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EMPIRICAL PROCESSES IN PROBABILISTIC NUMBER THEORY: THE LIL FOR THE DISCREPANCY OF $(n_k\omega)$ MOD 1

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Dedicated to the memory of Joseph L. Doob

Abstract. We prove a law of the iterated logarithm for the Kolmogorov-Smirnov statistic, or equivalently, the discrepancy of sequences $(n_k\omega)$ mod 1. Here (n_k) is a sequence of integers satisfying a sub-Hadamard growth condition and such that linear Diophantine equations in the variables n_k do not have too many solutions. The proof depends on a martingale embedding of the empirical process; the number-theoretic structure of (n_k) enters through the behavior of the square function of the martingale.

1. Introduction

Let $X_1, X_2, \ldots$, be i.i.d. random variables, uniformly distributed in $(0,1)$, and let

$$F_n(t) = \frac{1}{n}\,\mathrm{card}\{k \le n : X_k \le t\}$$

denote the empirical distribution function of the sample $X_1, \ldots, X_n$. The empirical process

$$\alpha_n(t) = \sqrt{n}\,(F_n(t) - t), \qquad 0 \le t \le 1,$$

captures important properties of the sequence $X_1, X_2, \ldots$, and plays an important role in probability theory and statistics. By the classical theory we have

$$\alpha_n(\cdot) \xrightarrow{\mathcal{D}[0,1]} B(\cdot), \tag{1.1}$$

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where B is the Brownian bridge and the convergence is weak convergence in the Skorohod space $\mathcal{D}[0,1]$. As a special case we have the well known results of Kolmogorov and Smirnov:

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le 1} |\alpha_n(t)| < x\right) = \sum_{k=-\infty}^{\infty} (-1)^k \exp(-2k^2x^2) \qquad (x>0)$$

and

$$\lim_{n\to\infty} P\left(\sup_{0\le t\le 1} \alpha_n(t) < x\right) = 1-\exp(-2x^2) \qquad (x>0).$$

The basic result (1.1) grew out of Doob's famous heuristics [9] for the Kolmogorov-Smirnov theorems; this was made precise by Donsker [8] some years later, leading to the development of the theory of weak convergence in metric spaces in the 1950's. The a.s. behavior of α_n is described by the Chung-Smirnov LIL

$$\limsup_{n\to\infty} (\log\log n)^{-1/2} \sup_{0\le t\le 1} |\alpha_n(t)| = 1/\sqrt{2} \qquad \text{a.s.} \tag{1.2}$$

A functional version of (1.2) was given by Finkelstein [14]. All these results are contained as special cases in Kiefer's [17] strong approximation theorem

$$\sup_{0\le s\le 1} |n(F_n(s)-s) - \sum_{j\le n} B_j(s)| = O(n^{1/3}(\log n)^{2/3}) \qquad \text{a.s.},$$

where $B_j(s)$, $j=1,2,\ldots$, are independent Brownian bridges.

There is an extensive literature dealing with generalizations of the above results for dependent random variables. The case of weak dependence is relatively simple: Berkes and Philipp [2] showed that Kiefer's approximation theorem remains valid, with a slightly weaker remainder term, for stationary mixing sequences (X_n) with a polynomial mixing rate. The only difference is that $B_j(s)$ should be replaced by $G_j(s)$, where $G_j(s)$, $0\le s\le 1$, are independent Gaussian processes with mean zero and covariance function Γ explicitly expressed by the covariances of (X_n). The case of strongly dependent (X_n) is radically different: as Dehling and Taqqu [7] showed, in the case when X_n are nonlinear functions of a strongly dependent Gaussian process (Y_n), the limit process in (1.1) will be a quasi-deterministic process, whose trajectories are random multiples of a fixed deterministic function. For a survey of the literature over such phenomena and empirical processes of dependent sequences see Dehling and Philipp [6].

Remarkably, empirical processes appeared much earlier in classical analysis and number theory, through their connection to the theory of uniform distribution mod 1, developed in a fundamental paper of H. Weyl [25]. For a finite numerical sequence $(x_1,\ldots,x_N)$ in $(0,1)$, the discrepancy D_N of the sequence, measuring its closeness to uniformity, was defined by Weyl as

$$D_N = D_N(x_1, \ldots, x_N) =: \sup_{0 \le s \le 1} \left| \frac{1}{N} \operatorname{card}(k \le N : x_k \le s) - s \right|.$$

An infinite sequence (x_n) in $(0,1)$ is called uniformly distributed in the Weyl sense if $D_N(x_1, \ldots, x_N) \to 0$ as $N \to \infty$. Clearly, this is equivalent to the uniform convergence of the empirical distribution function F_N of the sample $(x_1, \ldots, x_N)$ and $\sqrt{N} D_N$ is the Kolmogorov-Smirnov statistic of $(x_1, \ldots, x_N)$. There is an extensive literature on uniform distribution, for which we refer to Kuipers and Niederreiter [18] or Drmota and Tichy [11]. A much investigated problem in the theory is the study of the discrepancy of special sequences like $\{n_k \omega\}$, where n_k is an increasing sequence of integers and $\{\cdot\}$ denotes fractional part. By a classical result, $\{k\omega\}$ is uniformly distributed for any irrational ω, and the same holds for $\{n_k \omega\}$ for many concrete sequences n_k, e.g., polynomials with integer coefficients. For other sequences n_k (for example, $n_k = k!$) the result fails, but Weyl [25] proved that for any increasing sequence (n_k) of integers, $\{n_k \omega\}$ is uniformly distributed for almost all ω in the sense of Lebesgue measure. For notational simplicity, let

$$\eta_k = \eta_k(\omega) := n_k \omega \qquad (\mathrm{mod}\ 1)$$

and

$$D_N(\omega) = \sup_{0 \le s \le 1} \left| \frac{1}{N} \operatorname{card}(k \le N : \eta_k(\omega) \le s) - s \right|.$$

