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The functional central limit theorem for a family of GARCH observations with applications

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ABSTRACT

We consider polynomial GARCH(p, q) variables which define an important subclass of Duan's augmented GARCH(p, q) processes. We prove functional central limit theorems for the observations as well as for the volatility process under the assumption of finite second moments. The results imply the convergence of CUSUM, MOSUM and Dickey–Fuller statistics under optimal conditions.

1. Introduction

Statistical properties of financial data have received considerable attention in the literature during the last three decades. One of the most often used models is the GARCH(p, q) sequence introduced by Engle (1982) and Bollerslev (1986). We say that $\{y_k,\ k \in \mathbb{Z}\}$ is a GARCH(p, q) sequence if it satisfies the equations

$$y_k = \sigma_k \varepsilon_k \tag{1.1}$$

and

$$\sigma_k^2 = \omega + \sum_{1\le i\le p} \alpha_i y_{k-i}^2 + \sum_{1\le j\le q} \beta_j \sigma_{k-j}^2, \tag{1.2}$$

$$\omega > 0, \quad \alpha_i \ge 0, \quad 1 \le i \le p, \quad \beta_j \ge 0, \quad 1 \le j \le q. \tag{1.3}$$

The necessary and sufficient condition for the existence of a unique solution of (1.1) and (1.2) was obtained by Nelson (1990) for $p = q = 1$ and by Bougerol and Picard (1992a,b) in the general case. Without loss of generality we can assume that $\min(p, q) \ge 2$. To state the result of Bougerol and Picard (1992a,b) we put

$$\boldsymbol{\tau}_n = \left(\beta_1 + \alpha_1\varepsilon_n^2, \beta_2, \ldots, \beta_{q-1}\right) \in \mathbb{R}^{q-1}, \qquad \boldsymbol{\zeta}_n = \left(\varepsilon_n^2, 0, \ldots, 0\right) \in \mathbb{R}^{q-1}$$

and $\boldsymbol{\alpha} = (\alpha_2, \ldots, \alpha_{p-1}) \in \mathbb{R}^{p-2}$. Next we define a sequence of $(p+q-1) \times (p+q-1)$ matrices $\boldsymbol{A}_n$ written in block form as

$$\boldsymbol{A}_n = \begin{bmatrix} \boldsymbol{\tau}_n & \beta_q & \boldsymbol{\alpha} & \alpha_p \\ \boldsymbol{I}_{q-1} & 0 & 0 & 0 \\ \boldsymbol{\zeta}_n & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_{p-2} & 0 \end{bmatrix}$$

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where I_j is an identity matrix of size j. A norm of a $d \times d$ matrix $\boldsymbol{M}$ is given by

$$\|\boldsymbol{M}\| = \sup\left\{\|\boldsymbol{M}\boldsymbol{x}\|_d : \boldsymbol{x} \in R^d \text{ and } \|\boldsymbol{x}\|_d = 1\right\},$$

where $\|\cdot\|_d$ is the Euclidean norm in R^d. We say that a solution to (1.1) and (1.2) is non-anticipative if y_k is independent of $\sigma(\{\varepsilon_j, j > k\})$. We can now state the result of Bougerol and Picard (1992a,b).

Theorem 1.1. *If (1.3) holds, $E\log\|\boldsymbol{A}_0\| < \infty$ and*

$$\{\varepsilon_k,\ k \in \mathbb{Z}\} \quad \textit{are independent, identically distributed random variables,} \tag{1.4}$$

then (1.1) and (1.2) have a unique, non-anticipative, stationary and ergodic solution if and only if

$$\gamma_L = \inf_{0\le n<\infty} \frac{1}{n+1} E\log\|\boldsymbol{A}_0\boldsymbol{A}_1\dots\boldsymbol{A}_n\| < 0.$$

We note that γ_L is the top Lyapunov exponent of the sequence $\boldsymbol{A}_n$. For further discussion on the existence of a GARCH(p,q) sequence we refer the reader to Berkes et al. (2004). We also note that the moment condition $E\log\|\boldsymbol{A}_0\| < \infty$ in Theorem 1.1 is satisfied if $E\log(\varepsilon_0^2 + 1) < \infty$.

