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CONVERGENCE OF INTEGRAL FUNCTIONALS OF STOCHASTIC PROCESSES

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We investigate the convergence in distribution of integrals of stochastic processes satisfying a functional limit theorem. We allow a large class of continuous Gaussian processes in the limit. Depending on the continuity properties of the underlying process, local Lebesgue or Riemann integrability is required.

1. INTRODUCTION

Let $\{x_{k,n}, 1 \le k \le n, n = 1,2,\dots\}$ be a triangular array of random elements in $\mathcal{D}[0,1]$, the Skorokhod space of functions on $[0,1]$, and assume that

$$x_{[nt],n} \xrightarrow{\mathcal{D}[0,1]} \Gamma(t), \tag{1.1}$$

where

$$\Gamma(t) \text{ is a continuous Gaussian process on } [0,1]. \tag{1.2}$$

Condition (1.2) means that almost all sample paths of $\Gamma(t)$ are continuous on $[0,1]$. In many applications one needs the relation

$$\frac{1}{n} \sum_{1\le k\le n} T(x_{k,n}) \xrightarrow{\mathcal{D}} \int_0^1 T(\Gamma(s))\,ds \tag{1.3}$$

with some real-valued function T, where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution. This paper seeks to establish (1.3) for various processes $\{x_{k,n}, 1 \le k \le n\}$ under general conditions on the function $T(x)$.

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THEOREM 1.1. *If T is continuous on $(-\infty,\infty)$ and (1.1) and (1.2) are satisfied, then (1.3) holds.*

Under the conditions of Theorem 1.1, the sample paths of $T(\Gamma(s))$, $0 \le s \le 1$ are continuous with probability one, and thus the integral $\int_0^1 T(\Gamma(s))\,ds$ exists pathwise.

In the case when $\Gamma(t)$ is a Brownian motion, a short proof of Theorem 1.1 is provided by Pötscher (2004), using the continuous mapping theorem. His proof works also under (1.2). For the sake of completeness, in Section 3 we give a quick proof of Theorem 1.1 using the Skorokhod–Dudley–Wichura representation theorem.

Park and Phillips (1999), de Jong (2004), Pötscher (2004), and de Jong and Wang (2005) consider extensions of Theorem 1.1 for a larger class of functions T. The following result is due to Pötscher (2004).

THEOREM 1.2. *Let $\Gamma(t)$ be Brownian motion and assume that (1.1) holds and*

$$T \text{ is Borel measurable,} \tag{1.4}$$

$$T \text{ is locally Lebesgue integrable,} \tag{1.5}$$

$$x_{k,n}/(k/n)^{1/2} \text{ has a density function } h_{k,n} \text{ satisfying } h_{k,n}(x) \le K \text{ for all } x,\ 1 \le k \le n, \text{ and } n \ge 1 \text{ with some constant } K. \tag{1.6}$$

Then relation (1.3) holds.

Here the integral $\int_0^1 T(\Gamma(t))\,dt$ is defined pathwise; its existence is established in Karatzas and Shreve (1991, Prop. 6.27). Densities are always meant with respect to the Lebesgue measure. Local Lebesgue integrability of T means that $\int_{-N}^{N} T(x)\,dx$ exists for all N in the Lebesgue sense.

In this note we investigate two extensions of Theorem 1.2. First we consider the case when the limit in (1.1) is Gaussian, but not necessarily a Brownian motion, and then we study the case when the distribution of $x_{k,n}$ is not necessarily smooth, i.e., (1.6) may not hold.

2. RESULTS

Assuming only (1.2), (1.4), and (1.5) we want to define $\int_0^1 T(\Gamma(s))\,ds$ pathwise as a Lebesgue integral. Conditions (1.2) and (1.4) imply that the paths of $T(\Gamma(s))$, $0 \le s \le 1$ are Borel measurable with probability one, but as the next example shows, (1.5) in general is not enough for the existence of $\int_0^1 T(\Gamma(s))\,ds$.

Example 2.1

Let $T(x) = |x|^{-1/2}$ for $x \neq 0$ and $T(0) = 0$ and let $\Gamma(t) = t^2\xi$, $0 \leq t \leq 1$, where ξ is a standard normal random variable. (Or, alternatively, let T be as before and $\Gamma(t) = t^2 W(t)$, $0 \leq t \leq 1$, where $W(t)$, $0 \leq t \leq 1$ is a Brownian motion.) Then $\int_0^1 T(\Gamma(s))\,ds$ does not exist.

Our first result gives a sufficient condition for the existence of $\int_0^1 T(\Gamma(s))\,ds$.

THEOREM 2.1. *If (1.2) and (1.4) hold,*

$$E\Gamma(t) = 0, \qquad \sigma^2(t) = E\Gamma^2(t), \qquad 0 \leq t \leq 1, \tag{2.1}$$

$$\int_0^1 \frac{1}{\sigma^\alpha(t)}\,dt < \infty \quad \textit{with some } 0 < \alpha \leq 1, \tag{2.2}$$

and

$$|x|^{\alpha-1}T(x) \textit{ is locally Lebesgue integrable}, \tag{2.3}$$

then $\int_0^1 T(\Gamma(s))\,ds$ *exists with probability one.*

We note if $\Gamma(t)$ has stationary increments, $\Gamma(0) = 0$ and

$$\int_0^1 \frac{1}{\sigma(t)}\,dt < \infty, \tag{2.4}$$

then the local time of Γ exists with probability one (cf. Geman and Horowitz, 1980, Thm. 22.1). Geman and Horowitz (1980) also point out that under conditions like (2.4) the trajectories of $\Gamma(t)$ oscillate wildly. Hence (2.4) is never satisfied for smooth (e.g., differentiable) Gaussian processes.

