Permutation and bootstrap statistics under infinit variance

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Abstract Selection from a finit population is used in several procedures in statistics, among others in bootstrap and permutation methods. In this paper we give a survey of some recent results for selection in "nonstandard" situations, i.e. in cases when the negligibility condition of classical central limit theory is not satisfied These results enable us to describe the asymptotic properties of bootstrap and permutation statistics in case of infinit variances, when the limiting processes contain random coefficients We will also show that random limit distributions can be avoided by a suitable trimming of the sample, making bootstrap and permutation methods applicable for statistical inference under infinit variances.

1 Introduction

Selection from a finit population is a basic procedure in statistics and large sample properties of many classical tests and estimators are closely connected with the asymptotic behavior of sampling variables. Typical examples are bootstrap and permutation statistics, both of which assume a sample $X_1, X_2, \ldots, X_n$ of i.i.d. random variables with distribution function F and then drawing, with or without replacement, $m = m(n)$ elements from the finit set $\{X_1, \ldots, X_n\}$. The usefulness of this procedure is due to the fact that the asymptotic properties of many important func-

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tionals of the random variables $X_1^{(n)},\ldots,X_m^{(n)}$ obtained by resampling are similar to those of the functionals of the original sample $X_1,\ldots,X_n$. There is an extensive literature of bootstrap and permutation statistics in case of populations with finit variance; on the other hand, very little is known in the case of infinit variances. Athreya [2] showed, in the case when the underlying distribution is a stable distribution with parameter $0<\alpha<2$, that the normalized partial sums of bootstrap statistics converge weakly to a random limit distribution, i.e. to a distribution function containing random coefficients Recently, Aue, Berkes and Horváth [3] extended this result to permutation statistics. Note that the elements of a permuted sample are, in contrast to bootstrap, dependent, leading to a different limit distribution.

The purpose of the present paper is to give a survey of the asymptotics of permutation and bootstrap statistics in case of infinit variances, together with applications to statistical inference, e.g. for change point problems. In Section 2 we will show that resampling from an i.i.d. sample with infinit variance requires studying the limiting behavior of a triangular array of random variables violating the classical uniform asymptotic negligibility condition of central limit theory. Starting with the 1960's, classical central limit theory has been extended to cover such situations (see e.g. Bergström [4]). In this case the limit distribution is generally not Gaussian and depends on the non-negligible elements of the array. In the case of permutation statistics, the row elements of our triangular array are dependent random variables, making the situation considerably more complicated. Theorems 2.3 and 2.4 in Section 2 describe the new situation. As we will show in Section 3, the probabilistically interesting, but statistically undesirable phenomenon of random limit distributions can be avoided by trimmimg the sample, enabling one to extend a number of statistical procedures for observations with infinit variances.

Our paper is a survey of some recent results of the authors; the proofs will be given in our forthcoming papers [6] and [7].

2 Some general sampling theorems

For each $n\in\mathbb{N}$ let

$$x_{1,n}\le x_{2,n}\le\ldots\le x_{n,n}$$

be a sequence of real numbers and denote by $X_1^{(n)},X_2^{(n)},\ldots,X_m^{(n)}$ the random variables obtained by drawing, with or without replacement, $m=m(n)$ elements from the set $\{x_{1,n},\ldots,x_{n,n}\}$. Defin the partial sum process

$$Z_{n,m}(t)=\sum_{j=1}^{\lfloor mt\rfloor}X_j^{(n)}\qquad\text{for } 0\le t\le 1,\tag{1}$$

where $\lfloor \cdot \rfloor$ denotes integral part. Let $\xrightarrow{\mathscr{D}[0,1]}$ denote convergence in the space $\mathscr{D}[0,1]$ of càdlàg functions equipped with the Skorokhod J_1-topology. The following two results are well known.

Theorem 2.1. *Let*

$$\sum_{j=1}^{n} x_{j,n} = 0, \qquad \sum_{j=1}^{n} x_{j,n}^2 = 1 \tag{2}$$

and

$$\max_{1 \le j \le n} |x_{j,n}| \longrightarrow 0 \tag{3}$$

and draw $m = m(n)$ elements from the set $\{x_{1,n}, \dots, x_{n,n}\}$ with replacement, where

$$m/n \to c \qquad \text{for some } c > 0. \tag{4}$$

Then

$$Z_{n,m}(t) \xrightarrow{\mathscr{D}[0,1]} W(ct) \qquad \text{for } n \to \infty,$$

where $\{W(t), 0 \le t \le 1\}$ is a Wiener process.

Theorem 2.2. *Assume (2) and (3) and draw $m = m(n)$ elements from the set $\{x_{1,n}, \dots, x_{n,n}\}$ without replacement, where $m \le n$ and (4) holds. Then*

$$Z_{n,m}(t) \xrightarrow{\mathscr{D}[0,1]} B(ct) \qquad \text{for } n \to \infty,$$

where $\{B(t), 0 \le t \le 1\}$ is a Brownian bridge.

In the case of Theorem 2.1 the random variables $X_1^{(n)}, \dots, X_m^{(n)}$ are i.i.d. with mean 0 and variance $1/n$ and they satisfy the Lindeberg condition

$$\lim_{n\to\infty} \sum_{j=1}^{m} E[(X_j^{(n)})^2 I\{|X_j^{(n)}| \ge \varepsilon\}] = 0 \qquad \text{for any } \varepsilon > 0, \tag{5}$$

since the sum on the left hand side is 0 for $n \ge n_0(\varepsilon)$ by the uniform asymptotic negligibility condition (3). Thus Theorem 2.1 is an immediate consequence of the classical functional central limit theorem for sums of independent random variables (see e.g. Skorokhod [16]). Theorem 2.2, due to Rosén [15], describes a different situation: if we sample without replacement, the r.v.'s $X_1^{(n)}, \dots, X_m^{(n)}$ are dependent and the partial sum process $Z_{n,m}(t)$ converges weakly to a process with dependent (actually negatively correlated) increments.

