ON THE CONVERGENCE OF $\sum \mathbf{c_k f(n_k x)}$

Istvan Berkes and Michel Weber

Abstract. Let f be a periodic measurable function and (n_k) an increasing sequence of integers. We study conditions under which the series $\sum_{k=1}^\infty c_k f(n_k x)$ converges in mean and for almost every x. There is a wide classical literature on this problem going back to the 20's, but the results for general f are much less complete than in the trigonometric case $f(x) = \sin x$. As it turns out, the convergence properties of $\sum_{k=1}^\infty c_k f(n_k x)$ in the general case are determined by a delicate interplay between the coefficient sequence (c_k), the analytic properties of f and the growth speed and number-theoretic properties of (n_k). In this paper we give a general study of this convergence problem, prove several new results and improve a number of old results in the field. We also study the case when the n_k are random and investigate the discrepancy the sequence $\{n_k x\}$ mod 1 both in the random and nonrandom case.

1. Introduction and Mean Convergence.

Throughout this paper $\mathcal{N} = \{n_k, k \ge 1\}$ denotes an increasing sequence of positive numbers, and $\mathbf{c} = \{c_k, k \ge 1\}$ some element of ℓ^2. Let $\mathbf{T} = [0,1) = \mathbf{R}/\mathbf{Z}$ be the circle equipped with normalized Lebesgue measure λ, and let $f : \mathbf{T} \to \mathbf{R}$ be a Borel-measurable function. With these quantities in hand, one can formally define the series

$$\sum_{k\ge 1} c_k f(n_k x). \tag{1.1}$$

The aim of this paper is to study under which conditions the series (1.1) defines an element of $L^2(\mathbf{T})$ or converges for almost all x in $\mathbf{T}$. We are thus going to investigate

Mathematical Subject Classification: Primary 11F66, 11L03, 30B50, 11K38, 60G50 Secondary 11M41, 11L07.

Keywords: Dirichlet series, i.i.d. sums, discrepancy, mean convergence, almost everywhere convergence, martingales, Hadamard gap sequences.

the convergence problem of the sequence of partial sums

$$S_N^{\mathcal{N}}(\mathbf{c}, f) = \sum_{k=1}^{N} c_k f(n_k x), \qquad N = 1, 2 \ldots$$

in mean (namely in the space $L^2(\mathbf{T})$) or for almost all x in $\mathbf{T}$. In the trigonometric case this problem has been one of the central problems of harmonic analysis, investigated intensively from the 1920's, culminating in the celebrated theorem of Carleson [C], stating the the almost everywhere convergence of the series $\sum_{k=1}^{\infty} c_k \sin 2\pi kx$, $\sum_{k=1}^{\infty} c_k \cos 2\pi kx$ for all $\mathbf{c} \in \ell^2$. Starting from the 1930's, there has been also considerable interest in the convergence properties of the series (1.1) for general $f \in L^2(\mathbf{T})$, but the existing results are, even today, much less complete than in the trigonometric case. As it turned out, for general f the behavior of the series (1.1) is radically different from the trigonometric case: the terms of the series are usually far from orthogonal and the convergence properties of the sum depend sensitively on the coefficient sequence (c_k), the analytic properties of f, the growth speed and most importantly on the number-theoretic properties of the sequence n_k. As a result of the 'interference' between the behavior of the Fourier coefficients of f and the arithmetic properties of n_k, even the asymptotic computation of the integral

$$\int_{\mathbf{T}} \left(\sum_{k=1}^{N} c_k f(n_k x) \right)^2 dx \tag{1.2}$$

is generally a hard problem. The difficulties encountered in this field are clearly indicated by the long history of the Khinchin conjecture (for a survey see [RW]), a closely related problem dealing with the a.e. convergence of the averages

$$\frac{1}{N} \sum_{k=1}^{N} f(kx)$$

for general integrable f. The integral (1.2) (with $c_k = 1$ and with indicator functions f) is also fundamental in the metric theory of Diophantine approximation. The first insight into the nature of this integral and the closely related problem of mean convergence of (1.1) was given by the following result by Wintner [Wi2], connecting the convergence problem with Dirichlet series.

Theorem A. *Let $f \in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ and with Fourier series*

$$f \sim \sum_{k=1}^{\infty} (a_k \cos 2\pi kx + b_k \sin 2\pi kx). \tag{1.3}$$

Then the following statements are equivalent:

(a) The series $\sum_{k=1}^{\infty} c_k f(kx)$ *converges in* $L^2(\mathbf{T})$ *for any* $\mathbf{c} \in \ell^2$.

(b) There exists a constant $K > 0$ *such that for any* $n \geq 1$ *and any real* $\{c_k, 1 \leq k \leq n\}$ *we have*

$$\int_{\mathbf{T}} \left(\sum_{k=1}^{n} c_k f(kx) \right)^2 dx \leq K \left(\sum_{k=1}^{n} c_k^2 \right).$$

(c) The infinite matrix

$$\int_{\mathbf{T}} f(kt) f(\ell t) dt \qquad\qquad (k, \ell = 1, 2 \ldots)$$

defines a bounded operator on ℓ^2.

(d) The Dirichlet series

$$\sum_{n=1}^{\infty} a_n n^{-s}, \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n n^{-s} \tag{1.4}$$

are regular and bounded in the half-plane $\Re(s) > 0$.

The basic ingredient of Wintner's proof is Toeplitz's criterion [T] for the ℓ^2 boundedness of so called "D-matrices" in terms of Dirichlet series. The connection with the convergence problem in Theorem A is established by the Möbius transformation; see [Wi2] for the details. As a comparison, note that the assumption $f \in L^2(\mathbf{T})$, i.e. $\sum_{k=1}^{\infty}(a_k^2 + b_k^2) < \infty$ implies only that the sums in (1.4) are absolutely convergent for $\Re(s) > 1/2$.

Clearly, condition (a) of Theorem A implies that $\sum_{k=1}^{\infty} c_k f(kx)$ converges in measure for any $\mathbf{c} \in \ell^2$. By a remarkable result of Nikishin [Ni], the converse is also true, i.e. the convergence theory of $\sum_{k=1}^{\infty} c_k f(kx)$ is the same for L^2 convergence and convergence in measure.

Note that if condition (d) of Theorem A is not satisfied, the series $\sum_{k=1}^{\infty} c_k f(kx)$ can still converge for a large class of coefficient sequences (c_k). For example, Wintner noted that if $f \in L^2(\mathbf{T})$, $\int_{\mathbf{T}} f(t)dt = 0$ with Fourier series (1.3) then $\sum_{k=1}^{\infty} c_k f(kx)$ converges in $L^2(\mathbf{T})$ if $a_k = O(k^{-\gamma})$, $b_k = O(k^{-\gamma})$, $c_k = O(k^{-\gamma})$ for some $\gamma > 1/2$. Note that here the assumptions made on the Fourier coefficients of f do not in general imply the boundedness of the Dirichlet series in Theorem A and accordingly, the assumption made on the coefficient sequence (c_k) is stronger than $\mathbf{c} \in \ell^2$.

An application of the last remark is the series

$$\sum_{k=1}^{\infty} \frac{\psi(kx+\frac{1}{2})}{k}, \tag{1.5}$$

where

$$\psi(x) = \begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \neq [x], \\ 0 & \text{if } x = [x]. \end{cases}$$

This example has considerable historical interest, since it was used by Riemann [R] to illustrate the limitations of his own integration theory. He showed that both (1.5) and the trigonometric sum

$$\sum_{n=1}^{\infty} \frac{c(n)}{n} \sin 2\pi n x, \tag{1.6}$$

where

$$c(n) = \sum_{d|n} (-1)^d \tag{1.7}$$

converge if x is rational, and to the same limit. Moreover, he observed that the function defined by these series on the set of rational numbers is unbounded on any interval, and thus (1.6) cannot be the Fourier series of its sum in the Riemann sense. From the remark after Theorem A it follows that (1.5) converges in $L^2(\mathbf{T})$ and Wintner showed in [Wi1] that its sum belongs to $L^p(\mathbf{T})$ for any $p > 1$ and has (1.6)-(1.7) as its Fourier series in the Lebesgue sense.

Let us recall that a sequence of vectors $(x_n)_{n\in A}$ in a Hilbert space H is called a Riesz sequence if there exist positive constants C_1, C_2 such that

$$C_1 \left(\sum_{n\in A} |a_n|^2 \right) \leq \Big\| \sum_{n\in A} a_n x_n \Big\|^2 \leq C_2 \big(\sum_{n\in A} |a_n|^2 \big) \quad \text{for all sequences of scalars } (a_n)_{n\in A}.$$

Hedenmalm, Lindquist and Seip proved (see [HLS1], [HLS2]) that if

$$f \in L^2(\mathbf{T}),\ f(t) \sim \sum_{k=1}^{\infty} \varphi_k \cos 2\pi k t,$$

then $\{f(nx), n \geq 1\}$ is a Riesz sequence in $L^2(\mathbf{T})$ if and only if the Dirichlet series $\sum_{n=1}^{\infty} \varphi_n n^{-s}$ is analytic and bounded away from 0 and ∞ in the whole right half-plane $\Re z > 0$, i.e.

$$\delta \leq \Big| \sum_{n=1}^{\infty} \varphi_n n^{-\sigma - it} \Big| \leq \Delta, \qquad \text{for } \sigma > 0,$$

with some positive constants δ and Δ. See also the preceding work of Gosselin and Neuwirth in [GN], as well as the article by Ginsberg, Neuwirth and Newman in [GNN].

So far, we have considered the convergence in mean problem in the case $\mathcal{N} = \mathbb{N}$. In the general case the existing results in the literature are much less complete, due to number-theoretic difficulties. Given positive integers a, b, define

$$\langle a, b\rangle = \frac{(a,b)}{[a,b]},$$

where (a,b) and $[a,b]$ denote the greatest common divisor resp. least common multiple of a and b. The following theorem is an easy consequence of results of Wintner [Wi2].

Theorem B. *Let $f \in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ and Fourier series*

$$f \sim \sum_{k=1}^{\infty} (a_k \cos 2\pi kt + b_k \sin 2\pi kt),$$

where $a_k = O(k^{-\alpha})$, $b_k = O(k^{-\alpha})$, $\alpha > 1/2$. Let (n_k) be an increasing sequence of positive integers and (c_k) a real coefficient sequence. Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges in the mean provided

$$\sum_{k,l=1}^{\infty} |c_k||c_l| \langle n_k, n_l\rangle^{\alpha} < \infty. \tag{1.8}$$

To prove this, it suffices to consider the case when the Fourier series of f is a pure sine or cosine series. Now if $f \sim \sum_{k=1}^{\infty} a_k \cos 2\pi kt$, then by the assumption on the a_k and relation (52) of [Wi2] we have for any positive integers i, j

$$\int_{\mathbf{T}} f(it)f(jt)dt = \frac{1}{2}\sum_{h=1}^{\infty} a_{hi/(i,j)} a_{hj/(i,j)} \le C_1 \langle i, j\rangle^{\alpha} \sum_{h=1}^{\infty} h^{-2\alpha} \le C_2 \langle i, j\rangle^{\alpha}$$

for some constants C_1, C_2 and thus

$$\int_{\mathbf{T}} \left(\sum_{k=m}^{n} c_k f(n_k t)\right)^2 dt \le C_2 \sum_{m \le k, l \le n} |c_k||c_l| \langle n_k, n_l\rangle^{\alpha} \tag{1.9}$$

Hence Theorem B follows from (1.8).

In particular, under the assumptions made on f in Theorem B, $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges in $L^2(\mathbf{T})$ norm for any $\mathbf{c} \in \ell^2$ provided the quadratic form

$$\sum_{k,l=1}^{\infty} \langle n_k, n_l \rangle^{\alpha} x_k x_l \tag{1.10}$$

is bounded, i.e. there exists a constant $A > 0$ such that

$$\left| \sum_{k,l=1}^{N} \langle n_k, n_l \rangle^{\alpha} x_k x_l \right| \le A \sum_{n=1}^{N} x_n^2 \tag{1.11}$$

for any $N \ge 1$ and any real $x_1, \ldots, x_N$. This is equivalent, in turn, to the fact that the matrix $\langle n_k, n_l \rangle^{\alpha}$ $(k, l = 1, 2, \ldots)$ defines a bounded operator on ℓ^2. In the case $n_k = k$ this holds if and only if $\alpha > 1$, as it follows easily from Theorem A. Also, if the n_k are coprimes, then $\langle n_k, n_l \rangle = (n_k n_l)^{-1}$ and thus by Cauchy's inequality, (1.11) is satisfied if $\sum_{k=1}^{\infty} n_k^{-2\alpha} < \infty$. For general (n_k), a sufficient condition for (1.11) is (see e.g. Lemma 4.8 in [We1])

$$\sup_{k \ge 1} \sum_{l \ge k} \langle n_k, n_l \rangle^{\alpha} < \infty. \tag{1.12}$$

Unfortunately, computing the order of magnitude of the sums in (1.11), (1.12) for general (n_k) is a difficult number-theoretic problem. In a profound paper, Gál [G] showed that for any increasing (n_k) we have

$$\sum_{k,l=1}^{N} \langle n_k, n_l \rangle \le Cn(\log \log n)^2$$

and he constructed an (n_k) for which the bound $Cn(\log \log n)^2$ is actually attained. For this sequence (n_k), relation (1.11) clearly fails for $x_1 = \cdots = x_N = 1$. No sharp estimate for the left hand side of (1.11) is known for general $x_1, \ldots, x_N$.

The following theorem gives mean convergence criteria for $\sum_{k=1}^{\infty} c_k f(n_k x)$ in the case when (1.12) is not satisfied.

Theorem 1.1. *Let $f \in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ and Fourier series*

$$f(t) \sim \sum_{k=1}^{\infty} (a_k \cos 2\pi kt + b_k \sin 2\pi kt)$$

where $a_k = O(k^{-\alpha})$, $b_k = O(k^{-\alpha})$, $\alpha > 1/2$. Let (n_k) be an increasing sequence of positive integers and let (λ_n) be a positive nondecreasing sequence such that $\lambda_{2n}/\lambda_n = O(1)$ and

$$\sup_{1\le k\le N} \sum_{l=1}^{N} \langle n_k, n_l\rangle^{\alpha} \le \lambda_N. \tag{1.13}$$

Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges in $L^2(\mathbf{T})$ norm provided

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^{\gamma} \lambda_k < \infty \quad \textit{for some } \gamma > 1. \tag{1.14}$$

Note that in the case when $\lambda_N = O(1)$, $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges in the mean provided $\sum_{k=1} c_k^2 < \infty$ (see above), but condition (1.14) specialized to this case gives a more stringent condition. This is due to the fairly crude estimates we use for the quadratic form appearing in the argument.

We formulate a few corollaries of Theorem 1.1.

Corollary 1.1. *Assume f satisfies the assumptions of Theorem 1.1 and let $n_k = k^r$ where $r \ge 2$ is an integer. Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges in the mean provided $\sum_{k=1}^{\infty} c_k^2 < \infty$.*

Note that the assumptions made on the Fourier coefficients of f in Corollary 1.1 do not imply condition (d) of Theorem A, but $\sum_{k=1}^{\infty} c_k f(n_k x)$ still converges for all $\mathbf{c} \in \ell^2$. This is due to the speed and nice number-theoretic properties of n_k.

Corollary 1.2. *Assume f satisfies the assumptions of Theorem 1.1. Then the series $\sum_{k=1}^{\infty} c_k f(kx)$ converges in the mean provided*

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^{3+\varepsilon} < \infty, \qquad \textit{if } \alpha = 1$$

and

$$\sum_{k=1}^{\infty} c_k^2 k^{1-\alpha} < \infty, \qquad \textit{if } \alpha < 1.$$

Note that the case $\alpha > 1$ is uninteresting: in this case $\sum_{k=1}^\infty (|a_k| + b_k|) < \infty$ and thus

$$\|\sum_{k=1}^N c_k f(n_k x)\| \le \sum_{j=1}^\infty |a_j| \| \sum_{k=1}^N c_k \cos 2\pi n_k x\| + \sum_{j=1}^\infty |b_j| \| \sum_{k=1}^N c_k \sin 2\pi n_k x\| \le C(\sum_{k=1}^N c_k^2)^{1/2}$$

for some constant C and thus $\sum_{k=1}^\infty c_k f(n_k x)$ converges in the mean for any $\mathbf{c} \in \ell^2$. (Actually, the series converges almost everywhere also, see Gaposhkin [GA?].)

Corollary 1.3. *Let f satisfy the assumptions of Theorem 1.1 and let (n_k) be a sequence of integers such that for any $d \ge 1$ we have $\sum_{d|n_k} n_k^{-1} \le A/d$ with an absolute constant A. Then $\sum_{k=1}^\infty c_k f(n_k x)$ converges in the mean provided*

$$\sum_{k=1}^\infty c_k^2 (\log k)^\gamma (\log n_k) < \infty, \qquad \gamma > 1. \tag{1.15}$$

Condition (1.15) is satisfied if the sequence (n_k) is roughly unformly distributed among the residue classes mod d. If the n_k are coprimes, for any d the sum $\sum_{d|n_k} n_k^{-1}$ contains at most one term and thus the conditions of Corollary 1.3 are satisfied. The conditions are also satisfied for $n_k = k^r$, $r \ge 2$ but in this case Corollary 1.1 gives a better result. We see again that the number-theoretic properties of n_k play a crucial role in the convergence behavior of $\sum_{k=1}^\infty c_k f(n_k x)$, which can be anticipated from Theorem B. If (n_k) grows with a polynomial speed, then $\log n_k = O(\log k)$ and thus the convergence condition (1.15) reduces to $\sum c_k^2 (\log k)^{2+\varepsilon} < \infty$.

Proof of Theorem 1.1. By assumption (1.13), relation (1.9) and the Cauchy-Schwarz inequality we have

$$\begin{aligned} &\int_{\mathbf{T}} \left(\sum_{k=1}^N c_k f(n_k t) \right)^2 dt \\ &\le C_2 \sum_{k,l=1}^N |c_k||c_l| \langle n_k, n_l \rangle^\alpha \le C_2 \sum_{k,l=1}^N \frac{1}{2} (c_k^2 + c_l)^2 \langle n_k, n_l \rangle^\alpha \le C_2 \lambda_N \left(\sum_{k=1}^N c_k^2 \right) \end{aligned} \tag{1.16}$$

for any real $c_1, \ldots, c_N$. Assume now (1.14) and let $Z_\nu = \sum_{k=2^\nu+1}^{2^{\nu+1}} c_k f(n_k x)$. By the Cauchy-Schwarz inequality we have

$$\left(\sum_{k=2^m+1}^{2^n} c_k f(n_k x) \right)^2 = \left(\sum_{\nu=m}^n Z_\nu \right)^2 \le \left(\sum_{\nu=m}^n \nu^\gamma Z_\nu^2 \right) \left(\sum_{\nu=1}^\infty \nu^{-\gamma} \right). \tag{1.17}$$

Thus with some constant C we have, using (1.16),

$$\int_{\mathbf{T}} \left(\sum_{k=2^m+1}^{2^n} c_k f(n_k x) \right)^2 dx \le C \left(\sum_{\nu=m}^{n} \nu^\gamma \int_{\mathbf{T}} Z_\nu^2 dx \right)$$
$$\le C \sum_{\nu=m}^{n} \nu^\gamma \left(\sum_{k=2^\nu+1}^{2^{\nu+1}} c_k^2 \right) \lambda_{2^{\nu+1}} \le C' \sum_{k=2^m+1}^{2^n} c_k^2 (\log k)^\gamma \lambda_k. \tag{1.18}$$

Here the last expression tends to 0 as $m, n \to \infty$ and this remains valid if an arbitrary subset of the c_k's is replaced by 0's. Thus the L^2 norm of $\sum_{k=i}^{j} c_k f(n_k x)$ tends to 0 if $i, j \to \infty$, completing the proof of Theorem 1.1.

Note that in the case $n_k = k^r$, $r \in \mathbf{N}$ we have

$$\langle n_k, n_l \rangle^\alpha = \langle k, l \rangle^{r\alpha}$$

Thus in view of Theorem 1.1, for the proof of Corollaries 1.1 and 1.2 it suffices to prove the following

Lemma 1.1. *Let $\beta > 0$ and*

$$\lambda_N^* = \sup_{1 \le l \le N} \sum_{k=1}^{N} \langle k, l \rangle^\beta. \tag{1.19}$$

Then $\lambda_N^* = O(1)$, $\lambda_N^* = O(\log^2 N)$ and $\lambda_N^* = O(N^{1-\beta})$ according as $\beta > 1$, $\beta = 1$ or $\beta < 1$.

Proof. Fix $d|h$ and sum in (1.19) first for those k for which $(h, k) = d$. Then we get

$$\sum_{k \le n, (h,k)=d} \left(\frac{d}{[h,k]} \right)^\beta$$
$$= \sum_{k \le n, (h,k)=d} \left(\frac{d^2}{hk} \right)^\beta \le \left(\frac{d}{h} \right)^\beta \sum_{k \le n, d|k} \left(\frac{d}{k} \right)^\beta \le \left(\frac{d}{h} \right)^\beta \sum_{l=1}^{[n/d]} \frac{1}{l^\beta} \tag{1.20}$$

For $\beta = 1$ the last sum in (1.20) is at most $C^* \log n$ and thus summing for all $d|h$ and noting that the sum of all divisors of h is $\le Ch \log h$, we get the statement of the lemma. For $\beta > 1$ the last sum in (1.20) is $O(1)$ and

$$\sum_{d|h} d^\beta \le \sum_{d|h, d \le \sqrt{h}} d^\beta + \sum_{d|h, d \le \sqrt{h}} (h/d)^\beta.$$

Let $\varepsilon > 0$. Since the number of divisors of h is $O(h^\varepsilon)$, the first sum on the right hand side has $O(h^\varepsilon)$ terms and thus this sum is $O(h^{\beta/2+\varepsilon})$; the second sum on the right side is at most $h^\beta \sum_{j=1}^\infty j^{-\beta} = O(h^\beta)$. Choosing ε sufficiently small, we get the statement of the lemma in the case $\beta > 1$. Let finally $0 < \beta < 1$. Then the last expression in (1.20) is at most

$$\left(\frac{d}{h}\right)^\beta \left(\frac{n}{d}\right)^{1-\beta} = \frac{1}{h^\beta} n^{1-\beta} d^{2\beta-1}$$

and thus the λ_N^* is

$$\ll \frac{1}{h^\alpha} n^{1-\alpha} \sum_{d|h} d^{2\alpha-1}. \tag{1.21}$$

Let $0 < \varepsilon < \min(\alpha, 1-\alpha)$. Since the number of divisors of h is $O(h^\varepsilon)$, for $\alpha \geq 1/2$ the sum in (1.21) is $O(h^{2\alpha-1+\varepsilon})$ and thus the expression in (1.21) is

$$\ll \frac{1}{h^\alpha} n^{1-\alpha} h^{2\alpha-1+\varepsilon} = h^{\alpha-1+\varepsilon} n^{1-\alpha} \leq n^{1-\alpha}.$$

If $\alpha < 1/2$, then the sum in (1.21) is $O(h^\varepsilon)$ and thus the expression in (1.21) is

$$\ll \frac{1}{h^\alpha} n^{1-\alpha} h^\varepsilon \leq n^{1-\alpha}.$$

Thus in both cases the expression in (1.21) is $O(n^{1-\alpha})$, and thus the lemma is proved.

To prove Corollary 1.3, it suffices to show that

$$\sum_{k=1}^N \frac{(n_h, n_k)}{[n_h, n_k]} \leq C \log n_h. \tag{1.22}$$

Fix $d|n_h$ and compute the sum in (1.22) for those $h \leq k \leq N$ such that $(n_h, n_k) = d$. This restricted sum clearly cannot exceed, in view of the assumption of Corollary 1.3,

$$\sum_{1 \leq k \leq N, d|n_k} \frac{d^2}{n_h n_k} \leq \frac{d^2}{n_h} \frac{A}{d}.$$

Summing for all $d|n_h$, and using the fact that the sum of divisors of n_h is $O(n_h \log n_h)$, we get (1.22).

Our next theorem gives a necessary and sufficient condition for the mean convergence of the series $\sum_{k=1}^\infty c_k f(n_k x)$ in terms of the coefficients c_k and the Fourier coefficients of f. Despite its precise character, it is of mainly theoretical interest only since its number-theoretical character makes it difficult to apply in concrete cases.

Theorem 1.2. *Let $f \in L^2(\mathbf{T})$, $\int_{\mathbf{T}} f(t)dt = 0$ have complex Fourier series $f \sim \sum_{k\in Z, k\neq 0} \varphi_k e_k$ where $e_k(x) = \exp(2\pi i k x)$. Let (n_k) be an increasing sequence of positive integers. Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges in the mean if and only if the following conditions are fulfilled:*

$$\begin{aligned} &a) \qquad \lim_{R\to\infty} \sup_{P\geq R} \sum_{|n|>n_R} \Big(\sum_{\substack{n_k|n \\ k\leq P}} \varphi_{n/n_k} c_k \Big)^2 = 0, \\ &b) \qquad \sum_n (\sum_{n_k|n} \varphi_{n/n_k} c_k)^2 < \infty. \end{aligned} \tag{1.23}$$

If both sequences $\{\varphi_n, n \in \mathbf{Z}\}$ and $\mathbf{c}$ have constant signs then (1.23a) follows from (1.23b), so that the sequence $\{S_N^{\mathcal{N}}(\mathbf{c}, f), N \geq 1\}$ converges in mean if and only if condition (1.23b) holds. Also, if $\sum_n (\sum_{n_k|n} |\varphi_{n/n_k} c_k|)^2 < \infty$, then the sequence $\{S_N^{\mathcal{N}}(\mathbf{c}, f), N \geq 1\}$ converges in mean.

Proof. Observe that

$$S_N^{\mathcal{N}}(\mathbf{c}, f) = \sum_n e_n (\sum_{\substack{n_k|n \\ k\leq N}} \varphi_{n/n_k} c_k) = \sum_{|n|\leq n_N} e_n \sum_{n_k|n} \varphi_{n/n_k} c_k + \sum_{|n|>n_N} e_n \sum_{\substack{n_k|n \\ k\leq N}} \varphi_{n/n_k} c_k.$$

Let $M \geq N \geq R$. Then,

$$\begin{aligned} \langle S_N^{\mathcal{N}}(\mathbf{c}, f), S_M^{\mathcal{N}}(\mathbf{c}, f)\rangle &= \sum_n (\sum_{\substack{n_k|n \\ k\leq N}} \varphi_{n/n_k} c_k)(\sum_{\substack{n_k|n \\ k\leq M}} \varphi_{n/n_k} c_k) \\ &= \sum_{|n|\leq n_R} (\sum_{\substack{n_k|n \\ k\leq N}} \varphi_{n/n_k} c_k)^2 + \sum_{|n|>n_R} (\sum_{\substack{n_k|n \\ k\leq N}} \varphi_{n/n_k} c_k)(\sum_{\substack{n_k|n \\ k\leq M}} \varphi_{n/n_k} c_k). \end{aligned}$$

Thus

$$\begin{aligned} &\Big| \langle S_N^{\mathcal{N}}(\mathbf{c}, f), S_M^{\mathcal{N}}(\mathbf{c}, f)\rangle - \sum_{|n|\leq n_R} (\sum_{\substack{n_k|n \\ k\leq N}} \varphi_{n/n_k} c_k)^2 \Big| \\ &\qquad \leq \Big[\sum_{|n|>n_R} \big| \sum_{\substack{n_k|n \\ k\leq N}} \varphi_{n/n_k} c_k \big|^2 \Big]^{\frac{1}{2}} \Big[\sum_{|n|>n_R} \big| \sum_{\substack{n_k|n \\ k\leq M}} \varphi_{n/n_k} c_k \big|^2 \Big]^{\frac{1}{2}} \\ &\qquad \leq \sup_{P\geq R} \sum_{|n|>n_R} \big| \sum_{\substack{n_k|n \\ k\leq P}} \varphi_{n/n_k} c_k \big|^2 \\ &\qquad \to 0, \end{aligned}$$

as R tends to infinity by assumption. Consequently,

$$\lim_{R\to\infty} \sup_{M,N\geq R} \Big|\langle S_N^{\mathcal{N}}(\mathbf{c},f), S_M^{\mathcal{N}}(\mathbf{c},f)\rangle - \sum_{|n|\leq n_R} (\sum_{\substack{n_k|n\\ k\leq N}} \varphi_{n/n_k} c_k)^2\Big| = 0.$$

In other words,

$$\lim_{M,N\to\infty} \langle S_N^{\mathcal{N}}(\mathbf{c},f), S_M^{\mathcal{N}}(\mathbf{c},f)\rangle = A := \sum_n (\sum_{n_k|n} \varphi_{n/n_k} c_k)^2 < \infty.$$

And also

$$\lim_{N\to\infty} \|S_N^{\mathcal{N}}(\mathbf{c},f)\|_2^2 = A.$$

These two facts then imply that

$$\lim_{N,M\to\infty} \|S_N^{\mathcal{N}}(\mathbf{c},f) - S_M^{\mathcal{N}}(\mathbf{c},f)\|_2 = 0,$$

as required.