The quantity $\sqrt{N} D_N(\omega)$ is the Kolmogorov-Smirnov statistic of the sequence $\{n_k \omega\}$ and Weyl's theorem expresses the Glivenko-Cantelli theorem for the sequence $\{n_k \omega\}$. Probabilistically, $\{n_k \omega\}$ is a sequence of dependent random variables over the probability space $(0,1)$ equipped with Lebesgue measure; its dependence structure is very complicated and unusual from the probabilistic point of view. As a consequence, studying the asymptotic behavior of $D_N(\omega)$ is a hard problem and very few precise results are known. Cassels [5] and Erdős and Koksma [12] proved independently that for almost all $\omega \in [0,1)$

$$N D_N(\omega) = O(N^{1/2} (\log N)^{(5/2)+\varepsilon}), \qquad \varepsilon > 0.$$

The best result so far has been achieved by R.C. Baker [1] who reduced the exponent $5/2$ on the logarithm to $3/2$. The exact exponent of the logarithm is still an open problem, except for the fact that it cannot be less than $1/2$, as was shown by Berkes and Philipp [3].

The only case when a precise asymptotics for $D_N(\omega)$ is known is the case $n_k = k$, when its connection with the continued fraction expansion of ω makes the problem tractable. Kesten [15] showed that in this case

$$N D_N(\omega) \sim \frac{2}{\pi^2} \log N \log \log N$$

in measure. This asymptotics is not valid in the a.s. sense, but Khinchin [16] proved that for any positive nondecreasing function g on $(0, +\infty)$ the relation

$$ND_N(\omega) = O((\log N)g(\log\log N)) \qquad \text{a.s.}$$

holds if and only if $\sum_{n=1}^{\infty} g(n)^{-1}$ converges. Another case when the order of magnitude of the discrepancy of $\{n_k\omega\}$ is known (although less precisely than in the above case), is when n_k grows very rapidly. Philipp [20] proved that if (n_k) satisfies the Hadamard gap condition

$$\frac{n_{k+1}}{n_k} \geq 1 + \rho, \quad \rho > 0, \ k = 1, 2, \dots, \tag{1.3}$$

then we have for almost all $\omega \in [0, 1)$

$$\frac{1}{4} \leq \limsup_{N\to\infty} \frac{ND_N(\omega)}{\sqrt{N \log\log N}} \leq C(\varrho), \tag{1.4}$$

where $C(\varrho) \ll 1/\varrho$. Berkes and Philipp [3] constructed sequences (n_k) satisfying (1.3), for which the lower bound $1/4$ in (1.4) can be improved to $c \log\log 1/\varrho$ with an absolute constant c. Hence there cannot be an upper bound C in (1.4), independent of ϱ, that works for all sequences (n_k) satisfying a Hadamard gap condition (1.3). A similar construction (see Berkes and Philipp [3], Theorem 3) shows that for every $\varepsilon_k \downarrow 0$ there exists a sequence (n_k) of integers satisfying

$$n_{k+1}/n_k \geq 1 + \varepsilon_k, \quad k = 1, 2, \dots,$$

such that

$$\limsup_{N\to\infty} \frac{ND_N(\omega)}{\sqrt{N \log\log N}} = +\infty \qquad \text{a.e.}$$

Thus, under the slightest weakening of the Hadamard gap condition (1.3), the LIL (1.4) becomes generally false. No precise asymptotic result for $D_N(\omega)$ in the subexponential domain is known except an LIL, proved in Philipp [21] for the Hardy-Littlewood-Pólya sequences (n_k). These are defined as follows. Let $(q_1, q_2, \dots, q_\tau)$ be a finite set of coprime positive integers and let (n_k) be the multiplicative semigroup generated by $(q_1, q_2, \dots, q_\tau)$ and arranged in increasing order. Thus

$$(n_k)_{k=1}^{\infty} = (q_1^{\alpha_1} q_2^{\alpha_2} \dots q_\tau^{\alpha_\tau}, \ \alpha_i \geq 0, \ 1 \leq i \leq \tau).$$

Then relation (1.4) holds with a constant $C(r)$ on the right side depending only on the number r of primes involved in the prime factorization of $q_1, \dots, q_\tau$. (See Philipp [21].)

The purpose of this paper is to prove an LIL for the discrepancy $D_N(\omega)$ for a very large class of subexponentially growing sequences (n_k). Actually, we will see that in some sense 'almost all' sequences (n_k) in this domain will satisfy the LIL. Our main result will show that a sub-Hadamard growth condition plus certain Diophantine properties of (n_k) imply the LIL (1.4). To formulate our theorem, we introduce some terminology. We will say that a sequence (n_k) satisfies

Condition (B), if there exist constants $0 < \alpha < 1/2$ and $C > 0$ such that for each positive integer b and for each $R \geq 1$ the number of solutions (h, ν) of the Diophantine equation

$$hn_\nu = b$$

with $h \in \mathbb{N}$, $1 \leq h \leq R$, does not exceed CR^α.

Condition (C), if there exist constants $0 < \beta < 1/2$ and $C_0 > 0$ such that for each $N \geq 1$ and for fixed integers h_i with $0 < |h_i| \leq N^3$, $i = 1, 2, 3, 4$, the number of solutions $(\nu_1, \nu_2, \nu_3, \nu_4)$ of the Diophantine equation

$$h_1 n_{\nu_1} + h_2 n_{\nu_2} + h_3 n_{\nu_3} + h_4 n_{\nu_4} = 0 \tag{1.5}$$

subject to

$$1 \leq \nu_i \leq N, \qquad i = 1, 2, 3, 4, \tag{1.6}$$

does not exceed $C_0 N^{1+\beta}$, provided that no proper subsums in (1.5) vanish.

Condition (G), if there exists a constant $0 < \eta \leq 1$ such that for all $k \geq k_0(\eta)$ we have

$$n_{k+k^{1-\eta}}/n_k \geq k. \tag{1.7}$$

(Here, and in the sequel, n_j is meant as $n_{[j]}$ if j is not an integer.)