Typical applications of Theorem 2.1 and Theorem 2.2 include bootstrap and permutation statistics. Let $X_1, X_2, \dots$ be i.i.d. random variables with distribution function F with mean 0 and variance 1. Let $\{X_1^{(n)}, \dots, X_m^{(n)}\}$ be the bootstrap sample obtained by drawing $m = m(n)$ elements from the set $\{X_1, \dots, X_n\}$ with replacement. Clearly, $X_1^{(n)}, \dots, X_m^{(n)}$ are independent random variables with common distribution $F_n(t) = (1/n)\sum_{i=1}^{n} I\{X_i \le t\}$, the empirical distribution function of the sample $X_1, \dots, X_n$. Defin

$$\overline{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \quad \text{and} \quad \sigma_n^2 = \frac{1}{n}\sum_{k=1}^{n}(X_k - \overline{X}_n)^2$$

and apply Theorem 2.1 for the random finit set

$$\left\{ \frac{X_1 - \overline{X}_n}{\sigma_n \sqrt{n}}, \ldots, \frac{X_n - \overline{X}_n}{\sigma_n \sqrt{n}} \right\}, \tag{6}$$

where the selection process is independent of the sequence $X_1, X_2, \ldots$. It is easily checked that the conditions of Theorem 2.1 are satisfie and it follows that if (4) holds then conditionally on $\mathbf{X} = (X_1, X_2, \ldots)$, for almost all paths $(X_1, X_2, \ldots)$,

$$P_{\mathbf{X}}\left\{ \frac{1}{\sigma_n \sqrt{n}} \sum_{k=1}^{\lfloor mt \rfloor} (X_k^{(n)} - \overline{X}_n) \xrightarrow{\mathscr{D}[0,1]} W(ct) \right\} = 1.$$

This fundamental limit theorem for the bootstrap is due to Bickel and Freedman [8]. On the other hand, drawing n elements from the set $\{X_1, \ldots, X_n\}$ without replacement, we get a random permutation of $X_1, \ldots, X_n$ which we denote by $X_{\pi(1)}, \ldots, X_{\pi(n)}$. Again we assume that the selection process is independent of $X_1, X_2, \ldots$. It is clear that all $n!$ permutations of $(X_1, X_2, \ldots, X_n)$ are equally likely. Applying now Theorem 2.2 for the set (6), we get that for almost all paths $\mathbf{X} = (X_1, X_2, \ldots)$,

$$P_{\mathbf{X}}\left\{ \frac{1}{\sigma_n \sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} (X_{\pi(k)} - \overline{X}_n) \xrightarrow{\mathscr{D}[0,1]} B(t) \right\} = 1,$$

an important fact about permutation statistics.

As the above results show, the uniform asymptotic negligibility condition for the random set (6) is satisfie if $EX_1^2 < \infty$. It is easy to see that the converse is also true. Thus studying bootstrap and permutation statistics under infinit variances requires a model where uniform asymptotic negligibility fails, i.e. the elements of the set $\{x_{1,n}, \ldots, x_{n,n}\}$ are not any more ”small”. Clearly, in this case the limiting behavior of the partial sums of the selected elements will be quite different. If, for example, the largest element $x_{n,n}$ of the set $\{x_{1,n}, \ldots, x_{n,n}\}$ does not tend to 0 as $n \to \infty$, then the contribution of $x_{n,n}$ in the partial sums of a sample of size n taken from this set clearly will not be negligible and thus the limit distribution of such sums (if exists) will depend on this largest element. A similar effect is well known in classical central limit theory (see e.g. Bergström [4]), but the present situation will exhibit substantial additional difficulties Without loss of generality we may assume again that

$$\sum_{j=1}^{n} x_{j,n} = 0. \tag{7}$$

Next we will assume

$$x_{j,n} \longrightarrow y_j \qquad \text{and} \qquad x_{n-j+1,n} \longrightarrow z_j \tag{8}$$

for any fi ed j as $n \to \infty$ for some numbers $y_j, z_j, j \in \mathbb{N}$ and that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \sum_{j=K+1}^{n-K} x_{j,n}^2 = 0. \tag{9}$$

Condition (8) is no essential restriction of generality: if we assume only that the sequences $\{x_{j,n}, n \geq 1\}$, $\{x_{n-j+1,n}, n \geq 1\}$ are bounded for any fi ed j, then by a diagonal argument we can fin a subsequence of n's along which (8) holds. Then along this subsequence our theorems will apply and if the limiting numbers y_j, z_j are different along different subsequences, the processes $Z_{n,m}(t)$ will also have different limits along different subsequences. This seems to be rather pathological behavior, but it can happen even in simple i.i.d. situations, see Corollary 2.5 below. The role of condition (9) is to exclude a Wiener or Brownian bridge component in the limiting process, as it occurs in Theorem 2.1.

We formulate now our main sampling theorems. For the proof we refer to Berkes, Horváth and Schauer [6].

Theorem 2.3. *Let, for each* $n = 1, 2, \ldots$,

$$x_{1,n} \leq x_{2,n} \leq \ldots \leq x_{n,n} \tag{10}$$

be a finit set satisfying (7), (8), (9) *and*

$$\sum_{j=1}^{\infty} y_j^2 < \infty \qquad \text{and} \qquad \sum_{j=1}^{\infty} z_j^2 < \infty. \tag{11}$$

Let $X_1^{(n)}, \ldots, X_m^{(n)}$ *be the random elements obtained by drawing* $m = m(n) \leq n$ *elements from the set (10) without replacement, where (4) holds. Then for the processes* $Z_{n,m}(t)$ *define by (1) we have*

$$Z_{n,m}(t) \xrightarrow{\mathscr{D}[0,1]} R(ct) \qquad \text{for } n \to \infty,$$

where

$$R(t) = \sum_{j=1}^{\infty} y_j(\delta_j(t) - t) + \sum_{j=1}^{\infty} z_j(\delta_j^*(t) - t)$$

and $\{\delta_j(t), 0 \leq t \leq 1\}$, $\{\delta_j^*(t), 0 \leq t \leq 1\}$, $j = 1, 2, \ldots$ *are independent jump processes, each making a single jump from 0 to 1 at a random point uniformly distributed in* $(0, 1)$.

Theorem 2.4. *Let, for each* $n = 1, 2, \ldots$,

$$x_{1,n} \leq x_{2,n} \leq \ldots \leq x_{n,n} \tag{12}$$

be a finit set satisfying (7), (8), (9) *and (11). Let* $X_1^{(n)}, \ldots, X_m^{(n)}$ *be the random elements obtained by drawing* $m = m(n)$ *elements from the set (12) with replacement, where (4) holds. Then for the processes* $Z_{n,m}(t)$ *define by (1) we have*

$$Z_{n,m}(t) \xrightarrow{\mathscr{D}[0,1]} R(ct) \qquad \textit{for } n \to \infty$$

where

$$R(t) = \sum_{j=1}^{\infty} y_j(\delta_j(t) - t) + \sum_{j=1}^{\infty} z_j(\delta_j^*(t) - t)$$

and $\{\delta_j(t), t \geq 0\}$, $\{\delta_j^*(t), t \geq 0\}$, $j = 1, 2, \ldots$ *are independent Poisson processes with parameter 1.*

These results show that in the case when the asymptotic negligibility condition is not satisfied the limiting process will depend on the values of y_j and z_j, $j = 1, 2, \ldots$. Obviously the y_j's and z_j's form non-increasing sequences. The larger the differences between consecutive values are, the more the process $R(t)$ will be different from a Wiener process (or a Brownian bridge, respectively). Observe also that in the case of permutation statistics with $m = n$, the process $R(t)$ satisfie $R(0) = R(1) = 0$ and therefore it gives a non-Gaussian "bridge", having the same covariances (up to a constant) as a Brownian bridge. In the bootstrap case $R(t)$ has the same covariances (again up to a constant) as a scaled Wiener process.

