

# ESTIMATION IN NONSTATIONARY RANDOM COEFFICIENT AUTOREGRESSIVE MODELS

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*First Version received July 2007*

**Abstract.** We investigate the estimation of parameters in the random coefficient autoregressive (RCA) model  $X_k = (\varphi + b_k)X_{k-1} + e_k$ , where  $(\varphi, \omega^2, \sigma^2)$  is the parameter of the process,  $Eb_0^2 = \omega^2$ ,  $Ee_0^2 = \sigma^2$ . We consider a nonstationary RCA process satisfying  $E \log |\varphi + b_0| \geq 0$  and show that  $\sigma^2$  cannot be estimated by the quasi-maximum likelihood method. The asymptotic normality of the quasi-maximum likelihood estimator for  $(\varphi, \omega^2)$  is proven so that the unit root problem does not exist in the RCA model.

**Keywords.** random coefficient model; quasi-maximum likelihood; asymptotic normality; consistency; law of large numbers.

**AMS 2000 subject classification.** Primary 62F05; secondary 62M10.

## 1. INTRODUCTION

In this article, we are interested in the random coefficient autoregressive model (RCA) defined by the equations

$$X_k = (\varphi + b_k)X_{k-1} + e_k, \quad -\infty < k < \infty, \quad (1)$$

where  $\varphi$  is a real parameter. The RCA process was introduced by Andél (1976) who also studied its properties. For a detailed early study, we refer to Nicholls and Quinn (1982). Throughout this article, we assume that

$$\{(b_k, e_k)\} \text{ are i.i.d. random vectors.} \quad (2)$$

Let  $\log^+ x = \max\{\log x, 0\}$ . It follows from Aue *et al.* (2006) (cf. also Quinn, 1980, 1982) that under condition (2) and

$$E \log^+ |e_0| < \infty \text{ and } E \log^+ |\varphi + b_0| < \infty, \quad (3)$$

(1) has a stationary, nonanticipating (i.e.  $X_k$  is measurable with respect to the  $\sigma$ -algebra generated by  $(b_i, e_i)$ ,  $i \leq k$ ) solution if and only if

$$-\infty \leq E \log |\varphi + b_0| < 0. \quad (4)$$

Quinn and Nicholls (1981) started the study of the estimation of the parameter of the process in (1). Let  $\theta = (\varphi, \omega^2, \sigma^2)$ , where

JOURNAL OF TIME SERIES ANALYSIS

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$$Eb_0 = 0, \quad Eb_0^2 = \omega^2 > 0, \quad (5)$$

$$Ee_0 = 0, \quad Ee_0^2 = \sigma^2 > 0 \quad (6)$$

and

$$\text{cov}(b_0, e_0) = 0. \quad (7)$$

Aue *et al.* (2006) used the quasi-maximum likelihood method to estimate  $\theta$  when (4) holds. They established the strong consistency as well as the asymptotic normality of the quasi-maximum likelihood estimator under minimal conditions.

In this article, we consider the case when (4) does not hold. We assume

$$X_k = (\varphi + b_k)X_{k-1} + e_k, \quad 1 \leq k \leq n \quad (8)$$

and

$$E \log |\varphi + b_0| \geq 0, \quad (9)$$

i.e. we start the recursion in (8) from the initial value  $X_0$  and (9) guarantees that the solutions of (8) cannot converge. Throughout this article, we assume that  $X_0$  is a constant. Following the theory developed for the stationary case, we estimate the parameter  $\theta$  of the process in (8) using the quasi-likelihood method. Assuming that  $b_0$  and  $e_0$  are normally distributed, the conditional log-likelihood function (the constant terms are omitted) is given by

$$L_n(\mathbf{u}) = \sum_{k=1}^n \ell_k(\mathbf{u}) \text{ with } \ell_k(\mathbf{u}) = -\frac{1}{2} \left( \log(xX_{k-1}^2 + y) + \frac{(X_k - sX_{k-1})^2}{xX_{k-1}^2 + y} \right),$$

where  $\mathbf{u} = (s, x, y)$ . We show that

$$\frac{1}{n} L_n(\mathbf{u}) \xrightarrow{P} \infty$$

but

$$\frac{1}{n} (L_n(\mathbf{u}) - L_n(\theta)) \xrightarrow{P} f(s, x) \quad \text{for all } \mathbf{u} \text{ with } x > 0 \text{ and } y > 0,$$

where

$$f(s, x) = \frac{1}{2} \left\{ \log \frac{\omega^2}{x} + 1 - \frac{\omega^2}{x} - \frac{(\varphi - s)^2}{x} \right\}. \quad (10)$$

Since  $f(\cdot)$  does not depend on  $y$ , the quasi-maximum likelihood method cannot be used to estimate  $\sigma^2$ . Since  $|X_n| \xrightarrow{P} \infty$  ( $n \rightarrow \infty$ ) (cf. Lemma 1), so in (1)  $b_n X_{n-1}$  dominates  $e_n$ , which is the reason why the variance of  $e_0$  cannot be estimated by the quasi-likelihood method. Hence, we are interested in estimating  $\eta = (\varphi, \omega^2)$ . Now  $\hat{\eta}_n = \hat{\eta}_n(y) = (\hat{\eta}_{n,1}(y), \hat{\eta}_{n,2}(y))$  is defined by

$$\max_{\mathbf{z} \in \Gamma} L_n(\mathbf{z}, y) = L_n(\hat{\boldsymbol{\eta}}_n, y),$$

$\mathbf{z} = (s, x)$  and the set  $\Gamma$  satisfies

$$\Gamma = \{(s, x) : s_* \leq s \leq s^*, x_* \leq x \leq x^*\} \quad (11)$$

with some  $s_* < s^*$ ,  $0 < x_* < x^*$ . We prove the asymptotic consistency of  $\hat{\boldsymbol{\eta}}_n(y)$  for all  $y$  and consider the asymptotic normality of  $\hat{\boldsymbol{\eta}}_n$  under various conditions.

## 2. RESULTS

First, we study the asymptotic consistency of  $\hat{\boldsymbol{\eta}}_n(y)$ .

**THEOREM 1.** *If (2), (5)–(9) and (11) hold, then*

$$\hat{\boldsymbol{\eta}}_n(y) \xrightarrow{P} (\varphi, \omega^2) \quad (12)$$

for all  $y > 0$ .

Next, we consider the asymptotic normality of  $\hat{\boldsymbol{\eta}}_n(y)$ . Let

$$\Omega_0 = \begin{pmatrix} \omega^2 & \omega^2 E b_0^3 \\ \omega^2 E b_0^3 & \text{var}(b_0^2) \end{pmatrix}. \quad (13)$$

**THEOREM 2.** *If the conditions of Theorem 1 are satisfied and*

$$E e_0^4 < \infty \text{ and } E b_0^4 < \infty, \quad (14)$$

*then the distribution of  $n^{1/2}(\hat{\boldsymbol{\eta}}_n(\sigma^2) - (\varphi, \omega^2))$  converges to the bivariate normal distribution with mean  $\mathbf{0}$  and covariance matrix  $\Omega_0$ .*

We note that Theorems 1 and 2 were obtained by Ling and Li (2006) as a preliminary result for the study of nonstationary double AR(1) processes when  $b_0$  and  $e_0$  are normally distributed and independent. Their result implies that in the case of normal  $(b_0, e_0)$ ,  $\sigma^2$  cannot be estimated by the quasi-maximum likelihood method. A similar phenomenon was also observed by Jensen and Rahbek (2004a,b) in nonstationary ARCH models. Theorem 2 assumes that  $\sigma^2$  is known. We show in the next section that  $\hat{\boldsymbol{\eta}}_n(y)$  is asymptotically normal for all  $y > 0$  under the condition  $E \log |\varphi + b_0| > 0$ .