With these notations we can formulate now our main result.

Theorem 1. *Let (n_k) be an increasing sequence of positive integers satisfying conditions* (B), (C) *and* (G). *Then there is a constant D, depending only on the constants α, β, η, C and C_0 in these conditions, such that for almost all $\omega \in [0, 1)$*

$$\frac{1}{4} \leq \limsup_{N\to\infty} \frac{ND_N(\omega)}{\sqrt{N \log\log N}} \leq D.$$

In Section 2 we will discuss examples of applications of Theorem 1. In particular, we will show that conditions (B), (C) and (G) are satisfied not only for the Hardy-Littlewood-Pólya sequences, but, in some sense, for 'almost all' sequences (n_k) with a suitable minimal speed. For concrete sequences (n_k), verifying conditions (B) and (G) requires usually only elementary arguments, but condition (C) is different: for example, the proof that condition (C) holds for the Hardy-Littlewood-Pólya sequences requires the latest version of the subspace theorem of Evertse, Schlickewei and Schmidt [13].

It is worth pointing out that Theorem 1 does *not* contain the LIL (1.4) for Hadamard gap sequences. Indeed, while conditions (B) and (G) are trivially satisfied under (1.3), the example $n_k = 2^k - 1$ shows that condition (C) can fail under the Hadamard gap condition. There is in fact a basic difference between the Hadamard and sub-Hadamard cases: while under (1.3) the discrepancy of

$(n_k\omega)$ satisfies the LIL regardless of the arithmetic structure of (n_k), for sub-Hadamard sequences the asymptotic behavior of $D_N(\omega)$ depends sensitively on the number-theoretic properties of (n_k). Assuming 'nice' number-theoretic properties for a Hadamard sequence (n_k) will improve the constant $C(\varrho)$ in (1.4), but the exact value of the limsup in (1.4) seems to be unknown even in the simplest case $n_k = 2^k$.

The lower bound $1/4$ in Theorem 1 will be deduced from the following theorem, which gives a strong asymptotic result for the sums $\sum_{k\le N} \cos 2\pi n_k\omega$.

THEOREM 2. *Let (n_k) be an increasing sequence of positive integers satisfying condition* (G) *and condition* (C) *with $h_i = \pm 1$ only. Then there exists a sequence $\{Y_k(\omega_1,\omega_2)\}_{k=1}^\infty$ of independent standard Gaussian random variables defined on the unit square $[0,1)^2$, equipped with the two-dimensional Lebesgue measure, such that for almost all $(\omega_1,\omega_2) \in [0,1)^2$*

$$\sqrt{2}\sum_{k\le N} \cos 2\pi n_k\omega_1 - \sum_{k\le N} Y_k(\omega_1,\omega_2) \ll N^{1/2-\lambda} \tag{1.8}$$

for some $\lambda > 0$, depending on β, η and C_0 only.

Applying the classical law of the iterated logarithm for independent standard Gaussian random variables, relation (1.8) implies that for almost all $\omega_1 \in [0,1)$

$$\limsup_{N\to\infty} \frac{|\sum_{k\le N} \cos 2\pi n_k\omega_1|}{\sqrt{N\log\log N}} = 1. \tag{1.9}$$

Koksma's inequality [11, p. 11] or [18, p. 143] implies $1/(4\sqrt{2})$ as a lower bound in Theorem 1. However, the complex version of (1.9) that considers $e^{2\pi i n_k\omega_1}$ instead of $\cos 2\pi n_k\omega_1$ together with an improved version of Koksma's inequality will yield $1/4$ as a lower bound in Theorem 1, as claimed. For the details see Philipp [21].

Letting

$$F_n(t) = F_n(t,\omega) = \frac{1}{n}\,\mathrm{card}\,\{k \le n : \eta_k(\omega) \le t\}, \quad 0 \le t \le 1, \tag{1.10}$$

denote the empirical distribution function of the sequence $\{n_k\omega\}$, Theorem 1 provides a law of the iterated logarithm for the Kolmogorov-Smirnov statistic $\sqrt{N}\sup_{0\le t\le 1}|F_N(t)-t|$. The following stronger theorem will give information on the modulus of continuity of the process

$$\beta_N(t) = \sqrt{\frac{N}{\log\log N}}(F_N(t)-t), \quad 0 \le t \le 1.$$

THEOREM 3. *Let (n_k) be an increasing sequence of positive integers satisfying conditions* (B), (C) *and* (G). *Then there exist constants $\delta > 0$ and D, depending only on α, β, η, C and C_0, with the following property. For almost* *all $\omega \in [0,1)$ there is an $N_0 = N_0(\omega, \alpha, \beta, \eta, C, C_0)$ such that for all $N \geq N_0$ and all s and t with $0 \leq s < t \leq 1$*

$$\max_{n \leq N} n|F_n(t) - F_n(s) - (t-s)| \leq D(t-s)^{\delta}(N \log\log N)^{1/2} + N^{1/2}.$$

As a consequence, for each $\varepsilon > 0$ there exists with probability 1 a random index $N_0 = N_0(\varepsilon, \omega)$ such that

$$|\beta_N(t) - \beta_N(s)| \leq D|t-s|^{\delta} + \varepsilon \tag{1.11}$$

for all $0 \leq s < t \leq 1$ and all $N \geq N_0$. Here the constants $D > 0$ and $\delta > 0$ depend only on the parameters in conditions (B), (C) and (G). From (1.11) one can infer without difficulty (for the details see Philipp [20], Section 3.1) that the sequence $\beta_N(t)$ is with probability 1 relatively compact in the space $\mathcal{D}(0,1)$ endowed with the supremum norm. To identify the limit functions as the unit ball of the reproducing kernel Hilbert space associated with the covariance function of the Gaussian process accompanying the sequence $(\eta_k,\ k \geq 1)$ would require additional assumptions on the sequence of integers. Corollary 4.1 of Philipp [20] provides an example where the set of limit functions can be identified as such. We shall not pursue this issue in the present paper. Of course, (1.11) immediately implies Theorem 1.