Figure 1 shows the sample paths of the (appropriately normalized) limiting process $R(t)$ for permutations with $-y_j = z_j = c j^{-1/a}$ ($a \in \{0.5, 0.8, 1, 1.5\}$) and of a Brownian bridge $B(t)$. The pictures show the differences between the two limiting processes and that increasing the value of a makes $R(t)$ look closer to a Brownian bridge.

We now turn to applications of Theorems 2.3 and 2.4. The simplest situation with infinit variances is the case of i.i.d. random variables $X_1, X_2, \ldots$ belonging to the domain of attraction of a stable r.v. ξ_α with parameter $\alpha \in (0,2)$. This means that for some numerical sequences $\{a_n, n \in \mathbb{N}\}$, $\{b_n, n \in \mathbb{N}\}$

$$\frac{S_n - a_n}{b_n} \xrightarrow{d} \xi_\alpha, \tag{13}$$

where $S_n = \sum_{j=1}^n X_j$. The necessary and sufficien condition for this is

$$P(X_1 > t) \sim pL(t)t^{-\alpha} \quad \text{and} \quad P(X_1 < -t) \sim qL(t)t^{-\alpha} \quad \text{as } t \to \infty \tag{14}$$

for some numbers $p \geq 0$, $q \geq 0$, $p + q = 1$ and a slowly varying function $L(t)$. Let the ordered statistics of $(X_1, \ldots, X_n)$ be $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ and apply Theorem 2.3 and Theorem 2.4 to the random set

$$\left\{ \frac{X_{1,n} - \bar{X}_n}{T_n}, \ldots, \frac{X_{n,n} - \bar{X}_n}{T_n} \right\}, \tag{15}$$

where $\bar{X}_n = (1/n) \sum_{j=1}^n X_j$ is the sample mean and $T_n = \max_{1 \leq j \leq n} |X_j|$. The normalization T_n is used since the r.v.'s are outside the domain of attraction of a normal random variable. This leads to the following results from Aue, Berkes and Horváth [3] and Berkes, Horváth and Schauer [6].

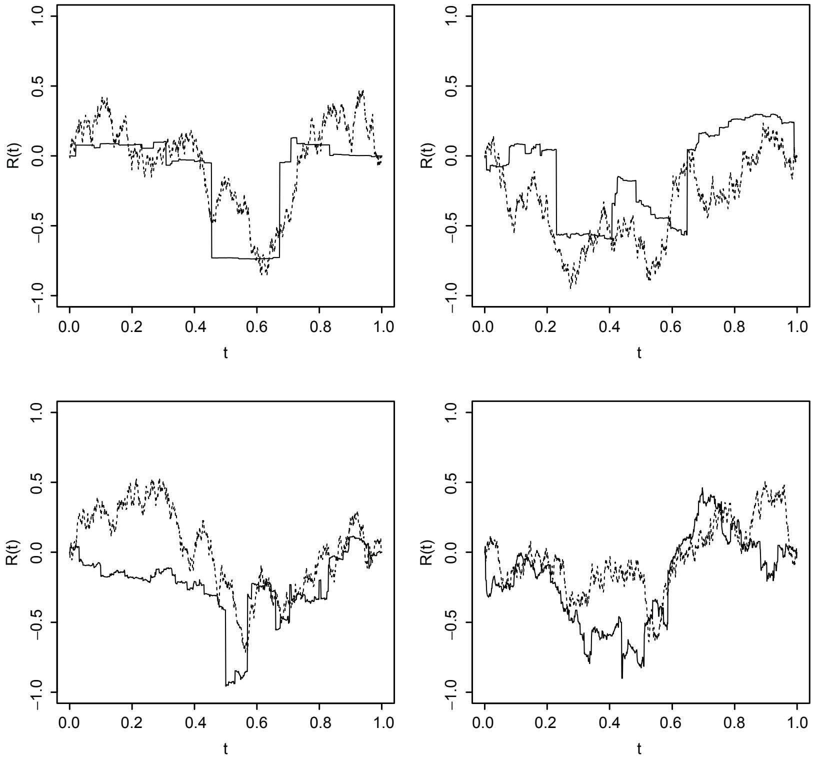

Fig. 1 Simulations of $R(t)$ (solid) and $B(t)$ (dashed) with $-y_j = z_j = const \cdot j^{-1/a}$ and $a = 0.5, 0.8, 1, 1.5$ (from top left to bottom right)

Corollary 2.1. *Let $X_1, X_2, \ldots$ be i.i.d. random variables with partial sums satisfying* (13) *with some $\{a_n, n \in \mathbb{N}\}$, $\{b_n, n \in \mathbb{N}\}$ and a stable random variable ξ_α, $\alpha \in (0,2)$. Furthermore let $X_1^{(n)}, \ldots, X_m^{(n)}$ be the variables obtained by drawing (independently of $X_1, X_2, \ldots$) $m = m(n) \leq n$ times without replacement from the set $\{X_1, \ldots, X_n\}$. Assume that* (4) *holds and defin the functional CUSUM statistic by*

$$Z^*_{n,m}(t) = \frac{1}{T_n} \sum_{j=1}^{\lfloor mt \rfloor} (X_j^{(n)} - \bar{X}_n) \qquad \text{for } t \in [0,1]. \tag{16}$$

Then

$$P_{\mathbf{X}}(Z^*_{n,m}(t) \leq x) \xrightarrow{d} P_{\mathbf{Z}}(R^*(ct) \leq x) \qquad \text{for } n \to \infty \tag{17}$$

for any real x, where

$$R^*(t) = \frac{1}{M}\left(-q^{1/\alpha}\sum_{j=1}^{\infty}\frac{1}{Z_j^{1/\alpha}}(\delta_j(t)-t) + p^{1/\alpha}\sum_{j=1}^{\infty}\frac{1}{(Z_j^*)^{1/\alpha}}(\delta_j^*(t)-t)\right).$$

Here $Z_j = \eta_1 + \ldots + \eta_j$, $Z_j^* = \eta_1^* + \ldots + \eta_j^*$, *where* $\{\eta_j, \eta_j^*, j \in \mathbb{N}\}$ *are independent exponential random variables with parameter* 1*,*

$$M = \max\{(q/Z_1)^{1/\alpha}, (p/Z_1^*)^{1/\alpha}\} \tag{18}$$

and $\{\delta_j(t), t \in [0,1]\}$, $\{\delta_j^*(t), t \in [0,1]\}$, $j \in \mathbb{N}$, *are independent jump processes, each making a single jump from* 0 *to* 1 *at a random point uniformly distributed in* $(0,1)$*, also independent of* $\{Z_j, Z_j^*, j \in \mathbb{N}\}$.