Duan (1997) defined the so-called augmented GARCH process which is a unification of several extensions of the GARCH model (1.1) and (1.2). We shall say that $\{y_k, -\infty < k < \infty\}$ is an augmented GARCH(p,q) sequence if (1.1) and the volatility equation

$$\Lambda(\sigma_k^2) = \sum_{i=1}^{p} g_i(\varepsilon_{k-i}) + \sum_{j=1}^{q} c_j(\varepsilon_{k-j})\Lambda(\sigma_{k-j}^2)$$

hold, where $\Lambda(\cdot)$, $g_i(\cdot)$ $(i = 1,\dots,p)$ and $c_j(\cdot)$ $(j = 1,\dots,q)$ are real valued measurable functions and Λ^{-1} exists. Motivated by the Box–Cox transformation of the original data, the function $\Lambda(\cdot)$ typically comprises the cases $\Lambda(x) = x^\delta$, $\delta > 0$, or $\Lambda(x) = \log x$.

In the present paper we consider augmented GARCH sequences with $\Lambda(x) = x^\delta$, i.e. where the volatility process is given by the equation

$$\sigma_k^{2\delta} = \sum_{i=1}^{p} g_i(\varepsilon_{k-i}) + \sum_{j=1}^{q} c_j(\varepsilon_{k-j})\sigma_{k-j}^{2\delta}, \quad \delta > 0. \tag{1.5}$$

We shall call such a process $\{y_k, -\infty < k < \infty\}$ a polynomial GARCH sequence. In order to assure that the left-hand side in (1.5) is non-negative we assume that

$$g_i(\varepsilon_0) \ge 0 \quad (i = 1,\dots,p) \quad \text{and} \quad c_j(\varepsilon_0) \ge 0 \quad (j = 1,\dots,q). \tag{1.6}$$

This framework includes GARCH(p,q), power GARCH(p,q) (cf. Carrasco and Chen (2002)) or asymmetric power GARCH(p,q) (cf. Ding et al. (1993)). For several further examples see also Carrasco and Chen (2002). Our goal is to establish functional central limit theorems (FCLTs) for the partial sum processes of the random variables

$$z_k^{(1)} = |y_k|^\delta \quad z_k^{(2)} = |y_k|^\delta \operatorname{sign}(y_k) \quad \text{and} \quad z_k^{(3)} = \sigma_k^\delta. \tag{1.7}$$

Our main result in Chapter 2 will show that

$$S_n^{(\ell)}(t) = \frac{1}{n^{1/2}} \sum_{1\le i\le nt} (z_i^{(\ell)} - Ez_0^{(\ell)}) \quad (\ell \in \{1,2,3\}, t \in [0,1])$$

converges weakly to a Brownian motion under second moment assumptions. The proof is short and relies on a basic theorem of Billingsley (1968) who extended the earlier work of Ibragimov (1962). The major advantage of our approach is that it frees the FCLT from the rather restrictive smoothness and moment conditions required by the earlier theory. In Section 2 we discuss the earlier approaches in this field. We will also give a simple necessary and sufficient criterion for the existence of a strictly stationary solution to (1.1) and (1.5) with finite second moments. Our results are stated in Section 2. In Section 3 we give applications of the FCLT for statistical inference on time series models such as change point detection and unit root testing. The proofs are given in Section 4.

2. Results

Lemma 2.1 gives a necessary and sufficient condition for the existence of a strictly stationary solution of (1.5) with $E\sigma_0^{2\delta} < \infty$. The proof is standard but for the sake of completeness it will be given in Section 4.

Lemma 2.1. *Assume that (1.4) and (1.6) hold. Then a unique non-anticipative and strictly stationary solution of (1.5) with* $E\sigma_0^{2\delta} < \infty$ *exists if and only if*

$$\sum_{i=1}^{p} Eg_i(\varepsilon_0) < \infty \quad \text{and} \quad \sum_{j=1}^{q} Ec_j(\varepsilon_0) < 1. \tag{2.1}$$

Assuming without loss of generality that $p = q$*, we have under (2.1)*

$$\sigma_k^{2\delta} = \sum_{m=1}^{\infty} \sum_{1\le l_1,\ldots,l_m\le p} g_{l_m}(\varepsilon_{k-l_1-l_2-\cdots-l_m}) \prod_{i=1}^{m-1} c_{l_i}(\varepsilon_{k-l_1-l_2-\cdots-l_i}) \quad \text{a.s.}$$

and

$$E\sigma_0^{2\delta} = \left(\sum_{s=1}^{p} Eg_s(\varepsilon_0)\right) \frac{1}{1-(Ec_1(\varepsilon_0)+\cdots+Ec_p(\varepsilon_0))}.$$

Remark 2.1. Using the independence of ε_0 and σ_0 it follows that a necessary and sufficient condition for $E|y_0|^{2\delta} < \infty$ is (2.1) and $E|\varepsilon_0|^{2\delta} < \infty$.

