Upper-Lower Class Tests for Weighted i.i.d. Sequences and Martingales

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Abstract We study the upper-lower class behavior of weighted sums $\sum_{k=1}^{n} a_k X_k$, where X_k are i.i.d. random variables with mean 0 and variance 1. In contrast to Feller's classical results in the case of bounded X_j, we show that the refined LIL behavior of such sums depends not on the growth properties of (a_n) but on its arithmetical distribution, permitting pathological behavior even for bounded (a_n). We prove analogous results for weighted sums of stationary martingale difference sequences. These are new even in the unweighted case and complement the sharp results of Einmahl and Mason obtained in the bounded case. Finally, we prove a general upper-lower class test for unbounded martingales, improving several earlier results in the literature.

Keywords Weighted i.i.d. sequences · Upper-lower class tests · Law of the iterated logarithm · Martingales

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1 Introduction and Results

Let $X_1, X_2, \ldots$ be independent random variables with mean 0 and finite variances, and let $S_n = \sum_{k=1}^n X_k$, $s_n^2 = \sum_{k=1}^n E X_k^2$. By Kolmogorov's law of the iterated logarithm (see [15]), if

$$|X_n| \leq \varepsilon_n s_n / \left(\log \log s_n^2\right)^{1/2} \tag{1.1}$$

with a positive numerical sequence $\varepsilon_n \to 0$, then

$$\limsup_{n \to \infty} \left(2 s_n^2 \log \log s_n^2\right)^{-1/2} S_n = 1 \quad \text{a.s.} \tag{1.2}$$

Here and in the sequel, $\log \log x$ is meant as $\log \log(x \vee e^e)$. Condition (1.1) is best possible: assuming only

$$|X_n| \leq K s_n / \left(\log \log s_n^2\right)^{1/2}$$

with some constant $K > 0$, relation (1.2) is generally false (see Marczinkiewicz and Zygmund [17] and Weiss [21]). A much more refined result was proved by Feller [8], who showed that if

$$|X_n| \leq K s_n / \left(\log \log s_n^2\right)^{3/2} \tag{1.3}$$

and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function, then

$$P\left\{S_n > s_n \varphi\left(s_n^2\right) \text{ i.o.}\right\} = 0 \quad \text{or} \quad 1 \tag{1.4}$$

according as

$$I(\varphi) := \int_1^\infty t^{-1} \varphi(t) e^{-\varphi(t)^2/2} \, dt < \infty \quad \text{or} \quad = \infty. \tag{1.5}$$

As customary, we say that φ belongs to the upper or lower class with respect to S_n according as the probability in (1.4) is 0 or 1. Condition (1.3) is also best possible: given any sequence $a_n \to \infty$, there exists a sequence (X_n) of independent zero-mean random variables satisfying (1.3) such that the sequence $a_n X_n$ fails the test (1.4)–(1.5). Specifically, Feller showed ([8], Theorem 5) that if

$$a_n = O\left(\left(\log \log s_n^2\right)^\alpha\right), \quad 0 < \alpha < 1,$$

then the test (1.4)–(1.5) remains valid, but the exponent $-\varphi(t)^2/2$ in the integral in (1.5) should be replaced by a polynomial of $\varphi(t)$ of degree $l + 1$, where l is the smallest positive integer with $l \geq 1/(1 - \alpha)$. If α approaches 1 (which means approaching the Kolmogorov condition (1.1)), then $l \to \infty$ and in the limiting case even the LIL breaks down.

For unbounded (X_n), the results are much less complete; for a sufficient condition for the integral test (1.4)–(1.5), see the Appendix of Feller [8]. For i.i.d. random variables (X_n) with $E X_1 = 0$ and $E X_1^2 = 1$, Feller [9] proved that $E X_1 = 0$, $E X_1^2 = 1$, and

$$E X_1^2 I\left(|X_1| > t\right) = O\left(\left(\log \log t\right)^{-1}\right) \tag{1.6}$$

imply (1.4)–(1.5) and the last condition is best possible. In particular, the test holds if

$$EX_1^2 \log \log |X_1| < \infty. \tag{1.7}$$

This condition is also optimal in the sense that given any function $\psi(x) = o(x^2 \log \log x)$, $x \to \infty$, there is an i.i.d. sequence (X_n) with $EX_1 = 0$, $EX_1^2 = 1$, and $E\psi(|X_1|) < +\infty$ such that the test (1.4)–(1.5) fails. Note that Feller's proof in [9] is valid only under the assumption that the distribution of X_1 is symmetric. Einmahl [4] was the first to remove this assumption; see Bai [2] for another proof of this result, using ideas from Einmahl [4].

The purpose of the present paper is to study the upper-lower class behavior of weighted sums $S_n = \sum_{k=1}^n a_k X_k$, where (X_n) is an i.i.d. sequence satisfying $EX_1 = 0$ and $EX_1^2 = 1$. We will assume (1.7) to guarantee that the test (1.4)–(1.5) is satisfied for $a_k = 1$. In contrast to the case of bounded (X_n) discussed above, we will see that in the weighted i.i.d. case the fine asymptotics of $S_n = \sum_{k=1}^n a_k X_k$ depends not on the growth speed of (a_n) but on its arithmetical distribution. For example, we will see that there exists a bounded sequence (a_n) such that the test (1.4)–(1.5) fails for $a_n X_n$. Our main result is the following:

Theorem 1 *Let (a_k) be a sequence of nonzero real numbers, $s_n^2 = \sum_{k=1}^n a_k^2$, and assume that $s_n \to \infty$ and $|a_n| = O(s_n^{1-\delta})$ for some $\delta > 0$. Let $M(x) = \#\{k \geq 1 : |s_k/a_k| \leq x\}$ for $x > 0$ and assume that*

$$M(x) \ll x^2 \quad as\ x \to \infty. \tag{1.8}$$

Then for any i.i.d. sequence (X_n) satisfying $EX_1 = 0$, $EX_1^2 = 1$, and (1.7), the weighted partial sums $S_n = \sum_{k=1}^n a_k X_k$ satisfy the test (1.4)–(1.5). Conversely, if the last statement is valid, we have

$$M(x) \ll x^2 (\log \log x)^2 \quad as\ x \to \infty. \tag{1.9}$$

Here and in the sequel, $\ll$ means the same as the O notation. A necessary and sufficient condition for the weighted strong law of large numbers in terms of the distributional properties of the coefficients was given by Jamison et al. [14]; a similar criterion for the weighted LIL is implicit in Fisher [10]. (See also Weber and Lin [20].) Adapting an argument of Jamison et al. [14], it is easy to show that there exists a bounded sequence (a_k) of positive numbers such that $\sum_{k=1}^\infty a_k^2 = \infty$ and

$$\limsup_{x \to \infty} M(x)/(x^2 \log x) \geq 1/2.$$

Thus, by the second half of Theorem 1, there exists an i.i.d. sequence X_n satisfying $EX_1 = 0$, $EX_1^2 = 1$, and (1.7) such that $a_n X_n$ fails the test (1.4)–(1.5) for some positive bounded sequence (a_n) with $s_n^2 = \sum_{k=1}^n a_k^2 \to \infty$.

