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On Functional Versions of the Arc-Sine Law

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Abstract Let $X_1, X_2, \dots$ be a sequence of random variables. Let $S_k = X_1 + \cdots + X_k$ and assume that S_k/b_k converges in distribution for some numerical sequence (b_k). We study the weak convergence of the random processes $\{\Lambda_n(z),\ z \in \mathbb{R}\}$, where

$$\Lambda_n(z) = \frac{1}{n} \sum_{k=1}^{n} I\left\{\frac{S_k}{b_k} \le z\right\}.$$

We consider the same problem when the normalized partial sums S_k/b_k are replaced by other functionals of the sequence (X_n). In particular, we investigate the case of sample extremes in detail.

Keywords Arc-sine law · Invariance principles · Partial sums · Extremes

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1 Introduction

One of the classical results on Brownian motion is the following arc-sine law of Lévy [14]. Let $\stackrel{d}{=}$ denote equality in distribution, and let A_ρ, $0<\rho<1$, denote the generalized arc-sine distribution with density

$$a_\rho(x)=\frac{\sin\pi\rho}{\pi}x^{\rho-1}(1-x)^{-\rho},\quad 0<x<1.$$

Theorem A (Lévy's first arc-sine law) *Let* $\{W(t),\ t\ge 0\}$ *be a standard Brownian motion. Then*

$$\int_0^1 I\big\{W(t)\le 0\big\}\,\mathrm{d}t \stackrel{d}{=} A_{1/2}.$$

If $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) random variables with $EX_1=0$ and $EX_1^2=1$, then by Donsker's invariance principle (cf. [3]) we obtain that

$$\frac{1}{n}\sum_{k=1}^{n} I\left\{S_k\le 0\right\}\stackrel{d}{\longrightarrow} A_{1/2},\tag{1.1}$$

where $S_k=X_1+\cdots+X_k$, and $\stackrel{d}{\longrightarrow}$ denotes convergence in distribution. Due to the strong dependence between the partial sums S_k, the law of large numbers for the averages of $I\{S_k\le 0\}$ does not hold. However, Erdős and Hunt [10] observed that if the Cesàro averages are replaced with logarithmic averages, then we have

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\{S_k\le 0\}\to 1/2\quad\text{a.s.}\tag{1.2}$$

Recently, the following more general result has been proved.

Theorem B (Almost sure central limit theorem) *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with* $EX_1=0$ *and* $EX_1^2=1$. *Then*

$$\sup_{z\in\mathbb{R}}\left|\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\left\{\frac{S_k}{\sqrt{k}}\le z\right\}-\Phi(z)\right|\to 0\quad\text{a.s.},\tag{1.3}$$

where $\Phi(z)$ *denotes the standard normal distribution function.*

Under $E|X_1|^{2+\delta}<\infty$, relation (1.3) was proved independently by Brosamler [6] and Schatte [17]. Assuming only finite second moments, (1.3) is due to Fisher [12] and Lacey and Philipp [13].

Theorem B was extended by Berkes and Csáki [2] to the logarithmic averages

$$\Lambda_n^*(z)=\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}I\big\{h_k(X_1,\ldots,X_k)\le z\big\},\tag{1.4}$$

where $h_k : \mathbb{R}^k \to \mathbb{R}$ are measurable functions such that

$$h_k(X_1, \dots, X_k) \xrightarrow{d} G \tag{1.5}$$

with a nondegenerate distribution G. They showed that $\Lambda_n^*(z)$ converges a.s. if the dependence of $h_k(X_1, \dots, X_k)$ on its initial $o(k)$ variables becomes negligible for $k \to \infty$. This condition is satisfied for normalized partial sums, extremal statistics, U-statistics, ordered samples, local times, etc. A simple example when the analogue of (1.3) fails is the Darling–Erdős theorem for the maximum of normalized partial sums of i.i.d. random variables.

Since the almost sure result (1.2) has a distributional analogue (1.1), the question arises if the almost sure central limit theorem (1.3) has a weak convergence version, i.e., if the sequence of processes

$$\frac{1}{n} \sum_{k=1}^{n} I\left\{ \frac{S_k}{\sqrt{k}} \le z \right\}$$

converges weakly. The same question arises for the averages

$$\Lambda_n(z) := \frac{1}{n} \sum_{k=1}^{n} I\left\{ h_k(X_1, \dots, X_k) \le z \right\}, \tag{1.6}$$

where $h_k : \mathbb{R}^k \to \mathbb{R}$ are functionals such that (1.5) holds. Since we can write

$$\Lambda_n(z) = \int_0^1 I\left\{ R_n(t) \le z \right\} \mathrm{d}t,$$

where

$$R_n(t) = h_k(X_1, \dots, X_k) \quad \text{if } t \in \big[k/n, (k+1)/n\big), \; 0 \le k \le n-1, \tag{1.7}$$

one might expect that the weak convergence of $R_n(t)$ to some limiting process $R(t)$ (i.e., the functional version of (1.5)) implies the weak convergence of $\Lambda_n(z)$. However, as we will show in Example 2.1 below, even the much stronger assumption $R_n(t) \to R(t)$ a.s. for all $t \in [0, 1]$ is in general not sufficient to imply the weak convergence of $\Lambda_n(z)$ to $\int_0^1 I\{R(t) \le z\} \, \mathrm{d}t$. A weak invariance principle for $R_n(t)$ implies only the convergence of the finite-dimensional distributions of $\Lambda_n(z)$, and to obtain the weak convergence of $\Lambda_n(z)$, we have to make additional conditions on the path properties of the limit process $R(t)$, see Theorem 2.2. For example, the weak convergence holds for normalized partial sums $h_k(X_1, \dots, X_k) = S_k/b_k$ of i.i.d. random variables, martingale difference sequences, and stationary Gaussian sequences, while Theorems 2.3 and 2.4 yield convergence criteria when the limit process $R(t)$ is a general Gaussian or normalized Lévy process. To give a nonlinear example, we show that $\Lambda_n(z)$ converges weakly in the case of extreme-value statistics of i.i.d. random variables.

Our paper is organized as follows. In Sect. 2 we state our main results and consider several examples. We also study the distribution of $\Lambda(z)$ when z is fixed. We provide a formula for the first and second moments of $\Lambda(z)$ when the weak limit of $R_n(t)$ is Gaussian. In case of extreme values, we provide explicit expressions for $E\Lambda^p(z)$, $p = 1, 2, \ldots$. Section 3 contains the proofs.

2 Results

Before we state our results, we introduce some basic notation.

