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Lacunary sequences and permutations

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Dedicated to the memory of Walter Philipp

Abstract By a classical principle of analysis, sufficiently thin subsequences of general sequences of functions behave like sequences of independent random variables. This observation not only explains the remarkable properties of lacunary trigonometric series, but also provides a powerful tool in many areas of analysis. In contrast to "true" random processes, however, the probabilistic structure of lacunary sequences is not permutation-invariant and the analytic properties of such sequences can change radically after rearrangement. The purpose of this paper is to survey some recent results of the authors on permuted function series. We will see that rearrangement properties of lacunary trigonometric series $\sum(a_k \cos n_k x + b_k \sin n_k x)$ and their nonharmonic analogues $\sum c_k f(n_k x)$ are intimately connected with the number theoretic properties of $(n_k)_{k\geq 1}$ and we will give a complete characterization of permutational invariance in terms of the Diophantine properties of $(n_k)_{k\geq 1}$. We will also see that in a certain statistical sense, permutational invariance is the "typical" behavior of lacunary sequences.

1 Introduction

Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1 \qquad (k = 1, 2, \ldots). \tag{1.1}$$

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Salem and Zygmund [31] proved that if $(a_k)_{k\geq 1}$ is a sequence of real numbers satisfying

$$A_N \to \infty \quad \text{and} \quad a_N = o(A_N) \quad \text{with} \quad A_N = \left(\frac{1}{2} \sum_{k=1}^{N} a_k^2 \right)^{1/2}, \tag{1.2}$$

then $(\cos 2\pi n_k x)_{k\geq 1}$ obeys the central limit theorem

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : A_N^{-1} \sum_{k=1}^{N} a_k \cos 2\pi n_k x \leq t \right\} = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2} du, \tag{1.3}$$

where λ denotes the Lebesgue measure. Under the same gap condition Weiss [40] proved (cf. also Salem and Zygmund [32], Erdős and Gál [12]) that if $(a_k)_{k\geq 1}$ satisfies

$$A_N \to \infty \quad \text{and} \quad a_N = o\big(A_N/(\log\log A_N)^{1/2}\big) \tag{1.4}$$

then $(\cos 2\pi n_k x)_{k\geq 1}$ obeys the law of the iterated logarithm

$$\limsup_{N\to\infty} (2A_N^2 \log\log A_N)^{-1/2} \sum_{k=1}^{N} a_k \cos 2\pi n_k x = 1 \qquad \text{a.e.} \tag{1.5}$$

Comparing these results with the classical forms of the central limit theorem and law of the iterated logarithm in probability theory, we see that under the gap condition (1.1) the functions $\cos 2\pi n_k x$ behave like independent random variables. Using martingale techniques, Philipp and Stout [30] proved that if instead of (1.2) we assume $a_N = o(A_N^{1-\delta})$ for some $\delta > 0$, then on the probability space $([0,1], \mathcal{B}, \lambda)$ there exists a Brownian motion process $\{W(t), t \geq 0\}$ such that

$$\sum_{k=1}^{N} a_k \cos 2\pi n_k x = W(A_N^2) + O\left(A_N^{1-\rho}\right) \qquad \text{a.s.} \tag{1.6}$$

for some $\rho > 0$. The last relation implies not only the CLT and LIL for $(\cos 2\pi n_k x)_{k\geq 1}$, but a whole class of further limit theorems for independent random variables; for examples and discussion we refer to [30].

The previous results extend, in a modified form, to lacunary subsequences of the system $\{f(nx)\}_{n\geq 1}$ where f is a periodic measurable function, but the asymptotic properties of this system are much more complicated than those of the trigonometric system. By a conjecture of Khinchin [23], if f has period 1 and is Lebesgue integrable on $(0,1)$, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(kx) = \int_0^1 f(t)\, dt \quad \text{a.e.} \tag{1.7}$$