For additional results on the connection between the asymptotic properties of $\{n_k\omega\}$ and the Diophantine properties of the sequence (n_k) we refer to Berkes, Philipp and Tichy [4].

The structure of our paper is the following. In Section 2 we give examples of applications of our theorems. In Sections 3–7 we give the proof of Theorem 3. Our argument makes substantial use of the ideas of Philipp [21], Theorem 2. In particular, we will streamline and improve upon the chaining argument presented in Section 3 of [21]. The bulk of the proof of Theorem 3 to be presented in Sections 3–7 consists of a martingale approximation of the sequence $\{F_k(t) - F_k(s) - (t-s)\}_{k=1}^{\infty}$. This involves blocking techniques, centering at conditional expectations, Doob's maximal inequalities for martingales and estimates of the conditional variances, in order to prepare for an application of an exponential bound for the martingale.

The proof of Theorem 2 is technically considerably simpler than those of Theorems 1 and 3 and can be modelled after the proof of Philipp [21], Theorem 2. We shall give it in Section 8 below. Finally, in the Appendix we formulate, for easier reference, some classical results on martingales and maximal inequalities that we will need for the proofs of our theorems. For further background reading we refer the interested reader to Dehling and Philipp [6] and Philipp [22].

2. Examples

The results of our paper establish a close connection between the asymptotic behavior of the discrepancy of $\{n_k\omega\}$ and the Diophantine properties of the sequence (n_k). The Diophantine property (B) is generally easy to handle; in contrast, verifying (C) in concrete cases usually requires very substantial tools from number theory. For example, the proof below that Hardy-Littlewood-Pólya sequences satisfy condition (C) requires the latest version of the subspace theorem of Evertse, Schlickewei and Schmidt [13]. On the other hand, random constructions provide a very large class of sequences satisfying the conditions of Theorems 1–3. For example, we will show below that almost all sequences (n_k) growing with a minimal speed specified by condition (G) satisfy the conclusion of Theorems 1–3.

We begin by showing that the Hardy-Littlewood-Pólya sequences satisfy conditions (B), (C) and (G).

Lemma 2.1. *Let (n_k) be a Hardy-Littlewood-Pólya sequence. Then there is a constant $C > 0$ such that for each positive integer b and for each $R \geq 1$ the number of solutions (h, ν) of the Diophantine equation*

$$hn_\nu = b \tag{2.1}$$

with $h \in \mathbb{N}$, $1 \leq h \leq R$, does not exceed $C(\log R)^r$. Here r is the number of primes involved in the prime factorization of $q_1, \dots, q_\tau$.

Proof. Let $p_1, \dots, p_r$ be the primes appearing in the prime factorization of $q_1, \dots, q_\tau$ and write $b = p_1^{\alpha_1} \dots p_r^{\alpha_r} M$, where $\alpha_i \geq 0$ are integers and M is not divisible by $p_1, \dots, p_r$. The number n_ν in (2.1) has the form $n_\nu = p_1^{\beta_1} \dots p_r^{\beta_r}$ with integers $\beta_i \geq 0$, and thus (2.1) implies that $\beta_i \leq \alpha_i$, $i = 1, \dots, r$, and

$$h = p_1^{\alpha_1 - \beta_1} \dots p_r^{\alpha_r - \beta_r} M = p_1^{\delta_1} \dots p_r^{\delta_r} M$$

with integers $\delta_i \geq 0$. Now $h \leq R$ implies $p_1^{\delta_1} \dots p_r^{\delta_r} \leq R/M \leq R$, and consequently $\delta_i \leq \log R / \log 2$, $i = 1, \dots, r$. Thus the number of r-tuples $(\delta_1, \dots, \delta_r)$, and consequently the number of h's that can possibly yield a candidate h for a solution (h, ν) of (2.1), is at most $(1 + \log R / \log 2)^r$. As h in (2.1) determines ν uniquely, Lemma 2.1 is proved. □

Lemma 2.2. *A Hardy-Littlewood-Pólya sequence satisfies condition* (C) *with any $\beta > 0$ and*

$$C_0 = \exp(18^9(\tau + 1)),$$

where τ denotes the number of generating elements of the sequence.

Proof. The number of choices for ν_4 in (1.5) is N, and thus the lemma follows from Theorem 1.1 of Evertse-Schlickewei-Schmidt [13] upon fixing ν_4 and dividing (1.5) by $h_4 n_{\nu_4}$. (Note that Hardy-Littlewood-Pólya sequences satisfy the Diophantine property in condition (C) even with $\beta = 0$. However, for technical reasons condition (C) was defined only for $\beta > 0$.) □

LEMMA 2.3. *A Hardy-Littlewood-Pólya sequence* (n_k) *satisfies condition* (G).

Proof. Let $q_1, \dots, q_\tau$ be the generating elements of (n_k). Clearly, an element $n_j = q_1^{\delta_1} \dots q_\tau^{\delta_\tau}$ of the sequence (n_k) satisfies $n_j \le R$ iff

$$\delta_1 \log q_1 + \dots + \delta_\tau \log q_\tau \le \log R,$$

and thus the number $A(R)$ of elements of (n_k) in $[0, R]$ equals the number of lattice points $(\delta_1, \dots, \delta_\tau)$ in the τ-dimensional 'tetrahedron'

$$x_1 \log q_1 + \dots + x_\tau \log q_\tau \le \log R, \qquad x_1 \ge 0, \dots, x_\tau \ge 0.$$