Remark 2.1. Although Theorem 2.1 covers a very large class of functions T, from a purely mathematical point of view the question arises what happens if instead of (1.4) we assume only that T is Lebesgue measurable, i.e., the level sets $\{T < c\}$ are Lebesgue measurable for any real c. (A set on the real line is called Lebesgue measurable if it is the union of a Borel set and a subset of a Borel set with measure 0. Unlike the class $\mathcal{B}$ of Borel sets, the class $\mathcal{L}$ of Lebesgue measurable sets has the property that given any set $A \in \mathcal{L}$ with measure 0, all subsets of A belong to $\mathcal{L}$.) If instead of (1.4) we assume only that T is Lebesgue measurable, the pathwise existence of $\int_0^1 T(\Gamma(s))\,ds$ becomes a delicate problem, because the composition of a Lebesgue measurable and a continuous function is in general not Lebesgue measurable. (See Halmos, 1950, p. 83.) If $\Gamma(t)$ has a continuous local time, this problem disappears because, as an argument in Pötscher (2004, p. 5) shows, in this case the integrand in (1.3) differs from a Borel-measurable function only on a set of Lebesgue measure 0, and hence it is Lebesgue measurable. However, the assumptions of Theorem 2.1 do not, in general, imply the existence of a local time for Γ, and thus the measurability of $T(\Gamma(t))$ remains open. To handle this case, let T_n be a sequence of functions such that T_n is continuous in the interval $[-n, n]$, $T_n(x) = 0$ for $|x| > n$, and $\int_{-n}^{n} |T_n(x) - T(x)|\, dx \le 1/n$. (Such a sequence T_n exists by Luzin's theorem (cf. Hewitt and Stromberg, 1969, Thm. 11.36).) As a trivial modification of the proof of Theorem 2.1 shows, in this case $\int_0^1 T_n(\Gamma(s))\, ds$ is a Cauchy sequence of random variables in L_1 norm and hence convergent in L_1. Defining its limit (which is easily shown to be independent of the approximating sequence T_n) as $\int_0^1 T(\Gamma(s))\, ds$, all results of our paper remain valid under the Lebesgue measurability of T. Note that in this case the definition of the integral $\int_0^1 T(\Gamma(s))\, ds$ is *not* pathwise.

Next we show an example where Theorem 2.1 can be used to establish the existence of $\int_0^1 T(\Gamma(s))\, ds$.

Example 2.2

Let $T(x) = |x|^{-1/2}$, $x \neq 0$, $T(0) = 0$, and

$$\Gamma(t) = \int_0^t W(s)\, ds,$$

where $W(t)$ is Brownian motion. Clearly, $\Gamma(t)$ is differentiable and $\sigma^2(t) = t^3/3$. Theorem 2.1 can be used with $\alpha = \frac{7}{12}$ to establish the existence of $\int_0^1 T(\Gamma(s))\, ds$ in this case.

After giving conditions for the existence of $\int_0^1 T(\Gamma(s))\, ds$, we turn to generalizations of Theorem 1.2.

THEOREM 2.2. *Assume that (1.1), (1.2), (1.4), (2.1), (2.2), and (2.3) are satisfied and there exist numbers $c_{i,k} > 0$, $1 \le k \le n$, $n \ge 1$ such that*

$$\lim_{n\to\infty} n c_{k,n}^{\alpha} = \infty \quad \textit{for any fixed } k, \tag{2.5}$$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k_0 \le k \le n} 1/c_{k,n}^{\alpha} < \infty \quad \textit{with some } k_0 \ge 1, \tag{2.6}$$

and

$$\begin{aligned} &x_{k,n}/c_{k,n} \textit{ has a density } h_{k,n}(x) \textit{ satisfying } |x|^{1-\alpha} h_{k,n}(x) \le K \\ &\textit{for all } x,\ 1 \le k \le n,\ n \ge 1 \quad \textit{with some constant } K. \end{aligned} \tag{2.7}$$

Then (1.3) holds.

We now give some applications of Theorem 2.2. Let e_i, $1 \le i < \infty$ be a stationary sequence of Gaussian random variables with $Ee_i = 0$ and $r(i-j) = Ee_i e_j$ such that

$$\lim_{m\to\infty} \frac{1}{(Cm^H L(m))^2} \sum_{1\le i,j\le m} r(i-j) = 1, \tag{2.8}$$

where

$$0 < H < 1,\ C > 0, \text{ and } L \text{ is a slowly varying function at } \infty. \tag{2.9}$$

Let

$$x_{k,n} = \frac{1}{Cn^H L(n)} \sum_{1\le i\le k} e_i, \qquad c_{k,n} = (Ex_{k,n}^2)^{1/2}$$

for any $1 \le k \le n$. According to Lemma 5.1 of Taqqu (1975),

$$x_{[nt],n} \xrightarrow{\mathcal{D}[0,1]} B_H(t),$$

where $B_H(t)$ is a fractional Brownian motion with parameter H. This means that $B_H(t)$ is a continuous Gaussian process with $EB_H(t) = 0$ and

$$EB_H(t)B_H(s) = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}.$$

Thus conditions (1.1), (1.2), (2.1), and (2.4) are satisfied, because $\sigma(t) = (EB_H^2(t))^{1/2} = t^H$, $0 \le t \le 1$. Clearly $x_{k,n}/c_{k,n}$ is standard normal and

$$Ex_{k,n}^2 = \frac{1}{(Cn^H L(n))^2} \sum_{1\le i,j\le k} r(i-j).$$

By $0 < H < 1$ we have for any fixed k

$$\lim_{n\to\infty} \frac{n}{Cn^H L(n)} \Big(\sum_{1\le i,j\le k} r(i-j) \Big)^{1/2} = \infty$$

provided that $\sum_{1\le i,j\le k} r(i-j) > 0$. On the other hand, if $\sum_{1\le i,j\le k} r(i-j) = 0$, then $P(x_{k,n} = 0) = 1$. By (2.8) this can happen only for finitely many k's, independently of n. Drop these 0 terms and use Theorem 2.2 for the rest of the array only. By (2.8) there is a constant k_0 such that

$$Ex_{k,n}^2 \ge \frac{1}{2}\left\{ \left(\frac{k}{n}\right)^H \frac{L(k)}{L(n)} \right\}^2, \quad \text{if } k \ge k_0.$$

Using the properties of slowly varying functions (cf. Bingham, Goldie, and Teugels, 1987, p. 26) we obtain

$$n^{H-1}L(n)\sum_{k_0\le k\le n}\frac{1}{k^H L(k)}\to\frac{1}{1-H}\qquad(n\to\infty).$$

Hence (2.6) holds with $\alpha = 1$, and therefore all conditions of Theorem 2.2 are established with $\alpha = 1$.