This remained open for almost 50 years until Marstrand [25] disproved it, but even today, no precise condition for the validity of (1.7) is known. Similarly, there is no analogue of Carleson's theorem [9] for the system $(f(nx))_{n\ge 1}$ and we do not know under what conditions the series $\sum_{k=1}^{\infty} c_k f(kx)$ converges almost everywhere. In the lacunary case, Kac [21] proved that if f satisfies a Lipschitz condition, then $f(2^k x)$ obeys a central limit theorem similar to (1.3) and not much later, Erdős and Fortet (see [22], p. 646) showed that the CLT fails for $f(n_k x)$ for $n_k = 2^k - 1$ even for some trigonometric polynomials f. Gaposhkin [18] proved that $f(n_k x)$ obeys the CLT if $n_{k+1}/n_k \to \alpha$ where α^r is irrational for $r = 1, 2 \dots$ and the same holds if all the fractions n_{k+1}/n_k are integers. He also showed (see [19]) that the validity of the CLT for $f(n_k x)$ is closely related to the number of solutions of the Diophantine equation

$$an_k + bn_\ell = c, \qquad 1 \le k, \ell \le N. \tag{1.8}$$

Improving these results, Aistleitner and Berkes [1] recently gave a necessary and sufficient Diophantine condition for the CLT for $f(n_k x)$. As the proofs of these results show, the asymptotic behavior of $f(n_k x)$ is determined by a complicated interplay between the arithmetic properties of $(n_k)_{k\ge 1}$ and the Fourier coefficients of f and the combination of probabilistic and number-theoretic effects leads to a unique, highly interesting asymptotic behavior. Let

$$D_N(x_1, \dots, x_N) := \sup_{0\le a<b<1} \left| \frac{\sum_{k=1}^N \mathbb{1}_{[a,b)}(x_k)}{N} - (b-a) \right|$$

denote the discrepancy (mod 1) of the finite sequence $(x_1, \dots, x_N)$, where $\mathbb{1}_{[a,b)}$ is the indicator function of the interval $[a,b)$, extended to $\mathbb{R}$ with period 1. Philipp [27], [28] proved that if $(n_k)_{k\ge 1}$ satisfies the Hadamard gap condition (1.1), then the discrepancy $D_N(n_k x)$ of the sequence $\{n_k x, 1 \le k \le N\}$ obeys the LIL

$$\frac{1}{4\sqrt{2}} \le \limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N \log\log N}} \le C_q \quad \text{a.e.,} \tag{1.9}$$

where C_q is a number depending on q. Note that if $(\xi_k)_{k\ge 1}$ is a sequence of independent random variables with uniform distribution over $(0,1)$, then

$$\limsup_{N\to\infty} \frac{ND_N(\xi_k)}{\sqrt{2N \log\log N}} = \frac{1}{2} \tag{1.10}$$

with probability one by the Chung–Smirnov LIL (see e.g. [33], p. 504). A comparison of (1.9) and (1.10) shows again that the sequence $(n_k x)_{k\ge 1}$ mod 1 behaves like a sequence of i.i.d. random variables. Surprisingly, however, the limsup in (1.9) can be different from the constant $1/2$ in (1.10) and, as Fukuyama [14] showed, it depends sensitively on $(n_k)_{k\ge 1}$. For example, for $n_k = a^k$, $a \ge 2$ the limsup Σ_a in (1.9) equals

$$\Sigma_a = \sqrt{42}/9 \qquad \text{if } \ a = 2,$$

$$\Sigma_a = \frac{\sqrt{(a+1)a(a-2)}}{2\sqrt{(a-1)^3}} \qquad \text{if } \ a \ge 4 \ \text{ is an even integer,}$$

$$\Sigma_a = \frac{\sqrt{a+1}}{2\sqrt{a-1}} \qquad \text{if } a \geq 3 \text{ is an odd integer.}$$

It is even more surprising that, as Fukuyama [15] showed, the limsup in (1.9) is not permutation-invariant and can change after a rearrangement of $(n_k)_{k\geq 1}$. Similarly,

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k=1}^{N} f(n_k x)$$

and the limiting variance in the CLT for $N^{-1/2}\sum_{k=1}^{N} f(n_k x)$ can change if we permute the sequence $(n_k)_{k\geq 1}$. These results show that even though lacunary subsequences of $(f(nx))_{n\geq 1}$ satisfy a large class of limit theorems for i.i.d. random variables and an i.i.d. sequence is a symmetric structure, the behavior of lacunary sequences is generally nonsymmetric. The purpose of the present paper is to give a detailed analysis of the probabilistic structure of $f(n_k x)$ and to clear up the effect of permutations on its asymptotic properties. The proofs of our results will be given in [3], [4], [5].