It is important to note that in Theorem 1 we work under the minimal moment condition (1.7). If we assume the stronger condition $EX_1^2 (\log |X_1|)^\gamma < \infty$ for some $\gamma > 1$, then it follows easily that $S_n = \sum_{k=1}^n a_k X_k$ satisfies the integral test (1.4)–(1.5), provided that $|a_n| = O(s_n^{1-\delta})$ for some $\delta > 0$, i.e., the arithmetic condition (1.8) in Theorem 1 becomes unnecessary.

In what follows, we give examples for irregular sequences (a_n) satisfying the arithmetic condition (1.8). A good source of such examples is number theory, where the irregularity of the sequence of primes provides the required properties of (a_n). A function f defined on the positive integers is called *strongly additive* if $f(mn) = f(m) + f(n)$ for coprime m and n and $f(p^r) = f(p)$ for prime p and $r = 1, 2, \ldots$. Clearly, $f(n) = \sum_{p|n} f(p)$ (here and in the sequel p denotes primes), which causes f to behave rather irregularly.

Example 1 Let $f \geq 0$ be a strongly additive number-theoretic function with $D_N^2 := \sum_{p \leq N} \frac{f^2(p)}{p} \to +\infty$ and assume that $f(p) = o(D_p)$, $p \to \infty$. Then the sequence $a_n = f(n)$ satisfies condition (1.8).

For example, we can choose $f(n) = \omega(n)$, the number of different prime factors of n. Additive functions f satisfying the assumptions of Example 1 play a prominent role in the probability theory of additive functions, see, e.g., Erdős and Kac [7] or Kubilius [16].

Another simple example for a suitable weight sequence in Theorem 1 is provided by the realizations of stationary ergodic sequences.

Example 2 Let $\{\xi_n, n \geq 1\}$ be a stationary ergodic sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ satisfying $0 < E|\xi_1|^p < \infty$ for some $p > 2$. Then for almost every $\omega \in \Omega$, the sequence $a_n = \xi_n(\omega)$ satisfies condition (1.8).

For example, let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function with period 1 satisfying $0 < \int_0^1 |f(x)|^p \, dx < \infty$ for some $p > 2$. Then $a_n = f(2^n x)$ satisfies (1.8) for almost every x in the sense of the Lebesgue measure.

Our next theorem extends Theorem 1 for weighted sums of stationary martingale difference sequences. Let $\{X_n, n \in \mathbb{Z}\}$ be a strictly stationary ergodic sequence satisfying $E[X_n|\mathcal{F}_{n-1}] = 0$ and $EX_0^2 = 1$, where $\mathcal{F}_n = \sigma(X_j, j \leq n)$. Let (a_n) be a sequence of nonzero real numbers, put $s_n^2 = \sum_{k=1}^n a_k^2 E(X_k^2|\mathcal{F}_{k-1})$, and define $M(x)$ as in Theorem 1. To keep the arithmetical condition $M(x) \ll x^2$ nonrandom, we will assume that $s_n^2 \sim B_n^2$ a.s. with $B_n^2 = \sum_{k=1}^n a_k^2$, or, alternatively,

$$B_n^{-2} \sum_{k=1}^n a_k^2 X_k^2 \longrightarrow 1 \quad \text{a.s.} \tag{1.10}$$

For $a_k = 1$, the validity of (1.10) is immediate from the ergodic theorem, and in the case where (X_n) is an i.i.d. sequence, a classical theorem of Jamison et al. [14] provides necessary and sufficient conditions for the weighted strong law (1.10). In the general stationary case, however, proving weighted strong laws is a difficult problem, and one needs restrictive conditions for such results. Let (w_k) be a sequence of positive reals and put $W_n = \sum_{k=1}^n w_k$. We say that $(w_k) \in \mathcal{W}$ if $W_n \to \infty$ and for every

stationary ergodic sequence (Z_k) with finite means, we have

$$W_n^{-1} \sum_{k=1}^{n} w_k Z_k \longrightarrow EZ_1 \quad \text{a.s.} \tag{1.11}$$

Sufficient criteria for (1.11) will be given below. Using this terminology, we can formulate now our result:

Theorem 2 *Let $\{X_n, \mathcal{F}_n, n \in \mathbb{Z}\}$ be a stationary ergodic martingale difference sequence satisfying $EX_1^2 = 1$ and (1.7). Let (a_k) be a sequence of nonzero real numbers with $(a_k^2) \in \mathcal{W}$ and assume that letting $B_n^2 = \sum_{k=1}^{n} a_k^2$ we have*

$$|a_n| = O\left(B_n^{1-\delta}\right) \quad \text{for some } \delta > 0.$$

Let $M(x) = \#\{k \geq 1 : |B_k/a_k| \leq x\}$ for $x > 0$ and assume that $M(x) \ll x^2$ as $x \to \infty$. Then the test (1.4)–(1.5) holds with

$$s_n^2 = \sum_{k=1}^{n} a_k^2 E\left[X_k^2 | \mathcal{F}_{k-1}\right].$$

Note that Theorem 2 is new even in the case $a_k = 1$. By the ergodic theorem $(a_k^2) \in \mathcal{W}$, and since $B_n^2 = n$, we have $M(x) = x^2$. Hence we get the following:

Corollary 1 *Let $\{X_n, \mathcal{F}_n, n \in \mathbb{Z}\}$ be a stationary ergodic martingale difference sequence satisfying $EX_1^2 = 1$ and*

$$EX_1^2 \log \log |X_1| < \infty.$$

Then the test (1.4)–(1.5) holds with $s_n^2 = \sum_{k=1}^{n} E[X_k^2 | \mathcal{F}_{k-1}]$.

In view of Feller's results in [9], the moment condition in Corollary 1 is best possible in the sense of the discussion following (1.7). Note that in Theorem 2 and Corollary 1 the scaling s_n in the upper-lower class test (1.4)–(1.5) is random, even though $s_n \sim B_n$ with a nonrandom B_n. This is a common feature of upper-lower class tests for martingales, see the remarks in Jain et al. [13], p. 127.

For general (a_k), a simple sufficient condition for the conclusion of Theorem 2 is

$$\frac{1}{B_n^2} \sum_{k=1}^{n-1} k|a_{k+1}^2 - a_k^2| = O(1), \tag{1.12}$$

as one can verify by simple calculations, using the ergodic theorem. In particular, (1.12) is satisfied if (a_n) is nonnegative, nondecreasing, and regularly varying.

Theorem 2 will be deduced from a general integral test (Theorem 3) for martingale difference sequences which has some own interest.