The next two examples are from Taqqu (1975).

Example 2.3

If the covariance functions satisfy

$$\lim_{k\to\infty}\frac{r(k)}{k^{2H-2}L_1(k)} = 1\quad\text{with some } \frac{1}{2} < H < 1$$

or

$$\lim_{k\to\infty}\frac{r(k)}{k^{2H-2}L_2(k)} = -1\quad\text{with some } 0 < H < \frac{1}{2} \text{ and } r(0) + 2\sum_{1\le k<\infty} r(k) = 0,$$

where $L_1(x)$ and $L_2(x)$ are slowly varying function at infinity, then (2.8) and (2.9) hold. For the proof we refer to Taqqu (1975).

Example 2.4

Let $\{\varepsilon_k, -\infty < k < \infty\}$ be a sequence of independent, identically distributed standard normal random variables and define

$$e_j = \sum_{1\le k<\infty} k^{H-3/2}\varepsilon_{j-k}\quad\text{with some } \frac{1}{2} < H < 1.$$

It is easy to see that $\{e_j, 1 \le j < \infty\}$ is a stationary Gaussian sequence with $Ee_j = 0$ and covariance function r satisfying

$$\lim_{k\to\infty}\frac{r(k)}{k^{2H-2}} = \int_1^\infty t^{H-3/2}(1-t)^{H-3/2}\,dt. \tag{2.10}$$

Thus (2.8) and (2.9) hold.

We note that the convergence relation (1.3) was also established by Jeganathan (2004) under the conditions of Example 2.4 assuming that T and T^2 are both Lebesgue integrable on the real line. The limit in Jeganathan (2004) is given as a functional of the local time of fractional Brownian motion.

Our next example is from Horváth and Kokoszka (1997).

Example 2.5

Consider the fractional ARIMA (p, d, q) process, which is a parametric model frequently used in modeling of long-memory time series (see, e.g., Brockwell and Davis, 1991, Sec. 13.2). Let $\{\varepsilon_k, -\infty < k < \infty\}$ be a sequence of independent, identically distributed normal random variables with $E\varepsilon_k = 0$ and $\sigma^2 = E\varepsilon_k^2 > 0$. Define the polynomials

$$\Phi_p(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p,$$

$$\Theta_q(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q,$$

with real coefficients ϕ_j, θ_j. As usual, we assume that Φ_p and Θ_q have no common roots and no roots in the closed unit disk. The fractional ARIMA (p, d, q) process is defined as the unique solution $\{e_n\}$ of the equations

$$\Phi_p(\mathcal{B})e_n = \Theta_q(\mathcal{B})(1 - \mathcal{B})^{-d}\varepsilon_n, \qquad -\infty < n < \infty, \tag{2.11}$$

where $\mathcal{B}$ denotes the backward shift operator defined by $\mathcal{B}e_n = e_{n-1}$ and $(1 - \mathcal{B})^{-d}$ is a linear time-invariant filter defined by

$$(1 - \mathcal{B})^{-d}\varepsilon_n = \sum_{0 \le j < \infty} b_j \varepsilon_{n-j}, \tag{2.12}$$

with $\{b_j, 0 \le j < \infty\}$ being the coefficients in the series expansion of $(1 - z)^{-d}$, $|z| < 1$. If $d < \frac{1}{2}$, then the infinite sum in (2.12) converges with probability one, and (2.11) has a unique moving-average solution

$$e_n = \sum_{0 \le j < \infty} c_j \varepsilon_{n-j},$$

with the weights c_j tending to zero at the rate j^{d-1}.

Theorem 13.2.2 of Brockwell and Davis (1991) and Theorem 4.10.1 of Bingham et al. (1987) yield

$$\lim_{k\to\infty} \frac{r(k)}{k^{2d-1}} = \sigma^2 \left| \frac{\Theta_q(1)}{\Phi_p(1)} \right|^2 \frac{\sin(\pi d)}{\pi} \int_0^\infty x^{-2d} e^{-x}\, dx,$$

where $r(k) = Ee_j e_{j+k}$. It is clear that (2.8) and (2.9) hold with $H = d + \frac{1}{2}$ and $L(x) = 1$.

Example 2.6

Let e_i, $-\infty < i < \infty$ be defined as in Example 2.4. It is easy to see that

$$\operatorname{var}\Bigg(\sum_{1 \le i \le k} \sum_{1 \le j \le i} e_j \Bigg) = \sum_{1 \le i, j \le k} ijr(i-j).$$

By (2.10) there is a constant $c_1 > 0$ such that

$$\lim_{n \to \infty} \frac{1}{n^{2H+2}} \sum_{1 \le i, j \le n} ijr(i-j) = c_1^2. \tag{2.13}$$

Let

$$c_{k,n} = \frac{1}{c_1 n^{H+1}} \Bigg(\sum_{1 \le i, j \le k} ijr(i-j) \Bigg)^{1/2}$$

and define

$$x_{k,n} = \frac{1}{c_1 n^{H+1}} \sum_{1 \le i \le k} \sum_{1 \le j \le i} e_j.$$