2 The trigonometric case

By Carleson's theorem [9], if $f \in L_2(0,2\pi)$ then its Fourier series

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{2.1}$$

converges almost everywhere. However, as was noted by Kolmogorov (see [24]), there exists an $f \in L_2(0,2\pi)$ whose Fourier series (2.1) diverges a.e. after a suitable permutation of its terms. This shows that the asymptotic properties of the trigonometric system $\{\cos kx, \sin kx\}_{k\geq 1}$ are not permutation-invariant. On the other hand, Erdős [10] proved (see also Zygmund [41]) that if $(n_k)_{k\geq 1}$ satisfies the Hadamard gap condition (1.1) and

$$\sum_{k=1}^{\infty} \left(a_k^2 + b_k^2\right) < \infty \tag{2.2}$$

then

$$\sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x) \tag{2.3}$$

converges almost everywhere after any rearrangement of its terms, giving a permutation-invariant property of lacunary trigonometric series. Our first result below states that under (1.1) the systems $(\cos n_k x)_{k\geq 1}$, $(\sin n_k x)_{k\geq 1}$ satisfy also the central limit theorem and law of the iterated logarithm in a permutation-invariant form. More precisely, we have

Theorem 2.1 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying* (1.1) *and let* $\sigma : \mathbb{N} \to \mathbb{N}$ *be a permutation of the positive integers. Then we have*

$$\lim_{N\to\infty} \lambda \left\{ x \in (0,1) : \sum_{k=1}^{N} \cos 2\pi n_{\sigma(k)} x \leq t\sqrt{N/2} \right\} = (2\pi)^{-1/2} \int_{-\infty}^{t} e^{-u^2/2} du \qquad (2.4)$$

and

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k=1}^{N} \cos 2\pi n_{\sigma(k)} x = 1 \qquad a.e. \qquad (2.5)$$

Note that for the unpermuted CLT and LIL we need much weaker gap conditions than (1.1). In fact, Takahashi [36], [37], [38] (cf. also Erdős [11]) showed that if a sequence $(n_k)_{k\geq 1}$ of integers satisfies

$$n_{k+1}/n_k \geq 1 + ck^{-\alpha}, \qquad 0 \leq \alpha < 1/2 \qquad (2.6)$$

then for any sequence $(a_k)_{k\geq 1}$ satisfying

$$A_N \to \infty \quad \text{and} \quad a_N = o(A_N N^{-\alpha}) \quad \text{with} \quad A_N = \left(\frac{1}{2} \sum_{k=1}^{N} a_k^2 \right)^{1/2}$$

we have the CLT (1.3) and under a slightly stronger coefficient condition also the LIL (1.5). Note, however, that (2.6) does not imply permutation-invariance and the following result shows that permutation-invariance fails under any gap condition weaker than (1.1).

Theorem 2.2 *For any positive sequence $(\varepsilon_k)_{k\geq 1}$ tending to* 0*, there exists a sequence $(n_k)_{k\geq 1}$ of positive integers satisfying*

$$n_{k+1}/n_k \geq 1 + \varepsilon_k, \qquad k \geq k_0$$

and a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ of the positive integers such that the permuted central limit theorem (2.4) *and the permuted law of the iterated logarithm* (2.5) *fail.*

By a theorem of Erdős [10], if $(n_k)_{k\geq 1}$ is any (not necessarily increasing) sequence of different positive integers such that for any integer $\nu > 0$ the number of solutions of the Diophantine equation

$$n_k \pm n_\ell = \nu, \qquad k, \ell \geq 1$$

is bounded by a constant C independent of ν, then the series (2.3) converges a.e. provided (2.2) holds. Since this Diophantine property is permutation-invariant, it implies the a.e. unconditional convergence of (2.3) as well. Note that Erdős' condition is much weaker than (1.1); in fact, it holds even for some polynomially growing sequences $(n_k)_{k\geq 1}$. How slowly a sequence $(n_k)_{k\geq 1}$ satisfying this condition can grow is a well known open problem in number theory; see Halberstam and Roth [20], pp. 84–97 and Ajtai et al. [6].

3 The system $f(nx)$

Let f be a measurable function satisfying

$$f(x+1) = f(x), \quad \int_0^1 f(x)\, dx = 0, \quad \int_0^1 f^2(x)\, dx < \infty \tag{3.1}$$

and let $(n_k)_{k \geq 1}$ be a sequence of integers satisfying the Hadamard gap condition (1.1). The central limit theorem for $f(n_k x)$ has a long history discussed in Section 1. To formulate criteria for the permutation-invariant CLT and LIL, let us say that a sequence $(n_k)_{k\geq 1}$ of positive integers satisfies

Condition $\mathbf{D}_2$, if for any fixed nonzero integers a, b, c the number of solutions of the Diophantine equation

$$an_k + bn_l = c \tag{3.2}$$

is bounded by a constant $K(a,b)$, independent of c.