Theorem 3 *Let $\{X_n, \mathcal{F}_n, n \in \mathbb{Z}\}$ be a martingale difference sequence with $E X_k^2 < \infty$ ($k \in \mathbb{Z}$). Assume that $B_n^2 := \sum_{k=1}^n E X_k^2 \to \infty$ and that for some $\delta > 0$, the following properties hold:*

(a) $\sum_{n=1}^{\infty} f(n)^{-4} E[X_n^4 I\{|X_n| \le \delta f(n)\}] < \infty$;
(b) $\sum_{n=1}^{\infty} f(n)^{-1} E[|X_n| I\{|X_n| > \delta f(n)\}] < \infty$;
(c) $B_n^{-2} \sum_{k=1}^n X_k^2 \to 1$ a.s.;
(d) $f^{-2}(n) \sum_{k=1}^n E[X_k^2 I\{|X_k| > \delta f(k)\}|\mathcal{F}_{k-1}] \to 0$,

where

$$f(k) = B_k/(\log \log B_k)^{1/2}.$$

Then the test (1.4)–(1.5) *is valid with* $s_n^2 = \sum_{k=1}^n E[X_k^2|\mathcal{F}_{k-1}]$.

The first to prove upper-lower class tests for martingales was Strassen [19], who proved that if $\{X_n, \mathcal{F}_n, n \in \mathbb{Z}\}$ is a martingale difference sequence with $E X_n^2 < \infty$ ($n \in \mathbb{Z}$), $S_n = X_1 + \cdots + X_n$, and $s_n^2 = \sum_{k=1}^n E[X_k^2|\mathcal{F}_{k-1}] \to \infty$ a.s., then the integral test (1.4)–(1.5) holds, provided that

$$|X_n| \le s_n/(\log s_n)^{\alpha}, \quad \alpha > 4. \tag{1.13}$$

Philipp and Stout [18] weakened (1.13) to

$$|X_n| \le s_n/(\log \log s_n)^{5/2}, \tag{1.14}$$

and Einmahl and Mason [6] showed that the test (1.4)–(1.5) actually holds under (1.3), a result which is optimal in view of Feller's theorem. The crucial new idea in [6] was to deduce the test (1.4)–(1.5) directly from the properties of the stopping times in the Skorokhod representation for the partial sums $\sum_{k=1}^n X_k$, bypassing the strong approximation of the partial sum process, which had been used in all earlier results in the field.

Results for unbounded (X_n) are far less satisfactory. Using the truncation method of Feller [8], pp. 399–401, from the result of Einmahl and Mason it follows that the test (1.4)–(1.5) holds provided that

$$\sum_{k=1}^{\infty} s_k^{-2} (\log \log s_k)^3 E[X_k^2 I\{|X_k| > K s_k/(\log \log s_k)^{3/2}\}|\mathcal{F}_{k-1}] < \infty \quad \text{a.s.} \tag{1.15}$$

for some constant $K > 0$. Similar conditions (and sets of conditions) were given in Jain et al. [13] and Philipp and Stout [18]. See also Einmahl and Mason [5] for analogous results for the Darling–Erdős theorem. While (1.15) is sharp in the case of bounded (X_n), it is far from optimal in typical unbounded situations such as stationary (X_n), where it requires $E X_n^2 (\log |X_n|)^{\gamma} < \infty$ for some $\gamma > 0$. Our Theorem 3 requires in the stationary case only (1.7), which is optimal in the sense discussed above. The improvement is achieved by using the method of Einmahl and Mason and employing, in contrast to [13], [18], and [19], a nonrandom truncation, an idea used first in the context of the LIL by Heyde and Scott [12].

2 Proofs

The following lemma is a consequence of the general theory of summation, see, e.g., Hardy [11].

Lemma 1 *Let $d_n \geq 0$, $n = 1, 2, \ldots$, and assume that $D_n := \sum_{k=1}^{n} d_k \to \infty$ and $d_n = o(D_n^{1-\delta})$ for some $\delta > 0$. If for some real sequence (x_n), the weighted averages $D_n^{-1} \sum_{k=1}^{n} d_k x_k$ converge to some $x \in \mathbb{R}$, then we also have*

$$\frac{\log \log D_n}{D_n} \sum_{k=1}^{n} \frac{d_k x_k}{\log \log D_k} \to x. \tag{2.1}$$

Proof of Theorem 1 We will deduce Theorem 1 from Theorem 3, whose proof will be given later. Assume that (X_n) is an i.i.d. sequence satisfying $EX_1 = 0$, $EX_1^2 = 1$, and the moment condition (1.7) and assume that the arithmetic condition (1.8) holds. Applying Theorem 3 for the martingale difference sequence $a_n X_n$, the quantities B_n^2 and s_n^2 in Theorem 3 reduce to the quantity $s_n^2 = \sum_{k=1}^{n} a_k^2$ in Theorem 1, and thus it suffices to verify the following four conditions:

$$\sum_{n=1}^{\infty} \frac{a_n^4}{s_n^4} (\log \log s_n)^2 E\left[X_1^4 I\{|X_1| \leq |s_n/a_n|(\log \log s_n)^{-1/2}\}\right] < \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{|a_n|}{s_n} (\log \log s_n)^{1/2} E\left[|X_1| I\{|X_1| \geq |s_n/a_n|(\log \log s_n)^{-1/2}\}\right] < \infty, \tag{2.3}$$

$$s_n^{-2} \sum_{k=1}^{n} a_k^2 X_k^2 \to 1 \quad \text{a.s.,} \tag{2.4}$$

$$\frac{\log \log s_n}{s_n^2} \sum_{k=1}^{n} a_k^2 E\left[X_1^2 I\{|X_1| \geq |s_k/a_k|(\log \log s_k)^{-1/2}\}\right] \to 0. \tag{2.5}$$

The validity of (2.4) is an immediate consequence of (1.8) and the strong law of large numbers of Jamison et al. [14]. To prove (2.2) and (2.3) introduce

$$N(x) = \#\{k \geq 1 : |s_k/a_k|(\log \log s_k)^{-1/2} \leq x\}.$$

Let us note that $|a_n| = O(s_n^{1-\delta})$ implies

$$\log \log s_n \ll \log \log |s_n/a_n|, \tag{2.6}$$

and consequently each of the inequalities

$$s_n^2/(a_n^2 \log \log s_n) \leq x^2, \qquad \frac{s_n^2/a_n^2}{\log \log(s_n^2/a_n^2)} \ll x^2, \qquad s_n^2/a_n^2 \ll x^2 \log \log x$$

implies the next one. Thus, (1.8) implies that

$$N(x) \ll x^2 \log \log x \quad \text{as } x \to \infty. \tag{2.7}$$

Clearly, the sum in (2.2) equals $\int_{-\infty}^{\infty} x^4 A(x)\, dF(x)$, where F is the common distribution function of the X_n's, and

$$A(x) = \sum_{n=1}^{\infty} \frac{a_n^4}{s_n^4} (\log\log s_n)^2 I\big(|x| \leq |s_n/a_n|(\log\log s_n)^{-1/2}\big) = \int_{|x|}^{\infty} \frac{dN(t)}{t^4}.$$

$A(x)$ is an even function, and integration by parts gives for $x > 0$,

$$\int_x^{\infty} \frac{dN(t)}{t^4} = -\frac{N(x)}{x^4} + 4\int_x^{\infty} \frac{N(t)}{t^5}\, dt. \tag{2.8}$$

