" type distribution result (1.6) and the almost sure asymptotic behavior of $\sum_{k=1}^{N} f(k)X_k$ with random X_k. As we are going to see, the key arithmetic property behind our results is a bound for the frequency of large values of f, and in fact a byproduct of our argument will be a large deviation result corresponding to (1.6).

Theorem 1.1. *Assume that $f \geq 0$ and condition* (1.8) *is satisfied. Then for any sequence $X, X_1, X_2, \dots$ of centered, independent, identically distributed, integrable random variables we have*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} f(n)X_n}{\sum_{n=1}^{N} f(n)} \overset{a.s.}{=} \mathbf{E}\, X. \tag{1.9}$$

Theorem 1.2. *Under the conditions of Theorem* 1.1*, for any sequence $X, X_1, X_2, \dots$ of centered, independent, identically distributed, square integrable random variables we have*

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} f(n)X_n}{A_N\sqrt{2N\log\log N}} \overset{a.s.}{=} \|X\|_2. \tag{1.10}$$

In view of the law of large numbers of Jamison-Orey-Pruitt [6], for the proof of Theorem 1.1 it would suffice to verify the arithmetical condition

$$\limsup_{t\to\infty} \frac{\#\big\{N : \sum_{n=1}^{N} f(n) \leq tf(N)\big\}}{t} < \infty. \tag{1.11}$$

Conversely, the validity of Theorem 1.1 implies (1.11). However, we could not find a direct argument for (1.11); instead we will use a suitable randomization of the function $f(n)$ in Theorems 1.1 and 1.2 and will obtain the theorems through studying the randomized function. This idea goes back to Weber [13], and our main tool will be Lemma 4.1 in [13] on the number of divisors of random sums S_n.

As we will see, (1.8) implies $B_n = o(A_n^2)$, and thus (1.6) describes the distribution of $f(n)$ in a short interval $(A_N(1-o(1)), A_N(1+o(1))$ around A_N. On the other hand, relation (1.11) can be equivalently written as

$$\#\{N : f(N) \geq cNA_N\} \ll 1/c$$

(see Lemma 2.1 below), and thus (1.11) is a large deviation result corresponding to (1.6). Our theorems show the interesting fact that such a result is valid under the same condition (1.8) required for the validity of the weak limit theorem (1.6).

Since (1.7) is the precise condition for the central limit theorem (1.6) for f, the question arises if (1.7) is also sufficient for the validity of Theorems 1.1 and 1.2. If $f(p)$ does not fluctuate too wildly, e.g. if

$$\sup_{n\leq p,p'\leq n^2} f(p)/f(p') = O(1), \tag{1.12}$$

then this is true, as the proofs of our theorems will show. It is possible that (1.7) is sufficient without regularity conditions, but this remains open.

2. Preparatory lemmas

Although Theorems 1.1 and 1.2 are formulated under condition (1.8), we will arrange the proofs to show that the Lindeberg condition (1.7) plus the mild regularity

condition (1.12) also suffice. For this reason, in the sequel we will assume (1.7) instead of (1.8) and will indicate the places where (1.8) or the additional condition (1.12) is required.

Lemma 2.1. *Under* (1.7) *we have*

$$\sum_{m=1}^{n} f(m) \sim nA_n, \qquad \sum_{m=1}^{n} f^2(m) \sim nA_n^2. \tag{2.1}$$

Proof. Fix $\varepsilon > 0$ and let us observe that by $f \geq 0$ and the Lindeberg condition (1.7) we have for $n \geq n_0(\varepsilon)$

$$\begin{aligned} B_n = \sum_{p<n} \frac{f^2(p)}{p} &= \sum_{\{p<n,\, f(p)\geq \varepsilon B_n^{1/2}\}} \frac{f^2(p)}{p} + \sum_{\{p<n,\, f(p)<\varepsilon B_n^{1/2}\}} \frac{f^2(p)}{p} \\ &\leq \varepsilon B_n + \varepsilon B_n^{1/2} \sum_{p<n} \frac{f(p)}{p} = \varepsilon B_n + \varepsilon B_n^{1/2} A_n \end{aligned}$$

and consequently

$$B_n \leq \left(\frac{\varepsilon}{1-\varepsilon}\right)^2 A_n^2.$$

Thus we proved

$$B_n = o(A_n^2). \tag{2.2}$$

Next we observe that

$$\sum_{p<n} \frac{1}{p} = \log\log n + c_0 + o(1)$$

for some constant c_0 and thus

$$\sum_{n^\alpha \leq p<n} \frac{1}{p} = O(1)$$

for any $0 < \alpha < 1$. Hence fixing $\varepsilon > 0$ and using (1.7) we get for $n \geq n_0(\varepsilon)$