Usually, the statistical inference is about  $\varphi$ , the expected value of the autoregressive coefficient. We show that  $\hat{\eta}_{n,1}(y)$  is asymptotically normal for all

$y$  so that there is no need to know  $\sigma^2$  to obtain asymptotic statistical inference about  $\varphi$ .

**THEOREM 3.** *We assume that the conditions of Theorem 1 are satisfied and (14) holds. Then for any  $y > 0$  the distribution of  $\sqrt{n}(\hat{\eta}_{n,1}(y) - \varphi)/\omega$  converges to the standard normal distribution and consequently the distribution of  $\sqrt{n}(\hat{\eta}_{n,1}(y) - \varphi)/\sqrt{\hat{\eta}_{n,2}(y)}$  converges also to the standard normal distribution.*

Next we are interested in the asymptotic distribution of  $\hat{\eta}_n(\sigma^2) - (\varphi, \omega^2)$  without assuming (14). The assumption  $Eb_0^4 < \infty$  will be replaced with the requirement that  $b_0^2$  is in the domain of attraction of a stable law. This means that

$$P\{b_0^2 > x\} = x^{-\alpha}L(x), \quad (15)$$

where  $1 < \alpha < 2$  and  $L$  is a slowly varying function at  $\infty$ .

Assumption  $\alpha > 1$  guarantees that  $Eb_0^2 = \omega^2$  exists. Let

$$a_n = \inf\{x : x^{-\alpha}L(x) \leq 1/n\}.$$

If (15) holds, then

$$\frac{1}{a_n} \sum_{1 \leq i \leq n} (b_i^2 - \omega^2) \xrightarrow{\mathcal{D}} \xi, \quad (16)$$

where  $\xi$  is a stable random variable with characteristic function

$$\exp\{-d|t|^\alpha(1 + \text{isign}(t) \tan(\pi\alpha/2))\}, \quad \text{if } 1 < \alpha < 2, \quad (17)$$

and  $d$  is a positive constant (Breiman, 1968, p. 204).

**THEOREM 4.** *We assume that the conditions of Theorem 1 are satisfied, (15) and*

$$E|e_0|^v < \infty \text{ with some } v > 2\alpha/(\alpha - 1) \quad (18)$$

*hold. Then  $n^{1/2}(\hat{\eta}_{n,1}(\sigma^2) - \varphi)$  and  $n(\hat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$  are asymptotically independent; the distribution of  $n^{1/2}(\hat{\eta}_{n,1}(\sigma^2) - \varphi)$  converges to the normal distribution with mean 0 and variance  $\omega^2$  and the distribution of  $n(\hat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$  converges to the stable distribution with the characteristic function given in (17).*

We note that if  $\{e_k\}$  and  $\{b_k\}$  are independent sequences, then (18) can be replaced by  $Ee_0^4 < \infty$ .

3. GROWTH OF  $X_n$ 

We will show in Section 4 (cf. Lemma 1) that under the conditions of Theorem 1,  $X_n \xrightarrow{P} \infty$ . Now we find the order of the growth of  $X_n$ . To state our results we need further notation. Let

$$\xi_i = \log |\varphi + b_i|, \quad S(i) = \xi_1 + \cdots + \xi_i, \quad \gamma_i = \prod_{1 \leq j \leq i} \text{sign}(\varphi + b_j).$$

Here, we consider the case when

$$E \log |\varphi + b_0| > 0. \quad (19)$$

**THEOREM 5.** *If (2), (3), (8) and (19) hold, then*

$$e^{-S(n)} \gamma_n X_n \longrightarrow X_0 + Y \quad a.s.$$

where

$$Y = \sum_{1 \leq i < \infty} e^{-S(i)} \gamma_i e_i.$$

The random normalization  $\exp(-S(n))$  is the correct one in Theorem 5, if the limit is nonzero with probability 1. The next result provides conditions for

$$P\{Y + X_0 \neq 0\} = 1. \quad (20)$$

**THEOREM 6.** *We assume that (2), (3), (8) and (19) hold.*

(i) *If*

$$P\{(\varphi + b_0)X_0 + e_0 = c\} = 0 \quad \text{for all } c, \quad (21)$$

*then (20) holds.*

(ii) *If*

$$\{b_k\} \text{ and } \{e_k\} \text{ are independent sequences} \quad (22)$$

*and*

$$P\{e_0 = c\} < 1 \quad \text{for all } c, \quad (23)$$

*then (20) holds.*

The first corollary says that under condition (19),  $X_n$  grows exponentially fast with probability 1.

COROLLARY 1. *If (2), (3), (8), (19) and (21) or (22) and (23) hold, then*

$$e^{-\tau n}|X_n| \longrightarrow \infty \text{ a.s. for all } 0 < \tau < E \log |\varphi + b_0|$$

and

$$e^{-\tau n}|X_n| \longrightarrow 0 \text{ a.s. for all } \tau > E \log |\varphi + b_0|.$$

The second corollary is the asymptotic normality of  $\hat{\eta}_n(y)$  without assuming that  $y = \sigma^2$ .

COROLLARY 2. *If (2), (5), (8), (11), (14), (19) and (21) or (22) and (23) hold, then for all  $y > 0$  the distribution of  $n^{1/2}(\hat{\eta}_n(\sigma^2) - (\varphi, \omega^2))$  converges to the bivariate normal distribution with mean  $\mathbf{0}$  and covariance matrix  $\Omega_0$ .*

Similarly, in the case of  $E \log |\varphi + b_0| > 0$ , we have the following generalization of Theorem 4.

COROLLARY 3. *If (2), (5)–(8), (11), (15), (18), (19) and (21) or (22) and (23) hold, then for all  $y > 0$ ,  $n^{1/2}(\hat{\eta}_{n,1}(\sigma^2) - \varphi)$  and  $n(\hat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$  are asymptotically independent; the distribution of  $n^{1/2}(\hat{\eta}_{n,1}(\sigma^2) - \varphi)$  converges to the normal distribution with mean 0 and variance  $\omega^2$  and the distribution of  $n(\hat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$  converges to the stable distribution with the characteristic function given in (17).*

#### 4. PROOFS OF THEOREMS 1–4

The proofs will use the following result:

LEMMA 1. *If (2) and (5)–(9) hold, then*

$$|X_n| \xrightarrow{P} \infty. \quad (24)$$

PROOF. We note that

$$P\{e_0 + c(\varphi + b_0) = c\} < 1 \text{ for all } c.$$

Indeed, if  $e_0 + c(\varphi + b_0) = c$  with probability 1, then multiplying this equation with  $e_0$  and taking expected values we obtain  $Ee_0^2 + c\varphi Ee_0 + cEb_0e_0 = cEe_0$ . Since  $Ee_0 = Ee_0b_0 = 0$ , we obtain  $Ee_0^2 = 0$ , which contradicts  $Ee_0^2 = \sigma^2 > 0$  (c.f. (6)). Since (9) implies  $P\{\varphi + b_0 = 0\} = 0$ , the result follows immediately from Remark 2.8 and Cor. 4.1 of Goldie and Maller (2000).  $\square$

We start with the study of the log-likelihood function.

LEMMA 2. If (2), (3) and (5)–(9) are satisfied, then

$$\sup_{\mathbf{u} \in \Gamma^*} \left| \frac{1}{n} (L_n(\mathbf{u}) - L_n(\boldsymbol{\theta})) - f(s, x) \right| \xrightarrow{P} 0, \quad (25)$$

where  $f(\cdot)$  is defined in (10) and

$$\Gamma^* = \{\mathbf{u} = (s, x, y) : s_* \leq s \leq s^*, x_* \leq x \leq x^*, y_* \leq y \leq y^*\},$$

with  $0 < x_*$  and  $0 < y_*$ .