(Since the total mass of N over $(0, \infty)$ is infinite, to make the proof of (2.8) precise, one has to apply integration by parts over (x, T) and then let $T \to \infty$, using the fact that $\int_x^{\infty} N(t)/t^5\, dt < \infty$ and $N(T)/T^4 \to 0$ by (2.7).) By (2.7) the right-hand side of (2.8) is $O(x^{-2}\log\log x)$, and thus the sum in (2.2) is

$$\int_{-\infty}^{\infty} x^4 A(x)\, dF(x) \ll \int_{-\infty}^{\infty} x^2 \log\log|x|\, dF(x) < \infty$$

by (1.7). Next, we prove (2.3). Let us note that by $|a_n| = O(s_n^{1-\delta})$ we have

$$|s_n/a_n|(\log\log s_n)^{-1/2} \geq s_n^{\delta/2} \tag{2.9}$$

for sufficiently large n, and thus we can change finitely many of the a_n's so that $|s_n/a_n|(\log\log s_n)^{-1/2} > 1$ for $n \geq 1$. Consequently, we can assume that $N(x) = 0$ for $0 \leq x \leq 1$. Similarly as before, the sum in (2.3) is $\int_{-\infty}^{\infty} |x| B(x)\, dF(x)$, where

$$B(x) = \sum_{n=1}^{\infty} \frac{|a_n|}{s_n} (\log\log s_n)^{1/2} I\big(|x| \geq |s_n/a_n|(\log\log s_n)^{-1/2}\big)$$

$$= \int_1^x \frac{dN(t)}{t} = \frac{N(x)}{x} + \int_1^x \frac{N(t)}{t^2}\, dt = O(x\log\log x).$$

Hence the sum in (2.3) is $\int_{-\infty}^{\infty} |x| B(x)\, dF(x) \ll \int_{-\infty}^{\infty} x^2 \log\log|x|\, dF(x) < \infty$ by (1.7). It remains to show (2.5). By (2.6) and (2.9) we get

$$\frac{\log\log s_n}{s_n^2} \sum_{k=1}^{n} a_k^2 E\big[X_1^2 I\{|X_1| \geq |s_k/a_k|(\log\log s_k)^{-1/2}\}\big]$$

$$\ll \frac{\log\log s_n}{s_n^2} \sum_{k=1}^{n} \frac{a_k^2}{\log\log s_k} E\big[X_1^2 \log\log X_1^2 I\{|X_1| \geq |s_k/a_k|(\log\log s_k)^{-1/2}\}\big].$$

Using (1.7) and (2.9), it follows that

$$E\big[X_1^2 \log\log X_1^2 I\{|X_1| \geq |s_k/a_k|(\log\log s_k)^{-1/2}\}\big] \to 0,$$

and thus (2.5) is an immediate consequence of Lemma 1.

To prove the converse part of Theorem 1, let (a_n) be a sequence of real numbers satisfying the assumptions of the theorem and assume that for any i.i.d. sequence (X_n) with

$$EX_1 = 0, \qquad EX_1^2 = 1, \qquad EX_1^2 \log\log|X_1| < \infty,$$

the partial sums $\sum_{k=1}^{n} a_k X_k$ satisfy the test (1.4)–(1.5), but (1.9) fails. Let

$$N^*(x) = \#\{k \geq 1 : |s_k/a_k|(\log\log s_k)^{1/2} \leq x\}.$$

Relation (2.6) shows that the inequality $|s_k/a_k| \leq x$ implies

$$|s_k/a_k|(\log\log s_k)^{1/2} \leq Cx(\log\log x)^{1/2}$$

for some constant C, and consequently

$$M(x) \leq N^*\big(Cx(\log\log x)^{1/2}\big).$$

Thus if (1.9) fails, then

$$N^*(x) \ll x^2 \log\log x$$

cannot be valid, either. Thus there exists an increasing sequence (x_k) of positive numbers with $x_1 = 1/100$ and $x_k \to \infty$ such that

$$N^*(x_k)/x_k^2 \log\log x_k \to \infty.$$

Then there exists a sequence (f_k) of positive numbers such that

$$\sum_{k=1}^{\infty} f_k x_k^2 \log\log x_k < \infty, \qquad \sum_{k=1}^{\infty} f_k N^*(x_k) = \infty.$$

In particular, $\sum_{k=1}^{\infty} f_k < \infty$, and thus by scaling (f_k) we can assume that $\sum_{k=1}^{\infty} f_k = 1$. For a fixed $r \geq 1$, define (f_n^*) so that $f_1^* = f_1 + \cdots + f_r$, $f_2^* = \cdots = f_r^* = 0$, and $f_n^* = f_n$ for $n > r$. Clearly $\sum_{n=1}^{\infty} f_n^* x_n^2 = (f_1 + \cdots + f_r)x_1^2 + \sum_{n>r} f_n x_n^2 \leq x_1^2 + \sum_{n>r} f_n x_n^2 \log\log x_n < 2 \cdot 10^{-4}$, provided that we choose r so large that $\sum_{n>r} f_n x_n^2 \log\log x_n \leq 10^{-4}$. Hence without loss of generality we may assume that $\sum_{n=1}^{\infty} f_n x_n^2 \leq 2 \cdot 10^{-4}$. Let (Y_n) be i.i.d. random variables with $P(Y_1 = x_k) = P(Y_1 = -x_k) = \frac{1}{2} f_k$ $(k = 1, 2, \ldots)$; clearly $EY_1 = 0$, $EY_1^2 \leq 2 \cdot 10^{-4}$, $EY_1^2 \log\log|Y_1| < \infty$, and $EN^*(|Y_1|) = \infty$. Let $X_n = cY_n$, where $c \geq 10$ is chosen so that $EX_1^2 = 1$; clearly $EX_1 = 0$ and $EX_1^2 \log\log|X_1| < \infty$. Let F denote the distribution function of Y_1. We claim that $a_n X_n$ cannot satisfy the LIL

$$\limsup_{n\to\infty} \big(2s_n^2 \log\log s_n\big)^{1/2} \sum_{k=1}^{n} a_k X_k = 1 \quad \text{a.s.,} \tag{2.10}$$

and thus the test (1.4)–(1.5) for $a_n X_n$ also fails. Indeed, if (2.10) were true, then using $s_{n+1}/s_n \to 1$ (which follows from $a_n/s_n \to 0$), we would have almost surely

for sufficiently large n,

$$\left|\sum_{k=1}^{n} a_k X_k\right| < 4\left(s_n^2 \log\log s_n\right)^{1/2}, \qquad \left|\sum_{k=1}^{n-1} a_k X_k\right| < 4\left(s_n^2 \log\log s_n\right)^{1/2},$$

and consequently

$$P\left(|a_n X_n| \geq 8\left(s_n^2 \log\log s_n\right)^{1/2} \text{ i.o.}\right) = 0.$$

Thus by the Borel–Cantelli lemma and $c \geq 10$

$$\sum_{n=1}^{\infty} P\left(|a_n X_n| \geq c\left(s_n^2 \log\log s_n\right)^{1/2}\right) < \infty. \tag{2.11}$$