Specializing to the GARCH(p, q) model we recover a result of Bollerslev (1986) and Bougerol and Picard (1992b), p. 122. They show that a strictly stationary solution to (1.1) and (1.2) satisfying $Ey_0^2 < \infty$ exists if and only if

$$(\alpha_1 + \cdots + \alpha_p)E\varepsilon_0^2 + (\beta_1 + \cdots + \beta_q) < 1.$$

For notational convenience we will write from now on z_k when we mean one of the random variables $z_k^{(\ell)}$ $(\ell = 1, 2, 3)$ in (1.7). The same convention will be used for $S_n^{(\ell)}(t)$.

Theorem 2.1. *Assume that (1.4) holds. If a strictly stationary solution of (1.1) and (1.5) with* $Ez_0^2 < \infty$ *exists, then the sum*

$$\tau^2 = \operatorname{Var} z_0 + 2 \sum_{1\le k<\infty} \operatorname{Cov}(z_0, z_k) \tag{2.2}$$

is convergent and

$$S_n(t) \xrightarrow{\mathcal{D}[0,1]} \tau W(t),$$

where $\{W(t),\ 0 \le t \le 1\}$ *is a standard Brownian motion.*

To the best of our knowledge, there is no functional CLT for this general class of heteroscedastic time series. However, for the convenience of the reader we shall recall some existing results for the GARCH(p, q) models below. The typical approach is to show that a specific dependence property (e.g. mixing, NED, association, etc.) holds and the results are deduced from a more general theory. However, the verification of many well-known dependence properties is rather delicate and requires unnecessary side conditions on the model such as smoothness and higher moments. Assuming more than two moments is clearly undesirable in applications since even in the case of GARCH(p, q) processes, computing the pth moment from the parameters is an open problem with the exception of some integral p's.

A possible method to prove a functional CLT is to use the Markov structure of the model to obtain mixing properties and then to employ the theory of mixing variables to deduce the desired asymptotics. A detailed study of β-mixing properties was given by Carrasco and Chen (2002). Their results cover many augmented GARCH(1,1) models and the PGARCH(p, q) processes. It remains, however, open whether β-mixing holds for the general model (1.1)–(1.5). Note also that the approach of Carrasco and Chen requires fairly restrictive conditions on the ε_k's, such as a continuous and positive density on the real line. Further it is known that β-mixing does not imply the CLT under finite second moments (cf. Bradley (1983)). An overview of different dependence structures occurring in econometric time series models can be found in Ango Nze and Doukhan (2004). Besides mixing, they consider e.g. NED (near-epoch dependence), association and the weak dependence concepts introduced by Doukhan and Louhichi (1999). An elaborate treatment of weak dependence with applications is presented in Dedecker et al. (2007).