PROOF. We write

$$\begin{aligned} L_n(\mathbf{u}) - L_n(\boldsymbol{\theta}) &= \frac{1}{2} \sum_{1 \leq k \leq n} \log \frac{\omega^2 X_{k-1}^2 + \sigma^2}{x X_{k-1}^2 + y} + \frac{1}{2} \sum_{1 \leq k \leq n} \frac{(X_{k-1} b_k + e_k)^2}{\omega^2 X_{k-1}^2 + \sigma^2} \\ &\quad - \frac{1}{2} \sum_{1 \leq k \leq n} \frac{((\varphi - s) X_{k-1} + X_{k-1} b_k + e_k)^2}{x X_{k-1}^2 + y}. \end{aligned} \quad (26)$$

Using the mean value theorem we conclude

$$\begin{aligned} \left| \log \frac{\omega^2 X_{k-1}^2 + \sigma^2}{x X_{k-1}^2 + y} - \log \frac{\omega^2}{x} \right| &\leq c_1 \left( \frac{x^*}{\omega^2} + \frac{x^* X_{k-1}^2 + y^*}{\omega^2 X_{k-1}^2 + \sigma^2} \right) \frac{1}{x_* X_{k-1}^2 + y_*} \\ &\leq c_2 \frac{1}{x_* X_{k-1}^2 + y_*}. \end{aligned}$$

By (24), we have that

$$E \frac{1}{x_* X_n^2 + y_*} \rightarrow 0 \quad (27)$$

and therefore by the Markov inequality

$$\frac{1}{n} \sum_{1 \leq k \leq n} \sup_{\mathbf{u} \in \Gamma^*} \left| \log \frac{\omega^2 X_{k-1}^2 + \sigma^2}{x X_{k-1}^2 + y} - \log \frac{\omega^2}{x} \right| \xrightarrow{P} 0. \quad (28)$$

Also,

$$\begin{aligned} \sum_{1 \leq k \leq n} \left\{ \frac{(X_{k-1} b_k + e_k)^2}{\omega^2 X_{k-1}^2 + \sigma^2} - 1 \right\} &= \sum_{1 \leq k \leq n} (b_k^2 - \omega^2) \frac{X_{k-1}^2}{\omega^2 X_{k-1}^2 + \sigma^2} + \sum_{1 \leq k \leq n} \frac{e_k^2}{\omega^2 X_{k-1}^2 + \sigma^2} \\ &\quad + \sum_{1 \leq k \leq n} b_k e_k \frac{2X_{k-1}}{\omega^2 X_{k-1}^2 + \sigma^2} - \sum_{1 \leq k \leq n} \frac{\sigma^2}{\omega^2 X_{k-1}^2 + \sigma^2}. \end{aligned}$$

Similarly to (27) we obtain

$$\frac{1}{n} \sum_{1 \leq k \leq n} \frac{\sigma^2}{\omega^2 X_{k-1}^2 + \sigma^2} \xrightarrow{P} 0.$$

Since by (24) and the independence of  $e_n$  and  $X_{n-1}$ , we have

$$E \frac{e_n^2}{\omega^2 X_{n-1}^2 + \sigma^2} \rightarrow 0.$$

So, we obtain

$$\frac{1}{n} \sum_{1 \leq k \leq n} \frac{e_k^2}{\omega^2 X_{k-1}^2 + \sigma^2} \xrightarrow{P} 0.$$

Now, we write

$$\begin{aligned} \sum_{1 \leq k \leq n} (b_k^2 - \omega^2) \frac{X_{k-1}^2}{\omega^2 X_{k-1}^2 + \sigma^2} &= \sum_{1 \leq k \leq n} (b_k^2 - \omega^2) \frac{1}{\omega^2} - \sum_{1 \leq k \leq n} b_k^2 \frac{\sigma^2}{\omega^2 (\omega^2 X_{k-1}^2 + \sigma^2)} \\ &\quad + \sum_{1 \leq k \leq n} \frac{\omega^2 \sigma^2}{\omega^2 (\omega^2 X_{k-1}^2 + \sigma^2)}. \end{aligned}$$

The weak law of large numbers yields

$$\frac{1}{n} \sum_{1 \leq k \leq n} (b_k^2 - \omega^2) \frac{1}{\omega^2} \xrightarrow{P} 0. \quad (29)$$

Using now the independence of  $b_k$  and  $X_{k-1}$  with (24), we obtain

$$E \frac{1}{n} \sum_{1 \leq k \leq n} b_k^2 \frac{\sigma^2}{\omega^2 (\omega^2 X_{k-1}^2 + \sigma^2)} \rightarrow 0 \quad \text{and} \quad E \frac{1}{n} \sum_{1 \leq k \leq n} \frac{\omega^2 \sigma^2}{\omega^2 (\omega^2 X_{k-1}^2 + \sigma^2)} \rightarrow 0$$

and therefore by the Markov inequality and (29) we conclude

$$\frac{1}{n} \sum_{1 \leq k \leq n} (b_k^2 - \omega^2) \frac{X_{k-1}^2}{\omega^2 X_{k-1}^2 + \sigma^2} \xrightarrow{P} 0. \quad (30)$$

By the independence of  $(b_k, e_k)$  and  $X_{k-1}$  we obtain

$$E \left| \frac{1}{n} \sum_{1 \leq k \leq n} b_k e_k \frac{X_{k-1}}{\omega^2 X_{k-1}^2 + \sigma^2} \right| \leq \frac{1}{n} \sum_{1 \leq k \leq n} E |b_k e_k| E \left| \frac{X_{k-1}}{\omega^2 X_{k-1}^2 + \sigma^2} \right| \rightarrow 0$$

on account of (24), resulting in

$$\frac{1}{n} \sum_{1 \leq k \leq n} b_k e_k \frac{X_{k-1}}{\omega^2 X_{k-1}^2 + \sigma^2} \xrightarrow{P} 0.$$



Hence, we proved that

$$\frac{1}{n} \sum_{1 \leq k \leq n} \left\{ \frac{(X_{k-1}b_k + e_k)^2}{\omega^2 X_{k-1}^2 + \sigma^2} - 1 \right\} \xrightarrow{P} 0. \quad (31)$$

Next, we write

$$\begin{aligned} \frac{(X_{k-1}(\varphi - s) + X_{k-1}b_k + e_k)^2}{xX_{k-1}^2 + y} &= (\varphi - s)^2 \frac{X_{k-1}^2}{xX_{k-1}^2 + y} + b_k^2 \frac{X_{k-1}^2}{xX_{k-1}^2 + y} + e_k^2 \frac{1}{xX_{k-1}^2 + y} \\ &\quad + 2(\varphi - s)b_k \frac{X_{k-1}}{xX_{k-1}^2 + y} + 2(\varphi - s)e_k \frac{X_{k-1}}{xX_{k-1}^2 + y} \\ &\quad + 2b_k e_k \frac{X_{k-1}}{xX_{k-1}^2 + y}. \end{aligned}$$