$$\begin{aligned} B_n - B_{[n^\alpha]} &= \sum_{\{[n^\alpha]\leq p<n,\, f(p)<\varepsilon B_n^{1/2}\}} \frac{f^2(p)}{p} + \sum_{\{[n^\alpha]\leq p<n,\, f(p)\geq \varepsilon B_n^{1/2}\}} \frac{f^2(p)}{p} \\ &\leq \varepsilon^2 B_n \sum_{[n^\alpha]\leq p<n} \frac{1}{p} + \varepsilon B_n \leq K\varepsilon B_n, \end{aligned}$$

where K is a constant depending only on α. Thus we have

$$B_n - B_{[n^\alpha]} = o(B_n),$$

and hence by a well-known result of Kubilius (see Theorem 12.1 of Elliott [2]) we get

$$\sum_{m=1}^{n} (f(m) - A_n) = o(nB_n^{1/2}), \qquad \sum_{m=1}^{n} (f(m) - A_n)^2 \sim nB_n,$$

whence (2.1) follows in view of (2.2).

The next lemma is Jamison-Orey-Pruitt's characterization [6, Theorem 3] of the weighted strong law of large numbers in L^1.

Lemma 2.2. *Let $\{w_k, k \geq 1\}$ be a sequence of positive reals and put $W_n = \sum_{k=1}^n w_k$. Then the relation*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N w_n X_n}{\sum_{n=1}^N w_n} \overset{a.s.}{=} \mathbf{E}\, X \tag{2.3}$$

holds for all sequences $\mathcal{X} = \{X, X_1, X_2, \ldots\}$ of centered, independent, identically distributed, integrable random variables if and only if

$$\limsup_{t\to\infty} \frac{N(t)}{t} < \infty, \tag{2.4}$$

where $N(t) = \#\{n : W_n \leq t w_n\}$.

Although (2.4) characterizes the weighted strong law of large numbers, we shall need a less elegant, but more adapted form of it. According to [6, Theorem 2, and the remark at the bottom of p. 41], the conditions

$$\mathbf{E}\, X^2 \int_{y\geq|X|} \frac{N(y)}{y^3} dy < \infty \tag{2.5}$$

and $\mathbf{E}\,|X| < \infty$ imply (2.3), hence (2.4). Also, by the first half of the proof of Theorem 3 in [6, p. 42], relation (2.4) and $\mathbf{E}\,|X| < \infty$ imply (2.5), so that:

Lemma 2.3. *Under the assumption $\mathbf{E}\,|X| < \infty$, (2.4) and (2.5) are equivalent.*

This observation will be crucial in the sequel. Trying to use the original characterization (2.4) we could only establish the results in the spaces $L \log^\varepsilon L$, ε arbitrarily small but strictly positive.

The next lemma comes from Weber [13, Lemma 2.1] and estimates (2.5) in Weber [13]. We also refer to Weber [14] for a precise study of the distribution of values of the divisors of i.i.d. random sums S_n, and to Weber [13] for the investigation of an extremal divisor case.

Lemma 2.4. *Let $\varepsilon = \{\varepsilon_i, i \geq 1\}$ be a Bernoulli sequence (i.e. a sequence of i.i.d. random variables with $\mathbf{P}\{\varepsilon_i = 0\} = \mathbf{P}\{\varepsilon_i = 1\} = 1/2$) and put $S_n = \varepsilon_1 + \ldots + \varepsilon_n$, $n \geq 1$. There exist two absolute constants $d_0 < \infty$ and $C < \infty$, such that for any $n \geq d \geq d_0$,*

$$\mathbf{E}\,\{\mathbf{1}_{d|S_n}\} \leq C(\tfrac{1}{n^{1/2}} + \tfrac{1}{d}).$$

For proving Theorem 1.2, we shall need a weighted version of the usual LIL. The following result is implicit in Fisher [4, see Corollary 3.4 and lines 8-11 on p. 178].

Lemma 2.5. *Let $\{w_k, k \geq 1\}$ be a sequence of positive reals and put $T_n = \sum_{k=1}^n w_k^2$. Assume that*

$$\limsup_{t\to\infty} \frac{1}{t} \#\{n : T_n \leq t w_n^2\} < \infty. \tag{2.6}$$

Then, for any sequence $\mathcal{X} = \{X, X_1, X_2, \ldots\}$ of centered, independent, identically distributed, square integrable random variables we have

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^N w_n X_n}{\sqrt{2 T_N \log\log T_N}} \overset{a.s.}{=} \|X\|_2. \tag{2.7}$$

One recognizes in (2.6) condition (2.4) for the weights w_n^2 and in view of Lemma 2.3 it follows that under $\mathbf{E}X^2 < \infty$, (2.6) is equivalent to

$$\mathbf{E}\, X^4 \int_{y \geq X^2} \frac{\Lambda(y)}{y^3} dy < \infty, \tag{2.8}$$

where $\Lambda(y) = \#\{n : T_n \leq y w_n^2\}$.