For GARCH(p, q) sequences various further dependence structures have been applied in order to obtain FCLTs by different authors and we shall review now the corresponding results. Let us first recall that if $E\varepsilon_0 = 0$, then $\{y_k, -\infty < k < \infty\}$ is a stationary and ergodic martingale difference sequence. Hence by Theorem 23.1 of Billingsley (1968) we obtain a functional CLT under $Ey_0^2 < \infty$. However, if $E\varepsilon_0 \ne 0$, or if we consider the absolute values $|y_k|$ or the conditional variances σ_k^2 the martingale structure no longer applies. Using NED, Davidson (2002) and Hansen (1991) obtained the weak convergence of $n^{-1/2}\sum_{k=1}^{[nt]} y_k$ to a Brownian motion if $Ey_0^4 < \infty$ instead of $Ey_0^2 < \infty$. The weak dependence concepts of Doukhan and Louhichi have been used in Berkes et al. (2004) to obtain a functional CLT for the squared observations y_k^2 under $E|y_0|^{8+\delta} < \infty$. A recent result of Doukhan and Wintenberger (2007) implies that the latter result holds under $E|y_0|^{4+\delta} < \infty$. Giraitis et al. (2007) observed that an ARCH(∞) sequence is associated. This follows from the fact that i.i.d. r.v.'s are associated and that association is inherited by coordinatewise non-decreasing transformations. Applying the well-known result of Newman and Wright (1981), they deduce a functional CLT for the partial sums processes of the squared observations y_k^2 under the sole assumption $Ey_0^4 < \infty$. However, whether association is applicable for the general model considered in this paper is unclear. Since we do not assume that the functions $g_i(\cdot)$ or $c_j(\cdot)$ in (1.5) are non-decreasing, we cannot assume that association is inherited from the driving i.i.d. process. Also the concept of association seems not to be applicable for the original GARCH variables, but only for the squared observations. These results show that the availability of a specific dependence structure often requires fairly restrictive regularity conditions on the underlying process. For example, in the case of the AR-GARCH model all existing Dickey–Fuller tests, as well as the underlying functional CLTs require the existence of 4 moments of the process. Technically, the requirement of the finiteness of higher moments of y_0 is a restriction on the functions $g_i(\cdot)$ and $c_j(\cdot)$ in (1.5). However, as we have already noted, even for the GARCH(p, q) model it is very difficult to connect the moments of y_0 with the specific model (cf. Ling and McAleer (2002)).

Our approach is based on a theorem due to Billingsley (1968). The crucial observation is that augmented GARCH(p, q) sequences can be well approximated with m-dependent random variables. Indeed, truncating the infinite series for $\sigma_k^{2\delta}$ at m (see Lemma 2.1) yields pm-dependent random variables $\sigma_{km}^{2\delta}$. According to Billingsley's result the FCLT follows if $E(\sigma_k^{\delta} - \sigma_{km}^{\delta})^2$ tends to zero sufficiently fast. We note that m-dependence was explored in Giraitis et al. (2000) to obtain a central limit theorem for ARCH(∞) sequences.

3. Applications

Example 3.1. The CUSUM (cumulative sum) statistics defined by

$$C_n = \max_{1\le k\le n} \left| \sum_{1\le i\le k} z_i - \frac{k}{n} \sum_{1\le i\le n} z_i \right|$$

is one of the most often used statistics to test for the stability of $\{z_i, 1 \le i \le n\}$. While with $z_k = \operatorname{sign}(y_k)|y_k|^{\delta}$ we test for stability in the mean, the choice $z_k = |y_k|^{\delta}$ or $z_k = \sigma_k^{\delta}$ is used for testing stability in the volatility. Under the assumptions of Theorem 2.1

$$\frac{C_n}{\tau n^{1/2}} \xrightarrow{\mathcal{D}} \sup_{0\le t\le 1} |B(t)|, \tag{3.1}$$

where $\{B(t),\ 0 \le t \le 1\}$ is a Brownian bridge and τ is defined by (2.2). The MOSUM (moving sum) version of C_n is

$$M_n = \max_{nh<k\le n} \left| \sum_{k-nh\le i\le k} z_i - h \sum_{1\le i\le n} z_i \right|,$$

where $0 < h < 1$. Under the conditions of Theorem 2.1

$$\frac{M_n}{\tau n^{1/2}} \xrightarrow{\mathcal{D}} \sup_{h\le t\le 1} |B(t) - B(t-h)|. \tag{3.2}$$

In order to use (3.1) and (3.2) we need to estimate τ. One could use, for example, the Bartlett estimator (cf. Giraitis et al. (2003)).

For a review of CUSUM and MOSUM we refer the reader to Csörgő and Horváth (1997). Zeileis et al. (2005) provides a comparison between CUSUM and MOSUM.