Conversely if the sequence $\{S_N^{\mathcal{N}}(\mathbf{c},f), N\geq 1\}$ converges in mean, it is then bounded in mean:

$$\sup_{N\geq 1} \|S_N^{\mathcal{N}}(\mathbf{c},f)\|_2 = B < \infty.$$

But as

$$\|S_N^{\mathcal{N}}(\mathbf{c},f)\|_2^2 = \sum_{|n|\leq n_N} \big(\sum_{n_k|n} \varphi_{n/n_k} c_k\big)^2 + \sum_{|n|>n_N} \big(\sum_{\substack{n_k|n\\ k\leq N}} \varphi_{n/n_k} c_k\big)^2,$$

this implies that $A \leq B$. Now let f^* denote the limit in mean of the sequence $\{S_N^{\mathcal{N}}(\mathbf{c},f), N\geq 1\}$. From

$$\Big|\langle f^*, e_n\rangle - \langle S_N^{\mathcal{N}}(\mathbf{c},f), e_n\rangle\Big| \leq \|f^* - S_N^{\mathcal{N}}(\mathbf{c},f)\|_2$$

we deduce

$$\langle f^*, e_n\rangle = \lim_{N\to\infty} \langle S_N^{\mathcal{N}}(\mathbf{c},f), e_n\rangle = \lim_{N\to\infty} \sum_{\substack{n_k|n\\ k\leq N}} \varphi_{n/n_k} c_k = \sum_{n_k|n} \varphi_{n/n_k} c_k.$$

Thus $f^* = \sum_{n\in\mathbf{Z}} e_n \sum_{n_k|n} \varphi_{n/n_k} c_k$. Let R be some positive integer and define $H_R =< e_n, |n|\leq n_R >$. Let p_R be the projection onto the orthogonal complement ${H_R}^\perp$ of H_R. Then,

$$\Big|\|p_R(f^*)\|_2 - \|p_R(S_N^{\mathcal{N}}(\mathbf{c},f))\|_2\Big| \leq \|p_R(f^*) - p_R(S_N^{\mathcal{N}}(\mathbf{c},f))\|_2 \leq \|f^* - S_N^{\mathcal{N}}(\mathbf{c},f)\|_2 \ \to\ 0,$$

as N tend to infinity. Thus,

$$\sup_{N\geq R}\Big|\|p_R(f^*)\|_2-\|p_R(S_N^{\mathcal{N}}(\mathbf{c},f))\|_2\Big|\leq \sup_{N\geq R}\|f^*-S_N^{\mathcal{N}}(\mathbf{c},f)\|_2\to\ 0,$$

as R tends to infinity. Now, by the triangle inequality,

$$\begin{aligned}\sup_{N\geq R}\|p_R(S_N^{\mathcal{N}}(\mathbf{c},f))\|_2 &= \sup_{N\geq R}\Big[\sum_{|n|>n_R}\Big(\sum_{\substack{n_k|n\\ k\leq N}}\varphi_{n/n_k}c_k\Big)^2\Big]^{\frac12}\\ &\leq \sup_{N\geq R}\Big|\|p_R(f^*)\|_2-\|p_R(S_N^{\mathcal{N}}(\mathbf{c},f))\|_2\Big|+\|p_R(f^*)\|_2\\ &\to\ 0,\end{aligned}$$

as R tends to infinity. This completes the proof.

2. Almost sure convergence–Sufficient conditions.

Let $f\in L^2(\mathbf{T})$ with $\int_{\mathbf{T}}f(t)dt=0$ and let $\mathcal{N}$ be an increasing sequence of positive integers. Using standard terminology, we call the pair $(f,\mathcal{N})$ (or, equivalently, the sequence $f(n_kx)$) a *convergence system* if for any $\mathbf{c}\in\ell^2$, $\sum_{k=1}^\infty c_kf(n_kx)$ converges for almost all $x\in(0,1)$. This is the simplest and strongest type of behavior of $\sum_{k=1}^\infty c_kf(n_kx)$, but it holds only in a few special situations. By Carleson's deep theorem [C], $\{\cos 2\pi nx\}$ and $\{\sin 2\pi nx\}$ are convergence systems. More generally, Gaposhkin [Gap4] proved (using Carleson's theorem) the following result:

Theorem C. *Let $f\in Lip_\alpha(\mathbf{T})$ for $\alpha>1/2$ and $\int_{\mathbf{T}}f(t)dt=0$. Then $\{f(nx),\ n=1,2,\ldots\}$ is a convergence system.*

Another classical result, proved by Kac [K1] for the Lipschitz class and extended substantially by Gaposhkin [Gap2] is the following

Theorem D. *Let $f\in L^2(\mathbf{T})$ with $\int_{\mathbf{T}}f(t)dt=0$ and assume that the square modulus of continuity $\omega_2(\delta,f)$ of f satisfies*

$$\omega_2(\delta,f)=\mathcal{O}\Big(\log\frac{1}{\delta}\Big)^{-\frac12-\varepsilon}\qquad(\varepsilon>0).\tag{2.1}$$

Let (n_k) be an sequence of positive reals satisfying the Hadamard gap condition

$$n_{k+1}/n_k\geq q>1\qquad k=1,2,\ldots\tag{2.2}$$

Then $f(n_k x)$ is a convergence system.

These theorems describe the known situations when $f(n_k x)$ is a convergence system; note that all conditions of these results are sharp. Gaposhkin [Gap1] showed that Theorem D becomes false if we assume (2.1) only for $\varepsilon = 0$ and Berkes [Be5] proved that the the condition $f \in \mathrm{Lip}_\alpha(\mathbf{T})$, $\alpha > 1/2$ in Theorem C and the Hadamard gap condition (2.2) in Theorem D are also best possible: there exists a function $f \in \mathrm{Lip}_{1/2}(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ such that for any positive sequence $\{\varepsilon_k, k \geq 1\}$ tending to 0, there exists an increasing sequence $\mathcal{N}$ of integers satisfying

$$n_{k+1}/n_k \geq 1 + \varepsilon_k, \qquad k = 1, 2, \dots \tag{2.3}$$

and $\mathbf{c} \in \ell^2$ such that the series $\sum_{k=1}^\infty c_k f(n_k x)$ diverges almost everywhere. Going beyond the conditions of Theorems C and D, the almost everywhere convergence behavior of $\sum_{k=1}^\infty c_k f(n_k x)$ becomes very complicated and and examples show that the properties of $\sum_{k=1}^\infty c_k f(n_k x)$ are determined by a delicate interplay between the coefficient sequence (c_k), the smoothness properties of f and the growth speed and number-theoretic properties of (n_k). In this section we give a detailed study of this behavior and prove several convergence results for such series. Our main interest will be to find convergence criteria of the type $\sum_{k=1}^\infty c_k^2 \omega(k) < \infty$ where $\omega(k) \to \infty$ is some positive sequence (called Weyl multiplyer) depending on f and (n_k).

Before formulating our results, we first give an equivalent reformulation of the convergence system property of $\sum_{k=1}^\infty c_k f(n_k x)$ in terms of maximal operators.

Proposition 2.1. *A pair $(f, \mathcal{N})$ is a convergence system if and only if there exists a constant C such that for any $\mathbf{c} \in \ell^2$ the following maximal inequality holds:*

$$\sup_{t \geq 0} t^2 \lambda \Big\{ \sup_{N \geq 1} \big| S_N^{\mathcal{N}}(\mathbf{c}, f) \big| > t \|\mathbf{c}\|_2 \Big\} \leq C.$$

Proof. Given a pair $(f, \mathcal{N})$, consider the $L^2(\mathbf{T})$–operators $S_N^{\mathcal{N}, f}$, $N = 1, 2 \dots$ defined via the isomorphism $\mathbf{c} \mapsto g$ if $g \sim \sum_k c_{|k|} e_k$ by

$$S_N^{\mathcal{N}, f}(g) = \sum_{k=1}^N c_k f(n_k .).$$

A first relevant observation in the proof will concern the following commutation property. Consider the family of pointwise measurable transformations of $\mathbf{T}$ defined for each positive integer j by

$$\tau_j x = jx \bmod 1.$$

For fixed j, the transformation $\tau = \tau_j$ preserves the normalized Lebesgue measure λ (see [Hal] p. 5-37), so that τ is an endomorphism of the torus. It has been proved in [P1] (Theorem 1 p. 112) that τ is also strongly mixing. Now, let $\mathcal{E}$ denotes the family of associated operators on $L^0(\mathbf{T})$:

$$T_j g = g \circ \tau_j.$$

That the T_j's are commuting positive L^2-isometries, preserving 1 is better viewed on Fourier expansion of g, since if $g \sim \sum_{m\in\mathbf{Z}} g_m e_m$, then $T_j f \sim \sum_{m\in\mathbf{Z}} g_m e_{mj}$, which readily implies

$$T_k(T_j g) = T_j(T_k g), \qquad (j,k = 1,2\ldots). \tag{2.4}$$

Proceeding next by approximation, we deduce that (2.4) hold for any $g \in L^p(\mathbf{T})$, $0 < p \le \infty$. This in particular implies that the sequence of operators $S_N^{\mathcal{N},f}$ commutes with $\mathcal{E}$: for any $g \in L^2(\mathbf{T})$,

$$S_N^{\mathcal{N},f}(T_j g) = T_j(S_N^{\mathcal{N},f} g), \qquad (N,j = 1,2\ldots) \tag{2.5}$$

Further, the family $\mathcal{E}$ verifies a mean ergodic theorem in $L^2(\mathbf{T})$: for any $g \in L^2(\mathbf{T})$,

$$\lim_{J\to\infty} \Big\| \frac{1}{J}\sum_{j=1}^J T_j g - \int_{\mathbf{T}} g d\lambda \Big\|_2 = 0.$$

Since strong convergence implies weak convergence, it follows for any $u, v \in L^2(\mathbf{T})$ that

$$\lim_{J\to\infty} \frac{1}{J}\sum_{j=1}^J \langle T_j u, v\rangle = \langle u, 1\rangle\langle v, 1\rangle.$$

Choosing $u = \chi\{A\}$, $v = \chi\{B\}$ where A, B are Borel sets of $\mathbf{T}$ and χ denotes indicator function, we deduce

$$\lim_{J\to\infty} \frac{1}{J}\sum_{j=1}^J \lambda(T_j^{-1}A \cap B) = \lambda(A)\lambda(B).$$

From this it follows easily that for any $a > 1$ and Borel sets A, B of $\mathbf{T}$, there exists $T \in \mathcal{E}$ such that

$$\lambda(T^{-1}A \cap B) \le a\lambda(A)\lambda(B). \tag{2.6}$$

Now Proposition 2.1 states, in terms of operators, that for any $g \in L^2(\mathbf{T})$ we have

$$\sup_{t\ge 0} t^2 \lambda\Big\{ \sup_{N\ge 1} \big|S_N^{\mathcal{N},f} g\big| > t\|g\|_2 \Big\} \le C.$$

And this is a consequence of the Continuity Principle [Gar]. ■

Proposition 2.1 implies that a pair $(f, \mathcal{N})$ is a convergence system only if the maximal operator

$$\sup_{N\geq 1} \left|S_N^{\mathcal{N}}(\mathbf{c}, f)\right|$$

belongs to $L^p(\mathbf{T})$ with $p < 2$. This has a consequence concerning convergence in mean. Say by analogy that a pair $(f, \mathcal{N})$ is an L^p-convergence system if for any $g \in L^2(\mathbf{T})$ the sequence $\{S_N^{\mathcal{N},f} g, N \geq 1\}$ converges in $L^p(\mathbf{T})$.

Corollary 2.1. *Assume that the pair $(\mathcal{N}, f)$ is a convergence system. Then, it is also an L^p-convergence system for any $p < 2$.*

Proof. Define $\omega_R = \sup_{N,M\geq R} \left|S_N^{\mathcal{N}}(\mathbf{c}, f) - S_M^{\mathcal{N}}(\mathbf{c}, f)\right|$. By assumption $\lim_{R\to\infty} \omega_R = 0$ a.e. And by the above remark $\omega_1 \in L^p(\mathbf{T})$, $p < 2$. Thus by Fatou's lemma

$$0 = \mathbf{E} \limsup_{N,M\to\infty} \left|S_N^{\mathcal{N}}(\mathbf{c}, f) - S_M^{\mathcal{N}}(\mathbf{c}, f)\right|^p \geq \limsup_{N,M\to\infty} \mathbf{E} \left|S_N^{\mathcal{N}}(\mathbf{c}, f) - S_M^{\mathcal{N}}(\mathbf{c}, f)\right|^p.$$

■

The previous results summarize the basic equivalence of a.e. convergence results and maximal inequalities for $f(n_k x)$. In Theorem 2.6 at the end of this section we will in fact prove a maximal inequality that leads to various a.e. convergence results for $\sum_{k=1}^\infty c_k f(n_k x)$. Except this result, however, our approach to a.e. convergence will be different and we will use a combination of martingale and quasi-orthogonality arguments to achieve our goal. Theorems C and D above show that the convergence properties of $\sum_{k=1}^\infty c_k f(n_k x)$ depend sensitively on the smoothness properties of f and we start with a few preliminary remarks concerning smoothness criteria. Let $f \in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ have Fourier series

$$f \sim \sum_{k=1}^{\infty} (a_k \cos 2\pi kx + b_k \sin 2\pi kx) \tag{2.7}$$

and let

$$r_f(N) = \sum_{k=N}^{\infty} (a_k^2 + b_k^2). \tag{2.8}$$

Given an integer $m \geq 1$, let $[f]_m$ denote the function in $[0, 1)$ which takes the constant value $m \int_{k/m}^{(k+1)/m} f(t)dt$ in the interval $[k/m, (k+1)/m)$, $(k = 0, 1, \ldots, m-1)$. In probabilistic terms, $[f]_m$ is the conditional expectation of f with respect to the σ-field generated by the intervals $[k/m, (k+1)/m)$. Let

$$r_f^*(N) = \|f - [f]_N\|. \tag{2.9}$$

The speed of convergence of $r_f^*(N)$ to zero clearly measures the smoothness of f; for example if f is a Lip (α) function then $r_f^*(N) = O(n^{-\alpha})$. A simple connection between $r_f(N)$ and $r_f^*(N)$ is given by the following lemma, due essentially to Ibragimov [Ibr]. Its proof will be given after the proofs of Theorems 2.1 and 2.2.

Lemma 2.1. *Let $\lambda > 1$ and $g(t) = f(\lambda t)$. Then we have for any $m \geq \lambda$*

$$\|g - [g]_m\| \leq C \left((m/\lambda)^{-1/2} + r_f((m/\lambda)^{1/3}) \right)$$

where C is a positive constant depending only on f.

In particular, for any $N \geq 1$ we have

$$r_f^*(N) \leq C(N^{-1/2} + r_f(N^{1/3})).$$

Thus if $r_f(N) = O(N^{-\alpha})$ for some $0 < \alpha \leq 1$, then $r_f^*(N) = O(N^{-\alpha/3})$.

Another connection between the smoothness properties of f and the quantity $r_f^*(N)$ is given by Lemma 2.2 below. Let $\delta : [0,1] \to [0,1]$ be concave increasing function with $\delta(0) = 0$, and let Ψ be a Young function.

Lemma 2.2. *Let $f \in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(x)dx = 0$ and assume that*

$$\int_{\mathbf{T}} \int_{\mathbf{T}} \Psi \Big(\frac{|f(u) - f(v)|}{\delta(|u - v|)} \Big) dudv < \infty. \tag{2.10}$$

Then for any $t > 0$

$$\lambda\{|f - [f]_n| \geq t\} \leq n^2 \Psi \left(\frac{ct}{\delta(n^{-1})} \right)^{-1}$$

where c is a constant depending on f.

Turning to the convergence behavior of $\sum c_k f(n_k x)$, we first study the lacunary case, i.e. we assume that (n_k) grows very rapidly. If (n_k) satisfies the Hadamard gap condition (2.2), then by Theorem D the system $f(n_k x)$ is a convergence system under mild smoothness conditions on f. We investigate now the case when (n_k) grows with a sub-exponential speed, i.e. it satisfies the gap condition

$$n_{k+1}/n_k \geq 1 + \varepsilon_k, \qquad k = 1, 2, \ldots$$

where ε_k tends to 0. A remarkable result on trigonometric series with sub-Hadamard gaps was proved by Erdős [E], who showed that if (n_k) is a sequence of positive integers satisfying

$$n_{k+1}/n_k \geq 1 + ck^{-\beta}, \qquad k = 1, 2, \ldots \tag{2.11}$$

for some $c > 0$, $\beta < 1/2$, then $\sin 2\pi n_k x$ satisfies the central limit theorem, i.e.

$$\lim_{N\to\infty} \lambda\{x \in (0,1) : (N/2)^{-1/2} \sum_{k=1}^{N} \sin 2\pi n_k x \leq t\} = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2} du.$$

Moreover, this result becomes false for $\beta = 1/2$. Thus, under (2.11) with $\beta < 1/2$ the sequence $\sin 2\pi n_k x$ behaves like a sequence of independent random variables, and this is no more valid if $\beta = 1/2$. Our next theorem gives a strong convergence property of series $\sum_{k=1}^{\infty} c_k f(n_k x)$ under the Erdős gap condition (2.11). Define, for any $\varrho > 0$

$$\tau_{k,\varrho}(\mathbf{c}) = \sup_{L \geq k^{\varrho+1}} \sum_{\ell=L}^{L+[k^{\varrho}]} |c_\ell|.$$

Theorem 2.1. *Let $f \in L^\infty(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ and $r_f(N) = O(N^{-\alpha})$ for some $\alpha > 0$. Let (n_k) be a sequence of positive integers satisfying the gap condition (2.11) with some $\beta < 1/2$, and let $\mathbf{c} \in \ell^2$ with $\tau_{k,\varrho}(\mathbf{c}) = o(1)$ for all $0 < \varrho < 1$. Assume that $\sum_{k=1}^{\infty} c_k f(n_k x)$ and all of its subseries converge in $L^2(\mathbf{T})$ norm. Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ also converges a.e.*

It seems likely that Theorem 2.1 remains valid without the technical condition $\tau_{k,\varrho}(\mathbf{c}) = o(1)$, but this remains open. This condition is certainly satisfied if $c_k = O(k^{-1/2})$ which, in turn, holds if $\mathbf{c} \in \ell^2$ and (c_k) is monotone.

Note that if X_k are independent r.v.'s then under suitable moment conditions, mean convergence of $\sum_{k=1}^{\infty} X_k$ implies a.e. convergence of the same series. Theorem 2.1 establishes a similar property for $\sum_{k=1}^{\infty} c_k f(n_k x)$. Note that the central limit theorem is in general not valid for $f(n_k x)$ under the gap condition (2.11) with $\beta < 1/2$, despite Erdős' theorem mentioned above. (See Kac [K2], p. 645).

Corollary 2.2. *Let $f \in L^\infty(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ and $r_f(N) = O(N^{-\alpha})$ for some $\alpha > 0$. Assume that the Dirichlet series*

$$\sum_{n=1}^{\infty} a_n n^{-s}, \quad \textit{and} \quad \sum_{n=1}^{\infty} b_n n^{-s} \tag{2.12}$$

are regular and bounded in the half-plane $\Re(s) > 0$. *Let* (n_k) *be a sequence of positive integers satisfying the gap condition (2.11) with some* $\beta < 1/2$. *Then* $\sum_{k=1}^\infty c_k f(n_k x)$ *converges a.e. provided* $\mathbf{c} \in \ell^2$ *and* $c_k = O(k^{-1/2})$.

Corollary 2.2 connects the a.e. convergence of lacunary series $\sum_{k=1}^\infty c_k f(n_k x)$ to the classical Wintner theory, showing that the boundedness of the associated Dirichlet series (2.12) implies not only mean, but actually a.e. convergence in the lacunary case. In Section 3 we will show that this result is best possible: if the boundedness condition on the Dirichlet series (2.12) is not satisfied, there exists a sequence (n_k) satisfying (2.11) for all $\beta < 1/2$, and a positive nonincreasing sequence $\mathbf{c} \in \boldsymbol{\ell^2}$ such that $\sum_{k=1}^\infty c_k f(n_k x)$ diverges almost everywhere. On the other hand, if we are interested in the a.e. convergence of $\sum_{k=1}^\infty c_k f(n_k x)$ under more stringent coefficient conditions $\sum_{k=1}^\infty c_k^2 \omega(k) < \infty$, $\omega(k) \to \infty$, then the condition on the Dirichlet series can be dropped, as the following result shows.

Lemma 2.3. *Let* $f \in \mathrm{Lip}_\alpha(\mathbf{T})$ *for some* $0 < \alpha \le 1$ *and assume that* $\int_{\mathbf{T}} f(t)dt = 0$. *Let* (n_k) *be an increasing sequence of positive integers and put*

$$\omega(j) := \max\left(\sum_{1\le \ell \le j} \left(\frac{n_\ell}{n_j}\right)^\alpha, \sum_{k \ge j} \left(\frac{n_j}{n_k}\right)^\alpha \right).$$

Then

$$\int_{\mathbf{T}} \left(\sum_{k=1}^N c_k f(n_k x) \right)^2 dx \le C \sum_{k=1}^N c_k^2 \omega(k)$$

with some constant C. *In particular, if* $\sum_{k=1}^\infty c_k^2 \omega(k) < \infty$, $\sum_{k=1}^\infty c_k f(n_k x)$ *converges in* L^2 *norm.*

In particular, if $n_k = [\exp(k/(\log k)^\tau)]$, then $\omega(j) = (\log j)^\rho$ and in the case $n_k = [\exp(k^\eta)]$, $0 < \eta < 1$, then $\omega(j) = j^{1-\eta}$.

We supplement Theorem 2.1 with another result reducing the almost everywhere convergence of $\sum_{k=1}^N c_k f(n_k x)$ to mean convergence under an additional assumption on the size of the tail sums $\sum_{k>N}^\infty c_k^2$, or, alternatively, under assuming $\sum_{k=1}^\infty c_k^2 \omega(k) < \infty$ for a suitable $\omega(k) \to \infty$.

Theorem 2.2. *Let* $f \in \mathrm{Lip}_\alpha(\mathbf{T})$ *for some* $0 < \alpha \le 1$ *and assume that* $\int_{\mathbf{T}} f(t)dt = 0$. *Let* (n_k) *be an increasing sequence of positive integers and* $\mathbf{c} \in \ell^2$. *Assume that*

$\sum_{k=1}^{\infty} c_k f(n_k x)$ converges in L^2 norm and

$$\lim_{R\to\infty} \Big(\sum_{k>R} c_k^2\Big)^{1/2} \Big(\sum_{k>R} n_k^{-2}\Big)^{1/2} \Big(\sum_{k=1}^{R} n_k^{\alpha}\Big)^{1/\alpha} = 0. \tag{2.13}$$

Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges almost everywhere.

If the sequence (n_k) satisfies the Hadamard gap condition (2.2), relation (2.13) trivially holds whenever $\mathbf{c} \in \ell^2$. If, on the other hand, (n_k) grows slower than exponentially, condition (2.13) imposes a restriction on the tail sums $\sum_{k>R} c_k^2$, which is very mild if (n_k) grows near exponentially. For example, if $n_k = [e^{k/(\log k)^\tau}]$ for some $\tau > 0$, then (2.13) reduces to

$$\sum_{k>R} c_k^2 = O\left((\log R)^{-\tau(1+2/\alpha)}\right).$$

If $n_k = [e^{k/(\log\log k)^\tau}]$, then (2.8) becomes

$$\sum_{k>R} c_k^2 = O\left((\log\log R)^{-\tau(1+2/\alpha)}\right),$$

and if $n_k = [e^{k^\gamma}]$, $0 < \gamma < 1$ then we get

$$\sum_{k>R} c_k^2 = O\left(R^{-(1-\gamma)(1+2/\alpha)}\right).$$

The latter case corresponds to the Erdős gap condition (2.11), and thus we see that the conditions of Theorem 2.2 are more restrictive than those of Theorem 2.1. On the other hand, in Theorem 2.2 we do not assume regularity conditions like $c_k = O(k^{-1/2})$.

Proof of Theorem 2.2. We follow Kac [K1]. For almost all points t_0

$$f^*(t_0) = \lim_{h\to 0} \frac{1}{h} \int_{t_0}^{t_0+h} f^*(u)du. \tag{2.14}$$

Now, since $\sum_{k\geq 1} c_k f(n_k.)$ converges in mean to f^*, by Parseval relation

$$\int_{t_0}^{t_0+h} f(u)^* du = \sum_{k\geq 1} c_k \int_{t_0}^{t_0+h} f(n_k u)du. \tag{2.15}$$

We shall use the following estimate: there exists a constant C such that for any $0 \leq a < b < 1$ and any positive integer k

$$\Big|\int_a^b f(n_k u)du\Big| \leq C n_k^{-1}. \tag{2.16}$$

Let χ be the characteristic function of the interval $[a,b]$, with period 1 extended onto the whole real line. Suppose that

$$\chi(x) = \sum_{m\in\mathbf{Z}} a_m e_m(x).$$

By Parseval's relation,

$$\int_a^b f(n_k u)du = \sum_{m\in\mathbf{Z}} \varphi_m \overline{a_{n_k m}}.$$

Since χ is of bounded variation, we have

$$a_m = \mathcal{O}\big(1/|m|\big),$$

(see Zygmund [Z] p. 323) and thus we get

$$\Big|\int_a^b f(n_k u)du\Big| \le \Big[\sum_{m\in\mathbf{Z}} |\varphi_m|^2\Big]^{1/2}\Big[\sum_{m\in\mathbf{Z}} |a_{n_k m}|^2\Big]^{1/2} \le C\|f\|_2/n_k.$$

Combining (2.15) with (2.16) gives

$$\Big|\int_{t_0}^{t_0+h} f^*(u)du - \sum_{k=1}^R c_k \int_{t_0}^{t_0+h} f(n_k u)du\Big| \le C\Big[\sum_{k>R} c_k^2\Big]^{1/2}\Big[\sum_{k>R}(\frac{1}{n_k})^2\Big]^{1/2}.$$

Since f belongs to $\mathrm{Lip}_\alpha(\mathbf{T})$,

$$\Big|\sum_{k=1}^R c_k \int_{t_0}^{t_0+h} \big[f(n_k u) - f(n_k t_0)\big]du\Big| \le C|h|^{1+\alpha}\sum_{k=1}^R |c_k| n_k^\alpha.$$

Therefore

$$\begin{aligned}&\Big|\frac{1}{h}\int_{t_0}^{t_0+h} f^*(u)du - \sum_{k=1}^R c_k f(n_k t_0)\Big| \\ &\le C\Big\{|h|^{-1}\Big[\sum_{k>R} c_k^2\Big]^{1/2}\Big[\sum_{k>R}(\frac{1}{n_k})^2\Big]^{1/2} + |h|^\alpha \sum_{k=1}^R |c_k| n_k^\alpha\Big\}.\end{aligned}$$

Choosing $h = h_R = \big(\sum_{k=1}^R n_k^\alpha\big)^{-1/\alpha}$ and observing that

$$|h_R|^\alpha \sum_{k=1}^R |c_k| n_k^\alpha = \frac{\sum_{k=1}^R |c_k| n_k^\alpha}{\sum_{k=1}^R n_k^\alpha} \to 0,$$

as R tend to infinity since c_k tend to 0 as k tend to infinity, finally shows in view of condition (2.13)

$$\lim_{R\to\infty}\Big|\frac{1}{h_R}\int_{t_0}^{t_0+h_R} f^*(u)du - \sum_{k=1}^R c_k f(n_k t_0)\Big| = 0.$$

The proof is completed by combining the above result with (2.14). ■

Proof of Lemma 2.1. From $f \in \mathrm{Lip}_\alpha(\mathbf{T})$ it follows (see Zygmund [Z] p. 324) that

$$\sum_{\ell=n+1}^{\infty} (a_\ell^2 + b_\ell^2) \le D n^{-2\alpha}.$$

Let $j \le k$ be fixed positive integers. Using Parseval's relation yields

$$\int_{\mathbf{T}} \varphi(n_j x)\varphi(n_k x)dx = \sum_{rn_j = sn_k} (a_r a_s + b_r b_s).$$

The relation $j \le k$ together with $rn_j = sn_k$ implies that $s \ge 1$ and $r \ge (n_k/n_j)$. Using the inequality $|a_r a_s + b_r b_s| \le (a_r^2 + b_r^2)^{1/2}(a_s^2 + b_s^2)^{1/2}$, and the Schwarz inequality we get

$$\Big|\int_{\mathbf{T}} \varphi(n_j x)\varphi(n_k x)dx\Big| \le \Big[\sum_{r \ge n_k/n_j} (a_r^2 + b_r^2)\Big]^{1/2}\Big[\sum_{s\ge 1}(a_s^2 + b_s^2)\Big]^{1/2} \le B(\frac{n_j}{n_k})^\alpha.$$

Thus

$$\begin{aligned}
&\Big|\int_{\mathbf{T}} \sum_{1\le j<k\le N} c_j c_k \varphi(n_j x)\varphi(n_k x)dx\Big| \\
&\le \sum_{1\le j<k\le N} |c_j||c_k|(\frac{n_j}{n_k})^\alpha \le \sum_{1\le j<k\le N} (\frac{|c_j|^2 + |c_k|^2}{2})(\frac{n_j}{n_k})^\alpha \\
&\le \frac{1}{2}\sum_{k=1}^{N} c_k^2 \sum_{1\le j<k} (\frac{n_j}{n_k})^\alpha + \frac{1}{2}\sum_{j=1}^{N} c_j^2 \sum_{j<k\le N} (\frac{n_j}{n_k})^\alpha \le \sum_{k=1}^{N} c_k^2 \omega(k),
\end{aligned}$$

proving Lemma 2.3.