Condition $\mathbf{D}_2^{(s)}$ (strong $\mathbf{D}_2$), if for any fixed integers $a \neq 0$, $b \neq 0$, c the number of solutions of the Diophantine equation (3.2) is bounded by a constant $K(a,b)$, independent of c, where for $c = 0$ we require also $k \neq l$.

Condition $\mathbf{D}_2^{(w)}$ (weak $\mathbf{D}_2$), if for any fixed nonzero integers a, b, c the number of solutions of the Diophantine equation

$$an_k + bn_l = c, \qquad 1 \leq k, l \leq N \tag{3.3}$$

is $o(N)$, uniformly in c.

Condition $\mathbf{D}_2$ is a variant of Sidon's $\mathbf{B}_2$ condition (see [34], [35]). Gaposhkin [19] proved that under mild smoothness assumptions on f, condition $\mathbf{D}_2$ implies the CLT for $f(n_k x)$ and Berkes and Philipp [8] showed that the same condition also implies a Wiener approximation for the partial sums of $f(n_k x)$, similar to (1.6). Recently, Aistleitner and Berkes [1] proved that the CLT holds for $f(n_k x)$ also under $\mathbf{D}_2^{(w)}$ and this condition is necessary. This settles the CLT problem for $f(n_k x)$, but, as we noted, the validity of the CLT does not imply permutation-invariant behavior of $f(n_k x)$. The purpose of this section is to give a precise description of the CLT and LIL behavior of permuted sums $\sum_{k=1}^N f(n_{\sigma(k)} x)$ and in particular, to obtain characterizations of permutation-invariance.

Our first result shows that if we assume the slightly stronger gap condition

$$n_{k+1}/n_k \to \infty \tag{3.4}$$

then the behavior of $f(n_k x)$ is permutation-invariant, regardless the number theoretic structure of $(n_k)_{k\geq 1}$. In what follows, let $\|\cdot\|$ denote the $L_2(0,1)$ norm.

Theorem 3.1 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the gap condition* (3.4)*. Then for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ of the integers and for any measurable function $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$f(x+1) = f(x), \quad \int_0^1 f(x)\, dx = 0, \quad \mathrm{Var}_{[0,1]} f < +\infty \tag{3.5}$$

we have

$$\frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(n_{\sigma(k)}x) \longrightarrow_d \mathcal{N}(0, \|f\|^2) \tag{3.6}$$

and

$$\limsup_{N\to\infty} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} f(n_{\sigma(k)}x) = \|f\| \quad \textit{a.e.} \tag{3.7}$$

Moreover, for any permutation σ of $\mathbb{N}$ we have

$$\limsup_{N\to\infty} \frac{ND_N(n_{\sigma(k)}x)}{\sqrt{2N \log\log N}} = \frac{1}{2} \quad \textit{a.e.} \tag{3.8}$$

Our next theorem shows that if we slightly strengthen (3.4), then not only the CLT and LIL, but a much larger class of limit theorems becomes permutation-invariant.

Theorem 3.2 *Let f be a function satisfying* (3.1) *and the Lipschitz α condition* $(0 < \alpha \leq 1)$*. Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers such that*

$$\sum_{k=1}^{\infty} (n_k/n_{k+1})^{\alpha} < \infty. \tag{3.9}$$

Then there exists a bounded i.i.d. sequence $(g_k)_{k\geq 1}$ of functions on $(0,1)$ such that

$$\sum_{k=1}^{\infty} |f(n_k x) - g_k(x)| < \infty \qquad \textit{a.e.} \tag{3.10}$$

Let σ be a permutation of $\mathbb{N}$. Relation (3.10) implies that

$$\sum_{k=1}^{\infty} \left| f(n_{\sigma(k)}x) - g_{\sigma(k)}(x) \right| < \infty \qquad \text{a.e.}$$

and consequently

$$\sum_{k=1}^{N} f(n_{\sigma(k)}x) - \sum_{k=1}^{N} g_{\sigma(k)}(x) = O(1) \qquad \text{a.e.} \tag{3.11}$$