As usual, $D[a,b]$ denotes the space of càdlàg functions defined on the interval $[a,b]$, i.e., all functions $x : [a,b] \to \mathbb{R}$ such that $\lim_{t\searrow t_0} x(t) = x(t_0)$ and $\lim_{t\nearrow t_0} x(t)$ exists. The symbol $\stackrel{D[a,b]}{\longrightarrow}$ indicates the weak convergence of random elements in $D[a,b]$ with respect to the Skorokhod topology (cf. [3]). Note that if $x \in D[a,b]$ and if $[c,d] \subset [a,b]$, then x can be considered as an element of $D[c,d]$ by restricting its domain of definition. If $x \in D[a+\varepsilon, b-\varepsilon]$ for every $0 < \varepsilon < (a+b)/2$, then we say that $x \in D(a,b)$. The convergence in $D(a,b)$ means the convergence in $D[a+\varepsilon, b-\varepsilon]$ for all $0 < \varepsilon < (a+b)/2$. Since $\Lambda_n(z)$ and $\Lambda(z)$ are defined on $\mathbb{R}$, we will also consider the convergence $\Lambda_n(z) \stackrel{D[\mathbb{R}]}{\longrightarrow} \Lambda(z)$. This means that there exists a strictly increasing and continuous distribution function T such that $\Lambda_n(T^{-1}(u)) \stackrel{D[0,1]}{\longrightarrow} \Lambda(T^{-1}(u))$. At the endpoints $u = 0$ and $u = 1$, we set $T^{-1}(0) = -\infty$ and $T^{-1}(1) = \infty$. Using the fact that Λ_n and Λ are (random) distribution functions, we assign the values 0 and 1 when the argument is $-\infty$ or ∞.

We can assume without loss of generality that all random variables and processes introduced so far (and later on) can be defined on a common probability space $(\Omega, \mathcal{A}, P)$. If $(S, \mathcal{S})$ is a measurable space and $R : \Omega \to S$ is $\mathcal{A} - \mathcal{S}$ measurable, then P induces a probability measure on $(S, \mathcal{S})$ denoted by PR^{-1} and defined as usual by $PR^{-1}(A) = P(R^{-1}A)$.

For some random processes $\{R_n(t),\ t \in (0,1)\}$ in $D(0,1)$, let

$$\Lambda_n(z) = \int_0^1 I\bigl\{R_n(t) \le z\bigr\}\,\mathrm{d}t. \tag{2.1}$$

If $R_n(t)$ is defined as in (1.7), then $\Lambda_n(z)$ reduces to the averages in (1.6). The basic assumption throughout this paper will be the existence of some limiting process $\{R(t),\ t \in (0,1)\}$ in $D(0,1)$ such that

$$R_n(t) \stackrel{D(0,1)}{\longrightarrow} R(t). \tag{2.2}$$

A typical limit is $R(t) = t^{-\alpha} W(t)$, where $\alpha \ge 1/2$, and $W(t)$ is standard Brownian motion. Note that this $R(t)$ is not an element in $D[0,1]$, and this is why we assume the weak convergence in $D(0,1)$. Put

$$\Lambda(z) = \int_0^1 I\bigl\{R(t) \le z\bigr\}\,\mathrm{d}t. \tag{2.3}$$

Our first theorem gives the convergence of the finite-dimensional distributions.

Theorem 2.1 *Let $\Lambda_n(z)$ and $\Lambda(z)$ be given as in (2.1) and (2.3). Assume that (2.2) holds, and let $z_1, \ldots, z_M$ be arbitrary real numbers. Then*

$$\bigl\{\Lambda_n(z_1), \Lambda_n(z_2), \ldots, \Lambda_n(z_M)\bigr\} \xrightarrow{d} \bigl\{\Lambda(z_1), \Lambda(z_2), \ldots, \Lambda(z_M)\bigr\} \tag{2.4}$$

whenever

$$P\bigl(R(t) = z_j\bigr) = 0 \quad \textit{for any } t \in (0,1) \textit{ and } 1 \le j \le M. \tag{2.5}$$

Theorem 2.1 provides conditions which imply that the finite-dimensional distributions of the processes $\Lambda_n(z)$ converge to $\Lambda(z)$. If we want to extend this result to $\Lambda_n(z) \xrightarrow{D[\mathbb{R}]} \Lambda(z)$, we need to show that the sequence $\{P\Lambda_n^{-1}, n \ge 1\}$ is tight. As the following example shows, relation (2.2) in general does *not* imply $\Lambda_n(z) \xrightarrow{D[\mathbb{R}]} \Lambda(z)$, even if the finite-dimensional distributions converge.

Example 2.1 In this example, R_n and R have values in $[0,1]$, hence it is sufficient to consider Λ_n and Λ as elements in $D[0,1]$ instead of $D[\mathbb{R}]$. Let ξ be uniform in $[1/3, 2/3]$ and define for $n \ge 3$,

$$R_n(t) = \begin{cases} 2t & \text{if } t \in [0, (\xi - \frac{1}{n})/2); \\ \xi - \frac{1}{n} & \text{if } t \in [(\xi - \frac{1}{n})/2, \xi); \\ \xi + \frac{1}{n} & \text{if } t \in [\xi, (\xi + \frac{1}{n} + 1)/2); \\ 2t - 1 & \text{if } t \in [(\xi + \frac{1}{n} + 1)/2, 1]. \end{cases}$$

Obviously $R_n(t)$ converges pointwise to $R(t)$, where

$$R(t) = \begin{cases} 2t & \text{if } t \in [0, \xi/2); \\ \xi & \text{if } t \in [\xi/2, (\xi+1)/2); \\ 2t - 1 & \text{if } t \in [(\xi+1)/2, 1]. \end{cases}$$

Now we have

$$\Lambda_n(z) = \int_0^1 I\bigl\{R_n(t) \le z\bigr\}\,\mathrm{d}t = \begin{cases} z/2 & \text{if } z \in [0, \xi - \frac{1}{n}); \\ \xi & \text{if } z \in [\xi - \frac{1}{n}, \xi + \frac{1}{n}); \\ (z+1)/2 & \text{if } z \in [\xi + \frac{1}{n}, 1]; \end{cases}$$

and

$$\Lambda(z) = \int_0^1 I\bigl\{R(t) \le z\bigr\}\,\mathrm{d}t = \begin{cases} z/2 & \text{if } z \in [0, \xi); \\ (z+1)/2 & \text{if } z \in [\xi, 1]. \end{cases}$$

It follows that $\Lambda_n(z) \xrightarrow{f.d.d.} \Lambda(z)$, but $\Lambda_n(z)$ does not converge in $D[0,1]$.

Due to the fact that the functions $\Lambda_n(z)$ and $\Lambda(z)$ are monotone and bounded, the tightness solely depends on properties of the limiting process $\{R(t),\ t \in (0,1)\}$. This observation is made more precise in the next theorem.

Theorem 2.2 *Let $\Lambda_n(z)$ and $\Lambda(z)$ be given as in (2.1) and (2.3). Assume that (2.2) holds and that*

$$P\big(R(t)=z\big)=0 \quad \text{for any } t \in (0,1) \text{ and } z \in \mathbb{R}. \tag{2.6}$$

If for all $0<\varepsilon<1/2$, the sample paths of

$$\Lambda^{\varepsilon}(z)=\int_{\varepsilon}^{1-\varepsilon} I\big\{R(t)\le z\big\}\,\mathrm{d}t \tag{2.7}$$

are continuous on $\mathbb{R}$, then $\Lambda_n(z) \xrightarrow{D[\mathbb{R}]} \Lambda(z)$.