The volume of the tetrahedron is $c_1(\log R)^\tau$, where

$$c_1 = c_1(\tau) = \frac{1}{\tau! \log q_1 \dots \log q_\tau},$$

and thus by a well known argument in analysis we have, as $R \to \infty$,

$$A(R) = c_1(\log R)^\tau + O((\log R)^{\tau-1}). \tag{2.2}$$

From (2.2) and the trivial relation $A(n_k) = k$ we get

$$\log n_k \sim \left(\frac{k}{c_1}\right)^{1/\tau}. \tag{2.3}$$

Formulas (2.2), (2.3) and $\log kn_k \sim \log n_k$ (which is a consequence of (2.3)) imply that the number of n_j's in the interval $[n_k, kn_k]$ is

$$\begin{aligned} c_1[(\log kn_k)^\tau - (\log n_k)^\tau] + O((\log kn_k)^{\tau-1}) &\sim c_1\tau(\log k)(\log n_k)^{\tau-1} \\ &\sim c_2 k^{(\tau-1)/\tau} \log k \end{aligned}$$

as $k \to \infty$. Thus for $k \ge k_0$ we have

$$n_{k+2c_2 k^{(\tau-1)/\tau} \log k} \ge kn_k,$$

and consequently (1.7) holds with any $\eta < 1/\tau$. □

We now show that, in some sense, almost all sequences (n_k) growing like a polynomial with a fixed large degree will satisfy conditions (B) and (C). We shall construct these sequences by induction. Let $n_1 = 1$ and suppose that $n_1 < n_2 < \dots < n_{k-1}$ have already been constructed and satisfy

$$(j-1)^{50} < n_j \le j^{50}, \quad j = 1, 2, \dots, k-1. \tag{2.4}$$

Then the cardinality of the set of integers of the form

$$a_1 n_{\mu_1} + a_2 n_{\mu_2} + a_3 n_{\mu_3}$$

with $1 \le \mu_1, \mu_2, \mu_3 \le k-1$, $|a_1|, |a_2|, |a_3| \le k^{11}$, is at most $(2k^{11}+1)^3(k-1)^3 = O(k^{36})$. Hence the cardinality of the set of integers included in the set of rational numbers

$$\frac{1}{a}(a_1 n_{\mu_1} + a_2 n_{\mu_2} + a_3 n_{\mu_3}), \quad a \in \mathbb{Z} - \{0\},\ |a| \le k^{11}, \tag{2.5}$$

subject to $1 \le \mu_1, \mu_2, \mu_3 \le k-1$, $|a_1|, |a_2|, |a_3| \le k^{11}$, is $O(k^{47})$. Thus, the interval $((k-1)^{50}, k^{50}]$ contains at most that many integers of the form (2.5). This number is at most $O(1/k^2)$ times the total number of integers in the interval. Calling these numbers "bad", we choose now n_k from the "good" integers (which constitute an overwhelming majority for k large), and note that (2.4) is satisfied also for $j = k$. This construction yields an infinite increasing sequence (n_k) with the property that for $k \ge k_0$ the Diophantine equation

$$a_1 n_{\mu_1} + a_2 n_{\mu_2} + a_3 n_{\mu_3} + a_4 n_{\mu_4} = 0 \tag{2.6}$$

with $1 \le \mu_i \le k$, $i = 1, 2, 3, 4$, and $\max(|a_i|, i = 1, 2, 3, 4) \le k^{11}$ has no solution if one of the indices, say μ_4, equals k and the corresponding factor $a_4 \ne 0$, while the other three indices μ_i are strictly less than k. Call this property (NS) (for "no solutions").

We now show that the constructed sequence (n_k) satisfies condition (C). Let $N \ge N_0$ be given and consider (1.5) subject to $0 < |h_i| \le N^4$, $i = 1, 2, 3, 4$, fixed and $1 \le \nu_1 \le \nu_2 \le \nu_3 \le \nu_4 \le N$. We can assume without loss of generality that $\nu_4 > 3N^{4/11}$, since otherwise (1.5) can have only $(3N^{4/11})^4 = O(N^{1+\beta})$ solutions, where $\beta = 5/11 < 1/2$. We now distinguish several cases according to the relative size of the indices ν_i. If $\nu_4 > \nu_3$, we set $k = \nu_4$. Then using property (NS) it follows that (1.5) has no solutions subject to $0 < |h_i| \le N^4$, since then $|h_i| \le N^4 < k^{11}$ by $k = \nu_4 > 3N^{4/11}$. (Note that the validity of (NS) has been established only for $k \ge k_0$, but by $k = \nu_4 > 3N^{4/11}$ this is satisfied if $N \ge N_0$. For the finitely many remaining values $1 \le N < N_0$ condition (C) is trivially satisfied.) If $\nu_4 = \nu_3 > \nu_2$, then (1.5) reduces to

$$h_1 n_{\nu_1} + h_2 n_{\nu_2} + h^* n_{\nu_4} = 0 \tag{2.7}$$

with $h^* = h_3 + h_4 \ne 0$, since otherwise the proper subsum $h_1 n_{\nu_1} + h_2 n_{\nu_2}$ would vanish. In (2.6) we set $a_1 = h_1, a_2 = h_2, a_3 = 0$, $a_4 = h^*$ and $\mu_4 = k$, and conclude that by property (NS), (2.7) has no solutions since $|h^*| \le 2N^4 < k^{11}$. If $\nu_4 = \nu_3 = \nu_2 > \nu_1$, then (1.5) reduces to

$$h_1 n_{\nu_1} + h^{**} n_{\nu_4} = 0 \tag{2.8}$$

with $h^{**} = h_2 + h_3 + h_4 \ne 0$, since otherwise we would have $h_1 = 0$, contrary to the assumption. In (2.6) we set $a_1 = h_1, a_2 = a_3 = 0, a_4 = h^{**}$ and $\mu_4 = k$, and conclude that (2.8) cannot have a solution since $|h^{**}| \le 3N^4 < k^{11}$. Finally, if $\nu_4 = \nu_3 = \nu_2 = \nu_1$, then there are only N possibilities for the

4-tuple $(\nu_1, \nu_2, \nu_3, \nu_4)$, and thus for fixed h_i the number of solutions of (1.5) is at most N, regardless of the restrictions on h_i.