However, the last sum equals

$$\sum_{n=1}^{\infty} E I\left\{|Y_n| \geq |s_n/a_n|(\log\log s_n)^{1/2}\right\}$$

$$= \int_{-\infty}^{\infty} \left[\sum_{n=1}^{\infty} I\left(|x| \geq |s_n/a_n|(\log\log s_n)^{1/2}\right)\right] dF(x)$$

$$= \int_{-\infty}^{\infty} N^*\left(|x|\right) dF(x) = E N^*\left(|Y_1|\right) = \infty,$$

contradicting to (2.11). □

Proof of Theorem 2 We will deduce Theorem 2 also from Theorem 3. The verification of conditions (a), (b) of Theorem 3 (which leads again to (2.2), (2.3)) remains unchanged. The validity of (c) follows from $(a_k^2) \in \mathcal{W}$. Finally, verifying condition (d) for $\delta = 1$ requires to show that

$$\frac{\log\log B_n}{B_n^2} \sum_{k=1}^{n} a_k^2 E\left[X_k^2 I\left\{|X_k| \geq |B_k/a_k|(\log\log B_k)^{-1/2}\right\}\big|\mathcal{F}_{k-1}\right] \to 0 \quad \text{a.s.} \tag{2.12}$$

Let

$$Z_k(m) := E\left[X_k^2 \log\log X_k^2 I\left\{|X_k| \geq |B_m/a_m|(\log\log B_m)^{-1/2}\right\}\big|\mathcal{F}_{k-1}\right].$$

By the assumption $|a_n| = O(B_n^{1-\delta})$, the inequality $|X_k| \geq |B_k/a_k|(\log\log B_k)^{-1/2}$ implies $|X_k| \geq c B_k^{\delta/2}$, and consequently $\log\log X_k^2 \geq c \log\log B_k$ with some constant $c > 0$. Thus the left-hand side of (2.12) is bounded by

$$\text{const} \cdot \frac{\log\log B_n}{B_n^2} \sum_{k=1}^{n} \frac{a_k^2}{\log\log B_k} Z_k(k)$$

with a constant independent of n. Observe that the sequence $\{Z_k(m), k \geq 1\}$ is stationary and ergodic for every fixed m, and thus by $(a_k^2) \in \mathcal{W}$ and Lemma 1 we have almost surely for every $\varepsilon > 0$,

$$\frac{\log \log B_n}{B_n^2} \sum_{k=1}^{n} \frac{a_k^2}{\log \log B_k} Z_k(m) \longrightarrow E Z_1(m) < \varepsilon$$

if m is chosen large enough. Clearly $\{Z_k(m), m \geq 1\}$ is nonincreasing for every k, and hence (2.12) follows immediately.

The fact that the sequence $a_n = f(n)$ in Example 1 satisfies condition (1.8) is implicit in Berkes and Weber [3], where it is proved that under the same conditions on f, the weighted i.i.d. sums $\sum_{k \leq N} f(k) X_k$ satisfy the LIL under $E X_1^2 = 0$, $E X_1^2 = 1$. Proving (1.8), i.e.,

$$M(t) := \#\left\{n \geq 1 : \sum_{k \leq n} f^2(k) \leq t f^2(n)\right\} \ll t^2 \quad \text{as } t \to \infty, \qquad (2.13)$$

directly in this case seems to be difficult, and in Berkes and Weber [3] an indirect argument was used, showing that (2.13) is equivalent to the validity of

$$E X^4 \int_{y \geq X^2} \frac{M(y)}{y^3} \, dy < \infty$$

for all r.v.'s X with $E X^2 < \infty$. The last relation can be verified by a randomization argument (see [3], pp. 1229–1231) using elementary arithmetic properties of additive functions.

To verify the statement in Example 2, let (ξ_n) be a stationary ergodic sequence with $0 < E|\xi_1|^p < \infty$ for some $p > 2$. Then one can find a probability space $(\Omega, \mathcal{A}, P)$, a function $f \in L^p(\Omega, \mathcal{A}, P)$, and an ergodic transformation $\tau : \Omega \to \Omega$ such that the sequence (η_n) of random variables defined by $\eta_n(\omega) = f(\tau^n \omega)$ has the same distribution as (ξ_n). By a result of Assani [1] we have, letting $g = f^2$,

$$\lim_{t \to \infty} \frac{\#\{n : g(\tau^n \omega)/n \geq 1/t\}}{t} = \int_{\Omega} g \, dP \quad \text{a.s.}$$

Using the ergodic theorem, the last relation implies, letting $D = \int_{\Omega} f^2 dP > 0$,

$$\lim_{t \to \infty} \frac{\#\{n : (\sum_{k=1}^{n} f^2(\tau^k \omega))^{1/2}/f(\tau^n \omega) \leq t\}}{t^2}$$

$$= \lim_{t \to \infty} \frac{\#\{n : Dn(1 + o(1))/f^2(\tau^n \omega) \leq t^2\}}{t^2} = 1 \quad \text{a.s.},$$

which shows that the sequence (η_n), and thus also (ξ_n), satisfies (1.8). On the other hand, $E|\xi_1|^p < \infty$, $p > 2$, and the Borel–Cantelli lemma imply that $|\xi_n| = O(n^{(1-\delta)/2})$ a.s. with a suitable $\delta > 0$, and thus, using the ergodic theorem and

$E\xi_1^2 > 0$, it follows that

$$|\xi_n| = O\left(\sum_{k=1}^{n} \xi_k^2\right)^{(1-\delta)/2} \quad \text{a.s.}$$

Hence (ξ_n) satisfies the coefficient condition $|a_n| = O(s_n^{1-\delta})$ of Theorem 1. $\square$

Proof of Theorem 3 We start with defining a truncated MDS $\{X_k^*, \mathcal{F}_k^*, k \geq 1\}$ as follows:

$$X_k^* = X_k I\{|X_k| \leq \delta f(k)\} - E\big[X_k I\{|X_k| \leq \delta f(k)\}\big|\mathcal{F}_{k-1}^*\big], \qquad (2.14)$$

where

$$f(k) = B_k/(\log\log B_k)^{1/2}$$

is the function defined in Theorem 3. Here $\mathcal{F}_k^* = \sigma(X_1^*, \ldots, X_k^*)$ if $k \geq 1$ and $\mathcal{F}_0^* = \{\emptyset, \Omega\}$. (Note that starting with $\mathcal{F}_0^*$, relation (2.14) determines successively $X_1^*, \mathcal{F}_1^*, X_2^*, \mathcal{F}_2^*, \ldots$.) Denote by S_n^* the partial sum $X_1^* + \cdots + X_n^*$ and, similarly to s_n^2, set

$$s_n^{*2} = \sum_{k=1}^{n} E\big[X_k^{*2}\big|\mathcal{F}_{k-1}^*\big].$$

Finally set $X_k^{**} = X_k - X_k^*$ and $S_n^{**} = X_1^{**} + \cdots + X_n^{**}$. Clearly,

$$\big|E\big[X_k I\{|X_k| \leq \delta f(k)\}\big|\mathcal{F}_{k-1}^*\big]\big| = \big|E\big[X_k I\{|X_k| > \delta f(k)\}\big|\mathcal{F}_{k-1}^*\big]\big| \quad \text{a.s.} \quad (2.15)$$

$\square$

Lemma 2 *Under condition* (b), *we have*

$$f(n)^{-2} \sum_{k=1}^{n} X_k^2 I\{|X_k| > \delta f(k)\} \to 0 \quad a.s.$$

Proof We have

$$\sum_{k=1}^{n} X_k^2 I\{|X_k| > \delta f(k)\} = \sum_{k=1}^{n} X_k^2 I^2\{|X_k| > \delta f(k)\}$$

$$\leq \left(\sum_{k=1}^{n} |X_k| I\{|X_k| > \delta f(k)\}\right)^2,$$

and thus the result follows from (b) and Kronecker's lemma. $\square$

In the sequel, $c_n \sim d_n$ means $\lim_{n\to\infty} c_n/d_n = 1$.