Because $x_{k,n}/c_{k,n}$ is standard normal, (2.7) holds for $0 < \alpha \le 1$. Also,

$$x_{[nt],n} \xrightarrow{\mathcal{D}[0,1]} \Gamma(t) = \int_0^t B_H(s)\, ds,$$

and hence (1.1) and (1.2) are satisfied. Using the covariance of $B_H(s)$ we get

$$\begin{aligned} \sigma^2(t) = E\Gamma^2(t) &= \int_0^t \int_0^t \frac{1}{2} \{ |u|^{2H} + |s|^{2H} - |u-s|^{2H} \}\, ds\, du \\ &= t^{2H+2} \int_0^1 \int_0^1 \frac{1}{2} \{ |u|^{2H} + |s|^{2H} - |u-s|^{2H} \}\, ds\, du, \end{aligned}$$

and therefore (2.2) holds for all $0 \le \alpha < 1/(1+H)$. By (2.13) we have (2.5) for all k_0 large enough. If

$$\sum_{1 \le i, j \le k} ijr(i-j) > 0, \tag{2.14}$$

then $nc_{k,n}^{\alpha} \to 0$ for any $0 \le \alpha < 1/(H+1)$. (We note that if (2.14) fails, then $P(x_{k,n} = 0) = 1$, so $T(x_{k,n}) = T(0)$ has no effect on the limit.) So all the conditions of Theorem 2.2 are satisfied, and therefore

$$\frac{1}{n} \sum_{1 \le k \le n} T(x_{k,n}) \xrightarrow{\mathcal{D}} \int_0^1 T(\Gamma(t))\, dt,$$

if $|x|^{\alpha-1} T(x)$ is locally Lebesgue integrable with some $0 \le \alpha < 1/(H+1)$.

In all our applications so far, the random variables $x_{k,n}$ in (1.3) were normal. The following example, which extends Examples 2.4 and 2.5, shows that the long memory linear processes

$$y_n = \sum_{0 \le k < \infty} a_k \varepsilon_{n-k}, \qquad n = 1,2,\dots$$

with weights $a_k \sim ck^{-\beta}$, $\frac{1}{2} < \beta < 1$ satisfy the assumptions of Theorem 2.2 even if the generating random variables ε_j are not Gaussian. This extends the results of Pötscher (2004, Sec. 3) to long memory processes.

Example 2.7

Let $\{\varepsilon_k, -\infty < k < \infty\}$ be a sequence of independent, identically distributed random variables with $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$, $E\varepsilon_0^4 < \infty$. Let $\{a_k, k \ge 0\}$ be a positive sequence satisfying

$$a_k \sim \text{const} \cdot k^{-\beta}, \qquad \tfrac{1}{2} < \beta < 1,$$

and let

$$y_j = \sum_{0 \le k < \infty} a_k \varepsilon_{j-k}, \qquad j = 1,2,\dots.$$

Because $\sum a_k^2 < \infty$, the sum defining y_j converges a.s. and $Ey_j = 0$, $Ey_j^2 < \infty$. An easy calculation shows that

$$E(y_1 + \cdots + y_n)^2 \sim An^{3-2\beta}$$

for some constant $A > 0$, and thus letting

$$x_{k,n} = A^{-1/2} n^{-(3/2-\beta)} \sum_{1 \le i \le k} y_i,$$

it follows from Davydov (1970, Thm. 2) that

$$x_{[nt],n} \xrightarrow{\mathcal{D}[0,1]} B_H(t), \qquad H = 3/2 - \beta.$$

We now show that if the characteristic function φ of ε_0 satisfies

$$|\varphi(t)| = O(|t|^{-\gamma}) \quad \text{as } |t| \to \infty$$

for some $\gamma > 0$, then the random variables $x_{n,n}$ have uniformly bounded densities, and thus Theorem 2.2 applies with $c_{k,n} = (Ex_{k,n}^2)^{1/2}$. Our argument follows Pötscher (2004). Let Ψ_n denote the characteristic function of $x_{n,n}$. As is seen from the proof of Lemma 3.1 of Pötscher (2004) (cf. formulas (3.3) and (B.1) there), we have

$$|\Psi_n(s)| \leq \prod_{j=1}^{n} |\varphi(A^{-1/2} n^{-(3/2-\beta)} c_{n-j} s)|,$$

where $c_j = \sum_{i=0}^{j} a_i$. By $E\varepsilon_0 = 0$, $E\varepsilon_0^2 = 1$ we have $\varphi(t) = 1 - t^2/2 + o(t^2)$ as $t \to 0$, and thus $|\varphi(t)| \leq (1 + t^2/4)^{-1}$ in a neighborhood of 0. We now claim that

$$|\varphi(t)| \leq (1 + ct^2)^{-\gamma/4} \quad \text{for all } t$$

with some positive constant c. By the previous remark and the bound $|\varphi(t)| = O(|t|^{-\gamma})$ (and assuming, without loss of generality, that $\gamma \leq 1$) the claimed inequality holds with $c = \frac{1}{4}$ for $|t| \leq t_0$ and $|t| \geq t_1$, provided t_0 is small enough and t_1 is large enough. To prove it for $t_0 < |t| < t_1$ note that $|\varphi(t)| < 1$ for $t \neq 0$ (otherwise $|\varphi|$ would be periodic) and thus by the continuity of φ there exists a constant $\varrho > 0$ such that $|\varphi(t)| \leq 1 - \varrho$ for $t_0 \leq |t| \leq t_1$. Hence choosing c small enough, the claimed inequality holds also for $t_0 \leq |t| \leq t_1$. Because $c_j \sim \text{const} \cdot j^{1-\beta}$ we get, using $|\varphi(t)| \leq 1$,

$$|\Psi_n(s)| \leq \prod_{j=1}^{[n/2]+1} |\varphi(A^{-1/2} n^{-(3/2-\beta)} c_{n-j} s)| \leq (1 + an^{-1}s^2)^{-n\gamma/8}$$

for some constant $a > 0$. Thus

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Psi_n(s)| ds &\leq \int_{-\infty}^{\infty} (1 + an^{-1}s^2)^{-n\gamma/8} ds \\
&= a^{-1/2} n^{1/2} \int_{-\infty}^{\infty} (1 + u^2)^{-n\gamma/8} du \\
&= a^{-1/2} n^{1/2} \left(\int_{|u| \leq 1} + \int_{|u|>1} \right) = a^{-1/2} n^{1/2} (I_1 + I_2), \quad \text{say.}
\end{aligned}$$