Clearly,

$$\begin{aligned} E \sup_{\mathbf{u} \in \Gamma^*} \left| \sum_{1 \leq k \leq n} e_k b_k \frac{X_{k-1}}{xX_{k-1}^2 + y} \right| &\leq \sum_{1 \leq k \leq n} E \left| b_k e_k \frac{X_{k-1}}{x_* X_{k-1}^2 + y_*} \right| \\ &= E|e_0 b_0| \sum_{1 \leq k \leq n} E \frac{|X_{k-1}|}{x_* X_{k-1}^2 + y_*} \end{aligned}$$

and since by (24)

$$E \frac{|X_n|}{x_* X_n^2 + y_*} \rightarrow 0,$$

the Markov inequality yields

$$\sup_{\mathbf{u} \in \Gamma^*} \left| \frac{1}{n} \sum_{1 \leq k \leq n} e_k b_k \frac{X_{k-1}}{xX_{k-1}^2 + y} \right| \xrightarrow{P} 0.$$

Similar arguments give

$$\sup_{\mathbf{u} \in \Gamma^*} \left| \frac{1}{n} |\varphi - s| \sum_{1 \leq k \leq n} e_k \frac{X_{k-1}}{xX_{k-1}^2 + y} \right| \xrightarrow{P} 0.$$

Next, we observe that

$$\begin{aligned} \sup_{\mathbf{u} \in \Gamma^*} \left| \sum_{1 \leq k \leq n} b_k \frac{X_{k-1}^2}{xX_{k-1}^2 + y} \right| &\leq \sup_{\mathbf{u} \in \Gamma^*} \frac{1}{x} \left| \sum_{1 \leq k \leq n} b_k \right| + \sup_{\mathbf{u} \in \Gamma^*} \left| \sum_{1 \leq k \leq n} b_k \frac{1}{x} \frac{y}{xX_{k-1}^2 + y} \right| \\ &\leq \frac{1}{x_*} \left| \sum_{1 \leq k \leq n} b_k \right| + \frac{y^*}{x_*} \sum_{1 \leq k \leq n} |b_k| \frac{1}{x_* X_{k-1}^2 + y_*}. \end{aligned}$$

By the law of large numbers, we have

$$\frac{1}{n} \sum_{1 \leq k \leq n} b_k \xrightarrow{P} 0$$

and the Markov inequality with (24) gives

$$\frac{1}{n} \sum_{1 \leq k \leq n} |b_k| \frac{1}{x_* X_{k-1}^2 + y_*} \xrightarrow{P} 0.$$

Similarly,

$$\sup_{\mathbf{u} \in \Gamma^*} \frac{1}{n} \left| \sum_{1 \leq k \leq n} e_k^2 \frac{1}{x X_{k-1}^2 + y} \right| \leq \frac{1}{n} \sum_{1 \leq k \leq n} e_k^2 \frac{1}{x_* X_{k-1}^2 + y_*} \xrightarrow{P} 0.$$

Now,

$$b_k^2 \frac{X_{k-1}^2}{x X_{k-1}^2 + y} - \frac{\omega^2}{x} = -b_k^2 \frac{y}{x(x X_{k-1}^2 + y)} + \frac{1}{x} (b_k^2 - \omega^2),$$

and therefore, arguing as before, we obtain

$$\sup_{\mathbf{u} \in \Gamma^*} \frac{1}{n} \left| \sum_{1 \leq k \leq n} \left( b_k^2 \frac{X_{k-1}^2}{x X_{k-1}^2 + y} - \frac{\omega^2}{x} \right) \right| \leq \frac{1}{x_*} \frac{y^*}{n} \sum_{1 \leq k \leq n} b_k^2 \frac{1}{x_* X_{k-1}^2 + y_*} + \frac{1}{x_*} \left| \sum_{1 \leq k \leq n} (b_k^2 - \omega^2) \right| \xrightarrow{P} 0.$$

Similarly,

$$\sup_{\mathbf{u} \in \Gamma^*} \frac{1}{n} \sum_{1 \leq k \leq n} \left( \frac{X_{k-1}^2}{x X_{k-1}^2 + y} - \frac{1}{x} \right) \xrightarrow{P} 0.$$

Thus, we proved

$$\sup_{\mathbf{u} \in \Gamma^*} \frac{1}{n} \sum_{1 \leq k \leq n} \frac{((\varphi - s)X_{k-1} + X_{k-1}b_k + e_k)^2}{x X_{k-1}^2 + y} - \left( \frac{(\varphi - s)^2}{x} + \frac{\omega^2}{x} \right) \xrightarrow{P} 0. \quad (32)$$

The result in Lemma 2 follows from (26), (28), (31) and (32).  $\square$

LEMMA 3. *If the conditions of Lemma 2 are satisfied and  $\boldsymbol{\eta} = (\varphi, \omega^2) \in \Gamma$ , then*

$$\sup_{y_* \leq y \leq y^*} |\widehat{\boldsymbol{\eta}}_n(y) - \boldsymbol{\eta}| \xrightarrow{P} 0 \quad \text{for all } 0 < y_* < y^*.$$

PROOF. It is easy to see that

$$f(s, x) \leq f(\boldsymbol{\eta}) \text{ for all } (s, x)$$

and we have equality if and only if  $(s, x) = \boldsymbol{\eta}$ . Since

$$\max_{\mathbf{z} \in \Gamma} (L_n(\mathbf{z}, y) - L_n(\boldsymbol{\theta})) = L_n(\hat{\boldsymbol{\eta}}_n, y) - L_n(\boldsymbol{\theta}),$$

$L_n(\mathbf{u})$ ,  $\mathbf{u} \in \Gamma^*$  is continuous on  $\Gamma^*$ , it converges uniformly to  $f(s, x)$ , standard arguments provide the result (Pfanztangl, 1969).

LEMMA 4. *If the conditions of Theorem 2 are satisfied, then for all  $0 < y$  we have*

$$\left| g_{1,n}(y) - \sum_{1 \leq k \leq n} \frac{b_k}{\omega^2} \right| = o_P(n^{1/2}) \quad (33)$$

and

$$\left| g_{2,n}(\sigma^2) - \sum_{1 \leq k \leq n} \frac{1}{2\omega^4} (b_k^2 - \omega^2) \right| = o_P(n^{1/2}), \quad (34)$$

where  $g_{1,n}(y)$  and  $g_{2,n}(y)$  are the partial derivatives of  $L_n(\mathbf{u})$  with respect to  $s$  and  $x$  at  $(\varphi, \omega^2, y)$ .

PROOF. Elementary calculations yield

$$\frac{\partial \ell_k(\mathbf{u})}{\partial s} = \frac{(X_k - sX_{k-1})X_{k-1}}{xX_{k-1}^2 + y}$$

and

$$\frac{\partial \ell_k(\mathbf{u})}{\partial x} = -\frac{1}{2} \left[ \frac{X_{k-1}^2}{xX_{k-1}^2 + y} - \frac{(X_k - sX_{k-1})^2 X_{k-1}^2}{(xX_{k-1}^2 + y)^2} \right]$$

and therefore

$$g_{1,n}(y) = \sum_{1 \leq k \leq n} \left\{ \frac{b_k X_{k-1}^2}{\omega^2 X_{k-1}^2 + y} + \frac{e_k X_{k-1}}{\omega^2 X_{k-1}^2 + y} \right\},$$

$$g_{2,n}(y) = \sum_{1 \leq k \leq n} -\frac{1}{2} \left\{ \frac{X_{k-1}^2}{\omega^2 X_{k-1}^2 + y} - \frac{(X_{k-1} b_k + e_k)^2 X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} \right\}.$$

Using the independence of  $(e_k, b_k)$  and  $X_{k-1}$ , we obtain

$$\text{var} \left( \frac{1}{n^{1/2}} \sum_{1 \leq k \leq n} \frac{e_k X_{k-1}}{\omega^2 X_{k-1}^2 + y} \right) = \frac{\sigma^2}{n} \sum_{1 \leq k \leq n} E \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} \rightarrow 0.$$