In order to apply Theorem 2.2, we need criteria to check the continuity of $\Lambda^{\varepsilon}(z)$. For example, using Kolmogorov's classical continuity criterion (see, e.g., [3]), it suffices to verify

$$\begin{aligned}&\sup_{z\in\mathbb{R}} E\big|\Lambda^{\varepsilon}(z+h)-\Lambda^{\varepsilon}(z)\big|^2\\ &\quad=\sup_{z\in\mathbb{R}} E\bigg(\int_{\varepsilon}^{1-\varepsilon} I\big\{R(t)\in[z,z+h]\big\}\bigg)^2 = o\big(h^{1+\beta}\big) \quad \text{for } h\to 0\ (\beta>0).\end{aligned} \tag{2.8}$$

Using Theorem 2.2 and this criterion, we obtain a simple integral criterion when $R(t)$ is a Gaussian process.

Theorem 2.3 *Let $\Lambda_n(z)$ and $\Lambda(z)$ be given as in (2.1) and (2.3). Assume that (2.2) holds. Further assume that the limiting process $\{R(t),\ t\in(0,1)\}$ in (2.2) is Gaussian with $\sigma_s=\sqrt{ER^2(s)}$ and $\rho(s,t)=\mathrm{Corr}(R(s),R(t))$. If for every $\varepsilon>0$,*

$$\int_{\varepsilon}^{1-\varepsilon}\int_{\varepsilon}^{t} \frac{\mathrm{d}s\,\mathrm{d}t}{\sigma_s\sigma_t\sqrt{1-\rho^2(s,t)}}<\infty, \tag{2.9}$$

then $\Lambda_n(z) \xrightarrow{D[\mathbb{R}]} \Lambda(z)$.

Theorem 2.3 contains many important examples. The following corollary highlights the most important special case, which motivated this paper.

Corollary 2.1 *Assume that $X_1, X_2, \ldots$ are i.i.d. with $EX_1=0$ and $EX_1^2=1$. Then*

$$\frac{1}{n}\sum_{k=1}^{n} I\bigg\{\frac{S_k}{\sqrt{k}}\le z\bigg\} \xrightarrow{D[\mathbb{R}]} \int_0^1 I\bigg\{\frac{W(t)}{\sqrt{t}}\le z\bigg\}\mathrm{d}t,$$

where $\{W(t),\ t\ge 0\}$ is a standard Brownian motion.

Corollary 2.1 can be extended to a large class of dependent random variables. We illustrate this fact by the following:

Corollary 2.2 *Assume that $X_1, X_2, \ldots$ is a martingale-difference sequence with finite second moments. Let $\mathcal{F}_n = \sigma(X_k, k \le n)$ and define $\sigma_n^2 = E[X_n^2 | \mathcal{F}_{n-1}]$. Further set $B_n^2 = \mathrm{Var}(S_n)$ and $s_n^2 = \sigma_1^2 + \cdots + \sigma_n^2$. Assume that $s_n^2 / B_n^2 \xrightarrow{P} 1$ and that $B_n^2 = n^\alpha L(n)$, where L is a slowly varying function at infinity, and α is a positive constant. Finally let $\{W(t), t \ge 0\}$ be a standard Brownian motion. If the Lindeberg condition*

$$B_n^{-2} \sum_{k=1}^{n} E X_k^2 I\{X_k^2 \ge \varepsilon B_k^2\} \to 0 \quad \text{for all } \varepsilon > 0$$

holds, then we have

$$\frac{1}{n} \sum_{k=1}^{n} I\left\{\frac{S_k}{B_k} \le z\right\} \xrightarrow{D[\mathbb{R}]} \int_0^1 I\left\{\frac{W(t^\alpha)}{t^{\alpha/2}} \le z\right\} \mathrm{d}t.$$

As another application of Theorem 2.3, we consider partial-sum processes of strongly dependent sequences converging to a fractional Brownian motion. Recall that a fractional Brownian motion is a Gaussian process $\{B_H(t), t \ge 0\}$ with $E B_H(t) = 0$ and

$$\mathrm{Cov}\big(B_H(s), B_H(t)\big) = \frac{1}{2}\big(t^{2H} + s^{2H} - (t-s)^{2H}\big), \quad 0 \le s \le t.$$

The constant $H \in (0, 1)$ is called the Hurst index. We note that $B_{1/2}(t)$ is a standard Brownian motion.

Corollary 2.3 *Let $X_1, X_2, \ldots$ be a stationary Gaussian sequence with $E X_1 = 0$ and assume that for some $0 < H < 1$,*

$$\mathrm{Var}(S_n) = n^{2H} L^2(n) \quad \text{for } n \to \infty,$$

where L is a slowly varying function at infinity. Then

$$\frac{1}{n} \sum_{k=1}^{n} I\left\{\frac{S_k}{k^H L(k)} \le z\right\} \xrightarrow{D[\mathbb{R}]} \int_0^1 I\left\{\frac{B_H(t)}{t^H} \le z\right\} \mathrm{d}t,$$

where $B_H(t)$ is a fractional Brownian motion with Hurst index H.

In all our examples so far, the limit process $R(t)$ was a Gaussian process. To consider different situations, let $X_1, X_2, \ldots$ be i.i.d. random variables belonging to the domain of attraction of a stable distribution G_α, i.e., assume that there are constants a_n and b_n such that $S_n / b_n - a_n \xrightarrow{d} G_\alpha$. As is known, the characteristic function φ of G_α has the form

$$\varphi(t) = \exp\left\{i\gamma t - c|t|^\alpha \left(1 + i\beta \frac{t}{|t|} \omega(t, \alpha)\right)\right\},$$

where

$$\omega(t,\alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{if } \alpha \neq 1, \\ \frac{2}{\pi}\log|t| & \text{if } \alpha = 1, \end{cases}$$

and c, α, β, and γ are constants (γ is a real, $c \geq 0,\ 0 < \alpha \leq 2,\ -1 \leq \beta \leq 1$). We refer to α as the characteristic exponent of the stable law. Note that if $\alpha \neq 1$, we can assume without loss of generality that $a_n = 0$ (cf. Feller [11], Theorem 3, p. 580), and hence $P(S_n \leq 0) \to G_\alpha(0)$. Letting $\rho = G_\alpha(0)$, it follows from a classical result of Spitzer [19] (see Doney [8] for an improvement) that $\frac{1}{n}\sum_{k=1}^n I\{S_k \leq 0\} \xrightarrow{d} A_\rho$, and thus the occupation time distribution for the sequence (X_n) is the generalized arc-sine distribution. By Skorohod [18] we have $S_{[nt]}/b_n \xrightarrow{D[0,1]} \mathcal{L}_\alpha(t)$, where $\{\mathcal{L}_\alpha(t),\ 0 \leq t \leq 1\}$ is a stable process with $\mathcal{L}_\alpha(1)$ having distribution G_α. (For the properties of stable processes, we refer to Borodin and Ibragimov [5].) As a simple calculation shows, in the case of the partial-sum functional $h_k(X_1, \ldots, X_k) = S_k/b_k$, the limiting process in (2.2) is the normalized stable process $\mathcal{L}_\alpha(t)/t^{1/\alpha}$. The following theorem extends Theorem 2.3 to the case where, instead of a Gaussian process, the limiting process $R(t)$ in (2.2) is a normalized Lévy process.