Example 3.2. CUSUM as well as MOSUM requires the estimation of τ^2. The estimation of τ^2 is not needed if ratio based statistics are used. Following Kim (2000) and Taylor (2005) we define

$$K_n(t) = \left(\frac{t}{1-t}\right)^{1/2} \max_{nt\le k\le n} \left| \sum_{nt<i\le k} z_i - \frac{k-nt}{n(1-t)} \sum_{nt\le i\le n} z_i \right| \Bigg/ \max_{1\le k\le nt} \left| \sum_{1\le i\le k} z_i - \frac{k}{nt} \sum_{1\le i\le nt} z_i \right|.$$

If the conditions of Theorem 2.1 are satisfied, then for any $0 < \delta < 1/2$

$$K_n(t) \xrightarrow{\mathcal{D}[\delta,1-\delta]} \left(\frac{t}{1-t}\right)^{1/2} \sup_{t\le s\le 1} \left| W(s) - W(t) - \frac{s-t}{1-t}(W(1) - W(t)) \right| \Bigg/ \sup_{0\le s\le t} \left| W(s) - \frac{s}{t} W(t) \right|.$$

Functionals of $K_n(t)$ can be considered as ratio based versions of the Kwiatowski et al. (1992) test.

Example 3.3. Starting with $x_0 = 0$ we define

$$x_k = \varrho x_{k-1} + z_k \quad 1 \le k < \infty,$$

where z_k is one of the $z_k^{(i)}$, $i = 1, 2, 3$, defined in (1.7). We assume that $E\varepsilon_0 = 0$. The least square estimator for ϱ is given by

$$\widehat{\varrho}_n = \sum_{1\le k\le n} z_k z_{k-1} \Big/ \sum_{1\le k\le n} z_{k-1}^2.$$

If $\varrho = 1$ (unit root), then under the conditions of Theorem 2.1

$$n(\widehat{\varrho}_n - 1) \xrightarrow{\mathcal{D}} \left(\int_0^1 W(s)\mathrm{d}W(s)\right) \Big/ \int_0^1 W^2(s)\mathrm{d}s. \tag{3.3}$$

The result in (3.3) is the asymptotics for the Dickey–Fuller test with augmented GARCH(p, q) errors. The same result was obtained by Ling et al. (2003) assuming $p = q = 1$, $E\varepsilon_0^4 < \infty$ and the existence of a symmetric density of ε_0.

4. Proofs

Proof of Lemma 2.1. We can assume without loss of generality that $p = q$. Repeated application of (1.5) yields

$$\sigma_k^{2\delta} = \sum_{1\le l_1\le p} g_{l_1}(\varepsilon_{k-l_1}) + \sum_{1\le l_1,l_2\le p} c_{l_1}(\varepsilon_{k-l_1}) g_{l_2}(\varepsilon_{k-l_1-l_2}) + \sum_{1\le l_1,l_2\le p} c_{l_1}(\varepsilon_{k-l_1}) c_{l_2}(\varepsilon_{k-l_1-l_2}) \sigma_{k-l_1-l_2}^{2\delta}$$

and more generally for $n \ge 2$

$$\sigma_k^{2\delta} = \sum_{m=1}^{n} \sum_{1\le l_1,\ldots,l_m\le p} g_{l_m}(\varepsilon_{k-l_1-l_2-\cdots-l_m}) \prod_{i=1}^{m-1} c_{l_i}(\varepsilon_{k-l_1-l_2-\cdots-l_i}) + \sum_{1\le l_1,\ldots,l_n\le p} \prod_{i=1}^{n} c_{l_i}(\varepsilon_{k-l_1-l_2-\cdots-l_i}) \sigma_{k-l_1-\cdots-l_n}^{2\delta}. \tag{4.1}$$

Due to (1.6) the random variable

$$X_k = \sum_{m=1}^{\infty} \sum_{1\le l_1,\ldots,l_m\le p} g_{l_m}(\varepsilon_{k-l_1-l_2-\cdots-l_m}) \prod_{i=1}^{m-1} c_{l_i}(\varepsilon_{k-l_1-l_2-\cdots-l_i}) \tag{4.2}$$

is well defined. These observations suggest setting $\sigma_k^{2\delta} = X_k$ as the solution of (1.5). It is easy to see that formally this defines a strictly stationary (and ergodic) solution of equation (1.5). It remains to show that X_k is a.s. finite, which will follow if we show that $EX_k < \infty$. Observe that by (1.4)

$$E\left(\sum_{1\le l_1,\ldots,l_m\le p} g_{l_m}(\varepsilon_{k-l_1-l_2-\cdots-l_m}) \prod_{i=1}^{m-1} c_{l_i}(\varepsilon_{k-l_1-l_2-\cdots-l_i})\right) = GC^{m-1}, \tag{4.3}$$