Proof of Theorem 2.1. As a first step, we approximate the functions $f(n_k x)$ by step-functions $\varphi_k(x)$ as follows. Let $2^\ell \le n_k < 2^{\ell+1}$, put $m = \big[\ell + 120\beta^{-1}\log k\big]$ and let $\varphi_k(x) = [f(n_k x)]_m$. By Lemma 2.1 we have

$$\|f(n_k x) - \varphi_k(x)\| \le C\left(\frac{2^m}{n_k}\right)^{-\beta/6} \le C2^{-20\log k} \le Ck^{-10}. \tag{2.17}$$

Choose ϱ so that $\frac{1}{2} \vee \frac{\gamma}{1-\gamma} < \varrho < 1$ (such a ϱ exists since $\gamma < 1/2$) and split the sequence of positive integers into consecutive blocks $\Delta_1, \Delta_1', \Delta_2, \Delta_2', \ldots$ so that

$$|\Delta_k| = |\Delta_k'| = [k^\varrho].$$

Set

$$T_k = \sum_{\nu\in\Delta_k} c_\nu f(n_\nu x), \qquad D_k = \sum_{\nu\in\Delta_k} c_\nu \varphi_\nu(x).$$

Clearly, each integer in Δ_k exceeds $(k-1)^\varrho \geq (k-1)^{1/2}$ and thus by (2.17)

$$\|T_k - D_k\| \leq C \sum_{\nu=(k-1)^{1/2}}^{\infty} \nu^{-10} \leq Ck^{-4}. \tag{2.18}$$

Next we show

Lemma 2.4. *We have*

$$\mathbf{P}\big\{|\mathbf{E}\,(D_k \mid \mathcal{F}_{k-1})| \geq k^{-2}\big\} \leq Ck^{-2}, \tag{2.19}$$

where $\mathcal{F}_{k-1}$ denotes the σ-field generated by $D_1, \dots, D_{k-1}$.

Proof. We first show that

$$|\mathbf{E}\,(T_k \mid \mathcal{F}_{k-1})| \leq Ck^{-2}. \tag{2.20}$$

To see this, let r and t denote the largest integer of Δ_{k-1} and the smallest integer of Δ_k, respectively. Let $2^\ell \leq n_r < 2^{\ell+1}$, $w = \left[\ell + \frac{120}{\beta}\log r\right]$. From the definition of φ_ν it is clear that every φ_ν, $1 \leq \nu \leq r$ takes a constant value on each interval of the form

$$A = [i2^{-w}, (i+1)2^{-w}), \quad 0 \leq i \leq 2^w - 1 \tag{2.21}$$

and thus each set of the σ-field $\mathcal{F}_{k-1}$ can be written as a union of intervals of the form (2.21). Thus to prove (2.20) it suffices to show that

$$|A|^{-1}\left|\int_A T_k\,dx\right| \leq Ck^{-2} \tag{2.22}$$

for any A of the form (2.21). Now for the set A in (2.21) we have

$$\begin{aligned} |A|^{-1}\left|\int_A T_k\,dx\right| &= 2^w \left|\int_{i2^{-w}}^{(i+1)2^{-w}} \sum_{\nu\in\Delta_k} c_\nu f(n_\nu x)dx\right| \\ &= \left|\int_i^{i+1} \sum_{\nu\in\Delta_k} c_\nu f(m_\nu t)dt\right| \end{aligned} \tag{2.23}$$

where $m_\nu = 2^{-w}n_\nu$. Using (2.11), $1 + x \geq \exp(x/2)$ for $0 \leq x \leq 1$ and the relations $r \sim t \sim Ck^{\varrho+1}$, $t - r \sim k^\varrho$ we get

$$\begin{aligned} \frac{1}{m_t} = \frac{2^w}{n_t} &\leq \frac{2^\ell r^{120/\beta}}{n_t} \leq r^{120/\beta}\frac{n_r}{n_t} \\ &\leq r^{120/\beta}\prod_{\nu=r}^{t-1}\left(1 + \frac{1}{\nu^\gamma}\right)^{-1} \leq r^{120/\beta}\left(1 + \frac{1}{t^\gamma}\right)^{-(t-r)} \\ &\leq r^{120/\beta}\exp\left(-\frac{t-r}{2t^\gamma}\right) \leq Ck^{240/\beta}\exp(-Ck^\tau) \\ &\leq Ck^{-3}, \end{aligned} \tag{2.24}$$

where $\tau = \varrho - (\varrho + 1)\gamma > 0$ by the choice of ϱ. By the periodicity of f and $\int_0^1 f dx = 0$ we clearly have for any real L and $\lambda \geq 1$

$$\Big| \int_L^{L+1} f(\lambda x) dx \Big| \leq \frac{2}{\lambda} \int_0^1 |f(x)| dx$$

and thus (2.24) shows that the last expression of (2.23) cannot exceed

$$\sum_{\nu \in \Delta_k} \frac{C|c_\nu|}{m_\nu} \leq C|\Delta_k| \frac{1}{m_t} \leq Ck^{-2}.$$

Hence we proved (2.22) and thus (2.20).

It is now easy to complete the proof of Lemma 2.4. By (2.18) and well-known properties of conditional expectations we have

$$\big\| \mathbf{E}\left(|D_k - T_k|^2 \mid \mathcal{F}_{k-1}\right) \big\|_1 = \mathbf{E}\,|D_k - T_k|^2 \leq Ck^{-8}$$

and thus by the Markov inequality

$$\mathbf{P}\big\{ \mathbf{E}\left(|D_k - T_k| \mid \mathcal{F}_{k-1}\right) \geq k^{-2} \big\} \leq \mathbf{P}\big\{ \mathbf{E}\left(|D_k - T_k|^2 \mid \mathcal{F}_{k-1}\right) \geq k^{-4} \big\} \leq Ck^{-4}.$$

Together with (2.20) this yields (2.19).

Set $\overline{D}_k = D_k - \mathbf{E}\left(D_k \mid \mathcal{F}_{k-1}\right)$; clearly $(\overline{D}_k, \mathcal{F}_k)$ is a martingale difference sequence and hence orthogonal. Also,

$$\begin{aligned} &\|\mathbf{E}(D_k \mid \mathcal{F}_{k-1})\| \leq \|\mathbf{E}((D_k - T_k) \mid \mathcal{F}_{k-1})\| + \|\mathbf{E}(T_k \mid \mathcal{F}_{k-1})\| \\ &\leq \|D_k - T_k\| + Ck^{-2} \leq C(k^{-4} + k^{-2}) \end{aligned} \tag{2.25}$$

by (2.18) and (2.20). By the assumptions of Theorem 2.1, $\sum_{k=1}^\infty T_k$ converges in $L_2(\mathbf{T})$ norm and thus

$$\| \sum_{k=m}^n T_k \| \to 0 \qquad \text{as} \quad m, n \to \infty.$$

Consequently, using the orthogonality of $\overline{D}_k$, (2.18) and (2.25) we get

$$\begin{aligned} &\left(\sum_{k=m}^n \mathbf{E}\overline{D}_k^2 \right)^{1/2} = \| \sum_{k=m}^n \overline{D}_k \| \leq \| \sum_{k=m}^n D_k \| + \| \sum_{k=m}^n \mathbf{E}(D_k \mid \mathcal{F}_{k-1}) \| \\ &\leq \| \sum_{k=m}^n D_k \| + C \sum_{k=m}^n k^{-2} \leq \| \sum_{k=m}^n T_k \| + C' \sum_{k=m}^n k^{-2} \longrightarrow 0 \end{aligned} \tag{2.26}$$

as $m, n \to \infty$. Thus $\sum_{k=1}^{\infty} \mathbf{E}\overline{D}_k^2 < \infty$ and thus the martingale convergence theorem implies that $\sum_k \overline{D}_k$ is a.e. convergent. Now $\sum_k \mathbf{E}\,(D_k \mid \mathcal{F}_{k-1})$ is a.e. convergent by Lemma 2.4 and the Borel–Cantelli lemma, further $\sum_k (T_k - D_k)$ is a.e. convergent by (2.18) and the Beppo Levi theorem. Thus $\sum_k T_k$ is a.e. convergent; for the same reason $\sum_k T_k'$ is also a.e. convergent, where

$$T_k' = \sum_{\nu \in \Delta_k'} c_\nu f(n_\nu x).$$

Hence setting

$$S_N = \sum_{\nu \le N} c_\nu f(n_\nu x), \quad N_k = 2 \sum_{i \le k} [i^\varrho]$$

we proved that S_{N_k} is a.e. convergent. To prove the theorem it remains to show that $M_k \to 0$ a.e. where

$$M_k = \max_{N_k \le N < N_{k+1}} |S_N - S_{N_k}|.$$

Let D denote a constant such that $|f| \le D$. Then by using $N_k \sim C k^{\varrho+1}$, $N_{k+1} - N_k \sim 2k^\varrho$ and $\tau_{k,\varrho}(\mathbf{c}) = o(1)$ we get

$$M_k \le D \sum_{\nu = N_k + 1}^{N_{k+1}} |c_\nu| \le C \tau_{k,\varrho}(\mathbf{c}) = o(1).$$

Hence Theorem 2.1 is proved. ■

Proof of Lemma 2.1. Let us write $f = f_1 + f_2$ where

$$f_1 = \sum_{k=1}^{N} (a_k \cos 2\pi k x + b_k \sin 2\pi k x), \quad f_2 = f - f_1,$$

N is an integer to be specified later. If $g(x) = f(\lambda x)$ then we have $g = g_1 + g_2$, where $g_1(x) = f_1(\lambda x)$, $g_2(x) = f_2(\lambda x)$. Evidently

$$|\cos \beta x - [\cos \beta x]_m| \le \beta / m, \quad |\sin \beta x - [\sin \beta x]_m| \le \beta / m$$

for any $\beta > 0$ and thus using

$$g_1(x) = \sum_{k=1}^{N} (a_k \cos 2\pi k \lambda x + b_k \sin 2\pi k \lambda x)$$

and the linearity of the operation $g \to [g]$ and the fact that $\|[g]_m\| \le \|g\|$ we get

$$\begin{aligned} |g_1 - [g_1]_m| &\le \sum_{k=1}^{N} 2\pi k\lambda(|a_k| + |b_k|)m^{-1} \\ &\le 2\pi\lambda m^{-1}\left(\sum_{k=1}^{N} k^2\right)^{1/2}\left[\left(\sum_{k=1}^{\infty} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{\infty} b_k^2\right)^{1/2}\right] \le C\lambda m^{-1}N^{3/2} \end{aligned} \tag{2.27}$$

with some constant C depending on f. Further, by the periodicity of f and f_1 we have

$$\|g_2 - [g_2]_m\|^2 \le 2\|g_2\|^2 = 4\int_0^1 f_2(\lambda x)^2 dx = 4\lambda^{-1}\int_0^{\lambda} f_2(t)^2 dt \le 4\lambda^{-1}\int_0^{[\lambda]+1} f_2(t)^2 dt$$

$$4\lambda^{-1}\int_0^{(}[\lambda]+1)\int_0^1 f_2(t)^2 dt \le 8\|f - f_2\|^2 = 8r(N). \tag{2.28}$$

Using relations (2.27)-(2.28) we get

$$\|g - [g]_m\| \le C(\lambda m^{-1}N^{3/2} + r(N)) \tag{2.29}$$

whence the statement of the lemma follows by choosing $N = [(m/\lambda)^{1/3}]$.

We turn now to the nonlacunary case, i.e. the case when no growth condition on (n_k) is assumed. As we already indicated, in this case the number-theoretic structure of the sequence (n_k) will play an important role in the convergence behavior of $\sum_{k=1}^{\infty} c_k f(n_k x)$. Before formulating our results, we first recall a useful notion from the theory of orthogonal series. Let (f_n) be a sequence of functions belonging to $L^2(0,1)$ and let $a_{j,k} = \int_0^1 f_j(x)f_k(x)dx$. We call (f_n) *quasi-orthogonal* when the quadratic form defined on $\ell^2(\mathbf{N})$ by: $(x_n)_n \mapsto \sum_{h,k} a_{h,k}x_h x_k$ is bounded. The important consequence of this property is that for any sequence $\mathbf{c} = \{c_n, n \ge 1\} \in \ell_2$, the series $\sum_n c_n f_n$ converges in $L^2(0,1)$.

A sequence $\mathbf{c} = \{c_n, n \ge 1\} \in \ell_2$ will be said *universal* if the series $\sum c_n\psi_n$ converges almost everywhere for every orthonormal system of functions $(\psi_n)_n$. By a theorem of Schur [O], p.56 if $\mathbf{c}$ is universal, then the series $\sum c_n f_n$ converges almost everywhere for any quasi-orthogonal system of functions (f_n). Typical examples of universal sequences are those produced by the general results from the theory of orthogonal series, like the Rademacher-Menshov theorem, implying that a sequence (c_n) is universal if $\sum_{k=1}^{\infty} c_k^2(\log k)^2 < \infty$. The notion of quasi-orthogonal system is therefore of particular relevance in the study of the convergence in mean and/or almost everywhere of series $\sum_n c_n f(n_k x)$. In this direction, we will establish the following general result. Here, and in the sequel, let $L(x) = \log(x \vee 1)$ for $x \in \mathbf{R}$.

Theorem 2.3. *Let $f \in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(x)dx = 0$. Let (n_k) be an increasing sequence of positive integers and assume that there exists a sequence (C_k) of positive integers such that*

$$\sum_{k=1}^{\infty} r_f^*(C_k)^2 < \infty \tag{2.30}$$

and

$$\sup_{h \geq 1} C_h \sum_{k > h} \frac{(n_h, n_k)}{n_k} L\left(\frac{(n_h, n_k) C_k}{n_h}\right) < \infty. \tag{2.31}$$

Then the series $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges a.e. for any universal sequence $\mathbf{c}$, *in particular if $\sum_{k=1}^{\infty} c_k^2 (\log k)^2 < \infty$.*

The following theorem describes what happens if condition (2.31) of Theorem 2.3 is not assumed.

Theorem 2.4. *Let $f \in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(x)dx = 0$. Let (n_k) be an increasing sequence of positive integers and assume that there exists a sequence (C_k) of positive integers and a positive nondecreasing sequence (λ_k) such that $\lambda_{2k}/\lambda_k = O(1)$ and*

$$\sum_{k=1}^{\infty} r_f^*(C_k)^2 / \lambda_k < \infty \tag{2.32}$$

$$\sup_{1 \leq h \leq N} C_h \sum_{h < k \leq N} \frac{(n_h, n_k)}{n_k} L\left(\frac{(n_h, n_k) C_k}{n_h}\right) \leq \lambda_N. \tag{2.33}$$

Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges a.e. provided $\sum_{k=1}^{\infty} c_k^2 (\log k)^2 \lambda_k < \infty$.

Choosing the sequences (C_k) and (λ_k) optimally in Theorem 2.4 requires a "balancing" act, but giving up a little accuracy, such sequences are easy to find: first choose (C_k) so that (2.32) holds with $\lambda_k = 1$ and then choose λ_k so that (2.33) holds.

The following analogue of Theorem 2.4 is easier to formulate and prove, but still have useful applications.

Theorem 2.5. *Let $f \in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(x)dx = 0$ and with Fourier coefficients satisfying $a_k = O(k^{-\alpha})$, $b_k = O(k^{-\alpha})$, $\alpha > 1/2$. Let (n_k) be an increasing sequence of integers and let (λ_k) be a positive nondecreasing sequence such that $\lambda_{2k}/\lambda_k = O(1)$ and*

$$\sup_{1 \leq h \leq N} \sum_{k=1}^{N} \langle n_h, n_k \rangle^{\alpha} \leq \lambda_N.$$

Then $\sum_{k=1}^\infty c_k f(n_k x)$ *converges a.e. provided* $\sum_{k=1}^\infty c_k^2 (\log k)^2 \lambda_k < \infty$.

Before proving Theorems 2.3-2.5, we give some applications.

Corollary 2.3. *Let* $f \in L^2(\mathbf{T})$ *with* $\int_{\mathbf{T}} f(x)dx = 0$ *and* $r_f(n) = O(n^{-\alpha})$. *Let* (n_k) *be an increasing sequence of coprime integers such that* $n_k \geq k^\beta$ *with some* $\beta > 1+1/(2\alpha)$. *Then* $\sum_{k=1}^\infty c_k f(n_k x)$ *converges a.e. for any universal* $\mathbf{c}$.

Corollary 2.3*. *Let* $f \in L^2(\mathbf{T})$ *with* $\int_{\mathbf{T}} f(x)dx = 0$ *and with Fourier coefficients satisfying* $a_k = O(k^{-\alpha})$, $b_k = O(k^{-\alpha})$, $\alpha > 1/2$. *Let* (n_k) *be an increasing sequence of pairwise coprime integers such that* $\sum_{k=1}^\infty n_k^{-\alpha} < \infty$. *Then* $\sum_{k=1}^\infty c_k f(n_k x)$ *converges a.e. for any universal* $\mathbf{c}$.

The assumptions of Corollaries 2.3 and 2.3* on f are different, but the conclusions are similar.

Corollary 2.4. *Let* $f \in L_2(\mathbf{T})$ *have Fourier-coefficients* $O(1/k)$ *(for example, let* $f \in BV(0,1)$*) and let* (n_k) *be a sequence of integers such that for any* $d \geq 1$ *we have* $\sum_{d|n_k} n_k^{-1} \leq A/d$ *with an absolute constant* A. *Then* $\sum_{k=1}^\infty c_k f(n_k x)$ *converges a.e. provided*

$$\sum_{k=1}^\infty c_k^2 (\log k)^2 \log n_k < \infty.$$

Corollary 2.5. *Let* $f \in L^2(\mathbf{T})$ *with* $\int_{\mathbf{T}} f(x)dx = 0$ *and* $r_f(n) = O(n^{-\alpha})$. *Then the series* $\sum_{k=1}^\infty c_k f(kx)$ *converges a.e. provided*

$$\sum_{k=1}^\infty c_k^2 k^\gamma < \infty \qquad \text{for some } \gamma > 1/(1+2\alpha).$$

Corollary 2.5*. *Let* $f \in L^2(\mathbf{T})$ *with* $\int_{\mathbf{T}} f(x)dx = 0$ *and with Fourier coefficients satisfying* $a_k = O(k^{-\alpha})$, $b_k = O(k^{-\alpha})$, $1/2 < \alpha < 1$. *Then* $\sum_{k=1}^\infty c_k f(kx)$ *converges a.e. provided*

$$\sum_{k=1}^\infty c_k^2 k^{1-\alpha} (\log k)^2 < \infty.$$

Corollary 2.6. *Let $f \in L_2(\mathbf{T})$ have Fourier-coefficients $O(1/k)$ (for example, let $f \in BV(0,1)$). Let $n_k = k^r$, $r \geq 2$. Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges a.e. provided*

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^2 < \infty.$$

Corollary 2.6*. *Let $f \in L_2(\mathbf{T})$ with $\int_{\mathbf{T}} f(x)dx = 0$ and with Fourier coefficients satisfying $a_k = O(k^{-\alpha})$, $b_k = O(k^{-\alpha})$, $\alpha > 1/2$. Let $n_k = k^r$, where r is an integer with $r > 1/\alpha$. Then $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges a.e. provided*

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^2 < \infty.$$

Proof of Theorem 2.5. We follow the proof of Theorem 1.1 with minor modifications, using the same notations. The assumption $\sum_{k=1}^{\infty} c_k^2 (\log k)^2 \lambda_k < \infty$ and the estimates in the second line of (1.18) with $\gamma = 2$ show that $\sum_{k=1}^{\infty} \nu^2 \int_{\mathbf{T}} Z_\nu^2 dx < \infty$ and thus $\sum_{k=1}^{\infty} \nu^2 Z_\nu^2 < \infty$ almost everywhere. Hence (1.17) implies that the first expression in (1.16) converges to 0 almost everywhere as $m, n \to \infty$, and thus the partial sums $\sum_{k=1}^{2^N} c_k f(n_k x)$ converge a.e. Now (1.16) and the Rademacher-Mensov inequality (see e.g. Zygmund [Z]) imply

$$\begin{aligned} &\int_{\mathbf{T}} \max_{2^N+1 \leq m \leq 2^{N+1}} \left(\sum_{k=2^N+1}^{m} c_k f(n_k t) \right)^2 dt \\ &\leq C_3 \lambda_{2^{N+1}} \left(\sum_{k=2^N+1}^{2^{N+1}} c_k^2 \right) (\log 2^N)^2 \leq C_4 \sum_{k=2^N+1}^{2^{N+1}} c_k^2 (\log k)^2 \lambda_k \end{aligned} \tag{2.34}$$

Summing these relations for $N = 1, 2, \ldots$ and using $\sum_{k=1}^{\infty} c_k^2 (\log k)^2 \lambda_k < \infty$, it follows that

$$\max_{2^N+1 \leq m \leq 2^{N+1}} \left(\sum_{k=2^N+1}^{m} c_k f(n_k t) \right)^2 \to 0 \qquad \text{a.e.}$$

completing the proof of Theorem 2.5.

Proof of Theorem 2.4. Let $f_k = [f]_{C_k}(n_k \cdot)$. By the Cauchy-Schwarz inequality we get

$$\begin{aligned} &\sum_{k=1}^{\infty} |c_k| \|f(n_k \cdot) - f_k(\cdot)\| = \sum_{k=1}^{\infty} |c_k| \|f(\cdot) - [f]_{C_k}(\cdot)\| = \sum_{k=1}^{\infty} |c_k| r_f^*(C_k) \\ &\leq \left(\sum_{k=1}^{\infty} c_k^2 \lambda_k \right)^{1/2} \left(\sum_{k=1}^{\infty} r_f^*(C_k)^2 / \lambda_k \right)^{1/2} < \infty \end{aligned}$$

by the assumptions of Theorem 2.4. It follows that $\sum_{k=1}^{\infty} |c_k| \|f(n_k t) - f_k(t)\|$ converges a.e. and thus the series $\sum_{k=1}^{\infty} c_k f(n_k t)$ converges almost everywhere if and only if the series $\sum_{k=1}^{\infty} c_k f_k(t)$ does. The problem thus reduces to the study of the last series, and to do this, we will analyse the correlation properties of the functions f_k. Define, for any non-empty interval $\dot{\pi}$ of $\mathbf{T}$,

$$f^{\dot{\pi}} = \frac{1}{|\dot{\pi}|} \int_{\dot{\pi}} f(u) du. \tag{2.35}$$

Then

$$[f]_{C_n}(x) = \sum_{\dot{\pi} \in \Pi_n} f^{\dot{\pi}} \chi(\dot{\pi})(x),$$

where Π_n denotes the partition of $[0, 1)$ defined by the subdivision

$$[(j-1)/C_n,\, j/C_n) \qquad j = 1, \ldots, C_n. \tag{2.36}$$

Since $\int_{\mathbf{T}} f(f) dt = 0$, we have

$$[f]_{C_n}(x) = \sum_{\dot{\pi} \in \Pi_n} f^{\dot{\pi}} \big[\chi(\dot{\pi})(x) - |\dot{\pi}|\big] \tag{2.37}$$

and consequently for $h \le k$ we get

$$\langle f_h, f_k \rangle = \sum_{\dot{\pi} \in \Pi_h} \sum_{\dot{\pi}' \in \Pi_k} f^{\dot{\pi}} f^{\dot{\pi}'} \langle \chi_{\dot{\pi}}(\{n_h y\}) - |\dot{\pi}|, \chi_{\dot{\pi}'}(\{n_k y\}) - |\dot{\pi}'| \rangle. \tag{2.38}$$

where the indicators are extended with period 1. Thus the calculation reduces to estimating the correlation for indicators of intervals. Let $0 \le a < b < 1$. It is classical to expand the indicator function $\chi([a, b))(x)$ in a Fourier series, and one gets

$$\begin{aligned}
\chi([a,b))(x) &= b - a + \sum_{n \in \mathbf{Z}^*} \Big(\frac{-1}{2i\pi n}\Big) \big\{e^{-2i\pi n b} - e^{-2i\pi n a}\big\} e^{2i\pi n x} \\
&= b - a \\
&+ \sum_{n=1}^{\infty} \frac{1}{\pi n} \Big\{ \sin 2\pi n x \big(\cos 2\pi n b - \cos 2\pi n a\big) + \cos 2\pi n x \big(\sin 2\pi n b - \sin 2\pi n a\big) \Big\},
\end{aligned} \tag{2.39}$$

for almost all x. Now, let $0 \le a < b < c < d < 1$. Put $\varphi = \chi([a, b))$, $\psi = \chi([c, d))$, and $\bar{\varphi} = \varphi - (b - a)$, $\bar{\psi} = \psi - (d - c)$. We study for given positive integers h and k the correlation of the functions $\bar{\varphi}_h = \bar{\varphi}(hx)$, $\bar{\psi}_k = \bar{\psi}(kx)$. Put for $u, v \in \mathbf{T}$ and integer n,

$$\delta_n(u, v) = e^{-2i\pi n v} - e^{-2i\pi n u}.$$

Then,

$$\bar\varphi(hx) = \sum_{n\in\mathbf{Z}^*} (\frac{-1}{2i\pi n}) e^{2i\pi nhx} \delta_n(a,b)$$

$$\bar\psi(kx) = \sum_{m\in\mathbf{Z}^*} (\frac{-1}{2i\pi m}) e^{2i\pi mkx} \delta_m(c,d),$$

so that

$$\begin{aligned}\langle \bar\varphi_h, \bar\psi_k\rangle &= \sum_{n\in\mathbf{Z}^*}\sum_{m\in\mathbf{Z}^*} \frac{1}{4\pi^2 mn}\delta_n(a,b)\delta_{-m}(c,d)\int_{\mathbf{T}} e^{2i\pi(nh-mk)x}dx \\ &= \sum_{\substack{m,n\in\mathbf{Z}^*\\ nh-mk=0}} \frac{1}{4\pi^2 mn}\delta_n(a,b)\delta_{-m}(c,d).\end{aligned}$$

The equation $nh-mk=0$ has solutions given by $n=\mu k/(h,k)$ and $m=\mu h/(h,k)$, $\mu = 1,2\ldots$. Thus,

$$\begin{aligned}&\langle \bar\varphi_h, \bar\psi_k\rangle \\ &= \frac{\langle h,k\rangle}{4\pi^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\big\{\delta_{\mu k/(h,k)}(a,b)\delta_{-\mu h/(h,k)}(c,d)+\delta_{-\mu k/(h,k)}(a,b)\delta_{\mu h/(h,k)}(c,d)\big\}.\end{aligned} \tag{2.40}$$

It remains to compute $\delta_n(a,b)\delta_{-m}(c,d)+\delta_{-n}(a,b)\delta_m(c,d)$. But, a plain calculation shows

$$\begin{aligned}&\delta_n(a,b)\delta_{-m}(c,d)+\delta_{-n}(a,b)\delta_m(c,d) \\ &= 2\Big\{\cos 2\pi(nb-md)-\cos 2\pi(nb-mc)-\cos 2\pi(na-md)+\cos 2\pi(na-mc)\Big\} \\ &= 2\sin 2\pi m(d-c)\big\{\sin 2\pi(2nb-m(c+d))-\sin 2\pi(2na-m(c+d))\big\} \\ &= 4\sin 2\pi m(d-c)\sin 2\pi n(b-a)\cos 2\pi(n(a+b)-m(c+d)).\end{aligned}$$