Theorem 2.4 *Let $\Lambda_n(z)$ and $\Lambda(z)$ be given as in* (2.1) *and* (2.3) *and assume that* (2.2) *holds. Further assume that $R(t) = \mathcal{L}(t)/b_t$, where $\mathcal{L}(t)$ is a Lévy process, and b_t is a normalizing function such that the distribution of $R(t)$ does not depend on t. Finally assume that $\ell(z) = \mathrm{d}P(R(t) \leq z)/\mathrm{d}z$ exists and is uniformly bounded. Then if for every $\varepsilon > 0$,*

$$\int_\varepsilon^{1-\varepsilon}\left(\int_\varepsilon^t \frac{b_t}{b_{t-s}}\,\mathrm{d}s\right)\mathrm{d}t < \infty, \tag{2.10}$$

we have $\Lambda_n(z) \xrightarrow{D[\mathbb{R}]} \Lambda(z)$.

Corollary 2.4 *Assume that $X_1, X_2, \ldots$ are i.i.d. random variables. Assume that S_n/b_n converges weakly to some stable distribution with characteristic exponent α. Then*

$$\frac{1}{n}\sum_{k=1}^n I\left\{\frac{S_k}{b_k} \leq z\right\} \xrightarrow{D[\mathbb{R}]} \int_0^1 I\left\{\frac{\mathcal{L}_\alpha(t)}{t^{1/\alpha}} \leq z\right\}\mathrm{d}t,$$

where $\mathcal{L}_\alpha(t)$ is a stable process with characteristic exponent α.

Motivated by the arc-sine law and the almost sure central limit theorem, all our examples so far were related to partial-sum processes. In what follows, we give an example involving nonlinear functionals of the sequence $\{X_n\}$. Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function $F(x)$ and set

$$M_n = \max\{X_1, X_2, \ldots, X_n\}.$$

If $P(M_n/b_n - a_n \le x)$ converges to some nondegenerate limiting distribution $G(x)$ ("is attracted to G"), then by the classical theory, G belongs to one of three different types of limiting distributions. As an illustration, we consider here the case where

$$G(x) = \begin{cases} 0, & x \le 0; \\ \exp(-\lambda x^{-\alpha}), & x > 0 \ (\lambda, \alpha > 0). \end{cases} \tag{2.11}$$

If M_n is attracted to the G above, then we can choose the constants as

$$a_n = 0 \quad \text{and} \quad b_n = \inf_{t \ge 0} \{1 - F(t) \le 1/n\} \tag{2.12}$$

(see, e.g., Bingham et al. [4], Sect. 8.13). Dwass [9] and Resnick [15] proved that

$$M_{\lfloor nt \rfloor}/b_n \xrightarrow{D[0,1]} E(t), \tag{2.13}$$

where $\{E(t), t \in [0,1]\}$ is the extremal process of type G. The finite-dimensional distributions of $E(t)$ can be characterized as follows: If $0 = t_0 < t_1 < t_2 < \cdots < t_k \le 1$, then

$$\big(E(t_1), \dots, E(t_k)\big) \stackrel{d}{=} \big(U_1, \max\{U_1, U_2\}, \dots, \max\{U_1, U_2 \dots, U_k\}\big), \tag{2.14}$$

where $U_1, \dots, U_k$ are independent with $P(U_i \le x) = G^{t_i - t_{i-1}}(x)$.

Theorem 2.5 *Assume that $X_1, X_2, \dots$ are i.i.d. random variables and define $M_n = \max\{X_1, \dots, X_n\}$. Assume further that $M_n/b_n \xrightarrow{d} G$, where G is given in* (2.11), *and b_n is given in* (2.12). *Then*

$$\frac{1}{n} \sum_{k=1}^{n} I\left\{\frac{M_k}{b_k} \le z\right\} \xrightarrow{D[\mathbb{R}]} \int_0^1 I\left\{\frac{E(t)}{t^{1/\alpha}} \le z\right\} \mathrm{d}t,$$

where $E(t)$ is an extremal process of type G.

2.1 Distribution of the Limiting Process

The purpose of this section is to investigate the distribution of the random variables $\Lambda(z)$ defined in (2.3). In the case of partial sums of i.i.d. random variables with mean 0 and variance 1, $R(t)$ in (2.2) equals $W(t)/\sqrt{t}$, and

$$\Lambda(z) = \int_0^1 I\big\{W(t)/\sqrt{t} \le z\big\}\, \mathrm{d}t. \tag{2.15}$$

By Lévy's arc-sine law we have $\Lambda(0) \stackrel{d}{=} A_{1/2}$, but computing the distribution of $\Lambda(z)$ for $z \ne 0$ seems to be a much harder problem. The combinatorial approaches used by Andersen [1] and Spitzer [19] seem not to carry over to this case.

In this section we will compute $E\Lambda^p(z)$ for some special functionals h_k in (1.7). Our first observation is that if the distribution of $R(t)$ does not depend on t, then $E\Lambda(z) = P(R(1/2) \le z)$.

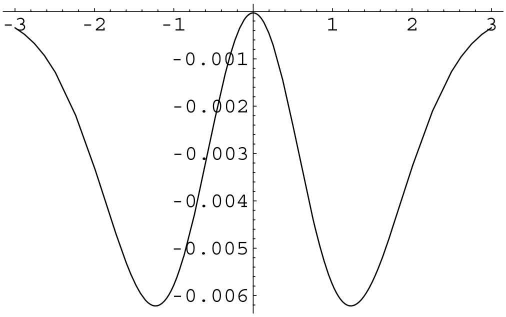

Fig. 1 The graph of $E\Lambda^2(z) - EA^2_{\Phi(z)}$ shows that for $z \neq 0$, $\Lambda(z)$ has not the same distribution as $A_{\Phi(z)}$, although the differences are very small

Lemma 2.1 *Assume that* $\{R(t), t \in (0,1)\}$ *is a Gaussian process with* $ER(t) = 0$, $R^2(t) = 1$, *and* $\mathrm{Corr}(R(s), R(t)) = \rho(s,t)$. *Then*

$$E\Lambda^2(z) = \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \big(\Phi^{(n)}(z)\big)^2,$$

where $\gamma_n = \int_0^1 \int_0^1 \rho^n(s,t)\, \mathrm{d}s\, \mathrm{d}t$, *and* $\Phi^{(n)}(z)$ *is the nth derivative of the standard normal distribution function.*

Returning to the special case (2.15), a reasonable conjecture is that $\Lambda(z)$ is generalized arc-sine. Since in this case we have $E\Lambda(z) = \Phi(z)$, this would mean that $\Lambda(z) \stackrel{d}{=} A_{\Phi(z)}$. In order to get evidence for this conjecture, we used Lemma 2.1 and computed $E\Lambda^2(z)$ and compared it with $EA^2_{\Phi(z)} = \Phi(z)(1+\Phi(z))/2$. If the conjecture were true, the functions have to coincide. However, the computations (see Fig. 1) disproves this conjecture, although the difference is very small.