3. Proofs

Proof of Theorem 1.1. Clearly $\sum_{[n/2] \leq p < n} 1/p \leq \sum_{[n/2] \leq j < n} 1/j = O(1)$, and thus using (1.7), (2.2) and the Cauchy-Schwarz inequality we get

$$\begin{aligned} A_n - A_{[n/2]} &= \sum_{\{[n/2] \leq p < n,\, f(p) \geq B_n^{1/2}\}} \frac{f(p)}{p} + \sum_{\{[n/2] \leq p < n,\, f(p) < B_n^{1/2}\}} \frac{f(p)}{p} \\ &\leq \left(\sum_{\{p < n,\, f(p) \geq B_n^{1/2}\}} \frac{f^2(p)}{p} \right)^{1/2} \left(\sum_{[n/2] \leq p < n} \frac{1}{p} \right)^{1/2} \\ &\quad + B_n^{1/2} \sum_{[n/2] \leq p < n} \frac{1}{p} = O(B_n^{1/2}) = o(A_n) \end{aligned} \tag{3.1}$$

which shows that A_n is slowly varying. Assume now $\mathbf{E}|X| < \infty$ and put

$$L(t) = \#\{n : \sum_{k=1}^{n} f(k) \leq t f(n)\}.$$

According to Lemmas 2.2 and 2.3, in order to prove Theorem 1.1, it suffices to prove

$$\mathbf{E}\, X^2 \int_{y \geq |X|} \frac{L(y)}{y^3} dy < \infty. \tag{3.2}$$

To establish (3.2), we use a probabilistic trick employed by Weber in [13]. Put

$$f_1(n) = \sum_{d_0 \leq p \leq n^{1/4}} f(p) \chi(p|n),$$

where d_0 is the same constant as in Lemma 2.4 of [13]. Using $f \geq 0$, (1.12), the slow variation of A_n and the asymptotics for $\sum_{p<n} p^{-1}$ used in the proof of Lemma 2.1 we get for sufficiently large n

$$A_n \geq \frac{1}{2} A_{2n} \geq \frac{1}{2} \sum_{n^{1/4} \leq p < 2n} \frac{f(p)}{p} \geq \frac{1}{2} \min_{n^{1/4} \leq p \leq n} f(p) \sum_{n^{1/4} \leq p \leq n} \frac{1}{p} \geq K \max_{n^{1/4} \leq p \leq n} f(p)$$

with some constant $K > 0$, and thus we get, using the fact that n has at most 4 different prime factors exceeding $n^{1/4}$,

$$|f(n) - f_1(n)| \leq \sum_{p \leq d_0} f(p) + 4 \max_{n^{1/4} \leq p \leq n} f(p) \ll A_n. \tag{3.3}$$

(Note that this is the only point in the proof where the smoothness condition (1.12) is used; if instead of the Lindeberg condition (1.7) plus (1.12) we assume

(1.8), relation (3.3) is obvious in view of (2.2).) Let $(\Omega, \mathcal{A}, \mathbf{P})$ be the probability space on which the sequence $X_1, X_2, \ldots$ is defined, and consider the product space $(\Omega, \mathcal{A}, \mathbf{P}) \times (\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbf{P}})$, where the second space supports a Bernoulli sequence $\{\varepsilon_i,\, i \geq 1\}$. Let $\widetilde{\mathbf{E}}$ denote expectation in $(\widetilde{\Omega}, \widetilde{\mathcal{A}}, \widetilde{\mathbf{P}})$ and set $S_n = \varepsilon_1 + \ldots + \varepsilon_n$, $n = 1, 2, \ldots$. Then by (2.1)

$$L(t) \leq \#\big\{n : nA_n \leq Ctf(n)\big\} \leq \#\big\{n : S_n A_{S_n} \leq Ctf(S_n)\big\}, \tag{3.4}$$

and this is true for any $t > 0$, simply because the graph of the random walk $\{S_n, n \geq 1\}$ replicates all positive integers with possible multiplicities. By the strong law of large numbers $\lim_{n\to\infty} S_n/n \overset{a.s.}{=} 1/2$, and thus