Similarly,

$$\begin{aligned} \text{var} \left( n^{-1/2} \sum_{1 \leq k \leq n} b_k \left\{ \frac{X_{k-1}^2}{\omega^2 X_{k-1}^2 + y} - \frac{1}{\omega^2} \right\} \right) &= \text{var} \left( n^{-1/2} \sum_{1 \leq k \leq n} b_k \frac{y}{\omega^2 (\omega^2 X_{k-1}^2 + y)} \right) \\ &= \frac{y^2}{\omega^2} \frac{1}{n} \sum_{1 \leq k \leq n} E \frac{1}{(\omega^2 X_{k-1}^2 + y)^2} \rightarrow 0, \end{aligned}$$

and thus an application of the Markov inequality completes the proof of (33). Write

$$\begin{aligned} \frac{X_{k-1}^2}{\omega^2 X_{k-1}^2 + y} - \frac{(X_{k-1} b_k + e_k)^2 X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} &= (\omega^2 - b_k^2) \frac{X_{k-1}^4}{(\omega^2 X_{k-1}^2 + y)^2} \\ &\quad + \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} (y - 2e_k b_k X_{k-1} - e_k^2) \\ &= (\omega^2 - b_k^2) \frac{X_{k-1}^4}{(\omega^2 X_{k-1}^2 + y)^2} + \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} (\sigma^2 - e_k^2) \\ &\quad - \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} 2e_k b_k X_{k-1} \\ &\quad + \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} (y - \sigma^2). \end{aligned} \quad (35)$$

One can easily verify

$$E \left( n^{-1/2} \sum_{1 \leq k \leq n} (\omega^2 - b_k^2) \left( \frac{X_{k-1}^4}{(\omega^2 X_{k-1}^2 + y)^2} - \frac{1}{\omega^4} \right) \right)^2 \rightarrow 0, \quad (36)$$

$$E \left( n^{-1/2} \sum_{1 \leq k \leq n} \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} (\sigma^2 - e_k^2) \right)^2 \rightarrow 0, \quad (37)$$

$$E \left( n^{-1/2} \sum_{1 \leq k \leq n} \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} 2e_k b_k X_{k-1} \right)^2 \rightarrow 0, \quad (38)$$

and since  $y = \sigma^2$  is assumed

$$\frac{1}{n^{1/2}} |y - \sigma^2| E \sum_{1 \leq k \leq n} \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} = 0, \quad (39)$$

and therefore (34) is proven.  $\square$

LEMMA 5. *If the conditions of Lemma 2 are satisfied, then*

$$\sup_{\mathbf{u} \in \Gamma^*} |g_{ij,n}(\mathbf{u}) - g_{ij}(\mathbf{u})| \xrightarrow{P} 0, \quad 1 \leq i, j \leq 2,$$

where

$$\begin{aligned} g_{11,n}(\mathbf{u}) &= \frac{\partial^2}{\partial s^2} \frac{1}{n} L_n(\mathbf{u}) = -\frac{1}{n} \sum_{1 \leq k \leq n} \frac{X_{k-1}^2}{xX_{k-1}^2 + y}, \\ g_{12,n}(\mathbf{u}) &= g_{21,n}(\mathbf{u}) = \frac{\partial^2}{\partial s \partial x} \frac{1}{n} L_n(\mathbf{u}) = \frac{\partial^2}{\partial x \partial s} \frac{1}{n} L_n(\mathbf{u}) \\ &= \frac{1}{n} \sum_{1 \leq k \leq n} -\frac{(X_k - sX_{k-1})X_{k-1}^3}{(xX_{k-1}^2 + y)^2}, \\ g_{22,n}(\mathbf{u}) &= \frac{\partial^2}{\partial x^2} \frac{1}{n} L_n(\mathbf{u}) = \frac{1}{n} \sum_{1 \leq k \leq n} \left\{ \frac{X_{k-1}^4}{2(xX_{k-1}^2 + y)^2} - \frac{(X_k - sX_{k-1})^2 X_{k-1}^4}{(xX_{k-1}^2 + y)^3} \right\} \end{aligned}$$

and

$$g_{11}(\mathbf{u}) = -\frac{1}{x}, \quad g_{12}(\mathbf{u}) = g_{21}(\mathbf{u}) = -\frac{(\varphi - s)}{x^2}, \quad g_{22}(\mathbf{u}) = \frac{1}{2x^2} - \frac{(\varphi - s)^2 + \omega^2}{x^3}.$$

PROOF. It can be proven along the lines of the proof of Lemma 2 and therefore the details are omitted.

PROOF OF THEOREM 2. Combining the central limit theorem for i.i.d. random vectors with Lemma 4, we obtain

$$n^{-1/2}(g_{1,n}(\sigma^2), g_{2,n}(\sigma^2)) \xrightarrow{D} N_2(\mathbf{0}, \Omega_*), \quad (40)$$

where

$$\Omega_* = \begin{pmatrix} \frac{1}{\omega^2} & \frac{Eb_0^3}{2\omega^4} \\ \frac{Eb_0^3}{2\omega^4} & \frac{\text{var } b_0^2}{4\omega^8} \end{pmatrix}.$$

Let  $\|\cdot\|$  denote the maximum norm of vectors. Let  $\nabla h(\mathbf{u}) = (\partial h(\mathbf{u})/\partial u_1, \partial h(\mathbf{u})/\partial u_2)^T$ . Applying the mean value theorem to the coordinates of  $\nabla L_n(\mathbf{u}, \sigma^2)$ , there are random vectors  $\xi_{n,1}$  and  $\xi_{n,2}$  such that  $\|\xi_{n,j} - \eta\| \leq \|\hat{\eta}_n - \eta\|, j = 1, 2$  and

$$0 = \frac{\partial L_n(\eta, \sigma^2)}{\partial u_j} + \left( \nabla \frac{\partial L_n(\xi_{n,j}, \sigma^2)}{\partial u_j} \right)^T (\hat{\eta}_n - \eta), \quad j = 1, 2. \quad (41)$$

Lemma 5 and Theorem 1 give that for all  $y > 0$

$$\left( \frac{1}{n} \nabla \frac{\partial L_n(\xi_{n,1}, y)}{\partial u_1}, \frac{1}{n} \nabla \frac{\partial L_n(\xi_{n,2}, y)}{\partial u_2} \right) \xrightarrow{P} \Omega_{**}, \quad (42)$$

where

$$\Omega_{**} = \begin{pmatrix} -\frac{1}{\omega^2} & 0 \\ 0 & -\frac{1}{2\omega^4} \end{pmatrix}.$$

Putting together (40)–(42), we conclude

$$n^{1/2}(\widehat{\eta}_n(\sigma^2) - \eta) \xrightarrow{\mathcal{D}} N_2(\mathbf{0}, \Omega_{**}^{-1} \Omega_* \Omega_{**}^{-1}).$$

Since  $\Omega_0 = \Omega_{**}^{-1} \Omega_* \Omega_{**}^{-1}$ , the proof of Theorem 2 is complete.  $\square$

The proof of Theorem 3 uses Lemma 6.