It remains open whether it is possible to extend Lemma 2.1 and get explicit representations for higher moments. The following lemma gives such a formula for the moments of $\Lambda(z) = \int_0^1 \{E(t)/t^{1/\alpha} \le z\}\, \mathrm{d}t$, when $E(t)$ is the extremal process of type G with G given in (2.11). In this case the moments uniquely determine the distribution of $\Lambda(z)$ (cf. Feller [11], Sect. VII.3]).

Lemma 2.2 *Let* G *be given as in* (2.11), *and let* $E(t)$ *be an extremal process of type* G. *Let* $\Lambda(z) = \int_0^1 I\{E(t)/t^{1/\alpha} \le z\}\, \mathrm{d}t$. *Then if* $p \ge 2$,

$$E\Lambda^p(z) = G^p(z)(p-1)! \prod_{k=0}^{p-2} \int_0^1 u^k e^{u\lambda z^{-\alpha}}\, \mathrm{d}u.$$

3 Proofs

3.1 Convergence of the Finite-Dimensional Distributions and Tightness

Lemma 3.1 *Assume that* (1.7) *and* (2.2) *hold. For* $0 < \varepsilon < 1/2$, *define* $\Lambda^\varepsilon(z)$ *as in* (2.7) *and set* $\Lambda_n^\varepsilon(z) = \int_\varepsilon^{1-\varepsilon} I\{R_n(t) \le z\}\, \mathrm{d}t$.

(A) *In order to prove* (2.4), *it is enough to show that, for all* $0 < \varepsilon < 1/2$,

$$\left\{\Lambda_n^\varepsilon(z_1), \Lambda_n^\varepsilon(z_2), \ldots, \Lambda_n^\varepsilon(z_M)\right\} \xrightarrow{d} \left\{\Lambda^\varepsilon(z_1), \Lambda^\varepsilon(z_2), \ldots, \Lambda^\varepsilon(z_M)\right\}. \tag{3.1}$$

(B) *In order to prove* $\Lambda_n(z) \xrightarrow{D[\mathbb{R}]} \Lambda(z)$, *it is enough to show that, for all* $0 < \varepsilon < 1/2$, $\Lambda_n^\varepsilon(z) \xrightarrow{D[\mathbb{R}]} \Lambda^\varepsilon(z)$.

Proof We will show part (B); the proof of part (A) is similar but easier. Recall that $\Lambda_n^\varepsilon(z) \xrightarrow{D[\mathbb{R}]} \Lambda^\varepsilon(z)$ means $\Lambda_n^\varepsilon(T^{-1}(u)) \xrightarrow{D[\mathbb{R}]} \Lambda^\varepsilon(T^{-1}(u))$ for some strictly increasing distribution function T. Obviously,

$$\sup_{u\in[0,1]} \left|\Lambda_n^\varepsilon\big(T^{-1}(u)\big) - \Lambda_n\big(T^{-1}(u)\big)\right| \le 2\varepsilon \quad \text{and}$$

$$\sup_{u\in[0,1]} \left|\Lambda^\varepsilon\big(T^{-1}(u)\big) - \Lambda\big(T^{-1}(u)\big)\right| \le 2\varepsilon.$$

Hence, for every bounded and uniformly continuous function $f : D[0,1] \to \mathbb{R}$ and every $\eta > 0$, there is some $\varepsilon > 0$ such that $Ef(\Lambda_n T^{-1}) \le \eta + Ef(\Lambda_n^\varepsilon T^{-1})$. Thus by the weak convergence of Λ_n^ε and the Portmanteau theorem we have $\limsup_{n\to\infty} Ef(\Lambda_n T^{-1}) \le \eta + Ef(\Lambda^\varepsilon T^{-1}) \le 2\eta + Ef(\Lambda T^{-1})$. Similarly we obtain $\liminf_{n\to\infty} Ef(\Lambda_n T^{-1}) \ge 2\eta + Ef(\Lambda T^{-1})$. Since these estimates hold for every $\eta > 0$, the statement is proven. □

Proof of Theorem 2.1 Let $(z_1, \ldots, z_M) \in \mathbb{R}^M$ be such that (2.5) holds, and let $(\lambda_1, \ldots, \lambda_M) \in \mathbb{R}^M$. Let $0 < \varepsilon < 1/2$ and consider the mapping $\tau_\varepsilon : D[\varepsilon, 1-\varepsilon] \to \mathbb{R}$ with

$$\tau_\varepsilon(x) = \sum_{i=1}^M \lambda_i \int_\varepsilon^{1-\varepsilon} I\left\{x(t) \le z_i\right\} \mathrm{d}t.$$

If x_n and x are elements of $D[\varepsilon, 1-\varepsilon]$ with $x_n \to x$ in the topology of $D[\varepsilon, 1-\varepsilon]$, we know that $x_n(t) \to x(t)$ for almost all $t \in [\varepsilon, 1-\varepsilon]$. Hence $I\{x_n(t) \le z\} \to I\{x(t) \le z\}$ for $\{t \in [\varepsilon, 1-\varepsilon] : t \in N^c \cap x(t) \ne z\}$ (N is a set of Lebesgue measure 0 depending on x_n and x, and N^c is its complement). This shows that $x_n \to x$ in the topology of $D[\varepsilon, 1-\varepsilon]$ implies $\tau_\varepsilon(x_n) \to \tau_\varepsilon(x)$, provided that the time $x(t)$ spends in $\{z_1, \ldots, z_M\}$ has Lebesgue measure 0, i.e.,

$$\int_\varepsilon^{1-\varepsilon} I\left\{x(t) \in \{z_1, \ldots, z_M\}\right\} \mathrm{d}t = 0.$$

Let ρ_ε denote the Skorokhod metric on $D[\varepsilon, 1-\varepsilon]$. Since $R \in D(0,1)$, we can consider R as an element of $D[\varepsilon, 1-\varepsilon]$ by restricting its domain of definition. We showed that τ_ε is PR^{-1} a.s. ρ_ε-continuous if

$$PR^{-1}\left\{x \in D[\varepsilon, 1-\varepsilon] \middle| \int_\varepsilon^{1-\varepsilon} I\left\{x(t) \in \{z_1, \ldots, z_M\}\right\} \mathrm{d}t > 0\right\} = 0,$$

which is

$$P\left\{\int_{\varepsilon}^{1-\varepsilon} I\left\{R(t)\in\{z_1,\dots,z_M\}\right\}\mathrm{d}t>0\right\}=0.$$