To verify that (n_k) also satisfies condition (B), let $R \geq 1$, $b \geq 1$ be given and consider the equation $hn_\nu = b$ with $h \in \mathbb{N}$, $1 \leq h \leq R$. If this equation has a solution (h, n_ν) at all, let (l, n_μ) denote its solution with the largest μ. This means that we have to study the Diophantine equation

$$hn_\nu - ln_\mu = 0 \tag{2.9}$$

subject to $\nu \leq \mu$ and $1 \leq h \leq R$. Set $k = \mu$. If $k \leq R^{1/4}$, then $\nu \leq R^{1/4}$, and since ν uniquely determines h in (2.9), in this case the number of solutions (h, n_ν) of (2.9) does not exceed $R^{1/4}$, regardless of the restriction on h. If $k > R^{1/4}$, then by property (NS), equation (2.9) has no solutions other than $h = l$, $\nu = \mu$, since from $\nu < \mu$, $1 \leq h \leq R$ it follows that $1 \leq l \leq h \leq R \leq k^4 \leq k^{11}$. (As we noted above, the validity of (NS) has been established only for $k \geq k_0$, but by $k > R^{1/4}$ this is satisfied if $R \geq R_0$. For the finitely many remaining values $1 \leq R < R_0$ condition (B) is trivially satisfied.) Thus the impossibility of a solution follows by setting in (2.6) $a_1 = a_2 = 0$, $a_3 = h$ and $a_4 = -l$.

If instead of (2.4) we require $n_j \in I_j$, where $I_1, I_2, \ldots$, are disjoint intervals on the positive line, each lying to the right of the preceding one, then the same construction will work as long as the length $|I_j|$ of the interval I_j satisfies $|I_j| \geq j^{49}$. Specifically, if $n_1 < n_2 < \cdots < n_{k-1}$ are given, the number of "bad" choices for n_k in the interval I_k is $O(1/k^2)$ times the total number of integers in the interval, and thus if we choose n_k at random, uniformly among all integers in the interval I_k, then the Borel-Cantelli lemma shows that with probability one, all choices for $k \geq k_0$ will be "good". Hence the above construction yields the following result.

Corollary. *Let $I_1, I_2, \ldots$ be disjoint intervals on the positive line, each lying to the right of the preceding one, such that $|I_k| \geq k^{49}$, $k = 1, 2, \ldots$, and let (n_k) be a random sequence such that n_k is uniformly distributed over the integers of the interval I_k. Then (n_k) satisfies conditions* (B) *and* (C) *with probability one.*

With a proper choice of the intervals I_k we can "regulate" the speed of growth of (n_k); in fact, we can guarantee an arbitrarily prescribed speed of growth provided this exceeds k^γ, γ large. Specifically, if $\phi(k)$, $k \geq 0$, is a sequence of integers with $\phi(0) = 0$ and $\phi(k) - \phi(k-1) > 2k^{49}$, $k = 1, 2, \ldots$, then choosing $I_k = [\phi(k) - k^{49}, \phi(k) + k^{49}]$ will imply $n_k \sim \phi(k)$. In particular, we can guarantee the validity of Condition (G) as well.

3. Preliminary lemmas

For technical reasons, in the proof of our theorems we will need condition (C) in a slightly different form. We say that a sequence (n_k) satisfies

Condition (C*), if there exist constants $0 < \beta < 1/2$, $\gamma > 0$ and $C_0 > 0$ such that for each $N \geq 1$ and for fixed integers h_i with $0 < |h_i| \leq N^{3(1+\gamma)}$, $i = 1, 2, 3, 4$, the number of solutions $(\nu_1, \nu_2, \nu_3, \nu_4)$ of the Diophantine equation

$$h_1 n_{\nu_1} + h_2 n_{\nu_2} + h_3 n_{\nu_3} + h_4 n_{\nu_4} = 0 \tag{3.1}$$

subject to

$$1 \leq \nu_i \leq N^{1+\gamma}, \quad i = 1, 2, 3, 4, \tag{3.2}$$

does not exceed $C_0 N^{1+\beta}$, provided that no proper subsums in (3.1) vanish.

Choosing $\gamma > 0$ so small that

$$(1+\beta)(1+\gamma) < 3/2, \tag{3.3}$$

and applying condition (C) with $N^{1+\gamma}$ instead of N, we see that condition (C) implies condition (C*), and thus the two conditions are actually equivalent. It is also clear that if condition (C*) is satisfied with constants β, γ then it is also satisfied with β, γ' for arbitrary $0 < \gamma' < \gamma$, and thus without loss of generality we can always assume that γ is small, e.g., relation (3.3) holds.

Lemma 3.1. *Assume that condition* (G) *holds. Then for any integer $A \geq 1$ we have for $k \geq k_0(A)$*

$$n_{k(1+k^{-\eta})^A} / n_k \geq k^A,$$

and consequently

$$n_{k+2Ak^{1-\eta}} / n_k \geq k^A.$$

Proof. We use induction on A. We apply (G) to $k(1+k^{-\eta})^A$ instead of k and obtain, writing $n(k)$ instead of n_k, in order to avoid subscripts,

$$\frac{n(k(1+k^{-\eta})^{A+1})}{n(k)} \geq \frac{n(k(1+k^{-\eta})^A)(1+(k(1+k^{-\eta})^A)^{-\eta})}{n(k(1+k^{-\eta})^A)} \cdot \frac{n(k(1+k^{-\eta})^A)}{n(k)}$$

$$\geq k(1+k^{-\eta})^A \cdot k^A \geq k^{A+1},$$

since the sequence (n_k) is increasing. □

For fixed s and t with $0 \leq s < t \leq 1$ we write

$$x_\nu = x_\nu(s,t) = \mathbf{1}(s \leq \eta_\nu(\omega) < t) - (t-s).$$

LEMMA 3.2. *Let $\mathcal{N}$ be a finite set of positive integers with card $\mathcal{N} = N$. Then for $0 \le s < t \le 1$*

$$E\left(\sum_{\nu\in\mathcal{N}} x_\nu(s,t)\right)^2 \ll (t-s)^\varepsilon N,$$

where

$$\varepsilon = 1 - \alpha\left(\frac{5}{4} - \frac{1}{2}\alpha\right)^2 > \frac{1}{2} \tag{3.4}$$

and where the constant implied by $\ll$ depends only on α and C in condition (B).