LEMMA 6. *If the conditions of Lemma 2 are satisfied, then for all  $y > 0$*

$$\sup_{\mathbf{u} \in \Gamma} \left| g_{12,n}(\mathbf{u}) - (s - \varphi) \frac{1}{n} \sum_{1 \leq k \leq n} \frac{X_{k-1}^4}{(xX_{k-1}^2 + y)^2} \right| = \mathcal{O}_P(n^{-1/2}).$$

PROOF. Using the expression for  $g_{12,n}(\mathbf{u})$  in Lemma 5, we obtain

$$g_{12,n}(\mathbf{u}) = -\frac{1}{n} \sum_{1 \leq k \leq n} \frac{(\varphi - s)X_{k-1}^4}{(xX_{k-1}^2 + y)^2} - \frac{1}{n} \sum_{1 \leq k \leq n} \frac{b_k X_{k-1}^4}{(xX_{k-1}^2 + y)^2} - \frac{1}{n} \sum_{1 \leq k \leq n} \frac{e_k X_{k-1}^3}{(xX_{k-1}^2 + y)^2}.$$

Also,

$$\frac{1}{n} \sum_{1 \leq k \leq n} \frac{b_k X_{k-1}^4}{(xX_{k-1}^2 + y)^2} = \frac{1}{nx^2} \sum_{1 \leq k \leq n} b_k - \frac{2y}{xn} \sum_{1 \leq k \leq n} \frac{b_k X_{k-1}^2}{(xX_{k-1}^2 + y)^2} - \frac{1}{nx^2} \sum_{1 \leq k \leq n} \frac{b_k}{(xX_{k-1}^2 + y)^2}.$$

The central limit theorem yields

$$\sup_{x_* \leq x \leq x^*} \left| \frac{1}{nx^2} \sum_{1 \leq k \leq n} b_k \right| = \mathcal{O}_P(n^{-1/2}).$$

Next, we show that

$$\sup_{x_* \leq x \leq x^*} |A_n(x)| = o_P(1), \quad (43)$$

where

$$A_n(x) = \frac{1}{n^{1/2}} \sum_{1 \leq k \leq n} \frac{b_k X_{k-1}^2}{(xX_{k-1}^2 + y)^2}.$$

Since for any  $x \in [x_*, x^*]$

$$EA_n(x)^2 = \frac{\omega^2}{n} \sum_{1 \leq k \leq n} E \left( \frac{X_{k-1}^2}{(xX_{k-1}^2 + y)^2} \right)^2 \rightarrow 0,$$

the finite dimensional distributions of  $A_n(x)$  converge to 0. Similarly, for all  $x, x' \in [x_*, x^*]$  we have by the mean value theorem that

$$\begin{aligned} E(A_n(x) - A_n(x'))^2 &= \frac{\omega^2}{n} \sum_{1 \leq k \leq n} E \left( X_{k-1}^2 \left[ \frac{1}{(xX_{k-1}^2 + y)^2} - \frac{1}{(x'X_{k-1}^2 + y)^2} \right] \right)^2 \\ &\leq (x - x')^2 \frac{\omega^2}{n} \sum_{1 \leq k \leq n} E \left( 2X_{k-1}^4 \frac{1}{(x_*X_{k-1}^2 + y_*)^3} \right)^2 \\ &\leq (x - x')^2, \end{aligned}$$

for all  $n$  large enough. By Billingsley (1968, p. 96), the sequence  $A_n(x)$  is tight, and therefore  $A_n(x)$  converges in  $\mathcal{C}[x_*, x^*]$  to 0. Hence the proof of (43) is complete.

Repeating the arguments leading to (43), we conclude

$$\sup_{x_* \leq x \leq x^*} \left| \frac{1}{n} \frac{y^2}{x^2} \sum_{1 \leq k \leq n} \frac{b_k}{(xX_{k-1}^2 + y)^2} \right| + \sup_{x_* \leq x \leq x^*} \left| \frac{1}{n} \sum_{1 \leq k \leq n} \frac{e_k X_{k-1}^3}{(xX_{k-1}^2 + y)^2} \right| = \mathcal{O}_P(n^{-1/2}).$$

The proof of Lemma 6 is established now.  $\square$

PROOF OF THEOREM 3. Similarly to (41), we have

$$0 = \frac{\partial L_n(\boldsymbol{\eta}, y)}{\partial u_j} + \left( \nabla \frac{\partial L_n(\boldsymbol{\xi}_{n,j}, y)}{\partial u_j} \right)^T (\hat{\boldsymbol{\eta}}_n(y) - \boldsymbol{\eta}), \quad j = 1, 2,$$

where  $\boldsymbol{\xi}_{n,j}$  satisfies  $\|\boldsymbol{\xi}_{n,j} - \boldsymbol{\eta}\| \leq \|\hat{\boldsymbol{\eta}}_n(y) - \boldsymbol{\eta}\|$ ,  $j = 1, 2$ . This gives

$$\hat{\eta}_{n,1}(y) - \varphi = - \left( c_{11}(n) \frac{1}{n} g_{1,n}(y) + c_{12}(n) \frac{1}{n} g_{2,n}(y) \right), \quad (44)$$

where  $c_{ij}(n)$  are defined by

$$\left( \frac{1}{n} \nabla \frac{\partial L_n(\boldsymbol{\xi}_{n,1}, y)}{\partial u_1}, \frac{1}{n} \nabla \frac{\partial L_n(\boldsymbol{\xi}_{n,2}, y)}{\partial u_2} \right)^{-1} = \begin{pmatrix} c_{11}(n) & c_{12}(n) \\ c_{21}(n) & c_{22}(n) \end{pmatrix}. \quad (45)$$

Using (35)–(38) we obtain

$$g_{2,n}(y) = o_P(n). \quad (46)$$

Now (42) gives that  $c_{11}(n) \rightarrow -\omega^2$  in probability. Applying Lemma 6 and (42) we obtain

$$|c_{12}(n)| = |\hat{\eta}_{1,n}(y) - \varphi| \mathcal{O}_P(1) + \mathcal{O}_P(n^{-1/2}). \quad (47)$$

By (44)–(47), we conclude

$$\begin{aligned}\widehat{\eta}_{n,1}(y) - \varphi &= (-\omega^2 + o_P(1)) \frac{1}{n} g_{1,n}(y) + [|\widehat{\eta}_{n,1}(y) - \varphi| \mathcal{O}_P(1) + \mathcal{O}_P(n^{-1/2})] o_P(1) \\ &= (-\omega^2 + o_P(1)) \frac{1}{n} g_{1,n}(y) + |\widehat{\eta}_{n,1}(y) - \varphi| o_P(1) + o_P(n^{-1/2}),\end{aligned}$$

which yields

$$\widehat{\eta}_{n,1}(y) - \varphi = (1 + o_P(1))^{-1} \left( (-\omega^2 + o_P(1)) \frac{1}{n} g_{1,n}(y) + o_P(n^{-1/2}) \right).$$

Now the first part of Theorem 3 follows from (33). The second part is an immediate consequence of the first part of Theorem 1 and Slutsky's lemma.  $\square$

The proof of Theorem 4 is based on the modification of Lemma 4, which yields Lemma 7.

LEMMA 7. *If the conditions of Theorem 4 are satisfied, then for all  $0 < y$  we have*

$$\left| g_{1,n}(y) - \sum_{1 \leq k \leq n} \frac{b_k}{\omega^2} \right| = o_P(n^{1/2}) \quad (48)$$

and

$$\left| g_{2,n}(\sigma^2) - \sum_{1 \leq k \leq n} \frac{1}{2\omega^4} (b_k^2 - \omega^2) \right| = o_P(a_n), \quad (49)$$

where  $g_{1,n}(y)$  and  $g_{2,n}(y)$  are the partial derivatives of  $L_n(\mathbf{u})$  with respect to  $s$  and  $x$  at  $(\varphi, \omega^2, y)$ .

PROOF. We follow the proof of Lemma 4. Since the proof of (33) required only that  $Eb_0^2 < \infty$ , we have (48).