Lemma 3 *Under conditions* (a)–(d), *we have*

$$s_n^{*2} \sim B_n^2 \sim s_n^2 \quad a.s.$$

Proof We will show that $s_n^{*2} \sim B_n^2$; the proof of the second relation is similar. An easy calculation gives

$$B_n^{-2} \sum_{k=1}^{n} E\big[X_k^{*2}|\mathcal{F}_{k-1}^*\big]$$

$$= B_n^{-2} \sum_{k=1}^{n} E\big[X_k^2 I\big\{|X_k| \le \delta f(k)\big\}\big|\mathcal{F}_{k-1}^*\big]$$

$$- B_n^{-2} \sum_{k=1}^{n} \big(E\big[X_k I\big\{|X_k| \le \delta f(k)\big\}\big|\mathcal{F}_{k-1}^*\big]\big)^2. \tag{2.16}$$

By (2.15) we obtain

$$B_k^{-2}\big(E\big[X_k I\big\{|X_k| \le \delta f(k)\big\}\big|\mathcal{F}_{k-1}^*\big]\big)^2$$

$$\le \delta\big(f(k)\log\log B_k\big)^{-1} E\big[|X_k| I\big\{|X_k| > \delta f(k)\big\}\big|\mathcal{F}_{k-1}^*\big],$$

and thus by Kronecker's lemma and (b) the second term in (2.16) tends to zero. Set

$$Y_k := X_k^2 I\big\{|X_k| \le \delta f(k)\big\} - E\big[X_k^2 I\big\{|X_k| \le \delta f(k)\big\}\big|\mathcal{F}_{k-1}^*\big].$$

Clearly (Y_k) is an MDS, and since by (a)

$$\sum_{k=1}^{\infty} B_k^{-4} E Y_k^2 \le \sum_{k=1}^{\infty} B_k^{-4} E\big[X_k^4 I\big\{|X_k| \le \delta f(k)\big\}\big] < \infty,$$

the martingale convergence theorem and Kronecker's lemma imply that $B_n^{-2} \sum_{k=1}^{n} Y_k \to 0$ almost surely, i.e.,

$$B_n^{-2} \sum_{k=1}^{n} \big(X_k^2 I\big\{|X_k| \le \delta f(k)\big\} - E\big[X_k^2 I\big\{|X_k| \le \delta f(k)\big\}\big|\mathcal{F}_{k-1}^*\big]\big) \to 0 \quad \text{a.s.}$$

In view of (c), (2.16), and the last relation, it remains to show that

$$B_n^{-2} \sum_{k=1}^{n} X_k^2 I\big\{|X_k| > \delta f(k)\big\} \to 0 \quad \text{a.s.},$$

which follows from Lemma 2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4 *Under conditions* (a)–(b), *we have*

$$\sum_{k=1}^{\infty} f(k)^{-4} E X_k^{*4} < \infty \quad a.s.$$

Proof By (2.15) we have

$$\sum_{k=1}^{\infty} f(k)^{-4} E X_k^{*4}$$

$$= \sum_{k=1}^{\infty} f(k)^{-4} E \big(X_k I \{ |X_k| \leq \delta f(k) \} - E \big[X_k I \{ |X_k| \leq \delta f(k) \} | \mathcal{F}_{k-1}^* \big] \big)^4$$

$$\leq \sum_{k=1}^{\infty} f(k)^{-4} \big(E \big[X_k^4 I \{ |X_k| \leq \delta f(k) \} \big]$$

$$+ 15 \delta^3 f(k)^3 E \big| E \big[X_k I \{ |X_k| \leq \delta f(k) \} | \mathcal{F}_{k-1}^* \big] \big| \big)$$

$$\leq \sum_{k=1}^{\infty} f(k)^{-4} \big(E \big[X_k^4 I \{ |X_k| \leq \delta f(k) \} \big] + 15 \delta^3 f(k)^3 E \big[|X_k| I \{ |X_k| > \delta f(k) \} \big] \big),$$

and the last sum is finite by (a) and (b). $\qquad\square$

Lemma 5 *Under conditions* (a)–(d), *we have*

$$\frac{\log \log s_n}{s_n^2} \big(s_n^2 - s_n^{*2} \big) \to 0 \quad a.s.$$

Proof By Lemma 3 it suffices to show that

$$f(n)^{-2} \big(s_n^2 - s_n^{*2} \big) \to 0 \quad \text{a.s.}$$

We can write

$$s_n^2 - s_n^{*2} = \sum_{k=1}^{n} \big(E \big[X_k^2 | \mathcal{F}_{k-1} \big] - E \big[X_k^{*2} | \mathcal{F}_{k-1} \big] \big)$$

$$+ \sum_{k=1}^{n} \big(E \big[X_k^{*2} | \mathcal{F}_{k-1} \big] - E \big[X_k^{*2} | \mathcal{F}_{k-1}^* \big] \big) := S_n^{(1)} + S_n^{(2)} \quad \text{(say)}.$$

A straightforward calculation shows (using again (2.15)) that

$$\big| E \big[X_k^2 - X_k^{*2} | \mathcal{F}_{k-1} \big] \big|$$

$$\leq E \big[X_k^2 I \{ |X_k| > \delta f(k) \} | \mathcal{F}_{k-1} \big] + \big(E \big[X_k I \{ |X_k| \leq \delta f(k) \} | \mathcal{F}_{k-1}^* \big] \big)^2$$

$$+ 2 \big| E \big[X_k I \{ |X_k| \leq \delta f(k) \} | \mathcal{F}_{k-1} \big] \big| \big| E \big[X_k I \{ |X_k| \leq \delta f(k) \} | \mathcal{F}_{k-1}^* \big] \big|$$

$$\leq E \big[X_k^2 I \{ |X_k| > \delta f(k) \} | \mathcal{F}_{k-1} \big] + 3 \delta f(k) E \big[|X_k| I \{ |X_k| > \delta f(k) \} | \mathcal{F}_{k-1}^* \big].$$

Thus, conditions (d) and (b), in connection with Kronecker's lemma and the Beppo Levi theorem, imply that

$$f(n)^{-2} S_n^{(1)} \to 0 \quad \text{a.s.}$$

It remains to prove

$$f(n)^{-2} S_n^{(2)} \to 0 \quad \text{a.s.,}$$

which follows from Lemma 4 by a similar argument, observing that $E(X_k^{*2}|\mathcal{F}_{k-1}) - E(X_k^{*2}|\mathcal{F}_{k-1}^*)$ is a martingale difference sequence and applying Kronecker's lemma and the martingale convergence theorem.   □