Corollary 2.2. *Corollary 2.1 remains valid if* $X_1^{(n)}, \ldots, X_m^{(n)}$ *are obtained by drawing with replacement from the set* $\{X_1, \ldots, X_n\}$. *In this case* $\{\delta_j(t), t \in [0,1]\}$, $\{\delta_j^*(t), t \in [0,1]\}$, $j \in \mathbb{N}$, *will be independent Poisson processes with parameter* 1.

Note that in both corollaries the right-hand side of (17) is a conditional probability given the random variables $Z_1, Z_1^*, Z_2, Z_2^*, \ldots$. This means that the limit distribution is a random distribution function, possibly define on a different probability space than $X_1, X_2, \ldots$. In the bootstrap case this phenomenon was firs observed by Athreya [2].

It is interesting to note that in the extreme case of $\alpha = 0$, i.e. in case of i.i.d. random variables $X_1, X_2, \ldots$ with slowly varying tails, Corollaries 2.1 and 2.2 remain valid. More precisely, we assume that

$$P(X_1 > t) \sim pL(t) \qquad \text{and} \qquad P(X_1 < -t) \sim qL(t) \tag{19}$$

for $t \to \infty$ and some non-increasing slowly varying function $L(t)$ that satisfie $\lim_{t\to\infty} L(t) = 0$. Using Theorems 2.3 and 2.4 yields the following results.

Corollary 2.3. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with slowly varying tails satisfying* (19)*. Furthermore let* $X_1^{(n)}, \ldots, X_m^{(n)}$ *be the variables obtained by drawing (independently of* $X_1, X_2, \ldots$*)* $m = m(n) \le n$ *times without replacement from the set* $\{X_1, \ldots, X_n\}$. *Assume that (4) holds. Defin* $Z_{n,m}^*(t)$ *as in* (16)*. Then*

$$P_{\mathbf{X}}(Z_{n,m}^*(t) \le x) \xrightarrow{d} P_{\mathrm{U}}(R^*(ct) \le x) \qquad \text{for } n \to \infty \tag{20}$$

for any real x, where

$$R^*(t) = -I(U > p)(\delta(t) - t) + I(U \le p)(\delta^*(t) - t).$$

Here U *is a uniform random variable on* $(0,1)$ *and* $\{\delta(t), t \in [0,1]\}$, $\{\delta^*(t), t \in [0,1]\}$ *are independent jump processes, each making a single jump from* 0 *to* 1 *at a random point uniformly distributed in* $(0,1)$*, independent of* U.

Corollary 2.4. *Corollary 2.3 remains valid if we sample* $X_1^{(n)}, \ldots, X_m^{(n)}$ *with replacement from the set* $\{X_1, \ldots, X_n\}$ *with* $m = m(n)$ *satisfying (4). Then, however,*

$\{\delta(t), t \in [0,1]\}$ *and* $\{\delta^*(t), t \in [0,1]\}$ *are independent Poisson processes with parameter* 1 *(independent of U).*

Our next corollary describes a situation when relation (8) fails, i.e. the sequences $x_{j,n}$ and $x_{n-j+1,n}$ do not converge for fi ed j. Let $X_1, X_2, \ldots$ be i.i.d. symmetric random variables with the distribution

$$P(X_1 = \pm 2^k) = 2^{-(k+1)} \qquad k = 1, 2, \ldots \tag{21}$$

This is the two-sided version of the St. Petersburg distribution. The distribution function $F(x)$ of X_1 satisfie

$$1 - F(x) = 2^{-k} \qquad \text{for } 2^{k-1} \le x < 2^k$$

which shows that $G(x) = x(1-F(x))$ is logarithmically periodic: if x runs through the interval $[2^k, 2^{k+1})$, then $G(x)$ runs through all values in $[1/2, 1)$ and $G(\log_2 x)$ is periodic with period 1. Thus (14) fails and consequently F does not belong to the domain of attraction of a stable law. The partial sums $S_k = \sum_{k=1}^n X_k$ have a remarkable behavior: for any fi ed $1 \le c < 2$, the normed sums $n^{-1} S_n$ converge weakly along the subsequence $n_k = \lfloor c 2^k \rfloor$ to an infinitel divisible distribution F_c such that $F_c = F_1^{*c}$ and $F_2 = F_1$. The class $\mathscr{F} = \{F_c,\ 1 \le c \le 2\}$ can be considered a circle, and in each interval $[2^k, 2^{k+1})$, the distribution of $n^{-1} S_n$ essentially runs around this circle in the sense that $n^{-1} S_n$ is close in distribution to F_c with $c = n/2^k$. This behavior was discovered by S. Csörgő [10], who called this quasiperiodic behavior 'merging'. As the following corollary shows, merging will also take place in the behavior of permutation and bootstrap statistics. For simplicity, we consider the case when we draw n elements from the sample $(X_1, \ldots, X_n)$. Let $\Psi(x)$, $0 < x < \infty$ denote the function which increases linearly from 1/2 to 1 on each interval $(2^j, 2^{j+1}]$, $j = 0, \pm 1, \pm 2, \ldots$.

Corollary 2.5. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with the distribution* (21)*. Let* $X_1^{(n)}, \ldots, X_n^{(n)}$ *be the elements obtained by drawing n times with replacement from the set* $\{X_1, \ldots, X_n\}$ *and let* $Z_n^*(t)$ *be define by (16) with* $m = n$*. Let* $1 \le c < 2$*. Then for* $n_k = \lfloor c 2^k \rfloor$ *we have*

$$P_{\mathbf{X}}(Z_{n_k}^*(t) \le x) \xrightarrow{d} P_{\mathbf{Z}}(R_c(t) \le x)$$

for any real x, where

$$R_c(t) = \frac{1}{M} \left[-\sum_{j=1}^{\infty} \frac{1}{Z_j} \Psi\left(\frac{Z_j}{c}\right) (\delta_j(t) - t) + \sum_{j=1}^{\infty} \frac{1}{Z_j^*} \Psi\left(\frac{Z_j^*}{c}\right) (\delta_j^*(t) - t) \right]$$

with

$$M = \max\left\{ \frac{\Psi(Z_1/c)}{Z_1}, \frac{\Psi(Z_1^*/c)}{Z_1^*} \right\}.$$

Here $Z_j = \eta_1 + \ldots + \eta_j$ *and* $Z_j^* = \eta_1^* + \ldots + \eta_j^*$, *where* $\{\eta_j, \eta_j^*, j \in \mathbb{N}\}$ *are i.i.d.* $\exp(1)$ *random variables and* $\{\delta_j(t), 0 \le t \le 1\}$, $\{\delta_j^*(t), 0 \le t \le 1\}$, $j = 1, 2, \ldots$ *are independent jump processes, each making a single jump from 0 to 1 at a uniformly distributed point in* $(0,1)$.