Therefore,

$$\begin{aligned}&\langle \bar\varphi_h, \bar\psi_k\rangle \\ &= \frac{\langle h,k\rangle}{\pi^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\sin 2\pi\frac{\mu h(d-c)}{(h,k)}\sin 2\pi\frac{\mu k(b-a)}{(h,k)}\cos 2\pi\frac{\mu(k(a+b)-h(c+d))}{(h,k)}.\end{aligned} \tag{2.41}$$

It follows that

$$\begin{aligned}\big|\langle \bar\varphi_h, \bar\psi_k\rangle\big| &\le \frac{\langle h,k\rangle}{\pi^2}\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}\big|\sin 2\pi\frac{\mu h(d-c)}{(h,k)}\big|\big|\sin 2\pi\frac{\mu k(b-a)}{(h,k)}\big| \\ &\le \frac{2}{\pi}\min\left\{\sum_{\mu=1}^{\infty}\frac{1}{\mu^2}(\frac{\mu h(d-c)}{[h,k]}\wedge\langle h,k\rangle), \sum_{\mu=1}^{\infty}\frac{1}{\mu^2}(\frac{\mu k(b-a)}{[h,k]}\wedge\langle h,k\rangle)\right\}.\end{aligned} \tag{2.42}$$

Now, if $\frac{(h,k)}{h(d-c)} > 1$

$$\sum_{\mu \leq \frac{(h,k)}{h(d-c)}} \frac{1}{\mu^2}(\frac{\mu h(d-c)}{[h,k]} \wedge \langle h,k\rangle) \leq \frac{h(d-c)}{[h,k]} \sum_{\mu \leq \frac{(h,k)}{h(d-c)}} \frac{1}{\mu} \leq C\frac{h(d-c)}{[h,k]} \log \frac{(h,k)}{h(d-c)},$$

and,

$$\sum_{\mu > \frac{(h,k)}{h(d-c)}} \frac{1}{\mu^2}(\frac{\mu h(d-c)}{[h,k]} \wedge \langle h,k\rangle) \leq \langle h,k\rangle \sum_{\mu > \frac{(h,k)}{h(d-c)}} \frac{1}{\mu^2} \leq C\langle h,k\rangle \frac{h(d-c)}{(h,k)} = C\frac{h(d-c)}{[h,k]}.$$

Thus

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu^2}(\frac{\mu h(d-c)}{[h,k]} \wedge \langle h,k\rangle) \leq C\frac{h(d-c)}{[h,k]} \log \frac{(h,k)}{h(d-c)}. \tag{2.43a}$$

If $\frac{(h,k)}{h(d-c)} \leq 1$, then $1 \leq \frac{h(d-c)}{(h,k)}$ and

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu^2}(\frac{\mu h(d-c)}{[h,k]} \wedge \langle h,k\rangle) \leq C\langle h,k\rangle \leq C\langle h,k\rangle \frac{h(d-c)}{(h,k)} = C\frac{h(d-c)}{[h,k]}. \tag{2.43b}$$

In both cases we get

$$\sum_{\mu=1}^{\infty} \frac{1}{\mu^2}(\frac{\mu h(d-c)}{[h,k]} \wedge \langle h,k\rangle) \leq C\frac{h(d-c)}{[h,k]} L\Big(\frac{(h,k)}{h(d-c)}\Big). \tag{2.44}$$

Therefore,

$$\big|\langle \bar\varphi_h, \bar\psi_k\rangle\big| \leq C \min\left\{\frac{h(d-c)}{[h,k]} L\Big(\frac{(h,k)}{h(d-c)}\Big), \frac{k(b-a)}{[h,k]} L\Big(\frac{(h,k)}{k(b-a)}\Big), \langle h,k\rangle\right\}. \tag{2.45}$$

Return now to (2.38). We deduce from (2.45)

$$\big|\langle \chi_{\dot\pi}(n_h y) - |\dot\pi|, \chi_{\dot\pi'}(n_k y) - |\dot\pi'|\rangle\big| \leq C\frac{n_h|\dot\pi'|}{[n_h,n_k]} L\Big(\frac{(n_h,n_k)}{n_h|\dot\pi'|}\Big),$$

so that

$$\begin{aligned} \big|\langle f_h, f_k\rangle\big| &\leq C \sum_{\dot\pi \in \Pi_h} \big|f^{\dot\pi}\big| \sum_{\dot\pi' \in \Pi_k} \big|f^{\dot\pi'}\big||\dot\pi'| \frac{n_h}{[n_h,n_k]} L\Big(\frac{(n_h,n_k)}{n_h|\dot\pi'|}\Big) \\ &\leq C \sum_{\dot\pi \in \Pi_h} \big|f^{\dot\pi}\big| \sum_{\dot\pi' \in \Pi_k} \Big|\int_{\dot\pi'} f(u)du\Big| \frac{n_h}{[n_h,n_k]} L\Big(\frac{(n_h,n_k)C_{N_k}}{n_h}\Big) \\ &\leq C\|f\|_1 \sum_{\dot\pi \in \Pi_h} \big|f^{\dot\pi}\big|(\frac{|\dot\pi|}{|\dot\pi|}) \frac{n_h}{[n_h,n_k]} L\Big(\frac{(n_h,n_k)C_k}{n_h}\Big) \\ &\leq C\|f\|_1^2 \frac{n_h C_h}{[n_h,n_k]} L\Big(\frac{(n_h,n_k)C_k}{n_h}\Big). \end{aligned} \tag{2.46}$$

Therefore, for $h \le k$

$$\big|\langle f_h, f_k\rangle\big| \le C\|f\|_1^2 \,\frac{(n_h,n_k)C_h}{n_k} L\Big(\frac{(n_h,n_k)C_k}{n_h}\Big). \tag{2.47}$$

Thus using (2.33) we get

$$\int_{\mathbf{T}} \left(\sum_{k=1}^N c_k f_k\right)^2 dx = \left|\sum_{h,k=1}^N \langle f_h, f_k\rangle c_h c_k\right| \le \sum_{h,k=1}^N \langle f_h, f_k\rangle \frac{1}{2}(c_h^2+c_k^2) \le \left(\sum_{k=1}^N c_k^2\right)\lambda_N$$

which corresponds to relation (1.16) in the proof of Theorem 1.1. The argument is now completed by following the proof of Theorem 2.5.

Proof of Theorem 2.3. This is a special case of the previous proof for $\lambda_n = O(1)$.

Proof of Corollaries 2.3, 2.3.* Since $\beta > 1 + 1/(2\alpha)$, we can choose $\gamma > 0$ such that $2\alpha\gamma > 1$ and $\beta > \gamma + 1$. Let $C_k = k^\gamma$, then (2.30) is trivially satisfied and the expression in (2.31) is at most

$$\sup_{h\ge 1} \sum_{k>h} \frac{1}{n_k} L(C_k) \le K h^\gamma \sum_{k>h} k^{-\beta}\log k = O(h^{(\gamma-\beta-1)}\log h) = O(1).$$

for some constant K. Hence Corollary 2.3 follows from Theorem 2.3. A similar calculation shows that Corollary 2.3* follows from Theorem 2.5.

Proof of Corollary 2.4. This is immediate from Theorem 2.5 and estimate (1.22) in Section 1.

Proof of Corollary 2.5. Let $C_n = n^\gamma$ where γ will be determined later. Observe that

$$C_h \sum_{k=h}^n \frac{(h,k)}{k} L\left(\frac{(h,k)C_k}{h}\right) \le \log C_n \sum_{k=h}^n \frac{(h,k)C_h}{k}. \tag{2.48}$$

Fix a $d|h$ and compute the last sum in (2.48) for those $h \le k \le n$ such that $(h,k) = d$. This restricted sum clearly cannot exceed

$$C_h \sum_{1\le k\le n, d|k} \frac{d}{k} \le C_n \sum_{l=1}^{[n/d]} \frac{1}{l} \le C_n \log n.$$

Now summing for all $d|h$, we have to multiply the result with the number of divisors of h, which is known to be at most $A(\varepsilon)h^\varepsilon \le A(\varepsilon)n^\varepsilon$, and thus the first sum in (2.48) is at most $A(\varepsilon)n^\varepsilon C_n \log C_n \log n = O(n^{\gamma+\varepsilon})$. Similarly,

$$\sum_{k<h} \frac{kC_k}{[h,k]} L\left(\frac{(h,k)C_h}{k}\right) = \sum_{k<h} \frac{(h,k)C_k}{h} L\left(\frac{(h,k)C_h}{k}\right) \le \log C_n \sum_{k<h} \frac{(h,k)C_k}{h}. \quad (2.49)$$

Fixing again $d|h$ and summing for k with $(h,k)=d$, the last sum contains at most h/d terms, all of which are $\le \frac{d}{h}C_n$, and thus the sum is $\le C_n$. Summing now for $d|h$ means again multiplying with at most $A(\varepsilon)h^\varepsilon \le A(\varepsilon)n^\varepsilon$ and thus the right side of (2.49) is at most $A(\varepsilon)n^\varepsilon C_n \log C_n = O(n^{\gamma+2\varepsilon})$. Thus choosing $\lambda_n = n^{\gamma+2\varepsilon}$, condition (2.32) of Theorem 2.4 is satisfied. Now $r_f(C_k) = O(C_k^{-\alpha}) = O(k^{-\gamma\alpha})$ and thus (2.30) will hold if $\gamma = 1/(1+2\alpha)$. As ε can be chosen arbitrarily small, Corollary 2.5 follows from Theorem 2.4.

Proof of Corollary 2.5.* This is immediate from Theorem 2.5 and the last statement of Lemma 1.1.

Proof of Corollaries 2.6, 2.6.* Let $n_k = k^r$ for some integer $r \ge 2$. Clearly $\langle n_k, n_l\rangle = \langle k,l\rangle^r$ and thus Corollary 2.6 follows from Theorem 2.5 and the first statement of Lemma 1.1. The proof of Corollary 2.6* is similar.

To conclude this chapter, we prove a maximal inequality providing a further way to prove a.e. convergence results for $\sum_{k=1}^\infty c_k f(n_k x)$.

Theorem 2.6. *Let $f \in L_2(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ and put*

$$S_N(x) = \sum_{k\le N} c_k f(kx).$$

Then for an arbitrary sequence (m_k) of positive integers we have

$$\int_0^1 \max_{M\le N} |S_M(x)|dx \le \sum_{k\le N} |c_k| r_f(m_k) + A \sum_{l=1}^{m_N} (|a_l|+|b_l|)(\sum_{k=d_l}^N c_k^2)^{1/2} \quad (2.50)$$

where $d_l = \inf\{k : m_k \ge l\}$ is the inverse function of m_k and A is an absolute constant.

If $f(x) = \sin 2\pi x$ or $\cos 2\pi x$, Theorem 2.6 reduces to Hunt's inequality [Hu]

$$\int_0^1 \max_{M\le N} |S_M(x)|dx \le C(\sum_{k=1}^N c_k^2)^{1/2} \quad (2.51)$$

and in fact Theorem 2.6 is easy consequence of Hunt's result. If the Fourier-series of f is absolutely convergent, i.e. $\sum_{l=1}^{\infty}(|a_l|+|b_l|)<\infty$, then choosing m_k so large that $\sum_{k=1}^{\infty} R(m_k)^2<\infty$, the right hand side of (2.50) is at most $C(\sum_{k=1}^{N} c_k^2)^{1/2}$, and thus the statement lemma reduces again to (2.51). In particular, it follows that if the Fourier-series of f is absolutely convergent (for example, if f belongs to the Lip 1/2 class), then $\sum_{k=1}^{\infty} c_k f(kx)$ converges a.e. provided $\mathbf{c}\in\ell^2$. This result is due to Gaposhkin [Gap4]. In contrast to Theorem 2.3, Theorem 2.6 loses the number-theoretic connection, but in the case $n_k=k$ it leads, despite the simplicity of its proof, to sharper results than the quasi-orthogonality method of Theorem 2.3, as the applications below will show.

Proof of Theorem 2.6. For simplicity we assume that the Fourier-expansion of f is a pure cosine series (i.e. $b_l=0$); the general case can be treated similarly. write $f=f_k+g_k$ where

$$f_k(x)=\sum_{l=1}^{m_k} a_l\cos 2\pi lx,\qquad g_k(x)=\sum_{l=m_k+1}^{\infty} a_l\cos 2\pi lx,$$

then

$$S_N(x)=T_N^{(1)}+T_N^{(2)}$$

where

$$T_N^{(1)}=\sum_{k\le N} c_k f_k(kx),\qquad T_N^{(2)}=\sum_{k\le N} c_k g_k(kx).$$

Clearly

$$|T_N^{(2)}|\le\sum_{k\le N}|c_k||g_k(kx)|$$

and thus

$$\max_{M\le N}|T_M^{(2)}|\le\sum_{k\le N}|c_k||g_k(kx)|.$$

Hence

$$\int_0^1\max_{M\le N}|T_M^{(2)}|dx\le\sum_{k\le N}|c_k|\|g_k(kx)\|_1\le\sum_{k\le N}|c_k|r_f(m_k).\tag{2.52}$$

On the other hand,

$$|T_N^{(1)}|=\left|\sum_{k\le N}c_k\sum_{l=1}^{m_k}a_l\cos 2\pi klx\right|=\left|\sum_{l=1}^{m_N}a_l\sum_{k=d_l}^{N}c_k\cos 2\pi klx\right|\le\sum_{l=1}^{m_N}|a_l|\left|\sum_{k=d_l}^{N}c_k\cos 2\pi klx\right|.$$

Thus

$$\max_{M\le N}|T_M^{(1)}|\le\sum_{l=1}^{m_N}|a_l|\max_{M\le N}\left|\sum_{k=d_l}^{M}c_k\cos 2\pi klx\right|$$

and thus using Hunt's inequality we get

$$\int_0^1 \max_{M\le N} |T_M^{(1)}|dx \le A \sum_{l=1}^{m_N} |a_l| \left(\sum_{k=d_l}^{N} c_k^2 \right)^{1/2} \tag{2.53}$$

where A is an absolute constant. The lemma now follows from (2.52) and (2.53).

We give now some corollaries of Theorem 2.6.

Corollary 2.7. *Let $f \in BV(0,1)$. Then $\sum_{k=1}^{\infty} c_k f(kx)$ converges a.e. provided*

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^{\beta} < \infty \quad \textit{for some } \beta > 2. \tag{2.54}$$

Corollary 2.8. *Let $f \in Lip_{\alpha}(\mathbf{T})$ for some $0 < \alpha < 1/2$ and let $\int_{\mathbf{T}} f(t)dt = 0$. Then $\sum_{k=1}^{\infty} c_k f(kx)$ converges a.e. provided*

$$\sum_{k=1}^{\infty} c_k^2 k^{1-2\alpha} (\log k)^{\beta} < \infty \qquad \textit{for some } \beta > 1 + 2\alpha. \tag{2.55}$$

Corollary 2.9. *Let $f \in Lip_{1/2}(\mathbf{T})$ and let $\int_{\mathbf{T}} f(t)dt = 0$. Then $\sum_{k=1}^{\infty} c_k f(kx)$ converges a.e. provided*

$$\sum_{k=1}^{\infty} c_k^2 (\log k)^{\beta} < \infty \quad \textit{for some } \beta > 2. \tag{2.56}$$

Corollary 2.8 was proved earlier by Gaposhkin [Gap2], while Corollary 2.9 improves Theorem 3 of Gaposhkin [Gap2].

Note that in the case $f \in \mathrm{Lip}_{\alpha}(\mathbf{T})$ the convergence condition is much stronger for $0 < \alpha < 1/2$. It is possible that in the case $0 < \alpha < 1/2$ a condition

$$\sum_{k=1}^{\infty} c_k (\log k)^{\gamma} < \infty \tag{2.57}$$

suffices for the a.e. convergence of $\sum_{k=1}^{\infty} c_k f(kx)$ but this remains open. On the other hand, Theorem 3 of [Be5] shows that for any $0 < \alpha < 1/2$ there exists $f \in \mathrm{Lip}_{\alpha}(\mathbf{T})$

with $\int_{\mathbf{T}} f(t)dt = 0$ and a real sequence (c_k) such that (2.72) holds for any $\gamma < 1 - 2\alpha$, but $\sum_{k=1}^{\infty} c_k f(kx)$ is a.e. divergent.

To prove the corollaries, assume first that $f \in \mathrm{Lip}_\alpha(\mathbf{T})$ with some $0 < \alpha \le 1/2$. (As we noted above, in the case $\alpha > 1/2$ the series $\sum_{k=1}^{\infty} c_k f(kx)$ converges a.e. for any $(c_k) \in \ell_2$ by Gaposhkin's theorem, so there is no convergence problem.) The Fourier coefficients of f satisfy (see Zygmund [Z] p. 241)

$$\sum_{k=2^n+1}^{2^{n+1}} (a_k^2 + b_k^2) \le C 2^{-2n\alpha}$$

whence it follows immediately that

$$\sum_{k=n}^{\infty} (a_k^2 + b_k^2) \le C n^{-2\alpha} \tag{2.58}$$

and

$$\sum_{k=1}^{\infty} (|a_k| + |b_k|) k^{\alpha - 1/2} (\log k)^{-\gamma} < \infty \qquad \text{for any } \gamma > 1. \tag{2.59}$$

The cases $0 < \alpha < 1/2$ and $\alpha = 1/2$ are treated differently, so we separate them.

(A) In the case $\alpha = 1/2$ we note that $r_f(n) = O(n^{-1/2})$ by (2.54) and thus by (2.56) and the Cauchy-Schwarz inequality the first term on the right side of (2.50) is bounded by

$$C \sum_{k \le N} |c_k| \frac{1}{\sqrt{m_k}} = C \sum_{k \le N} |c_k| (\log k)^{\beta/2} \frac{1}{\sqrt{m_k} (\log k)^{\beta/2}} \le C \left(\sum_{k \le N} \frac{1}{m_k (\log k)^{\beta}} \right)^{1/2}$$

which remains bounded if $m_k = k(\log k)^{1+\varepsilon-\beta}$, $\varepsilon > 0$. Then $d_l \sim l(\log l)^{-(1+\varepsilon-\beta)}$ and since by (2.56) we have $\sum_{k \ge N} c_k^2 \le C(\log N)^{-\beta}$, the second term on the right hand side of (2.50) is bounded by

$$C \sum_{l=1}^{m_N} (|a_l| + |b_l|)(\log d_l)^{-\beta/2}$$

which remains bounded by (2.55), since $\log d_l \sim \log l$ and $\beta > 2$.

Observe that if f is of bounded variation, then its Fourier coefficients satisfy $|a_k| = O(k^{-1})$, $|b_k| = O(k^{-1})$, and thus relations (2.54), (2.55) are satisfied with $\alpha = 1/2$. Hence the above proof also shows the validity of Corollary 2.7.

(B) In the case $0 < \alpha < 1/2$ we choose now $m_k = k(\log k)^\tau$ with τ to be determined later; then $d_l \sim l(\log l)^{-\tau}$. By (2.54) we have $R(n) = O(n^{-\alpha})$ and thus setting $\psi(k) = k^{1-2\alpha}(\log k)^\beta$, (2.58) and the Cauchy-Schwarz inequality shows that the first term on the right side of (2.50) is bounded by

$$C\sum_{k\le N}|c_k|\frac{1}{m_k^\alpha} = C\sum_{k\le N}|c_k|\psi(k)^{1/2}\frac{1}{m_k^\alpha\psi(k)^{1/2}} \le C\left(\sum_{k\le N}\frac{1}{m_k^{2\alpha}\psi(k)}\right)^{1/2}$$

which remains bounded, in view of the definitions of m_k and $\psi(k)$, if $\beta + 2\alpha\tau > 1$. On the other hand, $\sum_{k=1}^\infty c_k^2\psi(k) < \infty$ implies $\sum_{k\ge N} c_k^2 \le C\psi(N)^{-1}$, and thus the second term on the right hand side of (2.50) is bounded by

$$C\sum_{l=1}^{m_N}(|a_l| + |b_l|)\psi(d_l)^{-1/2}. \tag{2.60}$$

Substituting the values of $\psi(k)$ and d_l and using (2.55), we see that the sum in (2.60) remains bounded if $\beta-(1-2\alpha)\tau > 2$. We thus proved that if the sum in (2.58) converges and $m_k = k(\log k)^\tau$, then the left hand side of (2.55) remains bounded if

$$\beta > \max(2 + (1 - 2\alpha)\tau, 1 - 2\alpha\tau). \tag{2.61}$$

The right hand side (2.61) reaches its minimum for $\tau = -1$ with minimal value $1 + 2\alpha$, completing the proof.

3. Almost sure convergence–Necessary conditions.

Let $f \in L^2(\mathbf{T})$ with $\int_\mathbf{T} f(t)dt = 0$ and Fourier expansion

$$f \sim \sum_{k=1}^\infty (a_k \cos 2\pi kx + b_k \sin 2\pi kx).$$

Recall that by Wintner's theorem (Theorem A in Section 1), the series $\sum_n c_n f(nx)$ converges in the mean for all $(c_n) \in \ell^2$ iff

$$\sum_n \varphi_n/n^s \quad \text{and} \quad \sum_n \varphi_n/n^s \quad \text{are regular and bounded for } \Re s > 0. \tag{3.1}$$

In Section 2 we showed that (3.1) also implies the a.e. convergence of $\sum_{k=1}^\infty c_k f(n_k x)$ provided (n_k) satisfies the Erdős gap condition (2.11) with $\beta < 1/2$. The following result describes the situation when (3.1) fails.

Theorem 3.1. *Let $f \in \mathrm{Lip}_\alpha(\mathbf{T})$, $\int_{\mathbf{T}} f(t)dt = 0$ and assume that (3.1) is not valid. Then for any $\varepsilon_k \downarrow 0$ there exists $\mathbf{c} \in \ell^2$ and a sequence $\mathcal{N} = \{n_k, k \geq 1\}$ of positive integers satisfying*

$$n_{k+1}/n_k \geq 1 + \varepsilon_k \qquad (k \geq k_0)$$

such that the series $\sum_k c_k f(n_k x)$ is a.e. divergent.

This result is sharp: if (n_k) grows exponentially (i.e. $n_{k+1}/n_k \geq q > 1$) then $\sum_k c_k f(n_k x)$ converges a.e. for any $\mathbf{c} \in \ell^2$ by Kac's theorem (see Theorem D).

We note that the theorem remains valid, with minor modifications in the proof, if instead of $f \in \mathrm{Lip}_\alpha(\mathbf{T})$ we assume only $f \in L^2(\mathbf{T})$. However, as the positive result concerns the Lipschitz case, we will prove the converse also for that case.

For the proof we need two simple lemmas.

Lemma 3.1. *If (3.1) fails, then for any $N \geq 1$ there exist real numbers $\{a_j^{(N)}, j = 1, \ldots, N\}$ such that*

$$\int_0^1 \Big(\sum_{j=1}^N a_j^{(N)} f(jx)\Big)^2 dx \geq \Big(\sum_{j=1}^N (a_j^{(N)})^2\Big) L(N)$$

where $L(N) \to \infty$.

Proof. This is obvious, since by Wintner's theorem relation (3.1) is equivalent to the existence of a constant $C > 0$ such that for any $N \geq 1$ and any real sequence (a_j) we have

$$\int_0^1 \Big(\sum_{j=1}^N a_j f(jx)\Big)^2 dx \leq C\Big(\sum_{j=1}^N a_j^2\Big).$$

∎

Now, given $f \in \mathrm{Lip}_\alpha(\mathbf{T})$, choose the integer B so large that $(B-1)\alpha \geq 10$. Then we have

Lemma 3.2. *Let $1 \leq p_1 < q_1 < p_2 < q_2 < \ldots$ be integers such that $p_{k+1} \geq Bq_k$. Let $I_1, I_2, \ldots$ be sets of integers such that $I_k \subset [2^{p_k}, 2^{q_k}]$ and each element of I_k is divisible by 2^{p_k}. Let $b_j^{(k)}$, $j \in I_k$ be arbitrary coefficients with $|b_j^{(k)}| \leq 1$ and set*

$$X_k = X_k(\omega) = \sum_{j \in I_k} b_j^{(k)} f(j\omega) \qquad (k = 1, 2, \ldots, \ \omega \in \mathbf{T}).$$

Then there exist independent r.v.'s $Y_1, Y_2, \ldots$ on the probability space $(\mathbf{T}, \mathcal{B}, \lambda)$ such that $\mathbf{E}\, Y_k = 0$ and

$$|X_k - Y_k| \le 2^{-k} \qquad (k \ge k_0).$$

Proof. Let $\mathcal{F}_k$ denote the σ-field generated by the dyadic intervals

$$U_\nu = \left[\nu 2^{-Bq_k}, (\nu+1)2^{-Bq_k}\right] \qquad 0 \le \nu < 2^{Bq_k} \tag{3.2}$$

and set

$$\xi_j = \xi_j(\cdot) = \mathbf{E}\,(f(j\cdot)|\mathcal{F}_k), \qquad j \in I_k$$
$$Y_k = Y_k(\omega) = \sum_{j \in I_k} b_j^{(k)} \xi_j(\omega).$$

By $|f(x) - f(y)| \le C|x-y|^\alpha$ we have

$$|\xi_j(\omega) - f(j\omega)| \le C_1 2^{-(B-1)q_k\alpha} \le C_1 2^{-10q_k} \qquad j \in I_k$$

and since I_k has at most 2^{q_k} elements, we get

$$|X_k - Y_k| \le C_1 \cdot 2^{q_k} 2^{-10q_k} \le 2^{-k} \qquad for \qquad k \ge k_0.$$

Since $p_{k+1} \ge Bq_k$ and since each $j \in I_{k+1}$ is a multiple of $2^{p_{k+1}}$, each interval U_ν in (3.2) is a period interval for all $f(jx)$, $j \in I_{k+1}$ and thus also for ξ_j, $j \in I_{k+1}$. Hence Y_{k+1} is independent of the σ-field $\mathcal{F}_k$ and since $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots$ and Y_k is $\mathcal{F}_k$ measurable, the r.v.'s $Y_1, Y_2, \ldots$ are independent. Finally $E\xi_j = 0$ by $\int_{\mathbf{T}} f dx = 0$ and thus $EY_k = 0$.