By a standard argument in upper-lower class theory (see Feller [9], Lemma 1) we can assume that the function φ in the test (1.4)–(1.5) satisfies

$$a\sqrt{\log \log t} \leq \varphi(t) \leq b\sqrt{\log \log t} \qquad (2.17)$$

for some positive constants $a < b$, and in particular φ tends to ∞. Recall also that φ is a nondecreasing function. Clearly we can write

$$\varphi(t) = \tilde{\varphi}(t) + \frac{1}{\tilde{\varphi}(t)} \quad \text{with} \quad \tilde{\varphi}(t) = \frac{\varphi(t)}{2} + \left(\frac{\varphi^2(t)}{4} - 1 \right)^{1/2}, \qquad (2.18)$$

provided that $\varphi(t) \geq 2$. By another standard observation in the theory, $I(\varphi)$ is finite if and only if $I(\tilde{\varphi})$ is finite. The next lemma shows it is enough to consider only functions φ which are smooth in some sense.

Lemma 6 *Assume that $I(\varphi) = \infty$ $(< \infty)$. Then there is a nondecreasing function $\hat{\varphi} \geq \varphi$ $(\leq \varphi)$ and an absolute constant A such that $I(\hat{\varphi}) = \infty$ $(< \infty)$ and*

$$\left| \hat{\varphi}(x) - \hat{\varphi}(y) \right| \leq A \cdot \frac{\hat{\varphi}(x)}{x} |x - y| \quad \text{if } x < y \text{ and } [y, 2y] \cap [x, 2x] \neq \emptyset. \qquad (2.19)$$

Proof Assume that $I(\varphi) = \infty$. Simple calculations show that

$$\hat{\varphi}(x) = \frac{1}{x} \int_x^{2x} \varphi(t)\, dt \quad (x > x_0)$$

will work. The case $I(\varphi) < \infty$ can be treated similarly by defining $\hat{\varphi}(x) = \int_{x/2}^x \varphi(t)\, dt$.   □

Next we observe that

$$|X_n^*| \leq K_n s_n^*, \qquad (2.20)$$

where $K_n = 2\delta f(n)/s_n^* \sim 2\delta(\log \log s_n^*)^{-1/2}$ by Lemma 3; also K_n is $\mathcal{F}_{n-1}^*$-measurable. By the martingale version of the Skorokhod embedding theorem (Strassen [19]) we can assume that the sequence (X_n^*) is defined on a probability space together with a standard Wiener process $\{W(t),\ t \geq 0\}$ such that

$$S_n^* = W(T_n), \quad \text{where } T_n = \sum_{m=1}^n \tau_m, \qquad (2.21)$$

where τ_n are nonnegative r.v.'s, τ_n is $\mathcal{F}_n^*$-measurable $(n = 1, 2, \ldots)$, and

$$E\big[\tau_n | \mathcal{F}_{n-1}^*\big] = E\big[X_n^{*2} | \mathcal{F}_{n-1}^*\big] \quad \text{a.s.} \tag{2.22}$$

Also we have for any $r \geq 1$,

$$E\big[\tau_n^r | \mathcal{F}_{n-1}^*\big] \leq L_r E\big[X_n^{*2r} | \mathcal{F}_{n-1}^*\big] \quad \text{a.s.}, \tag{2.23}$$

where L_r is a constant depending only on r, and moreover, for $T_n \leq t \leq T_{n+1}$,

$$\big| W(t) - W(T_n) \big| \leq K_{n+1} s_{n+1}^*. \tag{2.24}$$

The last relation is a consequence of the construction of the Skorokhod stopping times and plays a crucial role in the following lemma, implicit in the proof of Theorem 1.1 of Einmahl and Mason [6].

Lemma 7 *Assume that $\{X_n^*, \mathcal{F}_n^*, n \geq 1\}$ is an MDS with finite variances such that $s_n^{*2} := \sum_{k=1}^n E[X_k^{*2} | \mathcal{F}_{k-1}^*] \to \infty$ a.s. Assume that (2.20)–(2.24) hold with some $K_n \sim \text{const} \cdot (\log \log s_n^*)^{-1/2}$ and K_n is $\mathcal{F}_{n-1}^*$-measurable. If there exists a positive constant K such that*

$$\limsup_{n \to \infty} \frac{\log \log s_n^{*2}}{s_n^{*2}} \big| T_n - s_n^{*2} \big| \leq K \quad a.s., \tag{2.25}$$

then for every positive and nondecreasing function φ,

$$P\big(S_k^* > s_k^* \varphi(s_k^{*2}) \text{ i.o.}\big) = \begin{cases} 1 & \text{if } I(\varphi) = \infty, \\ 0 & \text{if } I(\varphi) < \infty. \end{cases}$$

Note that the assumptions of Lemma 7 imply that $s_n^* \sim s_{n+1}^*$. Thus, it follows from (2.24) and (2.25) that, for $T_n \leq t \leq T_{n+1}$ and for sufficiently large n,

$$\big| W(t) - W(T_n) \big| \leq \text{const} \cdot \sqrt{t} / (\log \log t)^{1/2} \quad \text{a.s.} \tag{2.26}$$

Now the proof of Theorem 1.1 of Einmahl and Mason [6] can be followed almost verbatim, observing that the argument still goes through if their assumption (1.3) is replaced by (2.25).

The remainder of the proof of Theorem 3 will be divided into two steps. In the first step we will show that the integral test (1.4)–(1.5) holds for the truncated MDS $\{X_k^*, \mathcal{F}_k^*, k \geq 1\}$. Then we will show that if φ is monotone and satisfies the smoothness condition (2.19), then

$$P\big(S_k > s_k \varphi(s_k^2) \text{ i.o.}\big) = P\big(S_k^* > s_k^* \varphi(s_k^{*2}) \text{ i.o.}\big). \tag{2.27}$$

Now if $I(\varphi) = \infty$, the $\hat{\varphi}$ in Lemma 6 satisfies $\hat{\varphi} \geq \varphi$, $I(\hat{\varphi}) = \infty$, and thus

$$1 = P\big(S_k^* > s_k^* \hat{\varphi}(s_k^{*2}) \text{ i.o.}\big) = P\big(S_k > s_k \hat{\varphi}(s_k^2) \text{ i.o.}\big) \leq P\big(S_k > s_k \varphi(s_k^2) \text{ i.o.}\big).$$

An analogous result holds if $I(\varphi) < \infty$.