Just like in the case of partial sums, the class R_c of limiting processes is logarithmically periodic, namely $R_{2c} = R_c$ and for a fi ed n with $2^k \le n < 2^{k+1}$ the conditional distribution of $Z_n^*(t)$ is close to that of $R_c(t)$ with $c = n/2^k$.

Corollary 2.5 remains valid if we draw $X_1^{(n)}, \ldots, X_n^{(n)}$ without replacement from the set $\{X_1, \ldots, X_n\}$. Then $\delta_j(t)$ and $\delta_j^*(t)$ are independent Poisson processes with parameter 1.

3 Application to change point detection

Due to the random limit distributions in Corollaries 2.1 and 2.2, bootstrap and permutation methods cannot be directly used in statistical inference when the observations do not have finit variances. Let $X_1, X_2, \ldots$ be i.i.d. random variables belonging to the domain of normal attraction of a stable r.v. ξ_α with parameter $\alpha \in (0,2)$. Let $S_n = X_1 + \ldots + X_n$ and let $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ be the order statistics of the sample $(X_1, \ldots, X_n)$. It is well known (see e.g. Darling [11]) that for any fi ed j the ratios $X_{j,n}/S_n$, $X_{n-j,n}/S_n$ have nondegenerate limit distributions as $n \to \infty$, which means that the contribution of the extreme elements in the normed sum $n^{-1/\alpha}(X_1 + \ldots + X_n)$ is not negligible. As Corollaries 2.1 and 2.2 show, both in the permutation and bootstrap case we have

$$P_{\mathbf{X}}(Z_{n,m}^*(t) \le x) \xrightarrow{d} P_{\mathbf{Z}}(R^*(ct) \le x) \qquad \text{as } n \to \infty$$

for any real x, provided $m/n \to c > 0$, where

$$R^*(t) = \frac{1}{M}\left(-q^{1/\alpha} \sum_{j=1}^{\infty} \frac{1}{Z_j^{1/\alpha}}(\delta_j(t) - t) + p^{1/\alpha} \sum_{j=1}^{\infty} \frac{1}{(Z_j^*)^{1/\alpha}}(\delta_j^*(t) - t)\right).$$

Here the terms $Z_j^{-1/\alpha}(\delta_j(t) - t)$ and $(Z_j^*)^{-1/\alpha}(\delta_j^*(t) - t)$ are due to the extremal terms $X_{j,n}$ and $X_{n-j,n}$ in the sum S_n and thus to get a limit distribution not containing random coefficient we have to eliminate the effect of the extremal elements. Natural ideas are trimming the sample $(X_1, \ldots, X_n)$ before resampling, or to choose the sample size in resampling as $o(n)$, reducing the chance of the largest elements of the sample to get into the new sample. In this section we will show that after a suitable trimming, the limit distribution of the partial sums of the resampled elements will be normal, and thus bootstrap and permutation methods will work under infinit variances. We will illustrate this with an application to change point detection.

Consider the location model

$$X_j = \mu + \delta I(j > K) + e_j \qquad \text{for } j = 1, \ldots, n, \tag{22}$$

where $1 \le K \le n$, μ and $\delta = \delta_n \neq 0$ are unknown parameters. We assume that $|\delta| \le D$ with some $D > 0$ and that $e_1, \ldots, e_n$ are i.i.d. random variables with

$$Ee_1 = 0, \quad Ee_1^2 = \sigma^2 > 0 \quad \text{and } E|e_1|^\nu < \infty \text{ with some } \nu > 2. \tag{23}$$

We want to test the hypothesis $H_0 : K \ge n$ against $H_1 : K < n$. Common test statistics for this setting are the CUSUM statistics define by

$$T_n = \max_{1 \le k \le n} \frac{1}{n^{1/2} \widehat{\sigma}_n} \left| \sum_{j=1}^{k} (X_j - \bar{X}_n) \right|$$

and

$$T_{n,1} = \max_{1 \le k \le n} \sqrt{\frac{n}{k(n-k)}} \frac{1}{\widehat{\sigma}_n} \left| \sum_{j=1}^{k} (X_j - \bar{X}_n) \right|,$$

where

$$\bar{X}_n = \frac{1}{n} \sum_{j=1}^{n} X_j \quad \text{and} \quad \widehat{\sigma}_n^2 = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2.$$

The limit distributions of both statistics are known:

$$\lim_{n \to \infty} P_{H_0}(T_n \le x) = P\left(\sup_{0 \le t \le 1} |B(t)| \le x \right),$$

and

$$\lim_{n \to \infty} P_{H_0}\left(a(\log n) T_{n,1} \le x + b(\log n)\right) = \exp(-2\exp(-x)),$$

where $\{B(t), t \in [0,1]\}$ is a Brownian bridge,

$$a(x) = \sqrt{2 \log x} \qquad \text{and} \qquad b(x) = 2 \log x + \frac{1}{2} \log\log x - \frac{1}{2} \log \pi.$$

(see Csörgő and Horváth [9] and Antoch and Hušková [1]) and one can determine critical values based on these limit distributions. On the other hand, under the change point alternative if $n\delta_n^2 \to \infty$, then $T_n \to \infty$ in probability and if $n\delta_n^2 / \log\log n \to \infty$, then $T_n / \sqrt{\log\log n} \to \infty$ in probability. However, the convergence to the limit distribution is rather slow under the null hypothesis and thus the obtained critical values will work only for large sample sizes and lead to conservative tests otherwise. Antoch and Hušková [1] proposed the use of permutation statistics to get correct critical values for small and moderate size samples. Consider a sample $\mathbf{X} = (X_1, \ldots, X_n)$, let $\pi = (\pi(1), \ldots, \pi(n))$ be a random permutation of $(1, \ldots, n)$, independent of the sample $(X_1, \ldots, X_n)$, and let $\mathbf{X}_\pi = (X_{\pi(1)}, \ldots, X_{\pi(n)})$ be the permuted sample. Let $T_{n,1}^* = T_{n,1}(\mathbf{X}_\pi)$, where $\mathbf{X}$ is considered fi ed, and the randomness is in π. Antoch and Hušková [1] showed the following theorem:

Theorem 3.1. *If conditions* (22) *and* (23) *are satisfie and* $|\delta_n| \le D$ *for some* $D > 0$, *then for all* x

$$\lim_{n\to\infty} P_{\mathbf{X}}(T_n^* \le x) = P\left(\sup_{0\le t\le 1} |B(t)| \le x\right)$$

and

$$\lim_{n\to\infty} P_{\mathbf{X}}\left(a(\log n)T_{n,1}^* \le x + b(\log n)\right) = \exp(-2\exp(-x)),$$

for almost all realizations of $\mathbf{X}$ *as* $n \to \infty$.