Since the last relation is implied by (2.5), by the mapping theorem (see Billingsley [3], Theorem 2.7), we have $\tau_\varepsilon(R_n)\xrightarrow{d}\tau_\varepsilon(R)$. Since this is true for every $(\lambda_1,\dots,\lambda_M)\in\mathbb{R}^M$, we get (3.1) by the Cramér–Wold theorem. Hence the result follows from part (A) of Lemma 3.1. □

Proof of Theorem 2.2 Theorem 2.1, (2.6), and Lemma 3.1 yield that $\Lambda_n^\varepsilon(z)\xrightarrow{f.d.d.}\Lambda^\varepsilon(z)$. Hence, according to part (B) of Lemma 3.1, it is enough to show the tightness of $\Lambda_n^\varepsilon(\Phi^{-1}(u))$, $0\le u\le 1$, where $\Phi(x)$ is the standard normal distribution function. (Instead of $\Phi(x)$, we can use any continuous and strictly increasing distribution function.) The tightness will follow if we show that for every $\delta>0$,

$$\lim_{K\to\infty}\limsup_{n\to\infty}P\left\{\sup_{\substack{0\le u,v\le 1\\|u-v|\le 1/K}}\left|\Lambda_n^\varepsilon\left(\Phi^{-1}(u)\right)-\Lambda_n^\varepsilon\left(\Phi^{-1}(v)\right)\right|>\delta\right\}=0. \tag{3.2}$$

Since $\Lambda_n^\varepsilon(\Phi^{-1}(u))\to\Lambda_n^\varepsilon(\Phi^{-1}(0))=0$ a.s. $(u\to 0)$ for all $\eta>0$, there is $a\in(0,1)$ such that

$$P\left\{\Lambda_n^\varepsilon\left(\Phi^{-1}(a)\right)>\delta/6\right\}\le\eta. \tag{3.3}$$

Similarly, $\Lambda_n^\varepsilon(\Phi^{-1}(u))\to\Lambda_n^\varepsilon(\Phi^{-1}(1))=1-2\varepsilon$ a.s. $(u\to 1)$, and therefore, for all $\eta>0$, there is some $b\in(0,1)$ such that

$$P\left\{1-2\varepsilon-\Lambda_n^\varepsilon\left(\Phi^{-1}(b)\right)>\delta/6\right\}\le\eta. \tag{3.4}$$

We note that

$$\begin{aligned}\sup_{\substack{0\le u,v\le 1\\|u-v|\le 1/K}}&\left|\Lambda_n^\varepsilon\left(\Phi^{-1}(u)\right)-\Lambda_n^\varepsilon\left(\Phi^{-1}(v)\right)\right|\\ &\le 2\Lambda_n^\varepsilon\left(\Phi^{-1}(a)\right)+2\left(1-2\varepsilon-\Lambda_n^\varepsilon\left(\Phi^{-1}(u)\right)\right)\\ &\quad+2\sup_{\substack{a\le u,v\le b\\|u-v|\le 1/K}}\left|\Lambda_n^\varepsilon\left(\Phi^{-1}(u)\right)-\Lambda_n^\varepsilon\left(\Phi^{-1}(v)\right)\right|.\end{aligned}$$

Let $u_i=a+i(b-a)/K$, $0\le i\le K$. Now

$$\begin{aligned}\sup_{\substack{a\le u,v\le b\\|u-v|\le 1/K}}&\left|\Lambda_n^\varepsilon\left(\Phi^{-1}(u)\right)-\Lambda_n^\varepsilon(\Phi^{-1}(v))\right|\\ &\le 2\max_{0\le i\le K-1}\left|\Lambda_n^\varepsilon\left(\Phi^{-1}(u_{i+1})\right)-\Lambda_n^\varepsilon\left(\Phi^{-1}(u_i)\right)\right|.\end{aligned}$$

By the convergence of the finite-dimensional distributions we have

$$\Big\{\Lambda_n^\varepsilon\big(\Phi^{-1}(a)\big),\Lambda_n^\varepsilon\big(\Phi^{-1}(b)\big),\max_{1\le i\le K-1}\big|\Lambda_n^\varepsilon\big(\Phi^{-1}(u_{i+1})\big)-\Lambda_n^\varepsilon\big(\Phi^{-1}(u_i)\big)\big|\Big\}$$
$$\xrightarrow{d}\Big\{\Lambda^\varepsilon\big(\Phi^{-1}(a)\big),\Lambda^\varepsilon\big(\Phi^{-1}(b)\big),\max_{1\le i\le K-1}\big|\Lambda^\varepsilon\big(\Phi^{-1}(u_{i+1})\big)-\Lambda^\varepsilon\big(\Phi^{-1}(u_i)\big)\big|\Big\}.$$

By the almost sure continuity of $\Lambda^\varepsilon(z)$ on $[\Phi^{-1}(a),\Phi^{-1}(b)]$ we get that, for all $\delta>0$,

$$\lim_{K\to\infty}P\Big\{\max_{1\le i\le K-1}\big|\Lambda^\varepsilon\big(\Phi^{-1}(u_{i+1})\big)-\Lambda^\varepsilon\big(\Phi^{-1}(u_i)\big)\big|>\delta/6\Big\}=0.$$

Hence, combining the last relation with (3.3) and (3.4), we have

$$\lim_{K\to\infty}\limsup_{n\to\infty}P\Big\{\sup_{\substack{0\le u,v\le 1\\ |u-v|\le 1/K}}\big|\Lambda_n^\varepsilon\big(\Phi^{-1}(u)\big)-\Lambda_n^\varepsilon\big(\Phi^{-1}(v)\big)\big|>\delta\Big\}\le 2\eta.$$

Since this is true for any $\eta>0$, relation (3.2) is proven. □

3.2 Proof of the Main Results

Proof of Theorem 2.3 Since $R(t)$ is Gaussian, $P(R(t)\le z)$ is a continuous function of z for every $t\in(0,1)$. By Theorem 2.2 and the continuity condition (2.8) it is enough to show that

$$E\bigg(\int_\varepsilon^{1-\varepsilon}I\{R(t)\in[z,z+\delta]\}\,\mathrm{d}t\bigg)^2$$
$$=\int_\varepsilon^{1-\varepsilon}\int_\varepsilon^{1-\varepsilon}P\big(\big(R(s),R(t)\big)\in[z,z+\delta]^2\big)\,\mathrm{d}s\,\mathrm{d}t=o\big(\delta^{1+\eta}\big),\tag{3.5}$$

uniformly in z and with some $\eta>0$. The density $f_{s,t}(x,y)$ of the vector $(R(s),R(t))$ is bivariate normal, and hence

$$f_{s,t}(x,y)\le\frac{1}{2\pi\sigma_s\sigma_t\sqrt{1-\rho^2(s,t)}}\quad\text{for all }(x,y)\in\mathbb{R}^2.$$