with $G = Eg_1(\varepsilon_0) + \cdots + Eg_p(\varepsilon_0)$ and $C = Ec_1(\varepsilon_0) + \cdots + Ec_p(\varepsilon_0)$. By the monotone convergence theorem and (2.1) it follows that

$$EX_k = \frac{G}{1-C} < \infty.$$

Assume now that $\hat{\sigma}_k^{2\delta}$ is another stationary non-anticipative solution of (1.5), such that $E\hat{\sigma}_k^{2\delta} < \infty$. Obviously $\hat{\sigma}_k^{2\delta}$ has to satisfy (4.1) for every $n \ge 1$ and thus we get by (1.4), (1.6) and the stationarity of $\hat{\sigma}_k^{2\delta} - \sigma_k^{2\delta}$, that

$$E|\hat{\sigma}_k^{2\delta} - \sigma_k^{2\delta}| \le E\left(\sum_{1\le l_1,\ldots,l_n\le p} \prod_{i=1}^{n} c_{l_i}(\varepsilon_{k-l_1-l_2-\cdots-l_i}) |\hat{\sigma}_{k-l_1-\cdots-l_n}^{2\delta} - \sigma_{k-l_1-\cdots-l_n}^{2\delta}|\right) = C^n E|\hat{\sigma}_0^{2\delta} - \sigma_0^{2\delta}|.$$

Since $C < 1$ it follows that $E|\hat{\sigma}_k^{2\delta} - \sigma_k^{2\delta}| = 0$, and hence the solution is unique. The proof of Lemma 2.1 is now complete. □

Proof of Theorem 2.1. We show the functional CLT for the sequence $\{z_k^{(3)},\ k \ge 1\}$. The proof for $\{z_k^{(1)},\ k \ge 1\}$ and $\{z_k^{(2)},\ k \ge 1\}$ is similar. Let $\eta_k = \sigma_k^{\delta} - E\sigma_0^{\delta}$ and note that by Lemma 2.1 $\eta_k = f(\ldots, \varepsilon_{k-2}, \varepsilon_{k-1})$ where f is a measurable mapping from the space of infinite sequences into $\mathbb{R}$. According to Theorem 21.1 of Billingsley (1968), for the proof of Theorem 2.1 it is enough to find measurable mappings f_ℓ from $\mathbb{R}^\ell$ into $\mathbb{R}$ such that

$$\sum_{1\le \ell<\infty} \left(E(\eta_0 - \eta_{0\ell})^2\right)^{1/2} < \infty, \tag{4.4}$$

where $\eta_{0\ell} = f_\ell(\varepsilon_{-\ell}, \varepsilon_{-\ell+1}, \ldots, \varepsilon_{-1})$.

Every positive integer ℓ can be written as

$$\ell = pK + r, \tag{4.5}$$

where K, r are integers satisfying $K \geq 0$ and $0 \leq r \leq p-1$. Next we define

$$\sigma_{0\ell}^{2\delta} = \sum_{1\leq m\leq K} \sum_{1\leq l_1,\ldots,l_m\leq p} g_{l_m}(\varepsilon_{-l_1-l_2-\cdots-l_m}) \prod_{i=1}^{m-1} c_{l_i}(\varepsilon_{-l_1-l_2-\cdots-l_i}),$$

where K is defined in (4.5). (We define $\sigma_{0\ell} = 0$ if $K = 0$, i.e. $0 \leq \ell \leq p-1$.) Let

$$\eta_{0\ell} = \sigma_{0\ell}^{\delta} - E\sigma_0^{\delta}.$$

Then $\eta_{0\ell}$ is a function of $\varepsilon_{-\ell}, \ldots, \varepsilon_{-1}$ and we have

$$|\eta_0 - \eta_{0\ell}| \leq (\sigma_0^{2\delta} - \sigma_{0\ell}^{2\delta})^{1/2} = \left(\sum_{K+1\leq m<\infty} \sum_{1\leq l_1,\ldots,l_m\leq p} g_{l_m}(\varepsilon_{-l_1-l_2-\cdots-l_m}) \prod_{i=1}^{m-1} c_{l_i}(\varepsilon_{-l_1-l_2-\cdots-l_i}) \right)^{1/2}.$$

By (4.3) it follows that

$$E(\eta_0 - \eta_{0\ell})^2 \leq \frac{G}{1-C} C^K,$$

where K is given in (4.5). This shows (4.4). □

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