Note that Theorem 3.1 is valid under the null as well as under the alternative hypotheses. This shows that the critical values for the test based on $T_{n,1}$ can be replaced by the sample quantiles of its permutation version based on $T_{n,1}^*$, a procedure which is numerically quite convenient. For a given sample $(X_1,\ldots,X_n)$ we generate a large number of random permutations which leads to the empirical distributions of $T_n(\mathbf{X}_\pi)$ and $T_{n,1}(\mathbf{X}_\pi)$ and hence to the desired critical values. As the simulations in [1] show, these critical values are much more satisfactory than the critical values based on the limit distribution. As Hušková [14] pointed out, approximations using bootstrap versions of the test statistics would also work well.

Using N independent permutations, that is N values of T_n^*, denoted by $T_n^*(j)$, $j = 1,2,\ldots,N$, let

$$H_{n,N}(x) = \frac{1}{N}\sum_{j=1}^{N}\{T_n^*(j) \le x\},$$

be the empirical distribution function that can be used to approximate

$$H_n(x) = P_{H_0}(T_n \le x).$$

Defin $H_{n,N,1}$ and $H_{n,1}$ as the analogues to $H_{n,N}$ and H_n where $T_{n,1}$ replaces T_n. Berkes, Horváth, Hušková and Steinebach [5] showed that if conditions (22) and (23) are satisfied then we have

$$|H_{n,N}(x) - H_n(x)| = o_{P_{\mathbf{X}}}(1) \qquad \text{as } \min(n,N) \to \infty$$

for almost all realizations of $\mathbf{X}$. They also studied the rate of convergence and they proved

$$|H_{n,N}(x) - H_n(x)| = \mathcal{O}_{P_{\mathbf{X}}}\left(N^{-1/2} + n^{-(\nu-2)/(6\nu)}\right)$$

and

$$|H_{n,N,1}(x) - H_{n,1}(x)| = o_{P_{\mathbf{X}}}\left(N^{-1/2} + (\log\log n)^{-\nu/2}\right)$$

for almost all realizations of $\mathbf{X}$.

The previous results show that permutation and bootstrap statistics provide an effective way to detect a change of location in an i.i.d. sequence (X_n). Note, however, that for the validity of the limit distribution results above we need, by (23), the existence of $\nu > 2$ moments of the underlying variables. As we will see below, using a suitable trimming of the sample $(X_1,\ldots,X_n)$, the limiting processes in Corollaries 2.1 and 2.2 become Brownian motion, resp. Brownian bridge, and then the bootstrapped or permuted version of the CUSUM statistics will work without assuming the existence of second moments.

Fix a sequence $\omega_n \to \infty$ of integers with $\omega_n/n \to 0$. Put $m = n - 2\omega_n$ and let $(Y_1,\dots,Y_m)$ denote the part of the original sample $(X_1,\dots,X_n)$ obtained by removing the ω_n smallest and largest elements from the set. Let $Y_{1,m} \le \dots \le Y_{m,m}$ be the ordered sample of $(Y_1,\dots,Y_m)$. Draw m elements $Y_1^{(m)},\dots,Y_m^{(m)}$ from the set $\{Y_1,\dots,Y_m\}$ with or without replacement. Let $\varepsilon_j^{(m)}(t)$ count how many times $Y_{j,m}$ has been chosen among the firs $\lfloor mt \rfloor$ sampled elements:

$$\varepsilon_j^{(m)}(t) = k \ \text{ if } Y_{j,m} \text{ has been chosen } k \text{ times among the firs } \lfloor mt \rfloor \text{ elements,}$$

for $j = 1,\dots,m$. Clearly, in the case of selection without replacement k can only take the values 0, 1 while $k \in \{0,1,\dots,\lfloor mt \rfloor\}$ when drawing with replacement. Letting $\overline{Y}_m = (1/m)\sum_{j=1}^m Y_j = (1/m)\sum_{j=1}^m Y_{j,m}$, we have

$$\widehat{Z}_m(t) := \sum_{j=1}^{\lfloor mt \rfloor} (Y_j^{(m)} - \overline{Y}_m) = \sum_{j=1}^{m} (Y_{j,m} - \overline{Y}_m)\,\varepsilon_j^{(m)}(t) = \sum_{j=1}^{m} (Y_{j,m} - \overline{Y}_m)\,\overline{\varepsilon}_j^{(m)}(t),$$

where $\overline{\varepsilon}_j^{(m)}(t) = \varepsilon_j^{(m)}(t) - E\varepsilon_j^{(m)}(t)$ is the centered version of $\varepsilon_j^{(m)}(t)$.

Theorem 3.2. *In the case of selection without replacement, there exist independent, identically distributed indicator variables* $\delta_j^{(m)}(t)$*,* $j = 1,\dots,m$ *with* $P(\delta_j^{(m)}(t) = 1) = \lfloor mt \rfloor/m$ *such that for any* $0 < t < 1$

$$P\left(\varepsilon_j^{(m)}(t) \ne \delta_j^{(m)}(t) \quad \textit{for some } 1 \le j \le m^{1/3} \textit{ or } m - m^{1/3} < j \le m\right)$$
$$\le Ct^{-1}m^{-1/6}, \tag{24}$$

where C *is an absolute constant. Moreover, with probability 1*

$$\sum_{m^{1/3} < j \le m - m^{1/3}} (Y_{j,m} - \overline{Y}_m)\,\overline{\varepsilon}_j^{(m)}(t) = o_P\left(m^{1/\alpha}\omega_m^{1/2-1/\alpha}\right). \tag{25}$$

The statements of the theorem remain valid for selection with replacement, except that in this case the $\delta_j^{(m)}(t)$*,* $j = 1,\dots,m$ *are independent* $B(\lfloor mt \rfloor, 1/m)$ *random variables.*

Letting $\overline{\delta}_j^{(m)}(t) = \delta_j^{(m)}(t) - E\delta_j^{(m)}(t)$ and

$$A_m = \left(\sum_{j=1}^{m} (Y_{j,m} - \overline{Y}_m)^2\right)^{1/2} = \sqrt{m}\,\widehat{\sigma}_m \approx m^{1/\alpha}\omega_m^{1/2-1/\alpha},$$

relation (25) shows that the asymptotic behavior of $A_m^{-1}\widehat{Z}_m(t)$ is the same as that of

$$A_m^{-1} \sum_{j \in I_m} (Y_{j,m} - \bar{Y}_m) \bar{\varepsilon}_j^{(m)}(t) \tag{26}$$

with

$$I_m = \{k : 1 \le k \le \lfloor m^{1/3} \rfloor \text{ or } m - \lceil m^{1/3} \rceil < k \le m\}.$$