Here, and through Section 8, the symbols E and P respectively denote the expectation and probability defined in the probability space $[0,1)$, equipped with Lebesgue measure.

Proof. For simplicity we set

$$\varrho = \frac{1}{2} - \alpha, \quad \delta = \frac{1}{2}\alpha\varrho, \quad \Theta = \alpha + \delta = \alpha\left(1 + \frac{1}{2}\varrho\right).$$

Let $\sum_{h\neq 0} c_h e^{2\pi i h\omega}$ be the Fourier-expansion of $\mathbf{1}_{[s,t)}(\omega) - (t-s)$, assuming that the indicator $\mathbf{1}_{[s,t)}$ has been extended with period 1. Then

$$|c_h| \le \frac{1}{\pi|h|}, \quad h \neq 0. \tag{3.5}$$

We follow the proof of [21, Lemma 2.2] until the end of the first (unnumbered) display on p. 711. Applying condition (B) instead of the direct argument used to estimate the number of solutions of the Diophantine equation $hn_\nu = ln_\mu$, and using (3.5) we can continue the chain of inequalities and get that the L_2 norm of $\sum_{\nu\in\mathcal{N}} x_\nu(s,t)$ is at most

$$\begin{aligned} &2C^{1/2}N^{1/2}\sum_{u\ge 0} 2^{u\alpha/2}\left(\sum_{2^u\le|h|<2^{u+1}} |c_h|^2\right)^{1/2} \\ &\ll N^{1/2}\sum_{u\ge 0} 2^{-u\delta/2}\left(\sum_{2^u\le|h|<2^{u+1}} |c_h|^{2-\alpha-\delta}\right)^{1/2} \\ &\ll N^{1/2}\left(\sum_{u\ge 0} 2^{-u\delta}\right)^{1/2}\left(\sum_{u\ge 0}\sum_{2^u\le|h|<2^{u+1}} |c_h|^{2-\Theta}\right)^{1/2} \\ &\ll N^{1/2}\left(\sum_{h\neq 0} |c_h|^{2-\Theta}\right)^{1/2} \end{aligned} \tag{3.6}$$

$$\ll N^{1/2} \left(\sum_{h \neq 0} |c_h|^{2-(2+\varrho)\Theta} \cdot |h|^{-(1+\varrho)\Theta} \right)^{1/2}$$

$$\ll N^{1/2} \left(\sum_{h \neq 0} |c_h|^{(2-(2+\varrho)\Theta)p} \right)^{1/(2p)} \left(\sum_{h \neq 0} |h|^{-(1+\varrho)\Theta q} \right)^{1/(2q)}$$

with

$$\frac{1}{p} = 1 - \frac{1}{2}(2+\varrho)\Theta = 1 - \frac{1}{q},$$

so that

$$q = \frac{2}{(2+\varrho)\Theta} > 2.$$

Indeed,

$$\frac{1}{q} = \left(1 + \frac{1}{2}\varrho\right)\Theta = \left(1 + \frac{1}{2}\varrho\right)^2 \alpha = \left(1 + \frac{1}{2}\varrho\right)^2 \left(\frac{1}{2} - \varrho\right) < \frac{1}{2}$$

as $0 < \varrho < 1/2$. Since

$$(1+\varrho)\Theta q = (2+2\varrho)/(2+\varrho) > 1,$$

the second sum in the last line of (3.6) converges. In the first sum in the same line, the exponent of $|c_h|$ equals 2 by the choice of p, and thus this sum equals $\|\mathbf{1}_{[s,t)}(\cdot) - (t-s)\|_2^2 \leq 4(t-s)$. Hence we obtain the result observing that

$$\frac{1}{p} = 1 - \left(1 + \frac{1}{2}\varrho\right)^2 \alpha = 1 - \left(\frac{5}{4} - \frac{1}{2}\alpha\right)^2 \alpha = \varepsilon. \qquad \square$$

Lemma 3.3. *For $0 \leq s < t \leq 1$ and all $H \leq N^{1+\gamma}$ we have*

$$E\left(\sum_{\nu=H+1}^{H+N} x_\nu(s,t)\right)^4 \ll (t-s)^{2\varepsilon} N^2 + N^{1+\beta} \log^4 N,$$

where ε is defined by (3.4) *and the constant implied by $\ll$ depends only on C, C_0, α, β and γ.*

Remark. Of course, the factor $\log^4 N$ could be absorbed into the exponent $1+\beta$ on N. All that will be needed is that the exponent on N is less than $3/2$. However, keeping the logarithmic factors in the formulation and proof of the lemma facilitates reading as the individual steps become more transparent. The same applies for the logarithmic factors in the proof of later lemmas.