$$S_n A_{S_n} \sim (n/2)A_{n/2} \sim (n/2)A_n \quad \text{a.s.}$$

Here we used the fact that A_n is slowly varying and thus by the uniform convergence theorem for slowly varying functions (see e.g. Bingham et al. [1, Theorem 1.2.1]) if $\lambda_n \sim \mu_n$ are two integer sequences, then $A_{\lambda_n} \sim A_{\mu_n}$. Thus if $\Omega_\eta = \{S_n A_{S_n} \geq \eta n A_n, \forall n \geq 1\}$, then for some $\eta > 0$, $\mathbf{P}(\Omega_\eta) > 0$. Also, reading (3.4) on Ω_η gives

$$L(t) \leq \#\big\{n : nA_n \leq (Ct/\eta)f(S_n)\big\} \quad \text{on } \Omega_\eta, \text{ for all } t > 0. \tag{3.5}$$

But for all $t > 0$

$$\begin{aligned} \frac{1}{t}\#\big\{n : nA_n \leq tf(S_n)\big\} &\leq 1 + \frac{1}{t}\#\big\{n \geq t : nA_n \leq tf(S_n)\big\} \\ &= 1 + \frac{1}{t}\sum_{n\geq t} \chi\{\,(nA_n)^2 \leq t^2 f^2(S_n)\} \leq 1 + t\sum_{n\geq t}\frac{f^2(S_n)}{(nA_n)^2}. \end{aligned} \tag{3.6}$$

Further, by Lemma 2.4 there exists a constant C^* such that

$$\begin{aligned} \widetilde{\mathbf{E}}\, f_1^2(S_n) &= \widetilde{\mathbf{E}}\Big\{\sum_{d_0\leq p\leq (S_n)^{1/4}} f(p)\chi(p|S_n)\Big\}^2 \leq \widetilde{\mathbf{E}}\Big\{\sum_{d_0\leq p\leq n^{1/4}} f(p)\chi(p|S_n)\Big\}^2 \\ &= \sum_{d_0\leq p_{i_1}, p_{i_2}\leq n^{1/4}} f(p_{i_1})f(p_{i_2})\widetilde{\mathbf{E}}\,\chi(p_{i_1}p_{i_2}|S_n) \\ &\leq C^*\Big\{\sum_{p_{i_1}, p_{i_2}\leq n^{1/4}} \frac{f(p_{i_1})f(p_{i_2})}{p_{i_1}p_{i_2}}\Big\} \leq C^*\big\{\sum_{p\leq n^{1/4}}\frac{f(p)}{p}\big\}^2 \leq C^* A_n^2, \end{aligned} \tag{3.7}$$

provided n is sufficiently large, which from now on we assume. From (3.3), (3.7) and Minkowski's inequality it follows that

$$\widetilde{\mathbf{E}}\, f^2(S_n) \leq C' A_n^2. \tag{3.8}$$

Therefore

$$\widetilde{\mathbf{E}}\,\frac{1}{t}\#\big\{n : nA_n \leq tf(S_n)\big\} \leq 1 + t\sum_{n\geq t}\frac{\widetilde{\mathbf{E}}\, f^2(S_n)}{(nA_n)^2} \leq 1 + C't\sum_{n\geq t}\frac{1}{n^2} \leq C'' < \infty. \tag{3.9}$$

It follows that

$$\mathbf{E}\,\widetilde{\mathbf{E}}\, X^2 \int_{y\geq |X|} \frac{\#\big\{n : nA_n \leq yf(S_n)\big\}}{y^3} dy \leq C''\mathbf{E}\, X^2 \int_{y\geq |X|}\frac{1}{y^2}dy \leq C''\mathbf{E}\,|X| < \infty.$$

Also, in view of (3.5) and Fubini's Theorem

$$\widetilde{\mathbf{E}}\,\chi(\Omega_\eta)\cdot\mathbf{E}\,X^2\int_{y\ge|X|}\frac{L(y)}{y^3}dy\le\mathbf{E}\,\widetilde{\mathbf{E}}\,X^2\int_{y\ge|X|}\frac{\#\{n:nA_n\le(Cy/\eta)f(S_n)\}}{y^3}dy$$
$$\le C'''\mathbf{E}\,|X|<\infty.$$

As

$$\widetilde{\mathbf{E}}\,\chi(\Omega_\eta)\cdot\mathbf{E}\,X^2\int_{y\ge|X|}\frac{L(y)}{y^3}dy=\mathbf{P}\{\chi(\Omega_\eta)\}\mathbf{E}\,X^2\int_{y\ge|X|}\frac{L(y)}{y^3}dy,$$

condition (3.2) is obtained and Theorem 1.1 is thus proved. □

Remark. It is quite interesting to observe by the randomization argument we used in the above proof that the term "$\frac{1}{p}$" in the summand of A_n appears in (3.7) as the expectation of a random factor.