Using $1/(1 + u^2) \leq \exp(-c_1 u^2)$ for $|u| \leq 1$ and the substitution $v = n^{1/2} u$ we see that $I_1 \leq C_1 n^{-1/2}$ where C_1 is a constant depending only on γ. On the other hand, for $|u| > 1$ we have

$$(1 + u^2)^{-n\gamma/8} \leq (1 + u^2)^{-1} 2^{-n\gamma/8+1} \leq C_2 n^{-1/2} (1 + u^2)^{-1},$$

where C_2 is a constant depending only on γ. Hence

$$I_2 \leq C_2 n^{-1/2} \int_{-\infty}^{\infty} (1+u^2)^{-1}\, du = C_2 n^{-1/2}\pi,$$

and thus we proved that

$$\int_{-\infty}^{\infty} |\Psi_n(s)|\, ds \leq C \qquad n = 1,2,\ldots,$$

where C is a constant depending only on φ. By a well-known property of characteristic functions (see, e.g., Lukács, 1970, Thm. 3.2.2) it follows that the random variables $x_{n,n}$ have uniformly bounded densities, as claimed.

A minor variation of the preceding argument shows that the assumption $|\varphi(t)| = O(|t|^{-\gamma})$, $\gamma > 0$ can be weakened to

$$\int_{-\infty}^{\infty} |\varphi(t)|^r\, dt < \infty \quad \text{for some integer } r \geq 1,$$

which is the condition assumed in Pötscher (2004). Indeed, using $|\varphi(t)| \leq 1$ and $c_j \sim \mathrm{const}\cdot j^{1-\beta}$ we get, similarly as before,

$$|\Psi_n(s)| \leq \prod_{j=1}^{[n/2]} |\varphi(A^{-1/2} n^{-(3/2-\beta)} c_{n-j} s)| = \prod_{j=1}^{[n/2]} |\varphi(s n^{-1/2} a_{n,j})|,$$

where the $a_{n,j}$ are between positive bounds, independent of n, j. The last relation is very close to formula (B.1) in Pötscher (2004), and from there the proof can be completed by following his reasoning with minor changes.

So far we have replaced convergence to a Brownian motion in Pötscher's Theorem 1.2 with the convergence to a continuous Gaussian process. We showed that we still have the convergence in distribution of the integral functionals. Next we consider the case when (1.6) is not satisfied, i.e., if the distribution of $x_{k,n}$ is not necessarily smooth. The next example shows that (1.3) can fail if only (1.4) and (1.5) are assumed.

Example 2.8

Let $e_1, e_2, \ldots$ be independent, identically distributed random variables with $P(e_1 = 1) = P(e_1 = -1) = \frac{1}{2}$. Let $T(x) = 1$ if x is irrational and $T(x) = 0$ if x is rational. If n is a square number, then

$$\frac{1}{n} \sum_{1\leq k\leq n} T\left(\frac{1}{n^{1/2}} \sum_{1\leq i\leq k} e_i\right) = 0.$$

However,

$$\int_0^1 T(W(t))\,dt = 1 \quad \text{a.s.},$$

where W is a Brownian motion, so (1.3) cannot be true.

Our last result says that without assuming (1.6) or (2.7), the local Lebesgue integrability conditions in Theorems 1.2 and 2.2 should be replaced by the local Riemann integrability of $T(x)$ to have (1.3).

THEOREM 2.3. *If (1.1), (1.2), (1.4), (2.1), and (2.2) are satisfied and*

$$\lim_{h\to 0} \int_{-K}^{K} |x|^{\alpha-1} \sup_{|u|\le h} |T(x+u) - T(x)|\,dx = 0 \quad \text{for all } K > 0, \tag{2.15}$$

then (1.3) holds.

Note that the integral in (2.15) is finite if and only if T is locally bounded, i.e., bounded on bounded intervals. The sufficiency of the last condition is obvious from $0 < \alpha \le 1$; to see the necessity note that if there exists a point x_0 such that T is unbounded in any neighborhood of x_0, then for any fixed $h > 0$ the integrand in (2.15) equals $+\infty$ for $|x - x_0| < h$, and thus the integral is infinite. The integrand is undefined for $x = 0$, but because we mean (2.15) as a Lebesgue integral, this does not cause any problem.

We would like to point out that (2.15) cannot be replaced by

$$\lim_{h\to 0} \int_{-K}^{K} |x|^{\alpha-1} |T(x+h) - T(x)|\,dx = 0 \quad \text{for all } K > 0. \tag{2.16}$$

Indeed, the function T in Example 2.8 is bounded and Lebesgue measurable, and thus it satisfies

$$\lim_{h\to 0} \int_{-K}^{K} |T(x+h) - T(x)|^p\,dx = 0 \quad \text{for all } K > 0, \qquad p \ge 1$$

(see Hewitt and Stromberg, 1969, p. 199). From $0 < \alpha < 1$ and the Hölder inequality it follows that (2.16) is also valid, but according to Example 2.8, (1.3) cannot be true.

Remark 2.2. Condition (2.15) holds if and only if T is locally Riemann integrable, i.e., it is bounded and Riemann integrable on any bounded interval.