Turning to the proof of Theorem 3.1, let $\psi(k)$ grow so rapidly that $L(\psi(k)) \ge 2^k$ and let (r_k) be a nondecreasing sequence of integers to be chosen later. We define sets

$$I_1^{(1)}, I_2^{(1)}, \ldots, I_{r_1}^{(1)}, I_1^{(2)}, \ldots, I_{r_2}^{(2)}, \ldots, I_1^{(k)}, \ldots, I_{r_k}^{(k)}, \ldots \tag{3.3}$$

of positive integers by

$$I_j^{(k)} = 2^{c_j^{(k)}} \{1, 2, \ldots, \psi(k)\}, \qquad 1 \le j \le r_k,\ k \ge 1$$

where $c_j^{(k)}$ are suitable positive integers. (Here for any set $\{a, b, \ldots\} \subset \mathbf{R}$ and $\lambda \in \mathbf{R}$, $\lambda\{a, b, \ldots\}$ denotes the set $\{\lambda a, \lambda b, \ldots\}$.) Clearly we can choose the integers $c_j^{(k)}$ inductively so that the intervals in (3.3) satisfy the conditions of Lemma 3.2. By Lemma 3.1 there exist, for any $k \ge 1$, coefficients $\{a_\nu^{(k)},\ 1 \le \nu \le \psi(k)\}$, $\sum_{\nu=1}^{\psi(k)} a_\nu^{(k)2} = 1$ such that, setting

$$X^{(k)} = X^{(k)}(\omega) = \sum_{\nu=1}^{\psi(k)} a_\nu^{(k)} f(\nu\omega)$$

we have

$$\mathbf{E}\left(X^{(k)}\right)^2 \geq L\,\psi(k).$$

Let

$$X_j^{(k)}(\omega) = X^{(k)}(2^{c_j^{(k)}}\omega), \qquad 1 \leq j \leq r_k$$

Clearly the $X_j^{(k)}$ have the same distribution, and consequently

$$\mathbf{E}\left(X_j^{(k)}\right)^2 \geq L\,\psi(k).$$

By Lemma 3.1 there exist independent r.v.'s $Y_j^{(k)}$ $(1 \leq j \leq r_k,\ k = 1, 2, \ldots)$ such that $\mathbf{E}\,Y_j^{(k)} = 0$ and

$$\sum_{k,j} |X_j^{(k)} - Y_j^{(k)}| \leq K \tag{3.4}$$

for some constant $K > 0$. Hence by the Minkowski inequality

$$\mathbf{E}\,(Y_j^{(k)})^2 \geq \frac{1}{2} L(\psi(k)) \tag{3.5}$$

for $k \geq k_0$. Also $|Y_j^{(k)}| \leq |X_j^{(k)}| + K \leq \text{Const.}\psi(k)$ and thus setting

$$Z_k = \frac{1}{(r_k L\psi(k))^{1/2}} \sum_{j=1}^{r_k} Y_j^{(k)}$$

$$\sigma_k^2 = \mathbf{E}\Big(\sum_{j=1}^{r_k} Y_j^{(k)}\Big)^2 \geq \frac{1}{2} r_k L\,\psi(k)$$

we get from the central limit theorem with Ljapunov's remainder term

$$\begin{aligned}\mathbf{P}\big\{Z_k \geq 1\big\} \geq \mathbf{P}\Big\{\sum_{j=1}^{r_k} Y_j^{(k)} \geq 2\sigma_k\Big\} &\geq (1 - \Phi(2)) - C\frac{r_k}{\psi(k)^3}(r_k L\psi(k))^{3/2} \\ &\geq 1 - \Phi(2) - o(1) \geq 0.02, \qquad (k \geq k_0)\end{aligned}$$

provided r_k grows so rapidly that $r_k^{1/2} L(\psi(k))^{3/2} \geq \psi(k)^4$. Since the Z_k are independent, the Borel–Cantelli lemma implies $\mathbf{P}\big\{Z_k \geq 1\ i.o.\big\} = 1$, i.e. $\sum_{k\geq 1} Z_k$ is a.e. divergent, which, in view of (3.3), yields that

$$\sum_{k=1}^{\infty} \frac{1}{(r_k L(\psi(k))^{1/2}} \sum_{j=1}^{r_k} X_j^{(k)} \qquad \text{is a.e. divergent.} \tag{3.6}$$

Let now

$$\mathcal{N} := \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{r_k} I_j^{(k)}. \tag{3.7}$$

Then the sum in (3.6) is of the form $\sum_{i=1}^{\infty} c_i f(n_i x)$ where

$$\sum_{i=1}^{\infty} c_i^2 = \sum_{k=1}^{\infty} \frac{r_k}{r_k L(\psi(k))} = \sum_{k=1}^{\infty} \frac{1}{L(\psi(k))} < +\infty.$$

Finally, denote by $1+\rho_k$ the smallest of the ratios $(j+1)/j$, $1 \le j \le \psi(k)-1$; clearly $\rho_k > 0$. Given $\varepsilon_k \downarrow 0$ one can choose r_k growing so rapidly that

$$\rho_k \ge \varepsilon_{r_{k-1}} \qquad k = 1, 2, \dots. \tag{3.8}$$

Now if n_s and n_{s+1} belong to the same set $I_j^{(k)}$ then clearly $s \ge r_{k-1}$ and thus by (3.7) we get $n_{s+1}/n_s \ge 1+\rho_k \ge 1+\varepsilon_{r_{k-1}} \ge 1+\varepsilon_s$. Since $n_{s+1}/n_s \ge 2$ if n_s and n_{s+1} belong to different $I_j^{(k)}$'s, we proved that (n_k) satisfies

$$n_{k+1}/n_k \ge 1+\varepsilon_k \qquad (k \ge k_0). \tag{3.9}$$

This completes the proof of Theorem 3.1. ■

There are few results concerning the bounded case, namely the case when in the series $\sum_k c_k f(n_k x)$, f is not smooth but only bounded. We first consider the case of primes and prove the following result.

Theorem 3.2. *Let $\mathcal{P} := (P_k)$ be an increasing sequence of prime numbers. Let $\mathbf{c} = \{c_k, k \ge 1\}$ be a sequence of positive reals such that*

$$\sum_k c_k^2 < \infty, \qquad \sum_k c_k = \infty.$$

Then there exists a function $f \in L^\infty(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ such that the series $\sum_{k=1}^{\infty} c_k f(P_k x)$ diverges on a set with positive measure.

Theorem 3.2 will be deduced from the following

Theorem 3.3. *Let $\mathcal{P} := (P_k)$ be an increasing sequence of prime numbers. Let $\mathbf{c} = \{c_k, k \ge 1\}$ be a sequence of positive reals such that*

$$\sum_k c_k^2 < \infty, \qquad \sum_k c_k = \infty.$$

Put $C_n = \sum_{k\le n} c_k$ and consider the weighted sums

$$\mathcal{S}_n f = \frac{1}{C_n}\sum_{k\le n} c_k f(P_k x).$$

Then there exists a function $f \in L^\infty(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt = 0$ such that the sequence $\{\mathcal{S}_n f, n \ge 1\}$ diverges on a set with positive measure.

Proof of Theorem 3.2. Assuming that Theorem 3.3 is valid, there exists a bounded measurable function f such that $(\mathcal{S}_n f)_n$ does not converge almost everywhere. Then the partial sums $\sum_{k\le n} c_k f(P_k x)$ do not converge almost everywhere either. Otherwise, this would imply, in view of the assumption that the series $\sum_k c_k$ diverges, that $(\mathcal{S}_n f(x))_n$ tend to 0 almost everywhere, a contradiction. Hence the result. ■

To prove Theorem 3.2, we use Bourgain's entropy criterion in L^∞ which we recall here.

Lemma 3.3. ([Bo], Proposition 2]) *Let $\{S_n, n \ge 1\}$ be a sequence of $L^2(\mu) - L^\infty(\mu)$ contractions satisfying the following commutation assumption:*

(H) *There exists a family $\mathcal{E} = \{T_j, j \ge 1\}$ of μ-preserving measurable transformations of X, commuting with S_n ($S_n T_j(f) = T_j S_n(f)$) such that for any $g \in L^1(\mathbf{T})$,*

$$\lim_{J\to\infty} \Big\| \frac{1}{J}\sum_{j=1}^{J} T_j g - \int_{\mathbf{T}} g d\lambda \Big\|_1 = 0. \tag{3.10}$$

Moreover, assume that

$$\mu\{S_n(f) \text{ converges as } n \to \infty\} = 1 \quad \text{for all } f \in L^\infty(\mu). \tag{3.11}$$

For any $\delta > 0$, let $N_f(\delta)$ denotes the minimal number of $L^2(\mu)$-open balls centered in the set $\{S_n f, n \ge 1\}$ and enough to cover it. Then,

$$C(\delta) = \sup_{f\in L^\infty(\mu),\ ||f||_2=1} N_f(\delta) < \infty. \tag{3.12}$$

If the T_j's are defined as at the beginning of Section 2, we know that $S_n T_j(f) = T_j S_n(f)$ and for any $g \in L^2(\mathbf{T})$, $\lim_{J\to\infty} \big\| \frac{1}{J}\sum_{j=1}^{J} T_j g - \int_{\mathbf{T}} g d\lambda \big\|_2 = 0$. This, plus a plain approximation argument and the fact that barycenters of contractions are again contractions, finally imply (3.10). This means that assumption (**H**) is satisfied in our case. For proving Theorem 3.3, we will also need the lemma below, which is taken from [We1], (see Lemma 5.1.6 p. 76).

Lemma 3.4. *Let R, T, p be three positive integers such that $R \geq T(4p^2 - 3)$. Let $(H, ||.||)$ be a Hilbert space. Let $B = \{f_n,\ 1 \leq n \leq R\}$ be a finite subset of H such that $||f_n|| \leq 1$, $1 \leq n \leq R$ and $\Phi = \{\phi_n,\ 1 \leq n \leq R\}$ an orthonormal system of H. We assume*

$$\langle f_n, \phi_n \rangle \geq \frac{1}{p}, \qquad (1 \leq n \leq R) \tag{a}$$

Then B contains a subset B' satisfying

$$\mathrm{Card}\,(B') \geq T \qquad \text{and} \qquad \inf_{f,g \in B',\ f \neq g} ||f - g|| \geq \frac{1}{2p}. \tag{b}$$

Proof of Theorem 3.3. Let $\{T_N, N \geq 1\}$ be integers such that $T_N - T_{N-1}$ increases to infinity with N. Define

$$\Pi_N = \Big\{ u = P_{T_{N-1}+1}^{\alpha_{T_{N-1}+1}} \ldots P_{T_N}^{\alpha_{T_N}} : \alpha_i \in \{0,1\} \text{ and } (\alpha_{T_{N-1}+1}, \ldots, \alpha_{T_N}) \neq (0, \ldots, 0) \Big\},$$
$$f_N = \frac{1}{\big[2^{T_N - T_{N-1}} - 1\big]^{1/2}} \sum_{u \in \Pi_N} e_u. \tag{3.13}$$

Let $T_{N-1} < R \leq T_N$. Then,

$$\langle \mathcal{S}_R(f_N), f_N \rangle = \frac{1}{C_R} \frac{1}{\big[2^{T_N - T_{N-1}} - 1\big]^{1/2}} \sum_{u \in \Pi_N} \sum_{v \in \Pi_N} \sum_{k \leq R} c_k \langle e_{uP_k}, e_v \rangle.$$

Let $u, v \in \Pi_N$ and $k \leq R$. Then $\langle e_{uP_k}, e_v \rangle = 1$, if and only if $uP_k = v$. Noting $u = P_{T_{N-1}+1}^{\alpha_{T_{N-1}+1}} \ldots P_{T_N}^{\alpha_{T_N}}$, $v = P_{T_{N-1}+1}^{\beta_{T_{N-1}+1}} \ldots P_{T_N}^{\beta_{T_N}}$, this means that:

$$P_k P_{T_{N-1}+1}^{\alpha_{T_{N-1}+1}} \ldots P_{T_N}^{\alpha_{T_N}} = P_{T_{N-1}+1}^{\beta_{T_{N-1}+1}} \ldots P_{T_N}^{\beta_{T_N}}.$$

This equation has solutions if and only if k belongs to the interval $]T_{N-1}, T_N]$, and then the solutions are given by

$$\alpha_k = 0, \quad \beta_k = 1 \qquad \alpha_j = \beta_j, \qquad \textit{otherwise}.$$

Hence,

$$\langle f_N(P_k .), f_N \rangle = \frac{2^{T_\theta - T_{\theta-1} - 1} - 1}{2^{T_\theta - T_{\theta-1}} - 1} \geq \frac{1}{4}. \tag{3.14}$$

Consequently, for any integer $N \geq 1$ and any $T_{N-1} < R \leq T_N$

$$\langle S_R(f_N), f_N \rangle$$
$$= \frac{1}{C_R} \sum_{k \leq R} c_k \langle f_N(P_k .), f_N \rangle = \frac{1}{C_R} \sum_{\substack{k \leq R \\ k \in]T_{N-1}, T_N]}} c_k \langle f_N(P_k .), f_N \rangle \geq \frac{1}{4}. \tag{3.15}$$

The proof is completed by applying Lemma 3.4 and the entropy criterion in L^∞. ∎

The next two theorems will concern subsequences $\mathcal{N}$ generated by infinitely many primes.

Theorem 3.4. *Let $\mathcal{P} = \{P_1, P_2, \cdots\}$ be an increasing sequence of positive pairwise coprime integers, and denote by $\mathcal{C}(\mathcal{P})$ the infinite dimensional chain generated by $\mathcal{P}$. Let $\mathbf{c} = \{c_k, k \ge 1\}$ be a sequence of positive reals such that the series $\sum_{k=1}^{\infty} c_k$ diverges. Define for any measurable function $f : \mathbf{T} \to \mathbf{R}$ the weighted sums*

$$\mathcal{S}_n f(x) = \frac{1}{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,n]} c_j} \sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,n]} c_j f(jx).$$

Assume that

$$\limsup_{i\to\infty} \frac{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[\frac{1}{2}P_1^{2i},P_1^{2i}])} c_j}{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,P_1^{2i}])} c_j} > 0. \tag{3.16}$$

Then there exists a bounded measurable function f such that $(\mathcal{S}_n f)_n$ does not converge almost everywhere.

From Theorem 3.4 one can obtain

Theorem 3.5. *Let $\mathcal{P} = \{P_1, P_2, \cdots\}$ be an increasing sequence of positive pairwise coprime integers, and denote by $\mathcal{C}(\mathcal{P})$ the infnite dimensional chain generated by $\mathcal{P}$. Let $\mathbf{c} = \{c_k, k \ge 1\}$ be a sequence of positive reals such that*

$$\sum_k c_k^2 < \infty \qquad\qquad \sum_k c_k = \infty.$$

Assume that condition (3.16) is satisfied. Then, there exists a bounded measurable function f such that $(\sum_{k\le n} c_k f(P_k.))_n$ does not converge almost everywhere.

The proof of Theorem 3.5 is similar to the proof of Theorem 3.3, so it is omitted.

Proof of Theorem 3.4. The proof uses Bourgain's ideas in [Bo]. Let s be some fixed positive integer. Put for any integer $T \ge 0$

$$A_T = \left\{ n = P_1^{\alpha_1} \cdots P_s^{\alpha_s} \ : \ P_1^T \le n < P_1^{T+1},\ \alpha_i \ge 0,\ i = 1, \cdots, s \right\}. \tag{3.17}$$

By replacing α_1 by $\alpha_1 + 1$, one can easily verify that

$$\sharp(A_T) \le \sharp(A_{T+1}) \tag{3.18}$$

As for $n = P_1^{\alpha_1} \cdots P_s^{\alpha_s} \in A_T$, necessarily $0 \le \alpha_1 + \cdots + \alpha_s \le T$, we also deduce

$$\sharp(A_T) \le T^s. \tag{3.19}$$

Then, for any $d > 0$, there exists an integer $T > 0$ such that

$$\sharp(A_{T+d}) \le 2\sharp(A_T), \tag{3.20}$$

Indeed, otherwise, $\sharp(A_{T+d}) > 2\sharp(A_T)$ for any T, would imply for any integer n

$$\sharp(A_{nd}) > B2^n,$$

where B is some positive constant, which contradicts to (3.19). Choose d such that $P_1^d \le P_s$. Any element $j \in \mathcal{C}(\mathcal{P})$ such that $j \le P_1^d$ can be thus expressed as $j = P_1^{\alpha_1} \cdots P_r^{\alpha_r}$ with $r \le s$. Put for any $i = 0, \cdots, d$

$$f^{(i)}(x) = \frac{1}{\sharp(A_{T+i})^{\frac{1}{2}}} \sum_{n \in A_{T+i}} e^{2i\pi nx}, \tag{3.21}$$

and let

$$f = f^{(0)}.$$

Next, put for any $i = 0, \cdots, [\frac{d}{2}]$

$$\phi_i = \frac{f^{(2i-1)} + f^{(2i)}}{\sqrt{2}}, \tag{3.22}$$

and let for any integer j, $f_j(x) = f(jx)$. The set of functions $f^{(i)}$ is a sub-orthonormal system of L^2, the same property holds true for the system of functions ϕ_i. Moreover $||f_j|| = 1$ for any j.
Let $1 \le i \le [\frac{d}{2}]$, $j \in [P_1^{2i-1}, P_1^{2i}] \cap \mathcal{C}(\mathcal{P})$, and examine f_j. Let $n \in A_T$. Then nj may be written as follows $nj = P_1^{\beta_1} \cdots P_s^{\beta_s}$. Moreover

$$P_1^{T+2i-1} \le nj < P_1^{T+2i+1}.$$

It follows that we have the following implication

$$n \in A_T \textit{ and } j \in [P_1^{2i-1}, P_1^{2i}] \cap \mathcal{C}(\mathcal{P}) \quad \Rightarrow \quad nj \in A_{T+2i-1} \cup A_{T+2i}.$$

We may thus write

$$f_j(x) = \frac{1}{\sharp(D)^{\frac{1}{2}}} \sum_{m \in D} e^{2i\pi mx},$$

where $D \subset A_{T+2i-1} \cup A_{T+2i}$ and $\sharp(D) = \sharp(A_T)$. Hence,

$$\sqrt{2}\langle f_j, \phi_i\rangle = \frac{1}{[\sharp(A_T)\sharp(A_{T+2i-1})]^{\frac{1}{2}}} \sum_{m\in D\cap A_{T+2i-1}} 1 \; + \; \frac{1}{[\sharp(A_T)\sharp(A_{T+2i})]^{\frac{1}{2}}} \sum_{m\in D\cap A_{T+2i}} 1$$

$$\geq \frac{1}{\sharp(A_T)\sqrt{2}}.\sharp(A_T) = \frac{1}{\sqrt{2}},$$

and so for any $1 \leq i \leq [\frac{d}{2}]$, $P_1^{2i-1} \leq j \leq P_1^{2i}$

$$\langle f_j, \phi_i\rangle \geq \frac{1}{2}. \tag{3.23}$$

Further, $\langle f_j, \phi_k\rangle \geq 0$ for any j and k. Thus,

$$\begin{aligned}\langle S_{P_1^{2i}}(f), \phi_i\rangle &= \frac{1}{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,P_1^{2i}]} c_j} \sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,P_1^{2i}])} c_j\langle f_j, \phi_i\rangle \\ &\geq \frac{1}{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,P_1^{2i}]} c_j} \sum_{j\in\mathcal{C}(\mathcal{P})\cap[\frac{1}{2}P_1^{2i},P_1^{2i}])} c_j\langle f_j, \phi_i\rangle \\ &\geq \frac{1}{2}\frac{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[\frac{1}{2}P_1^{2i},P_1^{2i}]} c_j}{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,P_1^{2i}]} c_j}\end{aligned}$$

We have obtained for any $i = 1, \cdots, [\frac{d}{2}]$

$$\langle S_{P_1^{2i}}(f), \phi_i\rangle \; \geq \frac{1}{2}\frac{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[\frac{1}{2}P_1^{2i},P_1^{2i}]} c_j}{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,P_1^{2i}]} c_j}. \tag{3.24}$$

Now, by assumption

$$\limsup_{i\to\infty} \frac{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[\frac{1}{2}P_1^{2i},P_1^{2i}]} c_j}{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,P_1^{2i}]} c_j} > 0.$$

We may find an increasing sequence $(i_\lambda)_\lambda$ of integers as well as a positive real c, such that

$$\frac{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[\frac{1}{2}P_1^{2i_\lambda},P_1^{2i_\lambda}]} c_j}{\sum_{j\in\mathcal{C}(\mathcal{P})\cap[1,P_1^{2i_\lambda}]} c_j} \geq 2c \qquad\qquad (\lambda = 1, 2\ldots)$$

Consequently, for any λ such that $i_\lambda \leq d$,

$$\langle S_{P_1^{2i_\lambda}}(f), \phi_{i_\lambda}\rangle \geq c. \tag{3.25}$$

Let p be a positive integer such that $pc \geq 1$. Lemma 3.4 applied with the choices $R = [\frac{D}{2}]$, $T = \big[[\frac{D}{2}]/13\big]$ with $D = \sharp(\lambda \mid i_\lambda \leq d)$ and p shows that

$$N((S_{P_1^{2i}}(f), i \leq [\frac{D}{2}]), \frac{c}{2}) \geq T. \tag{3.26}$$

But d is arbitrary, thus

$$\sup_{f\in L^\infty \quad ||f||_2\leq 1} N((S_{P_1^{2i}}(f), i \geq 1), \frac{c}{2}) = \infty.$$

Applying now Bourgain's entropy criterion in L^∞ concludes the proof. ■

4. Random sequences.

In this section we investigate the convergence of the series $\sum_{k=1}^{\infty} c_k f(n_k x)$ where (n_k) is a random sequence of real numbers. Specifically, we will investigate the model when $n_k = X_1 + \ldots + X_k$, where the X_k are independent, identically distributed random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$. We will not assume that X_1 is integer valued or $X_1 > 0$; we assume only that the the distribution of X_1 is nondegenerate. If the random walk $\{\sum_{k=1}^{n} X_k,\ n \geq 1\}$ is transient, we have $|n_k| \to \infty$ a.s. On the other hand, if the random walk is recurrent and X_1 is nonlattice, (n_k) is dense in $\mathbb{R}$ with probability 1.

We begin our investigations with the study of random trigonometric sums of the form

$$\sum_{n=1}^{\infty} c_n e^{itS_n(\omega)} \tag{4.1}$$

where $(c_k) \in \ell_2$; the terms of this sum are functions defined on the product space $\Omega \times \mathbf{T}$, endowed with the product probability $\mathbf{P} \times \lambda$.

Theorem 4.1. *Let X_1 be nondegenerate with characteristic function φ and let $S_n = \sum_{k=1}^{n} X_k$ be the corresponding random walk. Then for any $\mathbf{c} \in \ell^2$ and any real t for which*

$$\rho = \max(|\varphi(t)|, |\varphi(2t)|, |\varphi(-t)|, |\varphi(-2t)|) < 1 \tag{4.2}$$

the series (4.1) converges with probability 1. Consequently, the series (4.1) converges for almost all $(t, \omega) \in \mathbf{T} \times \Omega$, provided $\mathbf{c} \in \ell^2$.

Since X_1 is nondegenerate, (4.2) holds for all but countably many t's. If X_1 is nonlattice, then $|\varphi(t)| < 1$ for all $t \neq 0$; otherwise there exists a $t_0 > 0$ such that $|\varphi(t)| = 1$ if and only if $t = kt_0$, $k \in \mathbf{Z}$. If X_1 is degenerate, then $S_n = cn$ with some constant c, and the statement of Theorem 4.1 reduces to Carleson's theorem, which is of course not contained in our result. But it is interesting to note that for all other random walks, the above formulated "random" version of Carleson's theorem is valid. This seems paradoxical at first sight, since the random walk S_n can be recurrent, e.g. it is possible that $S_n = 0$ for infinitely many n. However, by the theory of random walks the set $H = \{n : S_n = 0\}$ is thin (e.g. it has $O(n^{1/2})$ elements in the interval $[0, n]$) and Theorem 4.1 shows that $\sum_{k \in H} |c_k| < \infty$ even if $\sum_{k=1}^{\infty} |c_k| = \infty$.

For the proof of Theorem 4.1 we will need the following convergence result of probability theory, first observed by Stechkin in the context of orthogonal series (see e.g. Gaposhkin [Gap1 pp. 29-31]). For the present version, see Billingsley [Bi] p. 102, Problem 6; for an alternative proof see [We2], Theorem 2.1.

Lemma 4.1. *Let $\{\xi_i, i \geq 1\}$ be a sequence of random variables satisfying the assumption*

$$\mathbf{E}\left|\sum_{i\leq l\leq j}\xi_l\right|^\gamma \leq \left(\sum_{i\leq l\leq j}u_l\right)^\alpha, \qquad 0\leq i\leq j<\infty,$$

where $\{u_i, i\geq 1\}$ is a sequence of nonnegative reals such that the series $\sum_{l=1}^\infty u_l$ converges and $\alpha > 1$, $\gamma > 0$. Then the series $\sum_{l=1}^\infty \xi_l$ converges almost surely. Moreover, for $\alpha > 1$,we have

$$\| \sup_{i,j\geq 1} | \sum_{i\leq l\leq j} \xi_l| \|_\gamma \leq C(\sum_{l=1}^\infty u_l)^{\alpha/\gamma},$$

where the constant C depends on α only.

Applying Lemma 4.1 with $\gamma = 4$, $\alpha = 2$, $u_k = c_k^2$, for proof of Theorem 4.1 it suffices to prove the following

Lemma 4.2. *For any real $c_1, \ldots, c_N$ we have*

$$\mathbf{E}\left|\sum_{k=1}^N c_k e^{itS_k}\right|^4 \leq \frac{1}{(1-\rho)^2}\left(\sum_{k=1}^N c_k^2\right)^2. \tag{4.3}$$

where ρ is defined by (4.2).

Proof. In the case $\rho = 1$ the lemma is obvious, so we can assume $\rho < 1$. Clearly for any real $c_1, \ldots, c_N$ we have

$$\mathbf{E}|\sum_{k=1}^N c_k e^{itS_k}|^4 = \sum_{1\leq j,k,l,m\leq N} c_j c_k c_l c_m \mathbf{E}e^{it(S_j-S_k+S_l-S_m)}. \tag{4.4}$$

We now claim that

$$|\mathbf{E}e^{it(\pm S_j\pm S_k\pm S_l\pm S_m)}| \leq \rho^{(|j-k|+|l-m|)} \qquad (j\geq k\geq l\geq m). \tag{4.5}$$

provided in the last exponent there are two positive and two negative signs. Clearly we can assume that the sign of S_j in (4.5) is positive; otherwise we replace t by $-t$. There are 3 cases:

$$\begin{aligned}
(a)\quad |\mathbf{E}e^{it(S_j-S_k+S_l-S_m)}| &= |\mathbf{E}e^{it(S_j-S_k)}||\mathbf{E}e^{it(S_l-S_m)}| = |\varphi(t)|^{j-k}|\varphi(t)|^{l-m} \\
&\le \rho^{(|j-k|+|l-m|)}, \\
(b)\quad |\mathbf{E}e^{it(S_j-S_k-S_l+S_m)}| &= |\mathbf{E}e^{it(S_j-S_k)}||\mathbf{E}e^{-it(S_l-S_m)}| = |\varphi(t)|^{j-k}|\varphi(-t)|^{l-m} \\
&\le \rho^{(|j-k|+|l-m|)}, \\
(c)\quad |\mathbf{E}e^{it(S_j+S_k-S_l-S_m)}| &= |\mathbf{E}e^{it(S_j-S_k)+2it(S_k-S_l)+it(S_l-S_m)}| \\
&= |\varphi(t)|^{j-k}|\varphi(2t)|^{k-l}|\varphi(t)|^{l-m} \le \rho^{(|j-k|+|l-m|)},
\end{aligned}$$

proving (4.5). Thus splitting the sum on the right hand side of (4.4) into 24 subsums corresponding to a fixed relative order of j, k, l, m and in each such sum renaming the indices j, k, l, m so that they will be nonincreasing in the renamed order, we get

$$\mathbf{E}|\sum_{k=1}^{N} c_k e^{itS_k}|^4 \le 24 \sum_{N\ge j\ge k\ge l\ge m\ge 1} |c_j||c_k||c_l||c_m|\rho^{(|j-k|+|l-m|)}. \tag{4.6}$$

Summing the right hand side of (4.6) first for those indices (j, k, l, m) for which $j-k=r$ and $l-m=s$ are fixed, we get by Cauchy's inequality

$$\begin{aligned}
&\sum_{1\le k,k+r,m,m+s\le N} |c_k||c_{k+r}||c_m||c_{m+s}|\rho^{r+s} \\
&\le \rho^{r+s} \sum_{1\le k,k+r\le N} |c_k||c_{k+r}| \sum_{1\le m,m+s\le N} |c_m||c_{m+s}| \\
&\le \rho^{r+s} \left(\sum_{1\le k\le N} c_k^2\right)^{1/2} \left(\sum_{1\le k+r\le N} c_{k+r}^2\right)^{1/2} \left(\sum_{1\le m\le N} c_m^2\right)^{1/2} \left(\sum_{1\le m+s\le N} c_{m+s}^2\right)^{1/2} \\
&\le \rho^{r+s} \left(\sum_{1\le j\le N} c_j^2\right)^{2}.
\end{aligned}$$

Now summing for r and s we get Lemma 4.2.

We turn now to the convergence of the series (4.1) in $L^p(\mathbf{T}\times\Omega)$ for $p>2$. For simplicity, we consider the case $p=4$.