To prove (49), we use (35). It is assumed that  $Ee_0^4 < \infty$  and therefore (37) holds. Assumption (15) yields that  $E|b_0|^{2\tau} < \infty$  for all  $0 < \tau < \alpha$ , and therefore condition (18) with Hölder's inequality gives  $Ee_0^2 b_0^2 < \infty$ . Hence

$$E \left( n^{-1/2} \sum_{1 \leq k \leq n} \frac{X_{k-1}^2}{(\omega^2 X_{k-1}^2 + y)^2} 2e_k b_k X_{k-1} \right)^2 = \frac{4}{n} E(e_0 b_0)^2 \sum_{1 \leq k \leq n} \left( \frac{X_{k-1}^3}{(\omega^2 X_{k-1}^2 + y)^2} \right)^2 \rightarrow 0.$$

Clearly, (39) is satisfied. Thus, it is enough to show that

$$\sum_{1 \leq k \leq n} (\omega^2 - b_k^2) \left( \frac{X_{k-1}^4}{(\omega^2 X_{k-1}^2 + y)^2} - \frac{1}{\omega^4} \right) = o_P(a_n). \quad (50)$$



Let

$$\epsilon_k = b_k^2 - \omega^2 \text{ and } z_{k-1} = \frac{X_{k-1}^4}{(\omega^2 X_{k-1}^2 + y)^2} - \frac{1}{\omega^4}.$$

It is clear that  $|z_k| \leq c_1$  with some constant  $c_1$ . Also, according to Lemma 1,  $|z_k| \rightarrow 0$  in probability, as  $k \rightarrow \infty$ , and therefore

$$\delta_k = Ez_k^2 \rightarrow 0 \quad (k \rightarrow \infty). \quad (51)$$

Fix  $n$  and define

$$\epsilon_k^* = \epsilon_k I\{|\epsilon_k| \leq \tau_n a_n\} \text{ and } \epsilon_k^{**} = \epsilon_k^* - E\epsilon_k^*, \quad 1 \leq k \leq n,$$

where  $\tau_n$  is a numerical sequence (to be chosen later) tending to  $\infty$  and  $I\{\cdot\}$  denotes the indicator function. Let

$$A(t) = \int_{-t}^t x^2 dF(x),$$

where  $F$  denotes the distribution function of  $\epsilon_0$ . By the classical theory of the domain of attraction of stable laws (Feller, 1966, pp. 574–577), we have that

$$\lim_{t \rightarrow \infty} \frac{A(t)}{t^{2-\alpha} L(t)} = c_2 \quad (52)$$

with some  $0 < c_2 < \infty$ . Also, we note that by the definition of  $a_n$  and the properties of regularly varying functions we obtain

$$nL(a_n)/a_n^\alpha \rightarrow 1 \quad (n \rightarrow \infty). \quad (53)$$

We also need that for any  $\kappa > 0$  there is a constant  $0 < c_3 < \infty$  such that

$$\frac{L(\lambda x)}{L(x)} \leq c_3 \lambda^\kappa \quad \text{for all } \lambda \geq 1 \text{ and } x \geq 1. \quad (54)$$

The assertion in (54) is an immediate consequence of the monotone equivalence theorems in Bingham *et al.* (1987, p. 23). Indeed, there is a nonincreasing regularly varying function  $\psi$  such that

$$\lim_{x \rightarrow \infty} \frac{x^{-\kappa} L(x)}{\psi(x)} = 1,$$

and since  $\psi(\lambda x) \leq \psi(x)$  for all  $\lambda \geq 1$  and  $x \geq 1$ , (54) is proven.

Using the independence of  $\epsilon_k^{**}$  and  $z_{k-1}$ , we conclude

$$E(\epsilon_k^{**} z_{k-1})^2 = E(\epsilon_k^{**})^2 E z_{k-1}^2 \leq E(\epsilon_k^*)^2 E z_{k-1}^2 = A(\tau_n a_n) \delta_{k-1}$$

and the orthogonality of  $\{\epsilon_k^{**} z_{k-1}, k \geq 1\}$  yields

$$\text{var}\left(\frac{1}{a_n} \sum_{1 \leq k \leq n} \epsilon_k^{**} z_{k-1}\right) = \frac{1}{a_n^2} \sum_{1 \leq k \leq n} E(\epsilon_k^{**} z_{k-1})^2 \leq \frac{1}{a_n^2} A(\tau_n a_n) \sum_{1 \leq k \leq n} \delta_{k-1}.$$

Combining (52)–(54) we obtain

$$\begin{aligned} \text{var}\left(\frac{1}{a_n} \sum_{1 \leq k \leq n} \epsilon_k^{**} z_{k-1}\right) &= \mathcal{O}(1) a_n^{-2} (\tau_n a_n)^{2-\alpha} L(\tau_n a_n) \sum_{1 \leq k \leq n} \delta_{k-1} \\ &= \mathcal{O}(1) \tau_n^{2-\alpha} \frac{L(\tau_n a_n)}{L(a_n)} \frac{1}{n} \sum_{1 \leq k \leq n} \delta_{k-1} \\ &= \mathcal{O}(1) \tau_n^{2-\alpha+\kappa} \frac{1}{n} \sum_{1 \leq k \leq n} \delta_{k-1}. \end{aligned}$$

By (51), if  $\tau_n \rightarrow \infty$  slowly enough, then  $\tau_n^{2-\alpha+\kappa} \sum_{1 \leq k \leq n} \delta_{k-1}/n \rightarrow 0$ , showing that

$$\text{var}\left(\frac{1}{a_n} \sum_{1 \leq k \leq n} \epsilon_k^{**} z_{k-1}\right) = o(1).$$

Using the definitions of  $\epsilon_k^*$  and  $a_n$  together with (15), (53) and (54), we obtain that

$$\sum_{1 \leq k \leq n} P\{\epsilon_k^* \neq \epsilon_k\} = n(1 - F(\tau_n a_n)) \rightarrow 0 \quad (n \rightarrow \infty).$$

Next we observe that

$$\frac{n}{a_n} \int_{-\tau_n a_n}^{\tau_n a_n} x dF(x) \rightarrow 0 \quad (n \rightarrow \infty),$$

if  $\tau_n \rightarrow \infty$  slowly enough, so using  $z_k \rightarrow 0$  ( $k \rightarrow \infty$ ) we conclude that

$$\sum_{1 \leq k \leq n} z_{k-1} E \epsilon_k^* = \sum_{1 \leq k \leq n} z_{k-1} \int_{-\tau_n a_n}^{\tau_n a_n} x dF(x) = o(1) n \int_{-\tau_n a_n}^{\tau_n a_n} x dF(x) = o(a_n).$$

Now the proof of (50) is complete.  $\square$

PROOF OF THEOREM 4. Using (41), we obtain

$$\hat{\eta}_{n,1}(\sigma^2) - \varphi = -\left(c_{11}(n) \frac{1}{n} g_{1,n}(\sigma^2) + c_{12}(n) \frac{1}{n} g_{2,n}(\sigma^2)\right), \quad (55)$$

and

$$\hat{\eta}_{n,2}(\sigma^2) - \omega^2 = -\left(c_{21}(n) \frac{1}{n} g_{1,n}(\sigma^2) + c_{22}(n) \frac{1}{n} g_{2,n}(\sigma^2)\right), \quad (56)$$

where  $c_{ij}$  are defined in (45). By Lemma 7, (42) and (45) we obtain

$$\frac{n}{a_n}(\hat{\eta}_{n,2}(\sigma^2) - \omega^2) = \frac{1}{a_n} \sum_{1 \leq k \leq n} (b_k^2 - \omega^2) + o_P(1). \quad (57)$$