3. PROOFS

Proof of Theorem 1.1. By the Skorokhod–Dudley–Wichura representation theorem (cf. Shorack and Wellner, 1986, p. 47) there exist $x^*_{k,n}$, $1 \le k \le n$, and $\Gamma^*_n(t)$, $0 \le t \le 1$, such that

$$\{x_{k,n}, 1 \le k \le n\} \stackrel{\mathcal{D}}{=} \{x^*_{k,n}, 1 \le k \le n\} \quad \text{for each } n, \tag{3.1}$$

$$\{\Gamma(t), 0 \le t \le 1\} \stackrel{\mathcal{D}}{=} \{\Gamma^*_n(t), 0 \le t \le 1\} \quad \text{for each } n, \tag{3.2}$$

and

$$\sup_{0\le t\le 1} |x^*_{[nt],n} - \Gamma^*_n(t)| = o(1) \quad \text{a.s.} \tag{3.3}$$

For any $\varepsilon > 0$ there is N such that

$$P\left\{\sup_{0\le t\le 1} |\Gamma(t)| \ge N/2\right\} \le \varepsilon, \tag{3.4}$$

and therefore by (3.2) and (3.3) there is an integer n_0 such that

$$P\left\{\max_{1\le k\le n} |x^*_{k,n}| \ge N\right\} \le 2\varepsilon, \quad \text{if } n \ge n_0. \tag{3.5}$$

Let

$$T_N(x) = \begin{cases} T(x) & \text{if } |x| \le N \\ 0 & \text{if } |x| > N. \end{cases}$$

By (3.4) and (3.5) we have

$$P\left\{\frac{1}{n}\sum_{1\le k\le n} T(x^*_{k,n}) \ne \frac{1}{n}\sum_{1\le k\le n} T_N(x^*_{k,n})\right\} \le 2\varepsilon \quad \text{if } n \ge n_0$$

and

$$P\left\{\int_0^1 T(\Gamma^*_n(t))\,dt \ne \int_0^1 T_N(\Gamma^*_n(t))\,dt\right\} \le \varepsilon \quad \text{for all } n \ge 1.$$

Hence it is enough to prove that for any $N \ge 1$

$$\int_0^1 T_N(x^*_{[nt],n})\,dt - \int_0^1 T_N(\Gamma^*_n(t))\,dt \to 0 \quad \text{a.s.} \qquad (n \to \infty) \tag{3.6}$$

and

$$\frac{1}{n}\{|T_N(0)| + |T_N(x^*_{n,n})|\} \xrightarrow{P} 0 \qquad (n \to \infty), \tag{3.7}$$

because

$$\frac{1}{n}\sum_{1\le k\le n} T_N(x^*_{k,n}) = \int_0^1 T_N(x^*_{[nt],n})\,dt - \frac{1}{n}T_N(0) + \frac{1}{n}T_N(x^*_{n,n})$$

($x^*_{0,n} = 0$ by definition). We note that T_N is continuous on $[-N, N]$, and therefore it is uniformly continuous on $[-N, N]$. Hence (3.6) follows from (3.3). Relation (3.7) is obvious, because by (3.1)–(3.5) and the continuity of T_N on $[-N, N]$ we have $T_N(x^*_{n,n}) \xrightarrow{\mathcal{D}} T_N(\Gamma(1))$. ■

Proof of Theorem 2.1. Clearly, it is enough to consider the case $T \ge 0$. (Otherwise, write T as the difference between the positive and the negative parts and prove the existence for each part separately.) For any $\varepsilon > 0$ there is $N \ge 0$ such that

$$P\Big\{\sup_{0\le t\le 1} |\Gamma(t)| \ge N\Big\} \le \varepsilon. \tag{3.8}$$

Also, there is a constant c such that

$$\frac{1}{\sqrt{2\pi}}\,u^{1-\alpha}e^{-u^2/2} \le c \quad \text{for all } 0 \le u < \infty.$$

Because $\Gamma(t)$ is normal with $E\Gamma(t) = 0$ and $\sigma^2(t) = E\Gamma^2(t)$ we get, letting $T_N(x)$ denote the function defined in the previous proof,

$$\begin{aligned} E\int_0^1 T_N(\Gamma(s))\,ds &\le \int_0^1\int_{-N}^N T(x)\,\frac{1}{\sqrt{2\pi}\sigma(t)}\,e^{-(x^2/2\sigma^2(t))}\,dx\,dt \\ &\le c\int_0^1\int_{-N}^N T(x)\,\frac{1}{\sigma(t)}\left(\frac{x}{\sigma(t)}\right)^{-(1-\alpha)} dx\,dt \\ &= c\int_0^1 \frac{1}{\sigma^\alpha(t)}\,dt\int_{-N}^N |x|^{\alpha-1}T(x)\,dx < \infty. \end{aligned} \tag{3.9}$$

The proof is complete. ■

Proof of Theorem 2.2. For each $\varepsilon > 0$ there is $N \geq 0$ such that (3.4) holds. So by (1.1) there is an integer n_0 such that

$$P\left\{\max_{1\leq k\leq n} |x_{k,n}| \geq N\right\} \leq 2\varepsilon, \quad \text{if } n \geq n_0. \tag{3.10}$$

Thus

$$P\left\{\sum_{1\leq k\leq n} T(x_{k,n}) \neq \sum_{1\leq k\leq n} T_N(x_{k,n})\right\} \leq 2\varepsilon \quad \text{if } n \geq n_0$$

and

$$P\left\{\int_0^1 T(\Gamma(s))\,ds \neq \int_0^1 T_N(\Gamma(s))\,ds\right\} \leq \varepsilon,$$

where $T_N(x) = T(x)I\{|x| \leq N\}$. Next we show that for any k_0

$$\frac{1}{n}\sum_{1\leq k\leq k_0} T_N(x_{k,n}) \xrightarrow{P} 0. \tag{3.11}$$

Indeed,

$$\begin{aligned}
E|T_N(x_{k,n})| &= \int_{-N}^{N} |T(x)|\frac{1}{c_{k,n}}h_{k,n}\left(\frac{x}{c_{k,n}}\right)dx \\
&= c_{k,n}^{-\alpha}\int_{-N}^{N} |x|^{\alpha-1}|T(x)|\left(\frac{|x|}{c_{k,n}}\right)^{1-\alpha} h_{k,n}\left(\frac{x}{c_{k,n}}\right)dx \\
&\leq c_{k,n}^{-\alpha}K\int_{-N}^{N}|x|^{\alpha-1}|T(x)|\,dx,
\end{aligned}$$

where K is defined in condition (2.7). Using (2.5) we get

$$E\frac{1}{n}\left|\sum_{1\leq k\leq k_0} T_N(x_{k,n})\right| \leq c\sum_{1\leq k\leq k_0}\frac{1}{nc_{k,n}^{\alpha}} \to 0,$$

proving (3.11).