Proposition 4.1. *Let $\mathcal{X} = \{X, X_i, i\ge 1\}$ be a sequence of independent, identically distributed, lattice random variables defined on some probablility space $(\Omega, \mathcal{A}, \mathbf{P})$. We* *assume that the random walk $S_n = X_1 + \ldots + X_n$, $n \geq 1$ is transient. Then,*

$$
\begin{aligned}
&\mathbf{E}\int_{\mathbf{T}} \Big|\sum_{k=1}^n c_k\, e^{2\imath\pi\alpha S_k}\Big|^4 d\alpha \leq 4G(0,0)\Big(\sum_{k=1}^n |c_k|^2\Big)\\
&+6 \sum_{1\leq i\leq k<l\leq j\leq n} |c_i||c_j||c_k||c_l|\\
&\times\Big\{\mathbf{P}\big\{S_k - S_i = \pm(S_j - S_l)\big\} + \mathbf{P}\big\{S_k - S_i = (S_j - S_l) - 2(S_l - S_k)\big\}\Big\}
\end{aligned}
$$

Proof. Let $a_1, \ldots, a_n$ be complex numbers. Then,

$$
\begin{aligned}
\Big|\sum_{i=1}^n a_i\Big|^4 &= \Big(\sum_{i=1}^n\sum_{j=1}^n a_i\bar a_j\Big)\Big(\sum_{k=1}^n\sum_{l=1}^n a_k\bar a_l\Big)\\
&= \Big(\sum_{i=1}^n |a_i|^2 + \sum_{i=1}^n\sum_{j=i+1}^n (a_i\bar a_j + \bar a_i a_j)\Big)\Big(\sum_{k=1}^n |a_k|^2 + \sum_{k=1}^n\sum_{l=k+1}^n (a_k\bar a_l + \bar a_k a_l)\Big)\\
&= \Big(\sum_{k=1}^n |a_k|^2\Big)^2 + 2\Big(\sum_{k=1}^n |a_k|^2\Big)\sum_{k=1}^n\sum_{l=k+1}^n (a_k\bar a_l + \bar a_k a_l)\\
&\quad+\sum_{i=1}^n\sum_{j=i+1}^n\sum_{k=1}^n\sum_{l=k+1}^n (a_i\bar a_j + \bar a_i a_j)(a_k\bar a_l + \bar a_k a_l)\\
&:= \Big(\sum_{k=1}^n |a_k|^2\Big)^2 + A + B.
\end{aligned}
$$

Apply this in our case: $a_\ell = c_\ell e^{2\imath\pi\alpha S_\ell(\omega)}$, $i = 1, \ldots, n$. The sum A equals to

$$
A = 2\Big(\sum_{k=1}^n |c_k|^2\Big)\sum_{k=1}^n\sum_{l=k+1}^n \big(c_k\bar c_l e^{2\imath\pi\alpha(S_k(\omega)-S_l(\omega))} + \bar c_k c_l e^{2\imath\pi\alpha(S_l(\omega)-S_k(\omega))}\big).
$$

By integrating over $\Omega\times\mathbf{T}$, with respect to $\mathbf{P}\times m$, we obtain an expression, which is equal to

$$
\tilde A = 2\Big(\sum_{k=1}^n |c_k|^2\Big)\sum_{1\leq k<l\leq n}\Big(c_k\bar c_l\mathbf{P}\big\{S_l = S_k\big\} + \bar c_k c_l\mathbf{P}\big\{S_l = -S_k\big\}\Big).
$$

As $\langle f_n, f_m\rangle_{\mathbf{P}\times\lambda} = \mathbf{P}\{S_{|m-n|} = 0\}$ and since the system $\{f_n, n\geq 1\}$ is easily seen to be a quasi-orthogonal system, it follows that

$$
|\tilde A| \leq 4G(0,0)\Big(\sum_{k=1}^n |c_k|^2\Big). \tag{4.7}
$$

Now, the sum B equals to

$$\sum_{1\le i<j\le n}\sum_{1\le k<l\le n}\Big(\alpha_i\bar\alpha_j e^{2\imath\pi\alpha(S_i(\omega)-S_j(\omega))}+\bar\alpha_i\alpha_j e^{2\imath\pi\alpha(S_j(\omega)-S_i(\omega))}\Big)$$
$$\times\Big(\alpha_k\bar\alpha_l e^{2\imath\pi\alpha(S_k(\omega)-S_l(\omega))}+\bar\alpha_k\alpha_l e^{2\imath\pi\alpha(S_l(\omega)-S_k(\omega))}\Big).$$

Integrating this expression over $\Omega\times\mathbf{T}$, with respect to $\mathbf{P}\times m$, we find a sum of the type

$$\tilde B=\sum_{1\le i<j\le n}\sum_{1\le k<l\le n}\gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_l-S_k=\pm(S_j-S_i)\big\},$$

where $\gamma_i=\alpha_i$ or $\bar\alpha_i$. Consider six cases.

i) $(1\le k<l\le i)$ The sum differences S_j-S_i and S_l-S_k are independent, and we find in this case a contribution given by

$$\sum_{1\le i<j\le n}\sum_{1\le k<l\le i}\gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_j-S_i=\pm(S_l-S_k)\big\}.$$

ii) $(1\le k\le i<l<j)$ There are two subcases: $S_l-S_k=S_j-S_i$ and $S_l-S_k=-(S_j-S_i)$. Write $a=i-k$, $b=l-i$, $c=j-l$. This corresponds to $a+b=\pm(b+c)$.
— if $a+b=b+c$, then $S_i-S_k=S_j-S_l$, which are independent sum differences. Hence a contribution equal to

$$\sum_{1\le l<j\le n}\sum_{1\le k\le i<l}\gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_i-S_k=S_j-S_l\big\}.$$

— if $a+b=-b-c$, then $a=-c-2b$ and $S_i-S_k=-(S_j-S_l)-2(S_l-S_i)$, which are independent sum differences. Hence a contribution equal to

$$\sum_{1\le l<j\le n}\sum_{1\le k\le i\le l}\gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_i-S_k=-(S_j-S_l)-2(S_l-S_i)\big\}.$$

iii) $(1\le k\le i<j\le l\le n)$ Write $a=i-k$, $b=j-i$, $c=l-j$. The equation $S_j-S_i=\pm(S_l-S_k)$ corresponds to $a+b+c=\pm b$.
— if $a+b+c=b$, then $a+c=0$ and $(S_i-S_k)+(S_l-S_j)=0$ where S_i-S_k and S_l-S_j are independent sum differences. Therefore, this produces a contribution equal to

$$\sum_{1\le j\le l\le n}\sum_{1\le k\le i<j}\gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{(S_i-S_k)>S_l-S_j)\big\}.$$

— if $a+b+c=-b$, then $a=-c-2b$ or else $S_i - S_k = -(S_l - S_j) - 2(S_j - S_i)$ where $S_i - S_k$, $S_l - S_j$ and $S_j - S_i$ are independent sum differences. This produces a contribution equal to

$$\sum_{1\le j\le l\le n}\ \sum_{1\le k\le i<j} \gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_i - S_k = -(S_l - S_j) - 2(S_j - S_i)\big\}.$$

iv) $(1\le i<k<l\le j\le n)$ Write $a=k-i$, $b=l-k$, $c=j-l$. The equation $S_j - S_i = \pm(S_l - S_k)$ again corresponds to $a+b+c=\pm b$.

— if $a+b+c=b$, then $a+c=0$ and $(S_k - S_i)+(S_j - S_l)=0$ where $S_k - S_i$, $S_j - S_l$ are independent sum differences. This produces a contribution equal to

$$\sum_{1\le l\le j\le n}\ \sum_{1\le i<k<l} \gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{(S_k - S_i)+(S_j - S_l)=0\big\}.$$

— if $a+b+c=-b$, then $a=-c-2b$ and $S_k - S_i = -(S_j - S_l) - 2(S_l - S_k)$ where $S_k - S_i$, $S_j - S_l$ and $S_l - S_k$ are independent sum differences. This produces a contribution equal to

$$\sum_{1\le l\le j\le n}\ \sum_{1\le i<k<l} \gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_k - S_i = -(S_j - S_l) - 2(S_l - S_k)\big\}.$$

v) $(1\le i<k<j<l\le n)$ Write $a=k-i$, $b=j-k$, $c=l-j$. The equation $S_j - S_i = \pm(S_l - S_k)$ corresponds here to $a+b=\pm(b+c)$.

— if $a+b=b+c$, then $a=c$ and $S_k - S_i = S_l - S_j$ where $S_k - S_i$, $S_l - S_j$ are independent sum differences. This produces a contribution equal to

$$\sum_{1\le j<l\le n}\ \sum_{1\le i<k\le j} \gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_k - S_i = S_l - S_j\big\}.$$

— if $a+b=-b-c$, then $a=-2b-c$ and $S_k - S_i = -(S_l - S_j) - 2(S_j - S_k)$ where $S_k - S_i$, $S_l - S_j$ and $S_j - S_k$ are independent sum differences. This produces a contribution equal to

$$\sum_{1\le j<l\le n}\ \sum_{1\le i<k\le j} \gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_k - S_i = -(S_l - S_j) - 2(S_j - S_k)\big\}.$$

vi) $(1\le i<j\le k<l\le n)$ The sum differences $S_j - S_i$ and $S_l - S_k$ are independent, therefore in this case we find a contribution

$$\sum_{1\le i<j\le n}\ \sum_{1\le k<l\le i} \gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_j - S_i = \pm(S_l - S_k)\big\}.$$

Summarizing the above estimates, only two types of sums appear:

$$\sum_{1\le i\le k<l\le j\le n} \gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_k - S_i = \pm(S_j - S_l)\big\}. \tag{$S1$}$$

and

$$\sum_{1\le i\le k<l\le j\le n} \gamma_i\,\gamma_j\,\gamma_k\,\gamma_l\,\mathbf{P}\big\{S_k - S_i = (S_j - S_l) - 2(S_l - S_k)\big\} \tag{$S2$}$$

The proof is completed now by counting the number of occurences of these sums, and using (4.7). ■

We shall deduce from Proposition 4.1 a more explicit estimate of the fourth moment. We will use the following transform. Let $\gamma = \{\gamma_n, n \ge 1\}$ be a bounded sequence of non negative reals. Put for any $z \in \mathbf{Z}$

$$\gamma_h^{[z]} = \sum_{u\ge h} \gamma_u \mathbf{P}\{S_{u-h} = z\}.$$

By the transience assumption, these quantities are well defined since $\sum_{u\ge 0}\mathbf{P}\{S_u = z\} \le G(0,0)$ for any $z \in \mathbf{Z}$. In particular, if γ is nonincreasing, we get from the above equality:

$$\gamma_h^{[z]} \le G(0,0)\gamma_{z+h}.$$

In the case of the Bernoulli random walk, this is however read directly. As $\mathbf{P}\{S_{u-h} = z\} = 0$ if $z \le 0$ or $z > u-h$, one has

$$\begin{aligned}\gamma_h^{[z]} &= \sum_{u\ge z+h} \gamma_u \mathbf{P}\{S_{u-h} = z\} \overset{(u=v+z+h)}{=} \sum_{v=0}^{\infty} \gamma_{v+z+h}\mathbf{P}\{S_{v+z} = z\}\\ &= \sum_{v=0}^{\infty} \gamma_{v+z+h} 2^{-(v+z)} C_{v+z}^{z}.\end{aligned}$$

Using now the formula $\sum_{v=0}^{\infty} C_{v+z}^{z} x^v = \frac{1}{(1-x)^{z+1}}$ valid for $|x| < 1$, gives the relation

$$\sum_{v=0}^{\infty} 2^{-(v+z)} C_{v+z}^{z} = 2$$

for any $z \ge 0$.

Proposition 4.2. *Assume that* $\mathbf{P}\{X \ge 0\} = 1$. *Let* $a = \{a_k, k \ge 1\}$ *and* $c = \{c_k, k \ge 1\}$ *be two sequences of reals such that* $|a_k| \le c_k$ *for any* k *and* c *is nonincreasing. Then*

$$\mathbf{E}\int_{\mathbf{T}}\Big|\sum_{k=m}^{n+m}a_k\,e^{2\imath\pi\alpha S_k}\Big|^4d\alpha$$
$$\le 4G(0,0)\Big(\sum_{k=m}^{n+m}c_k^2\Big)+48\Big\{\sum_{l=m}^{m+n}c_l^2(\sum_{m\le i\le l}c_i)^2+(\sum_{i=m}^{m+n}c_i)^2\sum_{l\ge m+n}c_l^2\Big\}.$$

Corollary 4.1. *Assume $\mathbf{P}(X\ge 0)=1$. Then the series $\sum_{k=1}^{\infty}a_k\,e^{2\imath\pi\alpha S_k}$ converges in $L^4(\mathbf{P}\times\lambda)$, provided that the series*

$$\sum_{l\ge 1}c_l^2(\sum_{1\le i\le l}c_i)^2$$

converges. In particular the series $\sum_{k=1}^{\infty}k^{-a}\,e^{2\imath\pi\alpha S_k}$ converges in $L^4(\mathbf{P}\times\lambda)$ for any $a>3/4$.

Proof of Proposition 4.2. By Proposition 4.1

$$\mathbf{E}\int_{\mathbf{T}}\Big|\sum_{k=m}^{n+m}a_k\,e^{2\imath\pi\alpha S_k}\Big|^4d\alpha\le 4G(0,0)\Big(\sum_{k=m}^{n+m}c_k^2\Big)+6((S1)+(S2)),$$

where

$$(S1)=\sum_{m\le i\le k<l\le j\le m+n}c_ic_jc_kc_l\mathbf{P}\big\{S_k-S_i=\pm(S_j-S_l)\big\}$$
$$(S2)=\sum_{m\le i\le k<l\le j\le m+n}c_ic_jc_kc_l\mathbf{P}\big\{S_k-S_i=(S_j-S_l)-2(S_l-S_k)\big\}$$

Consider first the sums of type $(S1)$, the others will be in turn treated similarly. Write

$$\sum_{m\le i\le k<l\le j\le m+n}c_ic_jc_kc_l\mathbf{P}\big\{S_k-S_i=S_j-S_l\big\}=$$
$$=\sum_{z\in\mathbf{Z}}\sum_{m\le i\le k\le m+n}c_ic_k\mathbf{P}\big\{S_k-S_i=z\big\}\sum_{k<l\le j\le m+n}c_jc_l\mathbf{P}\big\{S_j-S_l=z\big\}$$
$$=\sum_{z\in\mathbf{Z}}\sum_{m\le i\le k\le m+n}c_ic_k\mathbf{P}\big\{S_{k-i}=z\big\}\sum_{k<l\le j\le m+n}c_jc_l\mathbf{P}\big\{S_{j-l}=z\big\}.$$

As

$$\sum_{k<l\le j\le m+n}c_jc_l\,\mathbf{P}\big\{S_{j-l}=z\big\}\le\sum_{m\le l\le m+n}c_l\Big(\sum_{j\ge l}c_j\mathbf{P}\big\{S_{j-l}=z\big\}\Big)=\sum_{m\le l\le m+n}c_lc_l^{[z]},$$

we get by putting this into the previous relation

$$\sum_{m\le i\le k<l\le j\le m+n} c_i c_j c_k c_l\,\mathbf{P}\big\{S_k-S_i=S_j-S_l\big\}\le$$

$$\sum_{z\in\mathbf{Z}}\Big(\sum_{l=m}^{m+n} c_l c_l^{[z]}\Big)\sum_{m\le i\le m+n} c_i\Big(\sum_{k\ge i} c_k\mathbf{P}\big\{S_{k-i}=z\big\}\Big)\le\sum_{z\in\mathbf{Z}}\Big(\sum_{l=m}^{m+n} c_l c_l^{[z]}\Big)\sum_{i=m}^{m+n} c_i c_i^{[z]}$$
$$\le\sum_{m\le i,l\le m+n} c_l c_i\sum_{z\in\mathbf{Z}} c_l^{[z]} c_i^{[z]}.$$

The sums related to the factor $\mathbf{P}\big\{S_k-S_i=-(S_j-S_l)\big\}$ are treated similarly; the latter probability being not 0 only if $\mathbf{P}\big\{S_k-S_i=0\big\}=\mathbf{P}\big\{S_j-S_l=0\big\}$, and its value is then $\mathbf{P}\big\{S_k-S_i=0\big\}\mathbf{P}\big\{S_j-S_l=0\big\}$. Thus

$$\begin{aligned}
&\sum_{m\le i\le k<l\le j\le m+n} c_i c_j c_k c_l\mathbf{P}\big\{S_k-S_i=-(S_j-S_l)\big\}\\
&=\sum_{m\le i\le k\le m+n} c_i c_k\mathbf{P}\big\{S_k-S_i=0\big\}\sum_{k<l\le j\le m+n} c_l c_j\mathbf{P}\big\{S_j-S_l=0\big\}\\
&\le\sum_{m\le i\le k\le m+n} c_i c_k\mathbf{P}\big\{S_k-S_i=0\big\}\sum_{k<l\le m+n} c_l\sum_{j\ge l} c_j\mathbf{P}\big\{S_j-S_l=0\big\}\\
&\le\sum_{m\le i\le k\le m+n} c_i c_k\mathbf{P}\big\{S_k-S_i=0\big\}\Big(\sum_{m\le l\le m+n} c_l c_l^{[0]}\Big)\\
&\le\sum_{i=m}^{m+n} c_i\Big(\sum_{k\ge i} c_k\mathbf{P}\big\{S_k-S_i=0\big\}\Big)\Big(\sum_{l=m}^{m+n} c_l c_l^{[0]}\Big)\le\sum_{i=m}^{m+n} c_i c_i^{[0]}\Big(\sum_{l=m}^{m+n} c_l c_l^{[0]}\Big)\\
&\le\Big(\sum_{i=m}^{m+n} c_i^2\Big)^2.
\end{aligned}$$

Now, consider the sums of type ($S2$):

$$\begin{aligned}
&\sum_{m\le i\le k<l\le j\le m+n} c_i c_j c_k c_l\,\mathbf{P}\big\{S_k-S_i=(S_j-S_l)-2(S_l-S_k)\big\}\\
&=\sum_{\substack{z_1\in\mathbf{Z}\\ z_2\in\mathbf{Z}}}\ \sum_{m\le i\le k<l\le j\le m+n} c_i\, c_j\, c_k\, c_l\mathbf{P}\big\{S_{k-i}=z_1\big\}\mathbf{P}\big\{S_{l-k}=z_2\big\}\mathbf{P}\big\{S_{j-l}=z_1+2z_2\big\}\\
&\le\sum_{\substack{z_1\in\mathbf{Z}\\ z_2\in\mathbf{Z}}}\sum_{i=m}^{n+m}|c_i|\\
&\times\Big\{\sum_{k=i}^{m+n} c_k\mathbf{P}\big\{S_{k-i}=z_1\big\}\sum_{l=k}^{m+n} c_l\Big(\sum_{j\ge l}|c_j|\mathbf{P}\big\{S_{j-l}=z_1+2z_2\big\}\Big)\mathbf{P}\big\{S_{l-k}=z_2\big\}\Big)\Big\}
\end{aligned}$$

$$\leq \sum_{\substack{z_1\in\mathbf{Z}\\ z_2\in\mathbf{Z}}} \sum_{i=m}^{n+m} c_i \Big\{ \sum_{i\leq k\leq n+m} c_k \mathbf{P}\big\{S_{k-i}=z_1\big\} \sum_{k\leq l\leq n+m} c_l c_l^{[z_1+2z_2]} \mathbf{P}\big\{S_{l-k}=z_2\big\}\Big)\Big\}.$$

Consider on $L^2(\mathbf{T})$, the operator U defined for $h \sim \sum_{z\in\mathbf{N}} h_z e_z$ by $Uh \sim \sum_{z\in\mathbf{N}} h_{z+1} e_z$. Let $g=\sum_{k=1}^{\infty} a_k e_k$. It follows that

$$\sum_{i,l=m}^{n+m} c_l c_i \sum_{z\in\mathbf{N}} c_l^{[z]} c_i^{[z]} \leq 4 \sum_{i,l=m}^{n+m} \sum_{z\in\mathbf{N}} c_l c_i c_{l+z} c_{i+z} = 4 \sum_{i,l=m}^{n+m} c_l c_i \langle U^l g, U^i g\rangle = 4 \Big\| \sum_{i=m}^{n+m} c_i U^i g \Big\|^2$$

and

$$\sum_{\substack{z_1\in\mathbf{Z}\\ z_2\in\mathbf{Z}}} \sum_{i=m}^{n+m} c_i \Big\{ \sum_{i\leq k\leq m+n} c_k \mathbf{P}\big\{S_{k-i}=z_1\big\} \sum_{k<l\leq m+n} c_l c_l^{[z_1+2z_2]} \mathbf{P}\big\{S_{l-k}=z_2\big\}\Big\}$$

$$\leq 2 \sum_{\substack{z_1\in\mathbf{Z}\\ z_2\in\mathbf{Z}}} \sum_{i=m}^{n+m} c_i \Big\{ \sum_{i\leq k\leq m+n} c_k \mathbf{P}\big\{S_{k-i}=z_1\big\} \sum_{k<l\leq m+n} c_l c_{l+z_1+2z_2} \mathbf{P}\big\{S_{l-k}=z_2\big\}\Big\}$$

$$\leq 2 \sum_{\substack{z_1\in\mathbf{Z}\\ z_2\in\mathbf{Z}}} \sum_{i=m}^{n+m} c_i \Big\{ \sum_{i\leq k\leq m+n} c_k \mathbf{P}\big\{S_{k-i}=z_1\big\} \sum_{l\geq k+z_2} c_l c_{l+z_1+2z_2} \mathbf{P}\big\{S_{l-k}=z_2\big\}\Big\}$$

$$\leq 4 \sum_{\substack{z_1\in\mathbf{N}\\ z_2\in\mathbf{N}}} \sum_{i=m}^{n+m} c_i \Big\{ \sum_{i+z_1\leq k\leq m+n} c_k \mathbf{P}\big\{S_{k-i}=z_1\big\} c_{k+z_2} c_{k+z_1+3z_2}\Big\}$$

$$\leq 4 \sum_{i=m}^{n+m} c_i \sum_{k=m}^{n+m} c_k \Big\{ \sum_{z_1\leq k-i} \mathbf{P}\big\{S_{k-i}=z_1\big\}\Big\} \sum_{z_2\in\mathbf{N}} c_{k+z_2} c_{i+z_2}$$

$$= 4 \sum_{i,k=m}^{n+m} c_i c_k \langle U^i g, U^k g\rangle = 4 \Big\| \sum_{m\leq i\leq m+n} c_i U^i g \Big\|^2.$$

Consequently,

$$\mathbf{E} \int_{\mathbf{T}} \Big| \sum_{k=m}^{n+m} a_k\, e^{2\imath\pi\alpha S_k} \Big|^4 d\alpha \leq 4G(0,0) \Big(\sum_{k=m}^{n+m} c_k^2 \Big) + 48 \Big\{ \Big(\sum_{k=m}^{n+m} c_k^2 \Big)^2 + \Big\| \sum_{m\leq i\leq m+n} c_i U^i g \Big\|^2 \Big\}.$$

As

$$\sum_{i=m}^{m+n} c_i U^i g = \sum_{i=m}^{m+n} \sum_{l\geq i} c_i c_l e_l = \sum_{l\geq m} e_l c_l \Big(\sum_{i=m}^{(m+n)\wedge l} c_i \Big)$$

$$= \sum_{l=m}^{m+n} e_l c_l \Big(\sum_{m\leq i\leq l} c_i \Big) + \Big(\sum_{m\leq i\leq m+n} c_i \Big) \sum_{l\geq m+n} e_l c_l,$$

one has

$$\Big\|\sum_{i=m}^{m+n} c_i U^i g\Big\|^2=\sum_{l=m}^{m+n} c_l^2\big(\sum_{m\le i\le l} c_i\big)^2+\big(\sum_{i=m}^{m+n} c_i\big)^2\sum_{l\ge m+n} c_l^2.$$

Hence

$$\mathbf{E}\int_{\mathbf{T}}\Big|\sum_{k=m}^{n+m} a_k\, e^{2\imath\pi\alpha S_k}\Big|^4 d\alpha$$
$$\le 4G(0,0)\Big(\sum_{k=m}^{n+m} c_k^2\Big)+48\Big\{\sum_{l=m}^{m+n} c_l^2\big(\sum_{m\le i\le l} c_i\big)^2+\big(\sum_{i=m}^{m+n} c_i\big)^2\sum_{l\ge m+n} c_l^2\Big\}.$$

■

We now turn to the study of convergence of $\sum_{k=1}^{\infty} c_k f(S_k x)$ for general $f\in L^2(\mathbf{T})$, $\int_{\mathbf{T}} f(t)dt=0$. In the case when the distribution of X_1 is absolutely continuous, the exact analogue of Theorem 4.1 holds, namely we have

Theorem 4.2. *Let X_1 have a bounded density concentrated on a finite interval. Let $f\in Lip_\alpha(\mathbf{T})$ for some $\alpha>0$ with $\int_{\mathbf{T}} f(t)dt=0$ and let $\mathbf{c}\in\ell^2$. Then for any fixed $x\neq 0$, $\sum_{k=1}^{\infty} c_k f(S_k x)$ converges with probability 1. Consequently, for almost every $\omega\in\Omega$, $\sum_{k=1}^{\infty} c_k f(S_k(\omega)x)$ converges for almost every x.*

In the case when X_1 has a lattice distribution, the situation is more complicated. The following theorem describes the Bernoulli case.

Theorem 4.3. *Let X_1 take the values 0 and 1 with probability $1/2$ each. Let $f\in L^2(\mathbf{T})$ with $\int_{\mathbf{T}} f(t)dt=0$ have Fourier series*

$$f\sim\sum_{k=1}^{\infty}(a_k\cos 2\pi kx+b_k\sin 2\pi kx)$$

and assume that the Dirichlet series $\sum_n a_n n^{-s}$, $\sum_n b_n n^{-s}$ are regular and bounded in the half-plane $\Re(s)>0$. Let

$$\tau_k(\mathbf{c}):=\sup_{\substack{L\ge k\\ u\le\gamma\log k}}\Big|\sum_{\ell=L}^{L+u} c_\ell\Big|.$$

Then the series $\sum_\ell c_\ell f(S_\ell(\omega)x)$ converges in mean $\mathbf{P}$-almost surely provided $\mathbf{c}\in\ell^2$ and $\tau_k(\mathbf{c})=o(1)$.

For the proof of Theorem 4.2, let

$$\psi(x) = \sup_{0\le x\le 1} |\mathbf{P}(S_n \le x) - x|$$

and note that by Theorem 1 of [Sc1] we have

$$\psi(n) \le Ce^{-\lambda n} \qquad (n \ge 1) \tag{4.8}$$

for some constants $C > 0$, $\lambda > 0$.

Lemma 4.3. *Let $k_0 < k_1 < \ldots < k_r$ be positive integers and let U be a uniform r.v. independent of the sequence $X_1, X_2, \ldots$. Then there exists a r.v. Δ with $|\Delta| \le \psi(k_1 - k_0)$ such that Δ is a function of U and $X_{k_0+1}, \ldots X_{k_1}$ and the vector $(S_{k_1} - \Delta, \ldots, S_{k_r} - \Delta)$ has uniform coordinates and is independent of $(X_1, \ldots, X_{k_0})$.*

This lemma is implicit in [Sc2] and can be obtained along the following lines. Let $Y = S_{k_1} - S_{k_0}$, then $|P(Y \le t) - t| \le \psi(k_1 - k_0)$ for all t and thus by Lemma 3 of [Sc2] there exists a uniform r.v. Y^*, which is a function of U and Y such that $|Y - Y^*| \le \psi(k_1 - k_0)$. Let $\Delta = Y - Y^*$, then

$$\begin{aligned}(S_{k_1} - \Delta, \ldots, S_{k_r} - \Delta) &= (S_{k_1} - Y + Y^*, \ldots, S_{k_r} - Y + Y^*)\\ &= (S_{k_1} - Y, \ldots, S_{k_r} - Y) + Y^* = Z + Y^*.\end{aligned}$$

Here the vector Z is obviously independent of $Y = X_{k_0+1} + \ldots + X_{k_1}$ and thus also of the uniform r.v. Y^*, which is a function of Y and U. Thus adding Y^* to the components of Z, we get a vector whose components are uniform (see Lemma 1 of [Sc2]), the independence of $Z + Y^*$ and $(X_1, \ldots, X_{k_0})$ follows also from Lemma 1 of [Sc2].

Proof of Theorem 4.2. We prove the statement for $x = 1$. Let $f \in \mathrm{Lip}_\alpha(\mathbf{T})$, $\alpha > 0$ with $\int_{\mathbf{T}} f(t)dt = 0$. By (4.8) we have

$$|\mathbf{E}f(S_n) - \int_0^1 f(x)dx| \le C_1 e^{-\lambda_1 n} \qquad (n \ge 1). \tag{4.9}$$

Set $\xi_k = f(S_k) - \mathbf{E}f(S_k)$. By (4.9), for any bounded sequence (c_k) the series $\sum c_k f(S_k)$ and $\sum c_k \xi_k$ are equiconvergent, and thus it suffices to prove that $\sum c_k \xi_k$ is a.s. convergent provided $(c_k) \in \ell_2$. In view of Lemma 4.1, this will follow if we show that

$$\mathbf{E}\Big(\sum_{k=1}^N c_k \xi_k\Big)^4 \le K\Big(\sum_{k=1}^N c_k^2\Big)^2 \tag{4.10}$$

for any real $(c_k)_{k=1}^N$ with a suitable constant K. We claim that

$$|\mathbf{E}(\xi_k\xi_l\xi_m\xi_n)| \le Ae^{-C(|l-k|+|n-m|)} \qquad (k \le l \le m \le n). \tag{4.11}$$

By Lemma 4.3, there exists a r.v. Δ with $|\Delta| \le \psi(l-k)$ such that the vector

$$(S_l - \Delta, S_m - \Delta, S_n - \Delta) =: (S_l', S_m', S_n')$$

is independent of S_k and thus the r.v.'s

$$X = f(S_k) - \mathbf{E}f(S_k) \quad \text{and} \quad Y = (f(S_l') - \mathbf{E}f(S_l))(f(S_m') - \mathbf{E}f(S_m))(f(S_n') - \mathbf{E}f(S_n))$$

are independent. Since $\mathbf{E}(X) = 0$, it follows that

$$\begin{aligned}\mathbf{E}\Big((f(S_k) - \mathbf{E}f(S_k))(f(S_l') - \mathbf{E}f(S_l))(f(S_m') - \mathbf{E}f(S_m))(f(S_n') - \mathbf{E}f(S_n)\Big) \\ = \mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y) = 0.\end{aligned} \tag{4.12}$$

In view of $|\Delta| \le \psi(l-k)$ and the boundedness and Lipschitz property of f it follows that replacing S_l', S_m', S_n' by S_l, S_m, S_n in the first expectation in (4.12) results in a change of at most $C\psi(l-k)$ of the expectation and thus we see that the expectation in (4.11) is at most $C\psi(l-k)$. A similar argument shows that the left hand side of (4.11) is at most $C\psi(n-m)$, and thus the left hand side of (4.11) is also bounded by $C(\psi(l-k)\psi(n-m))^{1/2}$, which proves (4.11) in view of (4.8).