Since (46) clearly holds, we also have (47) and from (49) we obtain that

$$\hat{\eta}_{n,1}(\sigma^2) - \varphi = c_{11}(n) \frac{1}{n} g_{1,n}(\sigma^2) + (|\hat{\eta}_{n,1} - \varphi| \mathcal{O}_P(1) + \mathcal{O}_P(n^{-1/2})) \mathcal{O}_P(a_n/n).$$

Hence by (42) and (48), we have

$$n^{1/2}(\hat{\eta}_{n,1}(\sigma^2) - \varphi) = n^{-1/2} \sum_{1 \leq k \leq n} b_k + o_P(1). \quad (58)$$

The convergence in distribution of  $n^{1/2}(\hat{\eta}_{n,1}(\sigma^2) - \varphi)$  and  $n(\hat{\eta}_{n,2}(\sigma^2) - \omega^2)/a_n$  now follows from (57) and (58); only the asymptotic independence must be established. Note that the vector  $(\sum_{1 \leq k \leq n} b_k/n^{1/2}, \sum_{1 \leq k \leq n} (b_k^2 - \omega^2)/a_n)$  converges in distribution (Sec. 10.1 in Meerschaert and Scheffler, 2001). The first coordinate of the limit is normal, the second does not contain normal component and therefore the coordinates of the limit distribution are independent (Meerschaert and Scheffler, 2001, p. 41).  $\square$

## 5. PROOFS OF THEOREM 5 AND COROLLARIES 1–3

Using (8), one can easily verify that

$$X_\ell = \sum_{i=1}^{\ell} e_i \prod_{j=i+1}^{\ell} (\varphi + b_j) + X_0 \prod_{j=1}^{\ell} (\varphi + b_j),$$

and therefore

$$\begin{aligned} \left( \prod_{j=1}^{\ell} (\varphi + b_j) \right)^{-1} X_\ell &= \sum_{i=1}^{\ell} e_i \left( \prod_{j=1}^i (\varphi + b_j) \right)^{-1} + X_0 \\ &= \sum_{i=1}^{\ell} e_i e^{-S(i)} \gamma_i + X_0. \end{aligned} \quad (59)$$

**PROOF OF THEOREM 5.** First we note that assumption (3) yields

$$|e_i| = \mathcal{O}(e^{ic_1}) \text{ a.s. for any } c_1 > 0$$

(Berkes *et al.*, 2003) and by the strong law of large numbers

$$e^{-S(i)} = o(e^{-ic_2}) \text{ a.s. for any } 0 < c_2 < E|\xi_0|.$$

Hence,  $Y$  is absolutely convergent with probability 1 and the result follows immediately from (59).  $\square$

The proof of the second part of Theorem 6 is based on Lemma 8.

LEMMA 8. *If (2), (3), (8), (19), (22) and (23) hold, then*

$$P\{Y = c\} = 0 \text{ for any } c.$$

PROOF. First, we show that for any sequence  $a_n$

$$\sum_{1 \leq i < \infty} P\{e^{-S(i)} \gamma_i e_i \neq a_i \mid \xi_j, -\infty < j < \infty\} = \infty \text{ a.s.} \quad (60)$$

Since  $E\xi_0$  exists, we obtain  $P\{\varphi + b_0 = 0\} = 0$ , so  $\gamma_i$  can be 0 only with probability 0. Hence (60) holds, if for any sequence  $d_n$

$$\sum_{1 \leq i < \infty} P\{e_i \neq d_i\} = \infty. \quad (61)$$

By (22), we have (61) if and only if

$$\sum_{1 \leq i < \infty} P\{e_0 \neq d_i\} = \sum_{1 \leq i < \infty} (1 - P\{e_0 = d_i\}) = \infty. \quad (62)$$

If  $P\{e_0 = b_i\} \rightarrow 1$ , then  $e_0$  must be a constant with probability 1, contradicting (23).

Using (60), we obtain that for any sequence  $a_n$

$$\sum_{1 \leq i < \infty} P\{e^{-S(i)} \gamma_i e_i \neq a_i\} = \infty, \quad (63)$$

and therefore Lemma 8 follows from Lévy (1931) (see also Breiman, 1968, p. 51).  $\square$

LEMMA 9. *If (2), (5)–(8), (11), (19) and (21) or (22) and (23) hold, then*

$$\left| g_{1,n}(y) - \sum_{1 \leq k \leq n} \frac{b_k}{\omega^2} \right| = \mathcal{O}(1) \text{ a.s.} \quad (64)$$

and

$$\left| g_{2,n}(y) - \sum_{1 \leq k \leq n} \frac{1}{2\omega^4} (\omega^2 - b_k^2) \right| = \mathcal{O}(1) \text{ a.s.} \quad (65)$$

where  $g_{1,n}(y)$  and  $g_{2,n}(y)$  are the partial derivatives of  $L_n(\mathbf{u})$  with respect to  $s$  and  $x$  at  $(\varphi, \omega^2, y)$ .

PROOF OF LEMMA 9. We return to the decompositions of  $g_{1,n}(y)$  and  $g_{2,n}(y)$  used in the proof of Lemma 4. Using Theorem 5 and Lemma 8, we obtain

$$\begin{aligned}
\left| \sum_{1 \leq k \leq n} \frac{e_k X_{k-1}}{\omega^2 X_{k-1}^2 + y} \right| &\leq \sum_{1 \leq k \leq n} |e_k| \frac{|X_{k-1}|}{\omega^2 X_{k-1}^2 + y} \\
&= \sum_{1 \leq k \leq n} |e_k| e^{-S(k-1)} (e^{-S(k-1)} |X_{k-1}|) (\omega^2 (e^{-S(k-1)} X_{k-1})^2 \\
&\quad + e^{-2S(k-1)} y)^{-1} \\
&\leq \left\{ \max_{1 \leq k \leq \infty} (e^{-S(k-1)} |X_{k-1}|) (\omega^2 (e^{-S(k-1)} X_{k-1})^2 + e^{-2S(k-1)} y)^{-1} \right\} \\
&\quad \times \sum_{1 \leq k \leq n} |e_k| e^{-S(k-1)} \\
&= \mathcal{O}(1) \quad \text{a.s.}
\end{aligned}$$

since by Berkes *et al.* (2003),  $\sum_{1 \leq k \leq n} |e_k| e^{-S(k-1)}$  is finite with probability 1. Similar arguments give

$$\left| \sum_{1 \leq k \leq n} b_k \left\{ \frac{X_{k-1}^2}{\omega^2 X_{k-1}^2 + y} - \frac{1}{\omega^2} \right\} \right| \leq \sum_{1 \leq k \leq n} |b_k| \frac{1}{\omega^2} \frac{|y|}{\omega^2 X_{k-1}^2 + y} = \mathcal{O}(1) \quad \text{a.s.},$$

completing the proof of (64).

The proof of (65) goes along the same lines and hence is omitted.  $\square$

PROOF OF COROLLARY 1. It is an immediate consequence of the strong law of large numbers and Theorems 5 and 6.  $\square$

PROOF OF COROLLARY 2. The proof of Theorem 2 can be repeated; only Lemma 4 must be replaced with Lemma 9.  $\square$

PROOF OF COROLLARY 3. Minor modifications of the proof of Theorem 4 are required only. Namely, one must use Lemma 9 instead of Lemma 7.  $\square$

#### ACKNOWLEDGEMENTS

Research by the first author was partially supported by OTKA grants T 43037 and K 61052; that by the second author was partially supported by NSF grant DMS 0604670 and grant RGC-HKUST 6428/06H and that by the third was partially supported by Hong Kong Research Grants Council.

## NOTE

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