Since the i.i.d. sequences $(g_k)_{k\geq 1}$ and $(g_{\sigma(k)})_{k\geq 1}$ are probabilistically equivalent, relation (3.11) implies that, up to an error term $O(1)$, the asymptotic properties of the partial sums $\sum_{k=1}^N f(n_{\sigma(k)}x)$ are the same for all σ. Thus Theorem 3.2 expresses a very strong form of permutation-invariance of the sequence $f(n_k x)$. Condition (3.9) is satisfied e.g. if $n_k = 2^{[ck \log_2 k]}$ with $c > 1/\alpha$; here $\log_2$ denotes logarithm with base 2.

The proof of Theorem 3.2 shows that the approximating i.i.d. sequence $(g_k)_{k\geq 1}$ can be chosen to satisfy

$$\mu\big\{x \in (0,1) : |f(n_k x) - g_k(x)| \geq \varepsilon_k\big\} \leq \varepsilon_k, \qquad k = 1,2,\dots \tag{3.12}$$

with $\varepsilon_k = (n_k/n_{k+1})^\alpha$. This gives more precise information than (3.10) if $(n_k)_{k\geq 1}$ grows very rapidly. Actually, the approximation given by (3.12) is best possible. Let $f(x) = \cos 2\pi x$ and let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers such that the ratios n_{k+1}/n_k are integers and $\sum_{k=1}^\infty (n_k/n_{k+1}) = \infty$. Then there exists no i.i.d. sequence $(g_n)_{n\geq 1}$ of functions on $[0,1]$ such that

$$\mu\big\{x \in (0,1) : |\cos 2\pi n_k x - g_k(x)| \geq \varepsilon_k\big\} \leq \varepsilon_k, \qquad k = 1,2,\dots. \tag{3.13}$$

with $\sum_{k=1}^\infty \varepsilon_k < \infty$.

So far, we investigated the permutational invariance of $f(n_k x)$ under the growth condition $n_{k+1}/n_k \to \infty$. Assuming only the Hadamard gap condition (1.1), the situation becomes more complex and the number theoretic structure of $(n_k)_{k\geq 1}$ comes into play. Our first result gives a necessary and sufficient condition for the permuted partial sums $\sum_{k=1}^N f(n_{\sigma(k)}x)$ to have only Gaussian limit distributions and gives precise criteria this to happen for a specific permutation σ.

Theorem 3.3 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition* (1.1) *and condition* $\mathbf{D}_2$. *Let f satisfy* (3.5) *and let σ be a permutation of $\mathbb{N}$. Then $N^{-1/2}\sum_{k=1}^N f(n_{\sigma(k)}x)$ has a limit distribution iff*

$$\gamma = \lim_{N\to\infty} N^{-1} \int_0^1 \left(\sum_{k=1}^N f(n_{\sigma(k)}x)\right)^2 dx \tag{3.14}$$

exists, and then

$$N^{-1/2} \sum_{k=1}^N f(n_{\sigma(k)}x) \to_d N(0,\gamma). \tag{3.15}$$

(If $\gamma = 0$ then the limit distribution is degenerate.)

Theorem 3.3 is best possible in the following sense:

Theorem 3.4 *If $(n_k)_{k\geq 1}$ satisfies* (1.1), *but condition* $\mathbf{D}_2$ *fails, then there exists a function f satisfying* (3.5) *and a permutation σ of $\mathbb{N}$ such that the limit in* (3.14) *exists, but the normed partial sums in* (3.15) *do not have a Gaussian limit distribution.*

In other words, under the Hadamard gap condition and condition $\mathbf{D}_2$, the limit distribution of $N^{-1/2}\sum_{k=1}^N f(n_{\sigma(k)}x)$ can only be Gaussian, but the variance of the limit distribution depends on the constant γ in (3.14) which, as simple examples show, is not permutation-invariant. For example, if $n_k = 2^k$ and σ is the identity permutation, then (3.14) holds with

$$\gamma = \gamma_f = \int_0^1 f^2(x)dx + 2\sum_{k=1}^{\infty}\int_0^1 f(x)f(2^k x)dx \tag{3.16}$$