By (25), the expression in (26) can be replaced by $A_m^{-1} \sum_{j \in I_m} (Y_{j,m} - \bar{Y}_m) \bar{\delta}_j^{(m)}(t)$, a normed sum of i.i.d. random variables. Using a tightness argument and the functional central limit theorem under Ljapunov's condition, we get

Corollary 3.1. *Assume H_0 and defin*

$$\widehat{T}_m = \max_{1 \le k \le m} \frac{1}{\sqrt{m}\widehat{\sigma}_m} \sum_{j=1}^{k} (Y_{\pi(j)} - \bar{Y}_m).$$

Then conditionally on **X***, for almost all paths*

$$P_{\mathbf{X}}(\widehat{T}_m \le x) \longrightarrow P\left(\sup_{0 \le t \le 1} |B(t)| \le x \right) \qquad \textit{for } n \to \infty.$$

A similar result holds for Darling-Erdős type functionals:

Corollary 3.2. *Assume H_0 and defin*

$$\widehat{T}_{m,1} := \max_{1 \le k \le m} \sqrt{\frac{m}{k(m-k)}} \frac{1}{\widehat{\sigma}_m} \left| \sum_{j=1}^{k} (Y_{\pi(j)} - \bar{Y}_m) \right|.$$

Then conditionally on **X***, for almost all paths*

$$P_{\mathbf{X}}\left((2 \log\log m)^{1/2} \widehat{T}_{m,1} \le x + 2 \log\log m + \frac{1}{2} \log\log\log m - \frac{1}{2} \log(\pi) \right) \\ \longrightarrow \exp(-2 \exp(-x)).$$

Corollaries 3.1 and 3.2 show that trimming and permuting the sample provides a satisfactory setup for change point problems under infinit variance. Just like in case of finit variances (cf. Theorem 3.1), Corollary 3.2 remains true under the change–point alternative. A small simulation study is given below, showing simulated critical values and the empirical power of the trimmed tests.

Consider the location model in (22) with

$$n \in \{100, 200\}, K \in \{n/4, n/2, 3n/4\}, \mu = 0, \delta \in \{0, 2, 4\}$$

and with i.i.d. errors e_j having distribution function

$$F(x) = \begin{cases} \frac{1}{2}(1-x)^{-1.5} & \text{for } x \le 0 \\ 1 - \frac{1}{2}(1+x)^{-1.5} & \text{for } x < 0. \end{cases}$$

We use trimming with $\omega_n = \lfloor n^\beta \rfloor$, $\beta \in \{0.2, 0.3, 0.4\}$. To simulate the critical values we generate a random sample $(X_1, \ldots, X_n)$ according to the above model and trim it to obtain $(Y_1, \ldots, Y_m)$. For $N = 10^5$ permutations of the integers $\{1, \ldots, m\}$ we calculate the values of $\widehat{T}_m$ and $\widehat{T}_{m,1}$ define in Corollaries 3.1 and 3.2, respectively. The computation of the empirical quantiles yields the desired critical values. Tables 1 and 3 summarize our results for $\widehat{T}_m$ and $\widehat{T}_{m,1}$, respectively.

Table 1 Simulated quantiles of $\widehat{T}_m$

			$\beta = 0.2$			$\beta = 0.3$			$\beta = 0.4$		
n	K	δ	10 %	5 %	1 %	10 %	5 %	1 %	10 %	5 %	1 %
100	-	0	1.115	1.225	1.429	1.144	1.271	1.514	1.142	1.264	1.495
100	25	2	1.127	1.239	1.467	1.157	1.288	1.541	1.153	1.283	1.535
100	25	4	1.144	1.264	1.501	1.161	1.292	1.542	1.164	1.295	1.556
100	50	2	1.137	1.258	1.489	1.150	1.280	1.523	1.159	1.289	1.539
100	50	4	1.154	1.276	1.514	1.159	1.289	1.539	1.167	1.298	1.553
100	75	2	1.133	1.255	1.483	1.153	1.281	1.528	1.155	1.283	1.536
100	75	4	1.148	1.276	1.514	1.163	1.293	1.548	1.159	1.291	1.550
200	-	0	1.166	1.293	1.549	1.144	1.268	1.498	1.169	1.303	1.560
200	50	2	1.176	1.306	1.561	1.157	1.284	1.525	1.183	1.314	1.577
200	50	4	1.179	1.311	1.566	1.172	1.300	1.549	1.182	1.313	1.582
200	100	2	1.168	1.300	1.551	1.171	1.298	1.553	1.169	1.299	1.558
200	100	4	1.179	1.307	1.561	1.182	1.314	1.576	1.181	1.315	1.574
200	150	2	1.149	1.270	1.503	1.154	1.279	1.530	1.176	1.307	1.569
200	150	4	1.164	1.289	1.531	1.158	1.284	1.537	1.181	1.309	1.569

Note that the differences between the estimated quantiles under the null ($\delta = 0$) and under the alternative hypotheses are small, just as in case of finit variances in Antoch and Hušková [1]. Comparing the simulated values with the asymptotic ones given in Tables 2 and 4, one can note relatively large differences for $n = 100$ (in particular in the case of $\widehat{T}_{m,1}$).

Table 2 Asymptotic critical values of T_m

10 %	5 %	1 %
1.224	1.358	1.628

The quantiles of $\widehat{T}_{m,1}$ show more fluctuation than those of $\widehat{T}_m$. Note that increasing β will stabilize the simulated quantiles.

Table 3 Simulated quantiles of $\widehat{T}_{m,1}$

			$\beta = 0.2$			$\beta = 0.3$			$\beta = 0.4$		
n	K	δ	10 %	5 %	1 %	10 %	5 %	1 %	10 %	5 %	1 %
100	-	0	2.835	3.174	3.961	2.900	3.199	3.540	2.719	2.945	3.453
100	25	2	2.793	3.143	3.511	2.827	3.032	3.465	2.661	2.898	3.371
100	25	4	2.719	2.945	3.415	2.694	2.946	3.441	2.660	2.898	3.385
100	50	2	2.939	3.328	3.960	2.703	2.978	3.386	2.650	2.881	3.335
100	50	4	2.747	3.034	3.625	2.640	2.872	3.342	2.635	2.869	3.346
100	75	2	2.926	3.339	3.808	2.875	3.353	3.765	2.764	3.012	3.438
100	75	4	2.864	3.091	3.667	2.731	2.950	3.466	2.685	2.921	3.411
200	-	0	3.065	3.431	4.317	3.065	3.469	4.560	2.924	3.286	3.994
200	50	2	3.035	3.393	4.501	3.009	3.433	4.527	2.896	3.243	3.844
200	50	4	2.959	3.242	4.071	2.929	3.249	4.078	2.851	3.110	3.541
200	100	2	3.027	3.406	4.500	2.983	3.317	4.004	2.868	3.179	3.775
200	100	4	2.886	3.188	3.799	2.892	3.112	3.751	2.771	3.019	3.508
200	150	2	3.024	3.519	5.092	2.951	3.356	4.125	2.832	3.070	3.570
200	150	4	2.949	3.323	4.386	2.874	3.183	3.812	2.802	3.051	3.547