Proof of Theorem 1.2. In view of Lemma 2.5 and the equivalence of (2.6) and (2.8), it suffices to verify that

$$\mathbf{E}\,X^4\int_{y\ge X^2}\frac{\Lambda(y)}{y^3}dy<\infty,$$

where

$$\Lambda(y)=\#\{n:\sum_{k=1}^{n}f^2(k)\le yf^2(n)\}.$$

The proof being very similar, we only mention the modifications. We replace $f_1(n)$ by

$$f_2(n)=\sum_{d_0\le p\le n^{1/8}}f(p)\chi(p|n),$$

where d_0 is the same constant as in Lemma 2.4. Similarly to (3.3), we get

$$|f(n)-f_2(n)|\ll A_n. \tag{3.10}$$

By using the same randomization argument as above and applying (2.1), we see that the set Ω_η^* defined by

$$\Omega_\eta^*=\{S_nA_{S_n}^2\ge\eta nA_n^2,\forall n\ge1\}$$

has positive probability for some $\eta>0$, and on Ω_η^* we have for all $y>0$

$$\Lambda(y)\le\#\{n:nA_n^2\le(C_0y/\eta)f^2(S_n)\}$$

for some positive constant C_0. Instead of (3.6) we have

$$\frac{1}{t}\#\{n:nA_n^2\le tf^2(S_n)\}\le1+\frac{1}{t}\#\{n\ge t:nA_n^2\le tf^2(S_n)\}$$
$$=1+\frac{1}{t}\sum_{n\ge t}\chi\{n^2A_n^4\le t^2f^4(S_n)\}\le1+t\sum_{n\ge t}\frac{f^4(S_n)}{n^2A_n^4}.$$

Now,

(3.11)

$$\begin{aligned}\widetilde{\mathbf{E}}\, f_2^4(S_n) &= \widetilde{\mathbf{E}}\Big\{\sum_{d_0\le p\le (S_n)^{1/8}} f(p)\chi(p|S_n)\Big\}^4 \le \widetilde{\mathbf{E}}\Big\{\sum_{d_0\le p\le n^{1/8}} f(p)\chi(p|S_n)\Big\}^4\\ &= \sum_{d_0\le p_{i_1},p_{i_2},p_{i_3},p_{i_4}\le n^{1/8}} f(p_{i_1})f(p_{i_2})f(p_{i_3})f(p_{i_4})\widetilde{\mathbf{E}}\,\chi(p_{i_1}p_{i_2}p_{i_3}p_{i_4}|S_n)\\ &\le C^*\Big\{\sum_{p_{i_1},p_{i_2},p_{i_3},p_{i_4}\le n^{1/8}} \frac{f(p_{i_1})f(p_{i_2})f(p_{i_3})f(p_{i_4})}{p_{i_1}p_{i_2}p_{i_3}p_{i_4}}\Big\} \le C^*\Big\{\sum_{p\le n^{1/8}} \frac{f(p)}{p}\Big\}^4\\ &\le C^* A_n^4,\end{aligned}$$

provided n is sufficiently large, which from now on we assume. From (3.10) and (3.11) it follows that

$$\widetilde{\mathbf{E}}\, f^4(S_n) \le C' A_n^4.$$

Thus instead of (3.9) we get

$$\widetilde{\mathbf{E}}\,\frac{1}{t}\#\big\{n : nA_n^2 \le t f^2(S_n)\big\} \le 1 + t\sum_{n\ge t}\frac{\widetilde{\mathbf{E}}\, f^4(S_n)}{n^2 A_n^4} \le 1 + C' t\sum_{n\ge t}\frac{1}{n^2} \le C'' < \infty.$$

To conclude, we now operate exactly as at the end of the previous proof. □

Remark. The reduction argument based on (3.3) we used to treat the case of additive arithmetical functions is no longer valid when passing to multiplicative functions, e.g. the usual divisor function d. However, the argument applies to the truncated divisor function $d_1(n) = \sum_{d\le n^{1/4}} \chi(d|n)$, and gives for any sequence $\mathcal{X}$ of centered, independent, identically distributed, integrable random variables

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N d_1(n) X_n}{\sum_{n=1}^N d_1(n)} \overset{a.s.}{=} \mathbf{E}\, X.$$

A similar result can be obtained for the LIL with the truncated divisor function $d_2(n) = \sum_{d\le n^{1/8}} \chi(d|n)$. We omit the details of proofs, which are quite similar to the above.

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