(see Kac [21]). Using an idea of Fukuyama [15], one can construct permutations σ of $\mathbb{N}$ such that

$$\lim_{N\to\infty}\frac{1}{N}\int_0^1\left(\sum_{k=1}^N f(n_{\sigma(k)}x)\right)^2 dx = \gamma_{\sigma,f} \tag{3.17}$$

with $\gamma_{\sigma,f} \neq \gamma_f$. If the Fourier coefficients of f are nonnegative, then $\gamma_f \geq \|f\|^2$ and the set of possible values $\gamma_{\sigma,f}$ belonging to all permutations σ is identical with the interval $[\|f\|^2, \gamma_f]$, see Aistleitner, Berkes and Tichy [2]. For general f it can happen that $\gamma_f < \|f\|^2$ and the set of limiting variances contains points outside of $[\gamma_f, \|f\|^2]$, see again [2].

Under the slightly stronger condition $\mathbf{D}_2^{(s)}$ we have permutation-invariance:

Theorem 3.5 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition* (1.1) *and condition $\mathbf{D}_2^{(s)}$. Let f satisfy* (3.5) *and let σ be a permutation of $\mathbb{N}$. Then the central limit theorem* (3.15) *holds with $\gamma = \|f\|^2$.*

We now pass to the problem of the LIL.

Theorem 3.6 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition* (1.1) *and condition $\mathbf{D}_2$. Let f be a measurable function satisfying* (3.5)*, let σ be a permutation of $\mathbb{N}$ and assume that the limit* (3.14) *exists. Then we have*

$$\limsup_{N\to\infty}\frac{\sum_{k=1}^N f(n_{\sigma(k)}x)}{\sqrt{2N\log\log N}} = \gamma^{1/2} \quad a.e. \tag{3.18}$$

Theorem 3.7 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition* (1.1) *and condition $\mathbf{D}_2^{(s)}$. Then for any measurable function f satisfying* (3.5) *and any permutation σ of $\mathbb{N}$ we have*

$$\limsup_{N\to\infty}\frac{\sum_{k=1}^N f(n_{\sigma(k)}x)}{\sqrt{2N\log\log N}} = \|f\| \quad a.e.$$

The proofs of Theorems 3.3–3.7 show that if f is a trigonometric polynomial of degree d, then in conditions $\mathbf{D}_2$ resp. $\mathbf{D}_2^{(s)}$ it suffices to assume the bound for the number of solutions of (3.2) for coefficients a, b satisfying $|a| \leq d, |b| \leq d$. Applying this with $d = 1$ and using the obvious fact that for a Hadamard lacunary sequence $(n_k)_{k\geq 1}$ and $c \in \mathbb{Z}$ the number of solutions of

$$n_k \pm n_l = c \qquad (k \neq l)$$

is bounded by a constant which is independent of c, we get Theorem 2.1 of the previous section.

Theorem 3.4 shows that condition $\mathbf{D}_2$ is best possible in Theorem 3.3. We were not able to decide whether this condition is also best possible in Theorem 3.6, but condition $\mathbf{D}_2$ is nearly best possible in Theorem 3.6 in the following sense: if there exist nonzero integers a, b, c such that the Diophantine equation

$$an_k + bn_l = c$$

has infinitely many solutions with $k \neq l$, then the LIL for $f(n_{\sigma(k)}x)$ fails to hold for a suitable permutation σ and a suitable trigonometric polynomial f.

Theorem 3.8 *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying* (1.1) *and condition $\mathbf{D}_2^{(s)}$. Then for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ we have*

$$\limsup_{N\to\infty} \frac{ND_N(n_{\sigma(k)}x)}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad a.e. \tag{3.19}$$

All the results formulated so far assumed the Hadamard gap condition (1.1) or the stronger condition (3.4). If we weaken (1.1), i.e. we allow subexponential sequences $(n_k)_{k\geq 1}$, we need much stronger Diophantine conditions even for the unpermuted CLT and LIL for $f(n_k x)$. Specifically, we need uniform bounds for the number of solutions of Diophantine equations of the type

$$a_1 n_{k_1} + \ldots + a_p n_{k_p} = b. \tag{3.20}$$

Call a solution of (3.20) *nondegenerate* if no subsum of the left hand side equals 0. Let us say that a sequence $(n_k)_{k\geq 1}$ of positive integers satisfies

Condition $\mathbf{A}_p$, if there exists a constant $C_p \geq 1$ such that for any integer $b \neq 0$ and any nonzero integers $a_1, \ldots, a_p$ the number of nondegenerate solutions of the Diophantine equation (3.20) is at most C_p.