Step 1. By Lemma 3 and Lemma 7 it suffices to show that

$$|T_n - s_n^{*2}| = o(f(n)^2) \quad \text{a.s.} \tag{2.28}$$

Using the definition of s_n^{*2} and (2.21), (2.22), we have

$$T_n - s_n^{*2} = \sum_{k=1}^{n} (\tau_k - E[X_k^{*2}|\mathcal{F}_{k-1}^*]) = \sum_{k=1}^{n} (\tau_k - E[\tau_k|\mathcal{F}_{k-1}^*]) \quad \text{a.s.}$$

By (2.23) and Lemma 4 we have

$$\sum_{k=1}^{\infty} f(k)^{-4} E(\tau_k - E[\tau_k|\mathcal{F}_{k-1}^*])^2$$

$$\leq \sum_{k=1}^{\infty} f(k)^{-4} E\tau_k^2 \leq L_2 \sum_{k=1}^{\infty} f(k)^{-4} EX_k^{*4} < \infty,$$

and thus by the martingale convergence theorem the series

$$\sum_{k=1}^{\infty} f(k)^{-2} (\tau_k - E[\tau_k|\mathcal{F}_{k-1}^*])$$

is a.s. convergent, implying (2.28) by the Kronecker lemma.

Step 2. Define $\tilde{\varphi}$ as in (2.18) and set

$$R_k = \frac{s_k}{\tilde{\varphi}(s_k^2)} + (\tilde{\varphi}(s_k^2)s_k - \varphi(s_k^{*2})s_k^*).$$

Then we have, on one hand,

$$P(S_k > s_k\varphi(s_k^2) \text{ i.o.})$$

$$= P(S_k^* + S_k^{**} > s_k^*\tilde{\varphi}(s_k^{*2}) + R_k \text{ i.o.})$$

$$\leq P(S_k^* > s_k^*\tilde{\varphi}(s_k^{*2}) \text{ i.o.}) + P(|S_k^{**}| > R_k \text{ i.o.})$$

and, on the other hand,

$$P(S_k > s_k\varphi(s_k^2) \text{ i.o.})$$

$$\geq P(S_k^* > s_k^*\tilde{\varphi}(s_k^{*2}) + |R_k - S_k^{**}| \text{ i.o.})$$

$$= P(S_k^* > s_k^*\tilde{\tilde{\varphi}}(s_k^{*2}) + s_k^*\tilde{\tilde{\varphi}}(s_k^{*2})^{-1} + |R_k - S_k^{**}| \text{ i.o.}).$$

We now show that for any $\varepsilon > 0$,

$$P(|S_k^{**}| > \varepsilon R_k \text{ i.o.}) = 0 \quad \text{and} \quad |R_k - S_k^{**}| \leq \frac{(\kappa - 1)}{2} s_k^*\tilde{\tilde{\varphi}}(s_k^{*2})^{-1} \quad \text{a.s.} \tag{2.29}$$

for some large enough κ, which implies in view of the forgoing estimates that

$$P\big(S_k^* > \tilde{\tilde{\varphi}}\big(s_k^{*2}\big)s_k^* + \kappa s_k^* \tilde{\tilde{\varphi}}\big(s_k^{*2}\big)^{-1} \text{ i.o.}\big)$$
$$\leq P\big(S_k > \varphi\big(s_k^2\big)s_k \text{ i.o.}\big) \leq P\big(S_k^* > \tilde{\varphi}\big(s_k^{*2}\big)s_k^* \text{ i.o.}\big).$$

(Here $\tilde{\tilde{\varphi}}$ is obtained by iterating the operation in (2.18).) Since $I(\varphi) = \infty$ if and only if $I(\tilde{\varphi}) = \infty$ and $I(\tilde{\varphi} + \kappa/\tilde{\varphi}) = \infty$, we get (2.27). To prove (2.29) we start with showing that the dominating part in R_k is $s_k/\tilde{\varphi}(s_k^2)$, i.e.,

$$\frac{\tilde{\varphi}(s_k^2)}{s_k}\big(\tilde{\varphi}\big(s_k^2\big)s_k - \tilde{\varphi}\big(s_k^{*2}\big)s_k^*\big) \to 0 \quad \text{a.s.} \tag{2.30}$$

In view of Lemma 3 and

$$\tilde{\varphi}\big(s_k^2\big)s_k - \tilde{\varphi}\big(s_k^{*2}\big)s_k^* = \frac{\tilde{\varphi}(s_k^2)(s_k^2 - s_k^{*2})}{s_k + s_k^*} + s_k^*\big(\tilde{\varphi}\big(s_k^2\big) - \tilde{\varphi}\big(s_k^{*2}\big)\big),$$

this will follow if

$$\frac{\tilde{\varphi}(s_k^2)^2}{s_k^2}\big(s_k^2 - s_k^{*2}\big) \to 0 \quad \text{a.s.} \quad \text{and} \quad \tilde{\varphi}\big(s_k^2\big)\big(\tilde{\varphi}\big(s_k^2\big) - \tilde{\varphi}\big(s_k^{*2}\big)\big) \to 0 \quad \text{a.s.} \tag{2.31}$$

Now the first relation in (2.31) follows from Lemma 5 and (2.17), since (2.17) remains valid for $\tilde{\varphi}$. Note that $\varphi(t) \sim \tilde{\varphi}(t)$ by (2.18). As we noted earlier, we can assume that $\tilde{\varphi}$ satisfies (2.19). Clearly, since $s_k^{*2} \sim s_k^2$, the intervals $[s_k^2, 2s_k^2]$ and $[s_k^{*2}, 2s_k^{*2}]$ will not be disjoint for any $k \geq k_0(\omega)$, where k_0 is almost surely finite. Thus we get from (2.19)

$$\tilde{\varphi}\big(s_k^2\big)\big(\tilde{\varphi}\big(s_k^2\big) - \tilde{\varphi}\big(s_k^{*2}\big)\big) \leq A\frac{\tilde{\varphi}(s_k^2)^2}{s_k^2}\big(s_k^2 - s_k^{*2}\big) \quad \text{for all } k \geq k_0$$

with an absolute constant A, and here the right-hand side tends to zero as we have already noted. Thus we proved that the second relation of (2.31) is also valid, and thus the dominating part of R_k is $s_k/\tilde{\varphi}(s_k^2)$. Hence to prove the first relation in (2.29) it suffices to show

$$|S_k^{**}| = o\left(\frac{s_k}{\tilde{\varphi}(s_k^2)}\right) \quad \text{a.s.,} \tag{2.32}$$

which by Lemma 3 and (2.17) will follow if

$$|S_k^{**}| = o\big(f(k)\big) \quad \text{a.s.} \tag{2.33}$$

It easy to see (cf. (2.15)) that $E|X_k^{**}| \leq 2 E[|X_k|I\{|X_k| > \delta f(k)\}]$, and hence (2.33) follows from condition (b) and Kronecker's lemma.

Since we proved that $|S_k^{**}| = o(R_k)$ a.s. and that the dominating part of R_k is $s_k / \tilde{\varphi}(s_k^2)$, we have in view of (2.17), Lemma 3, and $\varphi(t) \sim \tilde{\varphi}(t)$ $(t \to \infty)$,

$$|R_k - S_k^{**}| = |R_k|\big(1 + o(1)\big)$$

$$= \frac{s_k}{\tilde{\varphi}(s_k^2)}\big(1 + o(1)\big) \le \text{const} \cdot \frac{s_k^*}{\tilde{\tilde{\varphi}}(s_k^{*2})} \quad \text{a.s.}$$

This proves the second relation of (2.29).

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