By Luzin's theorem (cf. Hewitt and Stromberg, 1969, Thm. 11.36) and (2.3) for any $\delta > 0$ there is a continuous function T^* on $[-N, N]$ such that

$$\int_{-N}^{N}\left|\frac{T(x)}{|x|^{1-\alpha}} - T^*(x)\right|dx \leq \delta.$$

Let $T_N^*(x) = T^*(x)$ for $|x| \le N$ and 0 otherwise. Using again (2.7) we get

$$\begin{aligned}
&E\frac{1}{n}\left|\sum_{k_0\le k\le n}\{T_N(x_{k,n}) - T_N^*(x_{k,n})|x_{k,n}|^{1-\alpha}\}\right| \\
&\quad\le \frac{1}{n}\sum_{k_0\le k\le n}\frac{1}{c_{k,n}}\int_{-N}^{N}|T(x) - T^*(x)|x|^{1-\alpha}|h_{k,n}\left(\frac{x}{c_{k,n}}\right)dx \\
&\quad= \frac{1}{n}\sum_{k_0\le k\le n}\frac{1}{c_{k,n}}\int_{-N}^{N}\left|\frac{T(x)}{|x|^{1-\alpha}} - T^*(x)\right||x|^{1-\alpha}h_{k,n}\left(\frac{x}{c_{k,n}}\right)dx \\
&\quad= \frac{1}{n}\sum_{k_0\le k\le n}\frac{1}{c_{k,n}^{\alpha}}\int_{-N}^{N}\left|\frac{T(x)}{|x|^{1-\alpha}} - T^*(x)\right|\left|\frac{x}{c_{k,n}}\right|^{1-\alpha}h_{k,n}\left(\frac{x}{c_{k,n}}\right)dx \\
&\quad\le \delta K\frac{1}{n}\sum_{k_0\le k\le n}1/c_{k,n}^{\alpha}, \qquad\qquad \textbf{(3.12)}
\end{aligned}$$

where K is from condition (2.7). Also, (2.1) and (2.2) yield

$$\begin{aligned}
&E\left|\int_0^1\{T_N(\Gamma(s)) - T_N^*(\Gamma(s))|\Gamma(s)|^{1-\alpha}\}\,ds\right| \\
&\quad\le \int_0^1\int_{-N}^{N}|T(x) - T^*(x)|x|^{1-\alpha}|\,\frac{1}{(2\pi\sigma^2(t))^{1/2}}\,e^{-(x^2/2\sigma^2(t))}\,dx\,dt \\
&\quad= \frac{1}{\sqrt{2\pi}}\int_0^1\frac{1}{\sigma^{\alpha}(t)}\int_{-N}^{N}\left|\frac{T(x)}{|x|^{1-\alpha}} - T^*(x)\right|\left(\frac{|x|}{\sigma(t)}\right)^{1-\alpha}e^{-(x^2/2\sigma^2(t))}\,dx\,dt \\
&\quad\le c\int_0^1\frac{1}{\sigma^{\alpha}(t)}\,dt\int_{-N}^{N}\left|\frac{T(x)}{|x|^{1-\alpha}} - T^*(x)\right|dx \\
&\quad\le \delta c\int_0^1\frac{1}{\sigma^{\alpha}(t)}\,dt, \qquad\qquad \textbf{(3.13)}
\end{aligned}$$

where $c = \sup|u|^{1-\alpha}e^{-u^2/2}$. Similarly to (3.11) one can easily show that

$$\frac{1}{n}\sum_{1\le k\le k_0}T_N^*(x_{k,n})|x_{k,n}|^{1-\alpha}\xrightarrow{P}0 \quad \text{for all } k_0. \qquad \textbf{(3.14)}$$

In view of (3.11)–(3.14), it is enough to show that

$$\frac{1}{n}\sum_{1\le k\le n}T_N^*(x_{k,n})|x_{k,n}|^{1-\alpha}\xrightarrow{\mathcal{D}}\int_0^1 T_N^*(\Gamma(s))|\Gamma(s)|^{1-\alpha}\,ds. \qquad \textbf{(3.15)}$$

However, because of the continuity of $T_N^*(x)|x|^{1-\alpha}$ on $[-N,N]$, this is an immediate consequence of (3.1), (3.2), and (3.6). ■

Proof of Theorem 2.3. We use again the Skorokhod–Dudley–Wichura representation theorem, so we assume that (3.1)–(3.3) hold. By (1.1) and (1.2) we also have (3.4) and (3.5). Hence, following the argument in the proof of Theorem 1.1, it is enough to prove

$$\frac{1}{n}\sum_{1\le k\le n} T_N(x^*_{k,n}) \xrightarrow{\mathcal{D}} \int_0^1 T_N(\Gamma(s))\,ds \qquad (n\to\infty) \tag{3.16}$$

for all N, where $T_N(x) = T(x)I\{|x|\le N\}$. As in the proof of Theorem 1.1 we have

$$\frac{1}{n}\sum_{1\le k\le n} T_N(x^*_{k,n}) = \int_0^1 T_N(x^*_{[ns],n})\,ds - \frac{1}{n}T_N(0) + \frac{1}{n}T_N(x^*_{n,n}),$$

and thus

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{1\le k\le n} T_N(x^*_{k,n}) - \int_0^1 T_N(\Gamma^*_n(s))\,ds\right| \\
&\qquad\le \int_0^1 |T_N(x^*_{[ns],n}) - T_N(\Gamma^*_n(s))|\,ds + \frac{1}{n}|T_N(0)| \\
&\qquad\quad + \frac{1}{n}|T_N(x^*_{n,n}) - T_N(\Gamma^*_n(1))| \\
&\qquad\quad + \frac{1}{n}|T_N(\Gamma^*_n(1))|.
\end{aligned}$$