The following results are the analogues of Theorems 3.3–3.7 without growth conditions on $(n_k)_{k\geq 1}$.

Theorem 3.9 *Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers satisfying condition $\mathbf{A}_p$ for all $p \geq 2$. Let f satisfy* (3.5)*, let σ be a permutation of $\mathbb{N}$ and assume that the limit* (3.14) *exists. Then the permuted CLT* (3.15) *is valid.*

Theorem 3.10 *Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers satisfying condition $\mathbf{A}_p$ for all $p \geq 2$ with $C_p \leq \exp(Cp^\alpha)$ for some $\alpha > 0$. Moreover, assume that f satisfies* (3.5), σ *is a permutation of* $\mathbb{N}$ *and* (3.14) *holds. Then the permuted LIL* (3.18) *is valid.*

Note that for the validity of the LIL we require a specific bound for the constants C_p in condition $\mathbf{A}_p$. For subexponentially growing $(n_k)_{k\geq 1}$, verifying property $\mathbf{A}_p$ is a difficult number-theoretic problem. Classical examples of such sequences are the Hardy–Littlewood–Pólya sequences, i.e. increasing sequences $(n_k)_{k\geq 1}$ consisting of all positive integers of the form $q_1^{\alpha_1}\cdots q_\tau^{\alpha_\tau}$ $(\alpha_1,\ldots\alpha_\tau \geq 0)$, where $\{q_1,\ldots,q_\tau\}$ is a fixed set of coprime integers. Clearly, for $\tau \geq 2$ such sequences grow subexponentially; Tijdeman [39] proved that

$$n_{k+1} - n_k \geq \frac{n_k}{(\log n_k)^\alpha} \tag{3.21}$$

for some $\alpha > 0$, i.e. the growth of $(n_k)_{k\geq 1}$ is almost exponential. Hardy–Littlewood–Pólya sequences have remarkable probabilistic and ergodic properties. Nair [26] proved that if f is 1-periodic and integrable in $(0,1)$, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^N f(n_k x) = \int_0^1 f(t)dt \quad \text{a.e.}$$

Philipp [29] showed that the discrepancy of $\{n_k x\}$ satisfies the law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} \leq \limsup_{N\to\infty} \frac{ND_N(n_k x)}{\sqrt{2N\log\log N}} \leq C \quad \text{a.e.} \tag{3.22}$$

where C is a constant depending on the number of generators of $(n_k)_{k\geq 1}$. Recently, Fukuyama and Nakata [16] succeeded in computing the limsup in (3.22). Fukuyama and Petit [17] also showed that the central limit theorem

$$N^{-1/2}\sum_{k=1}^N f(n_k x) \to_d N(0,\gamma_f^*)$$

holds with

$$\gamma_f^* = \sum_{k,l:(n_k,n_l)=1} \int_0^1 f(n_k x) f(n_l x) dx. \tag{3.23}$$

The Diophantine properties of $(n_k)_{k\geq 1}$ have been studied in great detail in recent years; Amoroso and Viada [7] showed that Hardy–Littlewood–Pólya sequences satisfy condition $\mathbf{A}_p$ for any $p \geq 2$ with $C_p = \exp(p^6)$. This is a very deep result, involving a substantial sharpening of the subspace theorem of Evertse, Schlickewei and Schmidt (see [13]). Again, the limit γ in (3.14) depends on the permutation σ.

Since verifying condition $\mathbf{A}_p$ for a concrete subexponential sequence $(n_k)_{k\geq 1}$ is difficult, it is worth looking for Diophantine conditions which are strong enough to imply the permutation-invariant CLT and LIL, but which hold for a sufficiently large class of subexponential sequences. Such a Diophantine condition $\mathbf{A}_\omega$ will be given below. Actually, we will see that in a certain statistical sense, $\mathbf{A}_\omega$ is satisfied for "almost all" sequences $(n_k)_{k\geq 1}$ growing faster than polynomially and thus the permutation-invariant CLT and LIL are the "typical" behavior of sequences $f(n_k x)$ with superpolynomially growing $(n_k)_{k\geq 1}$. Given a nondecreasing sequence $\omega = (\omega_1, \omega_2, \dots)$ of positive numbers tending to $+\infty$, let us say that a sequence $(n_k)_{k\geq 1}$ of different positive integers satisfies

Condition $\mathbf{A}_\omega$, if for any $N \geq N_0$ the Diophantine equation

$$a_1 n_{k_1} + \ldots + a_r n_{k_r} = 0, \qquad 2 \leq r \leq \omega_N,\ 0 < |a_1|, \ldots, |a_r| \leq N^{\omega_N} \tag{3.24}$$

with different indices k_j and nonzero integer coefficients a_j has only such solutions where all n_{k_j} belong to the smallest N elements of the sequence $(n_k)_{k\geq 1}$.