This obviously implies that

$$P\big((R(s),R(t))\in[z,z+\delta]^2\big)\le\delta^2\Big(2\pi\sigma_s\sigma_t\sqrt{1-\rho^2(s,t)}\Big)^{-1}.$$

Hence, if (2.9) holds, then (3.5) follows if we choose $\eta<1/2$. Thus we have $\Lambda_n^\varepsilon(z)\xrightarrow{D[\mathbb{R}]}\Lambda^\varepsilon(z)$. Since the argument is true for all $0<\varepsilon<1/2$, Theorem 2.3 follows from Lemma 3.1. □

Proof of Theorem 2.4 We proceed as in the proof of Theorem 2.3. Assume that $s<t$. From our assumptions we get for $f_{s,t}(x,y)$, the joint density of $(R(s),R(t))$, that

$$f_{s,t}(x,y)=\ell(x)\frac{b_t}{b_{t-s}}\ell(b_t y/b_{t-s}-b_s x/b_{t-s})\le \sup_{x\in\mathbb{R}}\bigl|\ell(x)\bigr|^2\frac{b_t}{b_{t-s}}.$$

Now the result follows from (2.10). □

Proof of Theorem 2.5 First we note that $b_n=n^{1/\alpha}L(n)$, where $L(n)$ is slowly varying as $n\to\infty$ (cf. Bingham et al. [4], Theorem 8.13.2]). Hence $M_{\lfloor nt\rfloor}/b_{\lfloor nt\rfloor}\stackrel{D(0,1)}{\longrightarrow} E(t)/t^{1/\alpha}$. Note that the distribution function of the limiting process does not depend on t and is continuous. By Theorem 2.1 we get

$$\frac{1}{n}\sum_{k=1}^{n}I\{M_k/b_k\le z\}=\int_0^1 I\bigl\{M_{\lfloor nt\rfloor}/b_{\lfloor nt\rfloor}\le z\bigr\}\stackrel{f.d.d.}{\longrightarrow}\int_0^1 I\bigl\{E(t)/t^{1/\alpha}\le z\bigr\}\,\mathrm{d}t.$$

Our goal is again to establish (2.8) with $R(t)=E(t)/t^{1/\alpha}$. It can be easily seen that it is sufficient to show that, for every $h>0$ and for sufficiently small $\eta>0$,

$$\sup_{z\ge h}\int_0^1\int_0^t P\bigl(E(s)/s^{1/\alpha}\in[z,z+\delta],\,E(t)/t^{1/\alpha}\in[z,z+\delta]\bigr)\,\mathrm{d}s\,\mathrm{d}t=o\bigl(\delta^{1+\eta}\bigr)\tag{3.6}$$

as $\delta\to 0$. If $h=0$, then (2.8) trivially follows from (3.6) because of $R(t)\ge 0$. Since the distribution of $R(t)$ (which does not depend on t) is continuous, it is sufficient to show (3.6) for every $h>0$.

Relation (2.14) shows that, for $0\le s<t\le 1$, we have

$$P\bigl(E(s)\le x,E(t)\le y\bigr)=\begin{cases}G^s(x)G^{t-s}(y) & \text{if } x\le y;\\ G^s(y)G^{t-s}(y) & \text{if } x>y.\end{cases}$$

We assume throughout that $s\le t$. Since λ in (2.11) has no influence on the following arguments, we assume for simplicity that $\lambda=1$. Now we have to distinguish between the cases (i) $(z+\delta)s^{1/\alpha}>zt^{1/\alpha}$ and (ii) $(z+\delta)s^{1/\alpha}\le zt^{1/\alpha}$. Under assumption (i), we get by some algebra that

$$\begin{aligned}&P\bigl(E(s)/s^{1/\alpha}\in[z,z+\delta],E(t)/t^{1/\alpha}\in[z,z+\delta]\bigr)\\ &\quad=\bigl[G^{t-s}\bigl((z+\delta)t^{1/\alpha}\bigr)-G^{t-s}\bigl(zt^{1/\alpha}\bigr)\bigr]\times\bigl[G^s\bigl((z+\delta)s^{1/\alpha}\bigr)-G^s\bigl(zs^{1/\alpha}\bigr)\bigr]\\ &\qquad+\bigl[G^s\bigl((z+\delta)s^{1/\alpha}\bigr)-G^s\bigl(zt^{1/\alpha}\bigr)\bigr]G^{t-s}\bigl(zt^{1/\alpha}\bigr)\\ &\quad=:H_1(s,t,z,\delta)+H_2(s,t,z,\delta).\end{aligned}$$

Under (ii), we have the simpler relation

$$P\bigl(E(s)/s^{1/\alpha}\in[z,z+\delta],E(t)/t^{1/\alpha}\in[z,z+\delta]\bigr)=H_1(s,t,z,\delta).$$

Thus (3.6) will follow if, for every $h>0$,

$$\sup_{z\geq h}\int_0^1\left(\int_{r_i}^t H_i(s,t,z,\delta)\,\mathrm{d}s\right)\mathrm{d}t = o\big(\delta^{1+\eta}\big)\quad (i=1,2),\tag{3.7}$$

where $r_1=0$ and $r_2=t(\frac{z}{z+\delta})^\alpha$. Using the mean-value theorem and the explicit form of G, we obtain

$$\begin{aligned}\int_0^t H_1(s,t,z,\delta)\,\mathrm{d}t &\leq \delta\int_0^t e^{-(1-s/t)(z+\delta)^{-\alpha}} - e^{-(1-s/t)(z+\delta)^{-\alpha}}\,\mathrm{d}t\\ &= \delta t\big(L(z+\delta)-L(z)\big)\end{aligned}$$

with $L(z)=z^\alpha(1-G(z))$. Observe that $\sup_{z\geq 0}|L(z+\delta)-L(z)|=O(\delta^{\alpha\wedge 1})$. Thus, if η is chosen sufficiently small, we get (3.7) when $i=1$.

Next we consider H_2. A straightforward calculation shows that

$$\begin{aligned}&\int_{\{(z/(z+\delta))^\alpha\leq s/t\leq 1\}} H_2(s,t,z,\delta)\,\mathrm{d}s\\ &\quad = t\left[z^\alpha\big(G(z+\delta)-G(z)\big)-\left(1-\left(\frac{z}{z+\delta}\right)^\alpha\right)G(z)\right].\end{aligned}$$

First we assume that $\alpha\leq 1$. By the mean-value theorem we get that

$$1-\left(\frac{z}{z+\delta}\right)^\alpha \geq \alpha\delta(z+\delta)^{-1}$$

and

$$z^\alpha\big(G(z+\delta)-G(z)\big)\leq \delta\, z^\alpha G'(z)=\alpha\delta z^{-1}G(z).$$

Since $G(z)\leq 1$ and $z\geq h$, we obtain by combining our previous results that

$$\sup_{z\geq h}\int_0^1\int_{\{(z/(z+\delta))^\alpha\leq s/t\leq 1\}} H_2(s,t,z,\delta)\,\mathrm{d}s\,\mathrm{d}t\leq \frac{1}{2}\alpha\delta\big(z^{-1}-(z+\delta)^{-1}\big)=O\big(\delta^2\big).$$