Hence

$$\begin{aligned}
&P\left\{\left|\frac{1}{n}\sum_{1\le k\le n} T_N(x^*_{k,n}) - \int_0^1 T_N(\Gamma^*_n(s))\,ds\right| \ge \varepsilon\right\} \\
&\qquad\le P\left\{\sup_{0\le s\le 1}|x^*_{[ns],n} - \Gamma^*_n(s)| \ge h\right\} + P\left\{\frac{1}{n}|T_N(0)| \ge \varepsilon/4\right\} \\
&\qquad\quad + P\left\{\frac{1}{n}\sup_{|u|\le h}|T_N(\Gamma^*_n(1)+u) - T_N(\Gamma^*_n(1))| \ge \varepsilon/4\right\} \\
&\qquad\quad + P\left\{\frac{1}{n}|T_N(\Gamma^*_n(1))| \ge \varepsilon/4\right\} \\
&\qquad\quad + P\left\{\int_0^1 \sup_{|u|\le h}|T_N(\Gamma^*_n(s)+u) - T_N(\Gamma^*_n(s))|\,ds \ge \varepsilon/4\right\} \\
&\qquad= I_1 + I_2 + \cdots + I_5.
\end{aligned} \tag{3.17}$$

Because the distribution of $\Gamma_n^*(1)$ does not depend on n, we get

$$\lim_{n\to\infty} P\left\{\frac{1}{n}\,|T_N(\Gamma_n^*(1))| \geq \varepsilon/4\right\} = 0.$$

On the other hand,

$$\begin{aligned} E \sup_{|u|\leq h} &|T_N(\Gamma_n^*(1)+u) - T_N(\Gamma_n^*(1))| \\ &= \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2(1))^{1/2}} \sup_{|u|\leq h} |T_N(x+u) - T_N(x)|\, e^{-(x^2/2\sigma^2(1))}\, dx \\ &\leq \frac{1}{\sigma(1)} \int_{-\infty}^{\infty} \sup_{|u|\leq h} |T_N(x+u) - T_N(x)|\, dx, \end{aligned}$$

if $\sigma(1) > 0$, where the last integral exists for all h by condition (2.15). If $\sigma(1) = 0$, then $P\{\Gamma_n^*(1) = 0\} = 1$, and therefore

$$E \sup_{|u|\leq h} |T_N(\Gamma_n^*(1)+u) - T_N(\Gamma_n^*(1))| = \sup_{|u|\leq h} |T_N(u) - T_N(0)|.$$

Thus by the Markov inequality we have in both cases

$$\lim_{n\to\infty} P\left\{\frac{1}{n} \sup_{|u|\leq h} |T_N(\Gamma_n^*(1)+u) - T_N(\Gamma_n^*(1))| \geq \varepsilon/4\right\} = 0$$

for all h. Similarly to (3.13) we have

$$\begin{aligned} E\int_0^1 &\sup_{|u|\leq h} |T_N(\Gamma_n^*(s)+u) - T_N(\Gamma_n^*(s))|\, ds \\ &= \int_0^1 \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2(s))^{1/2}} \sup_{|u|\leq h} |T_N(x+u) - T_N(x)|\, e^{-(x^2/2\sigma^2(s))}\, dx\, ds \\ &\leq c\int_0^1 \frac{1}{\sigma^\alpha(s)}\, ds \int_{-\infty}^{\infty} |x|^{\alpha-1} \sup_{|u|\leq h} |T_N(x+u) - T_N(x)|\, dx. \end{aligned}$$

Here the last integral tends to 0 as $h \to 0$ by (2.15), and thus by (2.2) and the Markov inequality we have

$$\lim_{h\to 0} P\left\{\int_0^1 \sup_{|u|\leq h} |T_N(\Gamma_n^*(s)+u) - T_N(\Gamma_n^*(s))|\, ds \geq \varepsilon/4\right\} = 0$$

for all n. Now given $\delta > 0$ we can choose h so small that I_5 in (3.17) is at most δ for all n (note that the random variable in the probability I_5 does not depend on n). Then choosing n sufficiently large, (3.3) and the preceding estimates show that $I_1, \ldots, I_4$ will be less than δ. Thus (3.17) yields

$$\left| \frac{1}{n} \sum_{1 \le k \le n} T_N(x^*_{k,n}) - \int_0^1 T_N(\Gamma^*_n(s))\, ds \right| \xrightarrow{p} 0,$$

and the proof of (3.16) is complete. ∎

Proof of Remark 2.2. In view of the comments made after Theorem 2.3, we can assume that T is locally bounded. Let

$$g_h(x) = |x|^{\alpha-1} \sup_{|u| \le h} |T(x+u) - T(x)|$$

for $x \neq 0$ and $g_h(0) = 0$. The function $g_h(x)$ is Lebesgue integrable on $[-K, K]$ if h is small. For any fixed x, the sequence $g_h(x)$ is nonincreasing as $h \downarrow 0$, and thus it has a limit $g(x) \ge 0$. Hence by the monotone convergence theorem the limit in (2.15) equals $\int_{-K}^{K} g(x)\, dx$. So (2.15) holds if and only if $\int_{-K}^{K} g(x)\, dx = 0$ for all K. Because $g(x) \ge 0$, this is true if and only if $g(x) = 0$ almost everywhere. Clearly, $g(x_0) = 0$ for $x_0 \neq 0$ if and only if T is continuous at x_0. So we proved that (2.15) is equivalent with the almost everywhere continuity of T. Recalling that a function is Riemann integrable on an interval if and only if it is bounded and almost everywhere continuous on the interval (cf. Riesz and Szőkefalvi-Nagy, 1990, p. 23), Remark 2.2 is proved. ∎

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