Clearly, this property is permutation-invariant and it implies that for any fixed nonzero integer coefficients a_j the number of solutions of (3.24) with different indices k_j is at most N^r.

Theorem 3.11 *Let $\omega = (\omega_1, \omega_2, \dots)$ be a nondecreasing sequence tending to $+\infty$ and let $(n_k)_{k\geq 1}$ be a sequence of different positive integers satisfying condition $\mathbf{A}_\omega$. Then for any f satisfying* (3.5) *we have*

$$N^{-1/2} \sum_{k=1}^{N} f(n_k x) \longrightarrow_d \mathcal{N}(0, \|f\|^2). \tag{3.25}$$

If $\omega_k \geq (\log k)^\alpha$ for some $\alpha > 0$ and $k \geq k_0$, then we also have

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} f(n_k x)}{(2N \log\log N)^{1/2}} = \|f\| \quad \text{a.e.} \tag{3.26}$$

Condition $\mathbf{A}_\omega$ is different from the usual Diophantine conditions in lacunarity theory, which typically involve 4 or less terms. In contrast, $\mathbf{A}_\omega$ is an 'infinite order' condition, namely it involves equations with arbitrary large order. As is shown in [3], the usual Diophantine conditions do not suffice in Theorem 3.11. Given any $\omega_k \uparrow \infty$, it is not hard to see that any sufficiently rapidly growing sequence $(n_k)_{k\geq 1}$ satisfies $\mathbf{A}_\omega$; on the other hand, we do not have any "concrete" subexponential examples for $\mathbf{A}_\omega$. However, such examples not only exist, but we will show that, in a certain statistical sense, almost all sequences $(n_k)_{k\geq 1}$ growing faster than polynomially satisfy condition $\mathbf{A}_\omega$ for some appropriate ω. To make this precise requires defining a probability measure over the set of such sequences, or, equivalently, a natural random procedure to generate such sequences. A simple procedure is to choose n_k independently and uniformly from the integers in the interval

$$I_k = [a(k-1)^{\omega_{k-1}}, ak^{\omega_k}), \qquad k = 1, 2, \ldots. \tag{3.27}$$

Note that the length of I_k is at least $a\omega_k(k-1)^{\omega_k-1} \geq a\omega_1$ for $k = 2,3,\ldots$ and equals a for $k = 1$, and thus choosing a large enough, each I_k contains at least one integer. Let μ_ω be the distribution of the random sequence $(n_k)_{k\geq 1}$ in the product space $I_1 \times I_2 \times \ldots$.

Theorem 3.12 *Let $\omega_k \uparrow \infty$ and let f be a function satisfying* (3.5). *Then with probability one with respect to μ_ω the sequence $(f(n_k x))_{k\geq 1}$ satisfies the CLT* (3.25) *after any permutation of its terms, and if $\omega_k \geq (\log k)^\alpha$ for some $\alpha > 0$ and $k \geq k_0$, then $(f(n_k x))_{k\geq 1}$ also satisfies the LIL* (3.26) *after any permutation of its terms.*

The sequences $(n_k)_{k\geq 1}$ provided by μ_ω satisfy $n_k = O(k^{\omega_k})$; for slowly increasing ω_k the so obtained sequences grow much slower than exponentially, in fact they grow barely faster than with polynomial speed. If ω_k grows so slowly that $\omega_k - \omega_{k-1} = o((\log k)^{-1})$, then the so obtained sequence $(n_k)_{k\geq 1}$ has the precise speed $n_k \sim k^{\omega_k}$. We do not know if there exist polynomially growing sequences $(n_k)_{k\geq 1}$ satisfying the permutation-invariant CLT or LIL. As a simple combinatorial argument shows, sequences $(n_k)_{k\geq 1}$ satisfying $\mathbf{A}_p$ for all $p \geq 2$ cannot grow polynomially.

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