It is now easy to verify (4.10). By (4.11), the left hand side of (4.10) is bounded by

$$\Big|\sum_{1\le k\le l\le m\le n\le N} c_k c_l c_m c_n e^{-C(|l-k|+|n-m|)}\Big|. \tag{4.13}$$

Summing (4.13) first for those indices (k, l, m, n) for which $l - k = r$ and $n - m = s$ are fixed, we get by Cauchy's inequality

$$\begin{aligned}&\Big|\sum_{1\le k,k+r,m,m+s\le N} c_k c_{k+r} c_m c_{m+s} e^{-C(r+s)}\Big| \\ &\le e^{-C(r+s)}\Big|\sum_{1\le k\le k+r\le N} c_k c_{k+r}\Big|\Big|\sum_{1\le m\le m+s\le N} c_m c_{m+s}\Big| \\ &\le e^{-C(r+s)}\Big(\sum_{1\le k\le N} c_k^2\Big)^{1/2}\Big(\sum_{1\le k+r\le N} c_{k+r}^2\Big)^{1/2}\Big(\sum_{1\le m\le N} c_m^2\Big)^{1/2}\Big(\sum_{1\le m+s\le N} c_{m+s}^2\Big)^{1/2} \\ &\le e^{-C(r+s)}\Big(\sum_{1\le j\le N} c_j^2\Big)^2.\end{aligned}$$

Now summing for r and s we get (4.10). ∎

Proof of Theorem 4.3. Set $\delta_0 = 0$, $\Delta_0 = 0$ and for any integer $k \geq 1$,

$$\begin{aligned}\delta_k &= \inf\big\{n \geq 1 \ : \ X_{n+\delta_1+\ldots+\delta_{k-1}} = 1\big\},\\ \Delta_k &= \delta_1 + \ldots + \delta_k.\end{aligned}$$

Then, the random variables δ_k are iid and $\mathbf{P}\{\delta_k = m\} = 2^{-m}$ for all k and m. Further $\Delta_k = \inf\{\ell \geq 1 : S_l = k\}$. Let $\omega \in \Omega$, then

$$\sum_{\ell<\Delta_{k+1}(\omega)} c_\ell f(S_\ell(\omega)x) = \sum_{h=0}^{k} Y_h f(hx) \qquad\qquad (k = 0, 1\ldots)$$

where we put

$$Y_k = \sum_{\Delta_k\leq\ell<\Delta_{k+1}} c_\ell, \qquad\qquad (k = 0, 1\ldots) \tag{4.14}$$

and for $\Delta_k \leq L < \Delta_{k+1}$,

$$\sum_{\ell<L} c_\ell f(S_\ell(\omega)x) = \sum_{h=0}^{k-1} Y_h f(hx) + \sum_{\Delta_k\leq\ell\leq L} c_\ell f(kx) \qquad\qquad (k = 0, 1\ldots) \tag{4.15}$$

We first work the sums

$$\Theta_k(\mathbf{c}, x) := \sum_{h=0}^{k} Y_h f(hx) \qquad\qquad (k = 0, 1\ldots) \tag{4.16}$$

Let $y = \{Y_k, k \geq 0\}$. It follows from Theorem A in Section 1 that if f is such that the Dirichlet series (1.4) are regular and bounded in the half-plane $\Re(s) > 0$, the sequence $\{\Theta_k(\mathbf{c}, x), k \geq 0\}$ converges in mean, $\mathbf{P}$-almost surely provided that $\mathbf{P}\{y \in \ell^2\} = 1$. We shall prove the following Lemma.

Lemma 4.4. *For any $\mathbf{c} \in \ell^2$, the series $\sum_{k=0}^{\infty} \mathbf{E}Y_k^2$ converges.*

Proof. We introduce the discrete Laplace transform of the sequence $\{|c_\ell|, \ell \geq 0\}$. For any $\ell \geq 0$, we put:

$$b_\ell = \beta(|c_\ell|) := \sum_{k=0}^{\infty} |c_k| 2^{-|k-\ell|}, \tag{4.17}$$

and put $\mathbf{b} = \{b_\ell, \ell \geq 0\}$. From the definition it is clear that it is already defined for bounded sequences, in fact even for sequences growing less than geometrically. Clearly, $|c_\ell| \leq b_\ell$, $1/2 \leq \frac{b_{\ell+1}}{b_\ell} \leq 2$, and $\sum_{\ell=0}^{\infty} b_\ell \leq 3 \sum_{\ell=0}^{\infty} |c_\ell| \leq \infty$. Further, by convexity,

$$\beta(|c_\ell|)^2 \leq 3\beta(|c_\ell|^2). \tag{4.18}$$

Indeed, let $\theta := \sum_{k=0}^\infty 2^{-|k-\ell|}$. Then $\theta \le 3$ and by convexity

$$\beta(|c_\ell|)^2 = \theta^2 \big(\sum_{k=0}^\infty |c_k| \frac{2^{-|k-\ell|}}{\theta} \big)^2 \le \theta^2 \sum_{k=0}^\infty |c_k|^2 \frac{2^{-|k-\ell|}}{\theta} = \theta b(|c_\ell|^2) \le 3b(|c_\ell|^2).$$

A useful consequence of (4.18) is thus the implication: $\mathbf{c} \in \ell^2 \Rightarrow \mathbf{b} \in \ell^2$, since $\sum_{\ell=0}^\infty \beta(|c_\ell|)^2 \le 3 \sum_{\ell=0}^\infty \beta(|c_\ell|)^2 \le 9 \sum_{\ell=0}^\infty |c_\ell|^2$.

Computing $\mathbf{E} Y_k^2$ we get,

$$\begin{aligned}
\mathbf{E} Y_k^2 &= \sum_{m=k}^\infty \sum_{n=1}^\infty \mathbf{P}\big\{\Delta_k = m, \delta_{k+1} = n\big\} \Big(\sum_{m \le \ell < m+n} c_\ell \Big)^2, \\
&= \sum_{m=k}^\infty \sum_{n=1}^\infty \mathbf{P}\big\{\Delta_k = m, \delta_{k+1} = n\big\} \sum_{m \le \ell < m+n} c_\ell^2 \\
&+ 2 \sum_{m=k}^\infty \sum_{n=1}^\infty \mathbf{P}\big\{\Delta_k = m, \delta_{k+1} = n\big\} \Big(\sum_{m \le \ell < \lambda < m+n} c_\ell c_\lambda \Big), \\
&:= S_k^{(1)} + S_k^{(2)}.
\end{aligned}$$

Thereby,

$$\begin{aligned}
S_k^{(2)} &= 2 \sum_{m=k}^\infty \sum_{n=1}^\infty \mathbf{P}\big\{\Delta_k = m, \delta_{k+1} = n\big\} \Big(\sum_{m \le \ell < \lambda < m+n} c_\ell c_\lambda \Big), \\
&= 2 \sum_{k \le \ell < \lambda < \infty} c_\ell c_\lambda \sum_{m \le \ell} \mathbf{P}\big\{\Delta_k = m\big\} \big(\sum_{n > \lambda - m} 2^{-n} \big) \\
&= 2 \sum_{k \le \ell < \lambda < \infty} c_\ell c_\lambda \sum_{m \le \ell} \mathbf{P}\big\{\Delta_k = m\big\} 2^{-(\lambda - m)}.
\end{aligned}$$

Note that the random walk $\Delta = \{\Delta_k, k \ge 0\}$ is transient. Thus the Green function $G(0,x) = \sum_{k=0}^\infty \mathbf{P}\{\Delta_k = x\}$ is finite [S] for every $x \in \mathbf{Z}$. Moreover ([Br] Proposition 3.39 p. 56 and Theorems 3.33, 3.34 p. 54),

$$L := \sup_{x \ge 0} G(0,x) < \infty.$$

Consequently,

$$\sum_{k=0}^\infty S_k^{(2)} = 2 \sum_{k=0}^\infty \sum_{k \le \ell < \lambda < \infty} c_\ell c_\lambda \sum_{m \le \ell} \mathbf{P}\big\{\Delta_k = m\big\} 2^{-(\lambda - m)}$$

$$\leq 2 \sum_{0\leq \ell<\lambda<\infty} c_\ell c_\lambda \sum_{m\leq \ell} \sum_{k=0}^{\ell} \mathbf{P}\{\Delta_k = m\} 2^{-(\lambda-m)}$$
$$\leq 2 \sum_{0\leq \ell<\lambda<\infty} c_\ell c_\lambda \sum_{m\leq \ell} G(0,m) 2^{-(\lambda-m)} \leq 2L \sum_{0\leq \ell<\lambda<\infty} c_\ell c_\lambda \sum_{m\leq \ell} 2^{-(\lambda-m)}$$
$$\leq 4L \sum_{0\leq \ell<\lambda<\infty} c_\ell c_\lambda 2^{-(\lambda-\ell)} \leq 4L \sum_{0\leq \ell<\infty} c_\ell \sum_{\ell<\lambda<\infty} c_\lambda 2^{-(\lambda-\ell)} \leq 4L \sum_{0\leq \ell<\infty} b_\ell^2.$$

Therefore

$$\sum_{k=0}^{\infty} S_k^{(2)} \leq 4L \sum_{0\leq \ell<\infty} b_\ell^2.$$

Now,

$$S_k^{(1)} = \sum_{m=k}^{\infty} \sum_{n=1}^{\infty} \mathbf{P}\{\Delta_k = m, \delta_{k+1} = n\} \sum_{m\leq \ell<m+n} c_\ell^2,$$
$$= \sum_{\ell=k}^{\infty} c_\ell^2 \sum_{k\leq m\leq \ell} \mathbf{P}\{\Delta_k = m\} \Big(\sum_{n>\ell-m} 2^{-n} \Big)$$
$$= \sum_{\ell=k}^{\infty} c_\ell^2 \sum_{k\leq m\leq \ell} \mathbf{P}\{\Delta_k = m\} 2^{-(\ell-m)}.$$

Thus,

$$\sum_{k=0}^{\infty} S_k^{(1)} = \sum_{k=0}^{\infty} \sum_{\ell=k}^{\infty} c_\ell^2 \sum_{k\leq m\leq \ell} \mathbf{P}\{\Delta_k = m\} 2^{-(\ell-m)},$$
$$= \sum_{\ell=0}^{\infty} c_\ell^2 \sum_{k\leq \ell} \sum_{k\leq m\leq \ell} \mathbf{P}\{\Delta_k = m\} 2^{-(\ell-m)},$$
$$\leq 2L \sum_{\ell=0}^{\infty} c_\ell^2 \sum_{m\leq \ell} 2^{-(\ell-m)} \leq 4L \sum_{\ell=0}^{\infty} c_\ell^2.$$

Putting together the two last estimates gives

$$\sum_{k=0}^{\infty} \mathbf{E} Y_k^2 = \sum_{k=0}^{\infty} \big(S_k^{(1)} + S_k^{(2)} \big) \leq 16L \sum_{0\leq \ell<\infty} b_\ell^2.$$

Now, recall that $b_\ell = \beta(|c_\ell|)$. And by (4.18), $b_\ell^2 = \beta(|c_\ell|)^2 \leq 3\beta(|c_\ell|^2)$, so that $\mathbf{a} \in \ell_2$ implies $\mathbf{b} \in \ell_2$, whence we get the convergence of the series $\sum_{k=0}^{\infty} \mathbf{E} Y_k^2$. ■

The following intermediate result is a straightforward consequence of the preceding lemma.

Lemma 4.5. *Assume that f satisfies the assumptions of Theorem 4.3. Then for any $\mathbf{c} \in \ell^2$, the sequence $\{\Theta_k(\mathbf{c}, x), k \geq 0\}$ converges in mean, $\mathbf{P}$-almost surely.*

Now we pass to the study for $\Delta_k \le L < \Delta_{k+1}$, $k = 0, 1 \dots$ of the ratio

$$\Big(\sum_{\Delta_k \le \ell \le L} c_\ell\Big) f(kx).$$

By the strong law of large numbers $\Delta_k \sim k\mathbf{E}\delta_1$ almost surely and $\mathbf{E}\delta_1 = \sum_{m=1}^\infty m2^{-m}$. Further, by the Borel-Cantelli lemma

$$\mathbf{P}\{\delta_k \le \gamma \log k \quad \text{ultimately}\} = 1,$$

for some constant γ. It follows that with probability one

$$\max_{\Delta_k \le L < \Delta_{k+1}} \int_{\mathbf{T}} \Big|\sum_{\ell < L} c_\ell f(S_\ell(\omega)x) - \Theta_k(\mathbf{c}, x)\Big|^2 dx \le \tau_k^2(\mathbf{c})\|f\|_2^2, \qquad \text{ultimately}$$

and Theorem 4.3 is proved.

5. Discrepancy of random sequences $\{\mathbf{S_n x}\}$.

Given a sequence $\mathbf{s} = \{s_n, n \ge 1\}$ of real numbers, the discrepancy of $\mathbf{s}$ mod 1 is defined by

$$ND_N(\mathbf{s}) = \sup_{I \subset [0,1)} \Big|\sum_{\substack{n=1 \\ s_n \in I}}^N 1 - N|I|\Big|. \tag{5.1}$$

Clearly, $D_N(\mathbf{s})$ measures how far the distribution of $\mathbf{s}$ is from the uniform. In particular, $\mathbf{s}$ is unformly distributed mod 1 in the Weyl sense if and only if $D_N(\mathbf{s}) \to 0$ as $N \to \infty$. In this section we study the discrepancy of the sequence

$$\xi = \xi(\omega, x) = \big\{\{S_n(\omega)x\}, n \ge 1\big\},$$

where $\{u\}$ denotes the fractional part of of u, for almost all x and ω. As in the previous section, $S_n = \sum_{k=1}^n X_k$ is a random walk, where X_k are i.i.d. random variables. Letting $f_{a,b}(t) = I_{a,b}(t) - (b - a)$, the discrepancy of $\{S_n x\}$ can be written as

$$D_N = \frac{1}{N} \sup_{0 \le a < b \le 1} \Big|\sum_{k=1}^N f_{a,b}(S_k x)\Big|$$

which is closely related to the convergence problems studied in Section 4. As we will see, the behavior of D_N has a completely different character according as the distribution of X_1 is lattice or it has an absolutely continuous component.

Starting with the lattice case, a number of discrepancy estimates were proved in the recent work [We3]. In this case, the diophantine approximation properties of x naturally play a crucial role. In what follows, we will follow the approach in [We3] with significant simplifications due to the type of problem investigated. We first prove the following result:

Theorem 5.1. *Let $\mathcal{X} = \{X, X_i, i \geq 1\}$ be a sequence of independent, identically distributed, lattice random variables defined on some probablility space $(\Omega, \mathcal{A}, \mathbf{P})$. We assume that the random walk $S_n = X_1 + \ldots + X_n$, $n \geq 1$ is transient. Then, for any $\tau > 5/2$,*

$$D_N(\xi) \overset{a.s.}{=} \mathcal{O}\Big(N^{-1/2} \log^\tau N\Big). \tag{5.2}$$

To clarify the meaning of Theorem 5.1, recall that by classical results of Cassels [Cs] and Erdős and Koksma [EK], for any increasing sequence (n_k) of positive integers, the discrepancy of $\{n_k x\}$ is $O(N^{-1/2} \log^\tau N)$ for almost every x and for any $\tau > 5/2$. Of course, this implies Theorem 5.1 in the case $X > 0$ a.s. In the general transient case, n_k can be negative and $|n_k|$ is not necessarily increasing, but we have $|n_k| \to \infty$ a.s. and thus with probability one, every term of (n_k) is repeated only finitely many times. This is a situation similar to that in the results of Cassels [Cs] and Erdős and Koksma [EK], but one should observe that repetitions in a sequence of real numbers can change the discrepancy of the sequence drastically, even if we permit only finitely many repetitions of each term. The heuristic meaning of Theorem 5.1 is that repetitions in the sequence S_n are sufficiently limited so that the discrepancy behavior of $\{S_n x\}$ remains the same as in the strictly monotone case.

It is worth mentioning that the constant 5/2 in the theorems of Erdős, Cassels and Koksma has been improved to 3/2 by R.C. Baker [Ba]. Of course, this raises the question if Theorem 5.1 also holds with $\tau > 3/2$ instead of $\tau > 5/2$. In the remark after the proof of Theorem 5.2 we will show that the constant 5/2 can be improved to 7/4 if the characteristic function φ of X satisfies $|\varphi(t) - 1| \geq C|t|$ for $|t| \leq t_0$; this is satisfied e.g. if X has a finite, nonzero mean. The argument there can be easily generalized for other classes of random variables X. Whether Theorem 5.1 holds with $\tau > 3/2$ for all transient X remains open.

For the proof of Theorem 5.1, we need some lemmas. Put for any integers $N \geq 1$, $m \geq 0$:

$$\Theta_N(m, x) = \sum_{n=1}^{N} e^{2i\pi m S_n x}. \tag{5.3}$$

Lemma 5.1. *For any two integers $N \geq P \geq 1$, one has the following estimate:*

$$\mathbf{E}\int_{\mathbf{T}} \big|\Theta_N(m,x) - \Theta_P(m,x)\big|^2 dx \leq C_{\mathcal{X}}(N-P),$$

where the constant $C_{\mathcal{X}}$ depends on $\mathcal{X}$ only.

Proof. Since

$$\begin{aligned}
&\mathbf{E}\int_{\mathbf{T}} \big|\Theta_N(m,x) - \Theta_P(m,x)\big|^2 dx \\
&= \mathbf{E}\int_{\mathbf{T}} \sum_{P<k,\ell\leq N} e^{2i\pi m(S_k - S_\ell)x} dx = \sum_{P<k,\ell\leq N} \mathbf{P}\{S_k = S_\ell\},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{P<k,\ell\leq N} \mathbf{P}\{S_k = S_\ell\} \\
&= (N-P) + 2\sum_{P<k<\ell\leq N} \mathbf{P}\{S_{\ell-k} = 0\} \leq (N-P)\Big\{1 + 2\big(\sum_{\lambda\geq 1} \mathbf{P}\{S_\lambda = 0\}\big)\Big\},
\end{aligned}$$

the result follows from the transience assumption of the random walk. ■

Let $L : \mathbf{N} \to \mathbf{N}$ be increasing. Put for any positive integer n and $x \in \mathbf{T}$,

$$U_n(x) = \sum_{h=1}^{L(n)} \frac{1}{h}\big|\Theta_n(h,x)\big|. \tag{5.4}$$

Lemma 5.2. *For any two integers $n > \ell \geq 1$,*

$$\mathbf{E}\int_{\mathbf{T}} \big|U_n(x) - U_\ell(x)\big|^2 dx \leq C_{\mathcal{X}}\Big\{(n-\ell)\log^2 L(\ell) + n\log^2 \frac{L(n)}{L(\ell)}\Big\}. \tag{5.5}$$

Proof. Clearly,

$$U_\ell(x) - U_n(x) = \sum_{h=1}^{L(\ell)} \frac{1}{h}\big(|\Theta_\ell(h,x)| - |\Theta_n(h,x)|\big) - \sum_{h=L(\ell)+1}^{L(n)} \frac{1}{h}|\Theta_n(h,x)| := A - B.$$

By the Cauchy-Schwarz inequality and Lemma 5.1,

$$\mathbf{E}\int_{\mathbf{T}} A^2 d\lambda \leq \Big(\sum_{h=1}^{L(\ell)} \frac{1}{h}\Big)\Big(\sum_{h=1}^{L(\ell)} \frac{1}{h}\mathbf{E}\int_{\mathbf{T}} \big|\Theta_\ell(h,x) - \Theta_n(h,x)\big|^2 dx\Big) \leq C_{\mathcal{X}}(n-\ell)\log^2 L(\ell),$$

$$\mathbf{E}\int_{\mathbf{T}} B^2 d\lambda \leq \Big(\sum_{h=L(\ell)+1}^{L(n)} \frac{1}{h}\Big)\Big(\sum_{h=L(\ell)+1}^{L(n)} \frac{1}{h}\mathbf{E}\int_{\mathbf{T}} \big|\Theta_n(h,x)\big|^2 dx\Big) \leq C_{\mathcal{X}} n\log^2 \frac{L(n)}{L(\ell)}.$$

Lemma 5.2 thus follows. ■

Lemma 5.3. *Assume that $\ell \mapsto \log L(\ell)$ is concave. Then for any $\tau > 3/2$,*

$$U_n \overset{a.s.}{=} \mathcal{O}\Big(n^{1/2}(\log L(n)) \log^\tau n\Big). \tag{5.6}$$

Proof. The concavity assumption implies that for any $n > \ell \geq 1$ we have

$$\frac{\log L(n) - \log L(\ell)}{n-\ell} \leq \frac{\log L(n)}{n}.$$

Thus by Lemma 5.2

$$\begin{aligned} &\mathbf{E}\int_{\mathbf{T}} \big|U_n(x) - U_\ell(x)\big|^2 dx \\ &\leq C_{\mathcal{X}} \log L(n)\Big\{(n-\ell)\log L(\ell) + n \log \frac{L(n)}{L(\ell)}\Big\} \leq C_{\mathcal{X}}(n-\ell)\log^2 L(n). \end{aligned} \tag{5.7}$$

Hence,

$$\mathbf{E}\int_{\mathbf{T}} \big|U_n(x) - U_\ell(x)\big|^2 dx \leq C_{\mathcal{X}}(n-\ell)\log^2 L(n), \qquad \mathbf{E}\int_{\mathbf{T}} \big|U_n(x)\big|^2 dx \leq C_{\mathcal{X}} n \log^2 L(n).$$

Let $a > 1/2$. By the Chebysev inequality,

$$\mathbf{P}\times\lambda\big\{|U_{2^p}| > \big[2^p \log^2 L(2^p)\big]^{1/2} p^a\big\} \leq C_{\mathcal{X}} p^{-2a},$$

and thus the Borel-Cantelli Lemma yields

$$|U_{2^p}| \overset{a.s.}{=} \mathcal{O}(\big[2^p \log^2 L(2^p)\big]^{1/2} p^a)$$

Now, investigate the oscillation of U_n over the interval $[2^p, 2^{p+1})$. Put

$$U'_n = U_n / \big[\ 2^p \log^2 L(2^p)\big]^{1/2}.$$

Then

$$\mathbf{E}\big|U'_n - U'_\ell\big|^2 \leq C\,(\frac{n-\ell}{2^p}).$$

Applying Lemma 3.4 of [We2], gives

$$\big\| \sup_{2^p \leq n,m < 2^{p+1}} \big|U'_n - U'_\ell\big| \big\|_{2,\mathbf{P}\times\lambda} \leq C_{\mathcal{X}} p.$$

Let $\tau > 3/2$. By Tchebycheff inequality,

$$\mathbf{P}\big\{\sup_{2^p \leq n,m < 2^{p+1}} \big|U_n - U_l\big| > \big[2^p \log^2 L(2^p)\big]^{1/2} p^\tau\big\} \leq C\, p^{2-2\tau},$$

which implies by the Borel-Cantelli Lemma

$$\sup_{2^p \leq n,m < 2^{p+1}} \big|U_n - U_l\big| \overset{a.s.}{=} \mathcal{O}(\big[2^p \log^2 L(2^p)\big]^{1/2} p^\tau)$$

Combining our estimates easily gives the result. ∎

The next result is the classical Erdős-Turán inequality ([Har], Theorem 5.5, p. 129): there exists an absolute constant C such that for any positive integers L and N

$$ND_N(s) \le \frac{N}{L+1} + C\sum_{h=1}^{L} \frac{1}{h}\Big|\sum_{n=1}^{N} e^{2i\pi h s_n}\Big|.$$

We apply this inequality for $s = \xi$, $L = L(N)$, $n \le N$, and find

$$ND_N(\xi) \le N/L(N) + CU_N. \tag{5.8}$$

Applying Lemma 5.3 to (5.8) gives for any $\tau > 3/2$,

$$ND_N(\xi) \overset{a.s.}{=} \mathcal{O}\Big(N/L(N) + \big[N\log^2 L(N)\big]^{1/2}\log^\tau N\Big). \tag{5.9}$$

Choosing $L(N) = N$ establishes Theorem 5.1. ■

We pass now to another discrepancy result complementing Theorem 5.1.

Theorem 5.2. *Let $(X, X_i, i \ge 1)$ be an i.i.d. sequence with $\mathbf{E}\,|X| < \infty$ and characteristic function ϕ. Let $S_n = X_1 + \ldots + X_n$ and let $D_N(\alpha,\omega)$ denote the discrepancy of the sequence $\{S_k(\omega)\alpha\}_{k\le N}$. Let $L(N)$ be a nondecreasing function such that $N/L(N)$ is also nondecreasing and set*

$$G_N(\alpha) = \frac{N}{L(N)} + \sqrt{N}\sum_{h=1}^{L(N)} \frac{1}{h|1-\phi(h\alpha)|^{1/2}}.$$

Then for any fixed α

$$D_N(\alpha,\omega) = \mathcal{O}\Big(\frac{G_{2N}(\alpha)}{N}(\log N)^{1/4+\varepsilon}\Big) \qquad \textit{for a.e. } \omega.$$

Proof. Let $T_h(n,\omega,\alpha) = \sum_{k\le n} e^{2\pi i h S_k(\omega)\alpha}$. By the Erdős-Turán inequality we have for any $r \ge 1$

$$nD_n(\alpha,\omega) \le C\Big(\frac{n}{r} + \sum_{h=1}^{r}\frac{1}{h}|T_h(n,\omega,\alpha)|\Big)$$

and thus

$$nD_n(\alpha,\omega) \le C\Big(\frac{n}{L(n)} + \sum_{h=1}^{L(n)}\frac{1}{h}|T_h(n,\omega,\alpha)|\Big).$$

Consequently

$$\max_{1\le n\le 2^k} nD_n(\alpha,\omega) \le C\Big(\frac{2^k}{L(2^k)} + \sum_{h=1}^{L(2^k)} \frac{1}{h} \max_{1\le r\le 2^k} |T_h(r,\omega,\alpha)|\Big). \qquad (5.10)$$

By the fourth moment estimate in the first line of p. 364 of the paper of Blum and Cogburn [BC] we have

$$\mathbf{E}_\omega |T_h(n,\omega,\alpha)|^4 \le C\frac{1}{|1-\phi(h\alpha)|^2} n^2.$$

The same moment bound holds for the translated sums $\sum_{m+1\le k\le m+n} e^{2\pi i h S_k(\omega)\alpha}$ and thus applying Lemma 4.1 we get

$$\mathbf{E}_\omega \max_{1\le r\le 2^k} |T_h(r,\omega,\alpha)|^4 \le C\frac{1}{|1-\phi(h\alpha)|^2} 4^k$$

or equivalently

$$\|\max_{1\le r\le 2^k} |T_h(r,\omega,\alpha)\|_4 \le C\frac{1}{|1-\phi(h\alpha)|^{1/2}}$$

$$\|\max_{1\le r\le 4^k} |T_h(r,\omega,\alpha)\|_4 \le C\frac{1}{|1-\phi(h\alpha)|^{1/2}} 2^k.$$

Substituting this into (5.10) it follows that

$$\|\max_{1\le n\le 4^k} nD_n(\alpha,\omega)\|_4 \le C\Big(2^k + \sum_{h=1}^{2^k} \frac{1}{h|1-\phi(h\alpha)|^{1/2}} 2^k\Big) \le C2^k G_{2^k}(\alpha).$$

Thus

$$\mathbf{P}_\omega\{\max_{1\le n\le 4^k} nD_n(\alpha,\omega) \ge 2^k k^{1/4+\varepsilon} G_{2^k}(\alpha)\} \le \frac{\mathbf{E}_\omega\,(\max_{1\le n\le 4^k} nD_n(\alpha,\omega))^4}{16^k k^{1+4\varepsilon} G_{2^k}(\alpha)^4} \le Ck^{-(1+4\varepsilon)}.$$

Hence the theorem follows from the monotonicity of G_N and the Borel-Cantelli lemma. ■

Remarks. It is interesting to compare the bound obtained in Theorem 5.1 with the one obtained in [We3]. The two bounds are very similar: the only difference is that instead of the expression

$$\frac{1}{\sqrt{N}}\Big(\sum_{h=1}^{L(N)} \frac{1}{h|1-\phi(h\alpha)|}\Big)^{1/2} (\log N)^{3/2+\varepsilon} \qquad (A)$$

in the remainder term in [We3], the bound in Theorem 5.1 contains

$$\frac{1}{\sqrt{N}}\Big(\sum_{h=1}^{L(N)}\frac{1}{h|1-\phi(h\alpha)|^{1/2}}\Big)(\log N)^{1/4+\varepsilon}. \qquad (B)$$

The two expressions are incomparable: one can easily give examples when one is better than the other, and conversely. In the metric case (i.e. when we wish to estimate the discrepancy of $\{S_n(\omega)\alpha\}$ for almost every (ω,α)), the situation simplifies considerably, and both expressions can be easily evaluated. Assume for simplicity that the random variable X has a finite, nonzero mean c; then its characteristic function φ satisfies $\varphi(t) = 1 + ict + o(t)$ as $t \to 0$, and thus $|\varphi(t) - 1| \geq C|t|$ for $|t| \leq t_0$. Hence (A) can be bounded by

$$\frac{C}{\sqrt{N}}\Big(\sum_{h=1}^{L(N)}\frac{1}{h\langle h\alpha\rangle}\Big)^{1/2}(\log N)^{3/2+\varepsilon}. \qquad (5.11)$$

By a well known theorem of Khinchin, almost every α has type $< \psi$ with $\psi(x) = (\log x)^{1+\varepsilon}$ (see [KN] for definitions and the exact formulation) and thus choosing $L(N) = N^{1/2}$ and using Excercise 3.12 on page 131 of [KN], we get that

$$D_N(\alpha,\omega) = \mathcal{O}(N^{-1/2}(\log N)^{5/2+\varepsilon}) \quad \text{for almost every } (\omega,\alpha) \qquad (5.12)$$

which is exactly the bound in Theorem 5.1. (Of course, Theorem 5.1 is more general, since it assumes only the transience of S_n.) In case of the bound (B), the estimate in [KN] cannot be directly used, but a trivial modification of the proof of Lemma 3.3 in [KN] shows that if the type of α is $< \psi$, then

$$\sum_{h=1}^{m}\frac{1}{h\langle h\alpha\rangle^{1/2}} = \mathcal{O}\Big(\sqrt{\psi(2m)} + \sum_{h=1}^{m}\frac{\sqrt{\psi(2h)}}{h}\Big).$$

Using this, (B) yields the metric bound

$$D_N(\alpha,\omega) = \mathcal{O}(N^{-1/2}(\log N)^{7/4+\varepsilon}) \quad \text{for almost every } (\omega,\alpha) \qquad (5.13)$$

which is better than (5.12), but is very likely far from optimal.