The case $\alpha>1$ can be treated similarly. □

Proof of Corollaries 2.1 and 2.2 We only have to prove Corollary 2.2, which is the more general one. We set $B_0=0$ and define

$$\xi_n(t)=\frac{S_k}{B_n}\quad \text{if } t\in\big[B_k^2/B_n^2,\, B_{k+1}^2/B_n^2\big),\ 0\leq k\leq n.$$

Brown [7] showed that under the assumptions of Corollary 2.2, the polygonal line process obtained by connecting the points of discontinuity of $\xi_n(t)$ with straight lines converges to a standard Brownian motion. The argument there can be easily transposed to $\xi_n(t)\overset{D[0,1]}{\longrightarrow} W(t)$. Let

$$\eta_n(t)=\frac{S_k}{B_n}\quad \text{if } t\in\big[k/n,\,(k+1)/n\big),\ 0\leq k\leq n.$$

Hence $\eta_n(t) = \xi_n(g_n(t))$, where $g_n(t) = B^2_{\lfloor tn \rfloor}/B^2_n = (\lfloor nt \rfloor/n)^\alpha L(\lfloor nt \rfloor)/L(n)$. By the uniform convergence theorem for regularly varying functions (cf. Bingham et al. [4], p. 6) $\sup_{t\in[0,1]} |g_n(t) - t^\alpha| \to 0$ $(n \to \infty)$. In order to prove that the measures generated by the processes $\{\xi_n(t), t \in [0, 1]\}$ are tight, Brown [7] showed that, for every $\varepsilon > 0$, we have

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P\Bigg(\sup_{\substack{0\le s<t\le 1 \\ |s-t|<\delta}} \big|\xi_n(t) - \xi_n(s)\big| > \varepsilon \Bigg) \to 0.$$

(The argument remains valid for the broken line process.) This immediately implies that $\sup_{t\in[0,1]} |\xi_n(t^\alpha) - \eta_n(t)| \xrightarrow{P} 0$, respectively $\eta_n(t) \xrightarrow{D[0,1]} W(t^\alpha)$. Now we set

$$R_n(t) = \eta_n(t)\frac{B_{\lfloor nt \rfloor}}{B_n} = \frac{S_k}{B_k} \quad \text{if } t \in \big[k/n, (k+1)/n\big),\ 0 \le k \le n.$$

It is clear that $R_n(t) \xrightarrow{D(0,1)} W(t^\alpha)/t^{\alpha/2}$. Hence the limiting process $R(t) = W(t^\alpha)/t^{\alpha/2}$ is Gaussian with $\sigma_s = 1$ and $\mathrm{Corr}^2(R(s), R(t)) = (s/t)^\alpha$, for $s \le t$. Since condition (2.9) holds for every $\alpha > 0$, the result follows from Theorem 2.3. □

Proof of Corollaries 2.3–2.4 The proofs are the same as the proof of Corollary 2.2 with a corresponding invariance principle for the partial-sum process. In order to get the weak convergence of the partial sums in Corollary 2.3, we use Taqqu [20, Lemma 5.1]. It remains to show (2.9). After a change of variables $s = t - h$ condition (2.9) is

$$\int_\varepsilon^{1-\varepsilon} \int_0^{t-\varepsilon} \frac{\mathrm{d}h\, \mathrm{d}t}{\sqrt{1-\rho^2(t-h,t)}} < \infty, \tag{3.8}$$

where

$$\rho(t-h,t) = \frac{1}{2}\Bigg[\bigg(1-\frac{h}{t}\bigg)^H + \bigg(1-\frac{h}{t}\bigg)^{-H} - \frac{h^{2H}}{(t-h)^H t^H}\Bigg].$$

It can be easily seen that the integrand in (3.8) is only large when $h \to 0$. By some routine analysis we obtain uniformly in $t \in [\varepsilon, 1-\varepsilon]$ that

$$1 - \rho^2(t-h,t) \sim \bigg(\frac{h}{t}\bigg)^{2H} \quad \text{as } h \to 0.$$

Hence (3.8) holds.

Finally, for Corollary 2.4 we use Theorems 2.4 and 2.4 in Borodin and Ibragimov [5]. The verification of (2.10) is elementary and therefore omitted. □

3.3 Distribution of the Limits

Proof of Lemma 2.1 Assume that (X, Y) are jointly Gaussian with $\mathrm{Corr}(X, Y) = \rho$. A convenient way to compute $Ef_1(X)f_2(Y)$ is to use the Hermite expansions of f_1

and f_2 (cf. Rozanov [16, p. 182]). We obtain

$$Ef_1(X)f_2(Y)=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\alpha_n^{(1)}\alpha_n^{(2)},$$

where $\alpha_n^{(i)}$ are the Hermite coefficients of the corresponding functions f_i, i.e.,

$$\alpha_n^{(i)}=\int_{-\infty}^{\infty}f_i(x)\phi(x)H_n(x)\,\mathrm{d}x\quad(i=1,2)$$

with

$$H_n(x)=(-1)^n e^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2/2},$$

and $\phi(x)$ is the standard normal density function. (Of course, we need to assume that the Hermite expansions of f_i exist.) Hence

$$EI\{R(s)\le z\}I\{R(t)\le z\}=\sum_{n\ge 0}\frac{\rho^n(s,t)}{n!}\big(\Phi^{(n)}(z)\big)^2.$$ □

Proof of Lemma 2.2 By (2.14) we have

$$\begin{aligned}E\Lambda^p(z)&=p!\int_0^1\int_0^{t_{p-1}}\dots\int_0^{t_1}P\big(E_{t_0}\le zt_0^{1/\alpha},E_{t_1}\le zt_1^{1/\alpha},\dots,E_{t_{p-1}}\le zt_{p-1}^{1/\alpha}\big)\\&\quad\times \mathrm{d}t_0\dots\mathrm{d}t_{p-1}\\&=p!\int_0^1\int_0^{t_{p-1}}\dots\int_0^{t_1}G^{t_0}\big(zt_0^{1/\alpha}\big)G^{t_1-t_0}\big(zt_1^{1/\alpha}\big)\dots G^{t_p-t_{p-1}}\big(zt_{p-1}^{1/\alpha}\big)\\&\quad\times \mathrm{d}t_0\dots\mathrm{d}t_{p-1}\\&=p!G(z)\int_0^1\int_0^{t_{p-1}}\dots\int_0^{t_1}e^{(t_0/t_1-1)\lambda z^{-\alpha}}\dots e^{(t_{p-2}/t_{p-1}-1)\lambda z^{-\alpha}}\,\mathrm{d}t_0\dots\mathrm{d}t_{p-1}.\end{aligned}$$

The result can be easily obtained by changing variables $s_i=t_{i-1}/t_i$. □

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