Table 4 Asymptotic critical values of $T_{m,1}$

n	β	10 %	5 %	1 %
100	.2	3.223	3.636	4.572
100	.3	3.222	3.636	4.572
100	.4	3.218	3.634	4.575
200	.2	3.264	3.658	4.552
200	.3	3.262	3.658	4.552
200	.4	3.260	3.656	4.553

Figures 2–5 show the empirical power of the test based on the (non permuted) test statistics T_m, $T_{m,1}$ for each $\delta \in \{-3, -2.9, \ldots, 2.9, 3\}$.

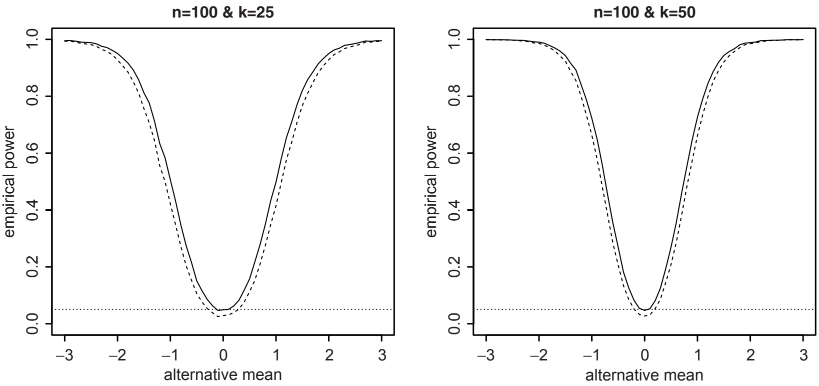

Fig. 2 Empirical power of T_m with empirical (dashed) and asymptotic (solid) critical values, $\alpha = 0.05, n = 100, \beta = 0.3$ and $K = 25, 50$ (left, right)

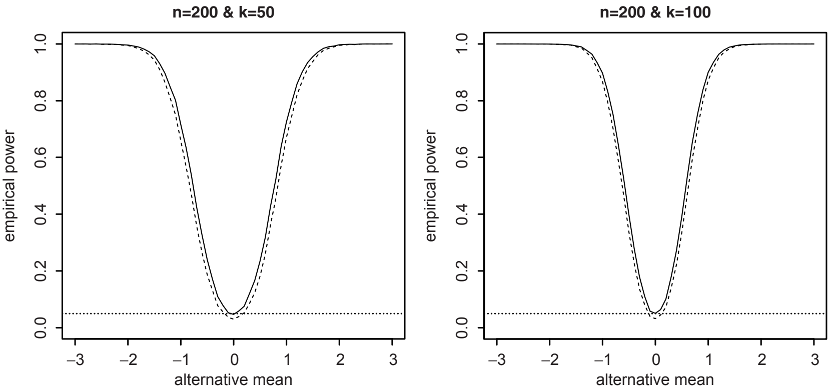

Fig. 3 Empirical power of T_m with empirical (dashed) and asymptotic solid) critical values, $\alpha = 0.05, n = 200, \beta = 0.3$ and $K = 50, 100$ (left, right)

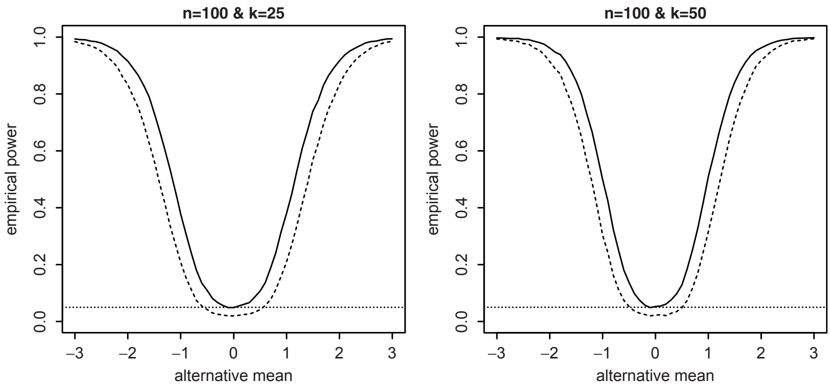

Fig. 4 Empirical power of $T_{m,1}$ with empirical (dashed) and asymptotic (solid) critical values $\alpha = 0.05, n = 100, \beta = 0.3$ and $K = 25, 50$ (left, right)

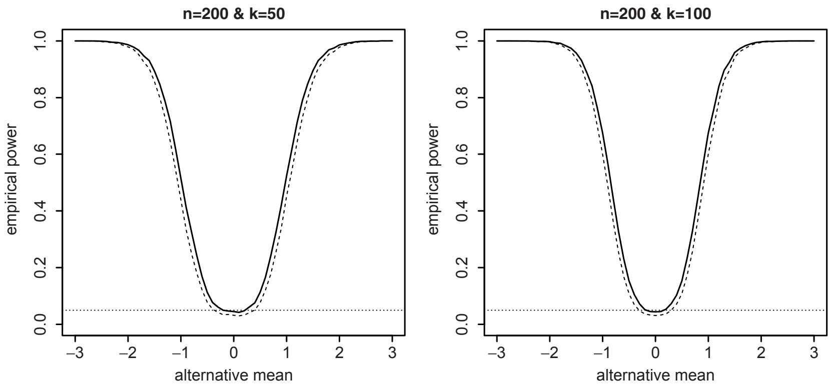

Fig. 5 Empirical power of $T_{m,1}$ with empirical (dashed) and asymptotic (solid) critical values $\alpha = 0.05, n = 200, \beta = 0.3$ and $K = 50, 100$ (left, right)

The figure show that the test based on $T_{m,1}$ is conservative when we use the asymptotic critical values.

In conclusion we note that the type of trimming we used above is not the only possibility to eliminate the large elements of the sample $(X_1,\ldots,X_n)$. Alternatively, we can remove from the sample the ω_n elements with the largest absolute values. In [7] we determined the asymptotic distribution of permuted and bootstrapped CUSUM statistics under this kind of trimming. While the limit distribution of $\widehat{T}_m$ remains the same in this case, note that, surprisingly, the asymptotic theory of the two trimming procedures is different, see [7], [12], [13] for further information.

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