We turn now to the study of the case when the random variables generating the random walk S_n are absolutely continuous. In this case Schatte [Sc2] proved that for any fixed x the discrepancy of $\{S_k x\}$ is $\mathcal{O}(\sqrt{\log N/N})$ a.s., and he also proved an LIL for the partial sums of $I(S_k x)$ where I is the indicator function of a fixed interval. These results obviously raise the question if the discrepancy of $\{S_k x\}$ is $\mathcal{O}(\sqrt{\log\log N/N})$ a.s., and below we show that the answer is affirmative.

Theorem 5.3. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with a bounded density concentrated on $[0,1]$ and let $S_n = X_1 + \ldots + X_n$ (mod 1). Then for every x and almost every ω, the discrepancy of the sequence $\{S_k(\omega)x\}$ is $\mathcal{O}\big(\sqrt{\log\log N/N}\big)$.*

For the proof we let, as in Section 4,

$$\psi(x) = \sup_{0\le x\le 1} |\mathbf{P}(S_n \le x) - x|$$

and note that by Theorem 1 of [Sc1] we have

$$\psi(n) \le Ce^{-\lambda n} \qquad (n \ge 1) \tag{5.14}$$

for some constants $C > 0$, $\lambda > 0$.

Lemma 5.4. *Let $f = I_{(a,b)} - (b-a)$ for some $0 \le a < b \le 1$. Then*

$$\mathbf{E}\bigg(\sum_{k=M+1}^{M+N} f(S_k)\bigg)^2 \le C\|f\|N \tag{5.15}$$

for any $M \ge 0$, $N \ge 1$ where $\|f\| = (\int_0^1 f^2(x)dx)^{1/2}$ and C is an absolute constant. The conclusion remains valid if f is a Lipschitz function with $\int_0^1 f(x)dx = 0$.

Proof. In what follows, C denotes positive absolute constants, possibly different at different places. We first show

$$|\mathbf{E}f(S_k)f(S_\ell)| \le C\psi(\ell - k)\|f\| \qquad (k < \ell). \tag{5.16}$$

Indeed, by Lemma 4.3 there exists a r.v. Δ with $|\Delta| \le \psi(\ell - k)$ such that $S_\ell - \Delta$ is a uniform r.v. independent of S_k. Hence

$$\mathbf{E}f(S_\ell - \Delta) = \int_0^1 f(x)dx = 0$$

and thus

$$\mathbf{E}f(S_k)f(S_\ell - \Delta) = \mathbf{E}f(S_k)\mathbf{E}f(S_\ell - \Delta) = 0. \tag{5.17}$$

On the other hand,

$$\begin{aligned}&|\mathbf{E}f(S_k)f(S_\ell)-\mathbf{E}f(S_k)f(S_\ell-\Delta)|\\&\le \mathbf{E}\big(|f(S_k)|\,|f(S_\ell)-f(S_\ell-\Delta)|\big)\le \big(\mathbf{E}f^2(S_k)\big)^{1/2}\big(\mathbf{E}|f(S_\ell)-f(S_\ell-\Delta)|^2\big)^{1/2}.\end{aligned}\tag{5.18}$$

Since X_1 has a bounded density, by Theorem 1 of [Sc1] the density φ_n of S_n exists for all $n\ge 1$ and satisfies $\varphi_n\to 1$ uniformly on $[0,1]$. Thus

$$\mathbf{P}\{S_n\in I\}\le C|I|\qquad (n\ge 1)\tag{5.19}$$

for some constant $C>0$, whence we get

$$\mathbf{E}f^2(S_k)\le C\int_0^1 f^2(x)dx=C\|f\|^2.\tag{5.20}$$

On the other hand,

$$\mathbf{E}|f(S_\ell)-f(S_\ell-\Delta)|^2=\mathbf{E}|I_{(a,b)}(S_\ell)-I_{(a,b)}(S_\ell-\Delta)|^2.\tag{5.21}$$

The difference on the right-hand side differs from zero only if one of S_ℓ and $S_\ell-\Delta$ is inside (a,b) and the other is outside of the interval. In this case S_ℓ is closer to the boundary of (a,b) than $|\Delta|$, and since $|\Delta|\le\psi(\ell-k)$, the probability of this event is at most $C\psi(\ell-k)$ by (5.19). Thus (5.21) yields

$$\mathbf{E}|f(S_\ell)-f(S_\ell-\Delta)|^2\le C\psi(\ell-k)\tag{5.22}$$

which, together with (5.18)–(5.21), gives

$$|\mathbf{E}f(S_k)f(S_\ell)-\mathbf{E}f(S_k)f(S_\ell-\Delta)|\le C\psi(\ell-k).$$

Thus using (5.17) we get (5.16). Now by (5.16) and (5.13)

$$\Big|\sum_{M+1\le k<\ell\le M+N}\mathbf{E}f(S_k)f(S_\ell)\Big|\le CN\|f\|\sum_{\ell\ge 1}\ell^{-2}\le CN\|f\|$$

which, together with (5.20), completes the proof of Lemma 5.4. For Lipschitz functions f the argument is similar.

Lemma 5.5. *Let $f=I_{(a,b)}-(b-a)$ for some $0\le a<b\le 1$. Then for any $M\ge 0$, $N\ge 1$, real $t\ge 1$ and $\|f\|\ge N^{-1/4}$ we have*

$$\mathbf{P}\Big\{\Big|\sum_{k=M+1}^{M+N}f(S_k)\Big|\ge t\|f\|^{1/4}(N\log\log N)^{1/2}\Big\}\le \exp\big(-C\frac{t\log\log N}{\|f\|^{1/2}}\big)+\frac{1}{t^2N}.\tag{5.23}$$

Proof. We divide the interval $[M+1, M+N]$ into subintervals $I_1, \dots, I_L$, with $L \sim N^{19/20}$, where each interval I_ν contains $\sim N^{1/20}$ terms. We set

$$\sum_{k=M+1}^{M+N} f(S_k) = \eta_1 + \ldots + \eta_L$$

where

$$\eta_\nu = \sum_{k \in I_\nu} f(S_k).$$

We deal with the sums $\sum \eta_{2j}$ and $\sum \eta_{2j+1}$ separately. Since there is a separation $\sim N^{1/20}$ between the even block sums η_{2j}, we can apply Lemma 4.3 to get

$$\eta_{2j} = \eta_{2j}^* + \eta_{2j}^{**}$$

where

$$\eta_{2j}^* = \sum_{k \in I_{2j}} f(S_k - \Delta_j)$$

$$\eta_{2j}^{**} = \sum_{k \in I_{2j}} (f(S_k) - f(S_k - \Delta_j))$$

where the Δ_j are r.v.'s with $|\Delta_j| \le \psi(N^{1/20}) \le N^{-10}$ and the r.v.'s η_{2j}^* $j = 1, 2, \ldots$ are independent. Relation (5.22) in the proof of Lemma 5.4 shows that the L_2 norm of each summand in η_{2j}^{**} is $\le C\psi(N^{1/20}) \le CN^{-10}$ and thus for $\|f\| \ge N^{-1/4}$ we have

$$\|\eta_{2j}^{**}\| \le CN^{-9} \le C\|f\|N^{-8}. \tag{5.24}$$

Thus

$$\Big\| \sum \eta_{2j}^{**} \Big\| \le C\|f\|N^{-7}$$

and therefore by the Markov inequality

$$\begin{aligned} &\mathbf{P}\left(\Big| \sum \eta_{2j}^{**} \Big| \ge t\|f\|^{1/4} (N \log\log N)^{1/2} \right) \\ &\le Ct^{-2}\|f\|^{-1/2}(N \log\log N)^{-1}\|f\|^2 N^{-14} \le t^{-2}N^{-1}. \end{aligned} \tag{5.25}$$

Let now $|\lambda| = \mathcal{O}(N^{-1/16})$, then $|\lambda \eta_{2j}^*| \le C|\lambda| N^{1/20} \le 1/2$ for $N \ge N_0$ and thus using $e^x \le 1 + x + x^2$ for $|x| \le 1/2$ we get, using $\mathbf{E}\eta_{2j}^* = 0$,

$$\begin{aligned} \mathbf{E}\left(\exp \lambda \left(\sum_j \eta_{2j}^* \right) \right) &= \prod_j \mathbf{E}\big(e^{\lambda \eta_{2j}^*}\big) \le \prod_j \mathbf{E}(1 + \lambda \eta_{2j}^* + \lambda^2 \eta_{2j}^{*2}) \\ &= \prod_j (1 + \lambda^2 \mathbf{E}\eta_{2j}^{*2}) \le \exp\left(\lambda^2 \sum_j E\eta_{2j}^{*2} \right). \end{aligned} \tag{5.26}$$

By Lemma 5.4

$$\|\eta_{2j}\| \le C\|f\|^{1/2}N^{1/40}$$

which, together with (5.24) and the Minkowski inequality, implies

$$\|\eta_{2j}^*\| \le C\|f\|^{1/2}N^{1/40}$$

and thus the last expression in (5.26) cannot exceed

$$\exp\Big(\lambda^2 C\|f\|\sum_j N^{1/20}\Big) \le \exp(\lambda^2 C\|f\|N).$$

Thus choosing

$$\lambda = (\log\log N/N)^{1/2}\|f\|^{-3/4}$$

(note that by $\|f\| \ge N^{-1/4}$ we have $|\lambda| = \mathcal{O}(N^{-1/6})$) and using the Minkowski inequality, we get

$$\begin{aligned}
&\mathbf{P}\Big\{\Big|\sum_j \eta_{2j}^*\Big| \ge t\|f\|^{1/4}(N\log\log N)^{1/2}\Big\}\\
&\le \exp\Big\{-\lambda t\|f\|^{1/4}(N\log\log N)^{1/2} + \lambda^2 C\|f\|N\Big\}\\
&= \exp(-\|f\|^{-1/2}t\log\log N + C\|f\|^{-1/2}\log\log N)\\
&\le \exp(-C'\|f\|^{-1/2}t\log\log N)
\end{aligned}$$

completing the proof of Lemma 5.5.

Using Lemma 5.5, the proof of Theorem 5.3 can be completed by a standard dyadic chaining argument. We will actually follow here an argument from [P2], which goes back, in turn, to Erdős and Gál. For any $h \ge 1$, $1 \le j \le 2^h$ let $\varphi_h^{(j)}$ denote the indicator function of the interval $[(j-1)2^{-h}, j2^h)$ and put

$$F(M,N,j,h) = \Big|\sum_{k=M+1}^{M+N}(\varphi_h^{(j)}(S_k) - 2^{-h})\Big|.$$

We note first that if $2^n \le N < 2^{n+1}$, then there exist integers m_ℓ with $0 \le m_\ell < 2^{n-\ell}$ $(1 \le \ell \le n)$ such that

$$F(0,N,j,h) \le F(0,2^n,j,h) + \sum_{\frac{1}{3}n\le\ell\le n} F(2^n + m_\ell 2^\ell, 2^{\ell-1}, j, h) + N^{1/3}. \tag{5.27}$$

Next we observe that if $0 \le a \le 1$ has the dyadic expansion

$$a = \sum_{j=1}^{\infty} \varepsilon_j 2^{-j} \qquad \varepsilon_j = 0, 1$$

and $H \ge 1$ is an arbitrary integer, then the indicator function g_a of $[0, a)$ satisfies

$$\sum_{h=1}^{H-1} \varrho_h(x) \le g_a(x) \le \sum_{h=1}^{H-1} \varrho_h(x) + \sigma_H(x)$$

where ϱ_h is the indicator function of $\Big[\sum\limits_{j=1}^{h} \varepsilon_j 2^{-j}, \sum\limits_{j=1}^{h+1} \varepsilon_j 2^{-j} \Big)$ and σ_H is the indicator function of $\Big[\sum\limits_{j=1}^{H} \varepsilon_j 2^{-j}, \sum\limits_{j=1}^{H} \varepsilon_j 2^{-j} + 2^{-H} \Big)$. For $\varepsilon_h = 0$ clearly $\varrho_h \equiv 0$ and for $\varepsilon_h = 1$, ϱ_h coincides with one of the $\varphi_h^{(j)}$. Also, σ_H coincides with some of the $\varphi_H^{(j)}$. Hence it follows that for any $N \ge 1$, $H \ge 1$ there exist suitable integers $1 \le j_h \le 2^h$, $1 \le h \le H$ such that

$$\begin{aligned} \Big| \sum_{k \le N} g_a(S_k) - a \Big| &\le h \le H \to \sum\nolimits^{*} \Big| \sum_{k \le N} \varphi_h^{(j_h)}(S_k) - 2^{-h} \Big| + N 2^{-H} = \\ &= h \le H \to \sum\nolimits^{*} F(0, N, j_h, h) + N 2^{-H} \end{aligned} \tag{5.28}$$

where the $(*)$ means that the summation is extended only for those $h < H$ such that $\varepsilon_h = 1$. Given now $N \ge 1$, define n by $2^n \le N < 2^{n+1}$, choose $H = 2^{n/2}$ in (5.28) and get, using also (5.27),

$$\begin{aligned} \Big| \sum_{k \le N} (g_a(S_k) - a) \Big| \le \sum_{h \le 2^{n/2}} \Big\{ & F(0, 2^n, j_h, h) + \\ & + \sum_{\frac{1}{3} n \le \ell \le n} F(2^n + m_\ell 2^\ell, 2^{\ell - 1}, j_h, h) \Big\} + 2\sqrt{N}. \end{aligned} \tag{5.29}$$

Formula (5.29) estimates the sum $\sum\limits_{k \le N} (g_a(S_k) - a)$ by means of the dyadic "building blocks" $F(0, 2^n, j_h, h)$, $F(2^n + m_\ell 2^\ell, 2^{\ell-1}, j_h, h)$ and thus it remains to estimate these quantities. Let

$$\varphi(n) = 10 (N \log \log N)^{1/2}$$

and introduce the events

$$\begin{aligned}
G(n,j,h) &= \Big\{F(0,2^n,j,h) \ge 2^{-h/8}\varphi(2^n)\Big\}\\
H(n,j,h,\ell,m) &= \Big\{F(2^n+m2^\ell,2^{\ell-1},j,h) \ge 2^{-h/8}2^{(\ell-n-3)/6}\varphi(2^n)\Big\}\\
G_n &= \bigcup_{h\le 2^{n/2}}\ \bigcup_{j\le 2^h} G(n,j,h)\\
H_n &= \bigcup_{h\le 2^{n/2}}\ \bigcup_{j\le 2^h}\ \bigcup_{\frac13 n\le \ell\le n}\ \bigcup_{m\le 2^{n-\ell}} H(n,j,h,\ell,m).
\end{aligned}$$

Note that

$$N^{-1/4} \le 2^{-(h+1)/2} \le \|\varphi_h^{(j)} - 2^{-h}\| \le 2^{-h/2}$$

and thus applying Lemma 5.6 with $M=0$, $N=2^n$ and $t=1$ we get

$$\mathbf{P}(G(n,h,j)) \le C\exp(-2^{h/4}\log n) + 2^{-h}.$$

Thus

$$\begin{aligned}
\mathbf{P}(G_n) &\le C\sum_{h\le 2^{n/2}} 2^h\exp(-2^{h/4}\log n) + C2^{-n}\sum_{h\le 2^{n/2}} 2^h \le\\
&\le C\exp(-2\log n) + C2^{-n/2} \le Cn^{-2}.
\end{aligned}$$

Similarly, using Lemma 5.5 with $M=2^n+m2^\ell$, $N=2^{\ell-1}$, $t=2^{(n-\ell)/3}$ we get

$$\mathbf{P}(H(n,j,h,\ell,m)) \le C\exp(-2^{h/4}2^{(n-\ell)/3}\log n) + 2^{2(\ell-n)/3}2^{-n}$$

whence

$$\mathbf{P}(H_n) \le Cn^{-2}$$

by a simple calculation. Hence the Borel–Cantelli lemma shows that there exists, for almost every ω, an index $n_0=n_0(\omega)$ such that for $n\ge n_0$ the expressions $F(0,2^n,j_h,h)$ and $F(2^n+m_\ell 2^\ell,2^{\ell-1},j_h,h)$ in (5.29) are bounded by the quantities $2^{-h/8}\varphi(2^n)$, resp. $2^{-h/8}2^{(\ell-n-3)/6}\varphi(2^n)$ (regardless the value of j_h and m_ℓ) and consequently

$$\begin{aligned}
\Big|\sum_{k\le N}(g_a(S_k)-a)\Big| &\le \sum_{h\le 2^{n/2}}\Big\{2^{-h/8} + \sum_{\frac13 n\le \ell\le n} 2^{-h/8}2^{(\ell-n-3)/6}\Big\}\varphi(2^n)+\\
&\quad + 2\sqrt{N} \le C\varphi(N)
\end{aligned}$$

completing the proof of Theorem 5.3. ∎

References

[Ba] Baker, R.C. [1981] *Metric number theory and the large sieve,* J. London Math. Soc. (2), **24**, pp.34-40.

[Be1] Berkes, I. [1976] *On the asymptotic behavior of* $\sum f(n_k x)$*-Main Theorems*, Z. Wahrscheinlichtkeitstheorie, **34**, Springer, Berlin, p.319-345.

[Be2] Berkes, I. [1976] *On the asymptotic behavior of* $\sum f(n_k x)$*-Applications*, Z. Wahrscheinlichtkeitstheorie, **34**, Springer, Berlin, p.347-365.

[Be3] Berkes, I. [1975] *An almost sure invariance principle for lacunary trigonometric series*, Acta Math. Acad. Scientiarum Hungaricae, **26**, no1-2, p.209-220.

[Be4] Berkes, I. [1993] *Critical behavior of the trigonometric system*, Trans. Amer. Math. Soc. **338** no2, p.553-585.

[Be5] Berkes, I. [1997] *On the convergence of* $\sum_n c_n f(nx)$ *and the Lip 1/2 class*, Trans. Amer. Math. Soc. **349** no10, p.4143-4158.

[Beu] Beurling A. [1989] *The collected works of Arne Beurling, Vol. 2: Harmonic Analysis*, Contemp. Math., Birkhaüser, Boston, p.378-380.

[Bi] Billingsley, P. [1968] *Convergence of probability measures*, Wiley, New York.

[Bo] Bourgain, J. [1988] *Almost sure convergence and bounded entropy*, Israel J. Math. **63**, p.79-95.

[BP1] Berkes I., Philipp W. [1993] *The size of trigonometric and Walsh series and uniform distribution modulo 1*, J. London Math. Soc. **50** no2, p.454-464.

[BP2] Berkes I., Philipp W. [1998] *A limit theorem for lacunary series* $\sum f(n_k x)$, Studia Scientitarum Mathem. Hungarica **34**, p.1-13.

[BC] Blum J.R., Cogburn R. [1975] *On ergodic sequences of measures*, Proc. Amer. Math. Soc. **51**, p.359-365.

[Br] Breiman L. [1992]. *Probability*, SIAM Ed.

[C] Carleson, L. [1966] *On convergence and growth of partial sums of Fourier series*, Acta Math. **116**, p.135-157.

[Cs] Cassels J.W.S. [1950] *Some metrical theorems in Diophantine approximation III*, Proc. Cambridge Philos. Soc. **46**, p.219-225.

[E] Erdős P. [1962] *On trigonometric sums with gaps*, Magyar Tud. Akad. Mat. Kut. Int. Közl. **7**, p. 37–42.

[EK] Erdős P., Koksma J.F. [1949] *On the uniform distribution modulo* 1 *of sequences* $(f(n,\theta))$, *Proc. Kon. Nederl. Akad. Wetensch.* **52**, *p.851-854.*

[G] Gál, I.S. [1949] A theorem concerning Diophantine approximations, Nieuw Archief voor Wiskunde **23**, *p.13–38.*

[Gap1] Gaposhkin, V. F. [1966] Lacunary series and independent functions, Uspekhi Mat. Nauk **21/6**, *p.1-82 (Russian), English translation: Russian Math. Surveys,* **21**, *p.3-82.*

[Gap2] Gaposhkin, V. F. [1966] On series by the system $\phi(nx)$, Math. Sbornik **69**, *p.328-353 (Russian), English translation: Amer. Math. Soc. Transl. (2),* **86** *(1970), p.167-197.*

[Gap3] Gaposhkin, V. F. [1967] A system of convergence, Math. Sbornik **74**, *p.93-99 (Russian), English translation: Math. USSR Sb.,* **3**, *p.83-90.*

[Gap4] Gaposhkin, V. F. [1968] On convergence and divergence systems, Mat. Zametki **4**, *p.253-260.*

[Gar] Garsia A. [1970] Topics in almost sure convergence, Lectures in Adv. Math. Markham Pub. Company.

[GRR] Garsia A., Rodemich E., Rumsey Jr H. [1970] A real variable lemma and the continuity of paths of some Gaussian processes, Indiana Univ. Math. J. **20** *p.565-578.*

[GNN] Ginsberg J., Newman D., Neuwirth J. [1970] Approximation by $\{f(kx)\}$, J. Funct. Anal. **5** *p.194-203.*

[GN] Gosselin R., Neuwirth J. [1968] On Paley-Wiener basis, J. Math. Mech. **18** *p.871-879.*

[Hal] Halmos, P. R. [1956] Lectures on ergodic theory, Math. Soc Jap.

[Har] Harman G., Metric number theory, London Math. Soc. Monographs New Series (1988).

[HLS1] Hedenmalm H., Lindqvist P., Seip K. [1997] A Hilbert space of Dirichlet series and systems of dilated functions in $L^2([0,1])$, Duke Math. J. **86**, *1-37.*

[HLS2] Hedenmalm H., Lindqvist P., Seip K. [1999] Addendum to "A Hilbert space of Dirichlet series and systems of dilated functions in $L^2([0,1])$", Duke Math. J. **99**, *175-178.*

[Hu] Hunt, R. [1967] On convergence of Fourier series, p.235-255.

[HT] Hellinger E., Toeplitz O. [1910] Grundlagen für eine Theorie der unendlichen Matrizen, Mathematische Annalen **69**, *p.289-330*

[Ibr] Ibragimov, I. A. [1960] Asymptotic distribution of values of certain sums, Vestnik Leningrad Univ. **5**, *p.55-69.*

[K1] Kac, M. [1943] Convergence of certain gap series, Annals of Math., **44** *no3, p.411-415.*

[K2] Kac, M. [1949] Probability methods in some problems of analysis and number theory, Bull. Amer. Math. Soc., **55**, *p.641-665.*

[KN] Kuipers, L., Niederreiter, H [1971] Uniform Distribution of sequences, J. Wiley Ed.

[KSZ] Kac M., Salem R., Zygmund A. [1948] A gap theorem, Trans. Amer. Math. Soc. **63**, *p.235-243.*

[M] Mitrinovic, D. S. [1970] Analytic inequalities, Springer Verlag **165**.

[Ni] Nikishin E.M. [1971] Resonance theorems and functions series, Math. Zametki **10/5**, *p.583–595.*

[O] Olevskii A. M. [1975] Fourier series with respect to general orthogonal systems, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **86**.

[P1] Philipp W. [1967] Some metrical theorems in number theory, Pacific J. of Math.. **20** *no1 p.109-127.*

[P2] Philipp W. [1975] Limit theorems for lacunary series and uniform distribution modulo 1, Acta Arithmetica. **XXVI** *p.242-251.*

[R] Riemann, B. [1892] Gesammelte mathematische Werke, Sd Edition, Leipzig.

[RW] Ruch J.J., Weber, M. [1997] Quelques résultats dans la théorie des sommes de Riemann, Expositiones. Math. **15**, *p.279–288.*

[Sc1] Schatte, P. (1984), On the asymptotic uniform distribution of sums reduced mod 1. Math. Nachr. 115, 275–281.

[Sc2] Schatte, P. (1988), On a law of the iterated logarithm for sums mod 1 with application to Benford's law. Prob. Theory Rel. Fields 77, 167–178.

[Sp] Spitzer F.L., Principles of random walks, Second Edition, Springer, New-York (1976).

[T] Toeplitz, O. [1938] zur Theorie der Dirichlet Reihen, Amer. Journal of Math. **60**, *p.880-888.*

[We1] Weber M. [1998] Entropie métrique et convergence presque partout, Coll. "Travaux en Cours" **58**, *Hermann, Paris.*

[We2] Weber, M. [2002] Some examples of application of the metric entropy method, to appear in Acta Math. Hungar. 2004, **105** *no 3.*

[We3] Weber, M. [2003] Discrepancy of randomly sampled sequences of integers, to appear in Mathematische Nachrichten.

[We4] Weber, M. [2003] A theorem related to the Marcinkiewicz-Salem conjecture, to appear in Results in math.

[Wi1] Wintner, A. [1937] On a trigonometrical series of Riemann, Amer. Journal of Math. **59**, *p.629-634.*

[Wi2] Wintner, A. [1944] Diophantine approximation and Hilbert space, Amer. Journal of Math. **66**, *p.564-578.*

[Wi3] Wintner, A. [1944] Random factorizations and Riemann's hypothesis, Duke Math. J. **11**, *267-275.*

[Z] Zygmund A. [1959] Trigonometric series, Cambridge University Press, Second Edition.

Istvan Berkes, Technical University Graz, Department of Statistics, Steyrergasse 17/IV, A-8010 Graz, Austria. E-mail: `berkes@stat.tu-graz.ac.at`

Michel Weber: Mathématique (IRMA), Université Louis-Pasteur et C.N.R.S., 7 rue René Descartes, 67084-Strasbourg Cedex, France. E-mail: `weber@math.u-strasbg.fr`