Proof. We expand again $\mathbf{1}_{[s,t)}(\omega) - (t-s)$ into a Fourier series

$$\sum_{h \neq 0} c_h \exp(2\pi i h \omega),$$

assuming that the indicator $\mathbf{1}_{[s,t)}$ has been extended with period 1. Substituting $n_\nu\omega$ into this series we obtain

$$x_\nu = x_\nu(s,t) = \mathbf{1}_{[s,t)}(n_\nu\omega) - (t-s) = x_\nu^* + x_\nu^{**},$$

where

$$x_\nu^* = \sum_{0<|h|\le N^{3(1+\gamma)}} c_h \exp(2\pi i h n_\nu \omega)$$

and

$$x_\nu^{**} = \sum_{|h|>N^{3(1+\gamma)}} c_h \exp(2\pi i h n_\nu \omega).$$

Note that for all ω and all ν

$$|x_\nu^{**}| \le |x_\nu| + |x_\nu^*| \le \mathbf{1}_{[s,t)}(n_\nu\omega) + (t-s) + \sum_{0<|h|\le N^{3(1+\gamma)}} |c_h| < 15 \log N \tag{3.7}$$

and

$$||x_\nu^{**}||_2^2 = \sum_{|h|>N^{3(1+\gamma)}} |c_h|^2 < N^{-3(1+\gamma)} \tag{3.8}$$

by (3.5). Thus by Minkowski's inequality

$$\begin{aligned} E\left|\sum_{\nu=H+1}^{H+N} x_\nu^{**}\right|^4 &\le (15N\log N)^2 \cdot E\left(\sum_{\nu=H+1}^{H+N} |x_\nu|^{**}\right)^2 \\ &\le 225N^2\log^2 N \cdot (N\cdot N^{-3/2})^2 \ll N\log^2 N. \end{aligned} \tag{3.9}$$

Using the multinomial theorem we expand

$$E\left|\sum_{\nu=H+1}^{H+N} x_\nu^*\right|^4 = \sum_{\substack{\nu_i=H+1\\ i=1,2,3,4}}^{H+N} E x_{\nu_1}^* \bar{x}_{\nu_2}^* x_{\nu_3}^* \bar{x}_{\nu_4}^* \tag{3.10}$$

$$= \sum_{0<|h_i|\le N^{3(1+\gamma)}} c_{h_1}\bar{c}_{h_2}c_{h_3}\bar{c}_{h_4} \sum_{\substack{\nu_i=H+1\\ i=1,2,3,4}}^{H+N} \mathbf{1}\{h_1 n_{\nu_1} - h_2 n_{\nu_2} + h_3 n_{\nu_3} - h_4 n_{\nu_4} = 0\}.$$

By condition (C*) and the assumption $H \le N^{1+\gamma}$ the inner sum in the second line of (3.10) does not exceed $C_0(2N)^{1+\beta}$, provided that no proper subsum vanishes. Hence by (3.5) the total contribution of these terms in the sum in the second line of (3.10) is $\ll N^{1+\beta}(\log N)^4$. If a 3-term subsum vanishes, e.g.,

$$h_1 n_{\nu_1} - h_2 n_{\nu_2} + h_3 n_{\nu_3} = 0,$$

then for the indicator in the inner sum in the second line of (3.10) not to vanish the term $h_4 n_{\nu_4}$ would have to be zero, which is impossible. If on the other hand

$$h_1 n_{\nu_1} - h_2 n_{\nu_2} = 0 \quad \text{and} \quad h_3 n_{\nu_3} - h_4 n_{\nu_4} = 0,$$

then we can rewrite the indicator as the product of the two indicators

$$\mathbf{1}\{h_1 n_{\nu_1} - h_2 n_{\nu_2} = 0\} \cdot \mathbf{1}\{h_3 n_{\nu_3} - h_4 n_{\nu_4} = 0\}$$

and then the corresponding terms add up to

$$\left(E \left| \sum_{\nu=H+1}^{H+N} x_\nu^* \right|^2 \right)^2 .$$

This last expression can be handled by an application of Lemma 3.2. Since $|a-b|^2 < 2|a|^2 + 2|b|^2$ for any complex numbers a, b, we have by $x_\nu = x_\nu^* + x_\nu^{**}$, (3.8) and Minkowski's inequality

$$\begin{aligned} E \left| \sum_{\nu=H+1}^{H+N} x_\nu^* \right|^2 &\ll E \left| \sum_{\nu=H+1}^{H+N} x_\nu \right|^2 + E \left| \sum_{\nu=H+1}^{H+N} x_\nu^{**} \right|^2 \\ &\ll (t-s)^\varepsilon N + \left(\sum_{\nu=H+1}^{H+N} ||x_\nu^{**}||_2 \right)^2 \\ &\ll (t-s)^\varepsilon N + (N \cdot N^{-3/2})^2 \ll (t-s)^\varepsilon N + N^{-1}. \end{aligned}$$

The square of this last term (which is clearly $\le (t-s)^{2\varepsilon} N^2 + 3$) is a bound for the total contribution of those terms for which the considered proper subsums vanish. A similar bound is obtained for the contribution of terms with other two-term subsums (e.g., $h_1 n_{\nu_1} + h_3 n_{\nu_3}$) vanishing. Collecting our estimates, we get Lemma 3.3. □

Lemma 3.4. *Assume (n_k) satisfies condition* (C*) *with $h_i = \pm 1$ only and assume, without loss of generality, that* (3.3) *holds. Then for all $N \ge 1$, $H \le N^{1+\gamma}$ we have*

$$E \sup_{M \le N} \sup_{0 \le s < t \le 1} \left| \sum_{\nu=H+1}^{H+M} x_\nu(s,t) \right|^4 \ll N^2 \log^8 N, \tag{3.11}$$

where the constant implied by $\ll$ depends only on C_0.

Proof. We follow the proof of [21, Lemma 2.1]. By the Erdős-Turán inequality [18, p. 112] we have for each $R \ge 1$ and each $\omega \in [0,1)$

$$\sup_{0 \le s \le t \le 1} \left| \sum_{\nu=H+1}^{H+M} x_\nu(s,t) \right| \le \frac{6M}{R} + 2 \sum_{r=1}^{R} \frac{1}{r} \left| \sum_{\nu=H+1}^{H+M} e(r n_\nu \omega) \right|,$$

where $e(x) = e^{2\pi i x}$. Choosing $R = M$ and using $|a+b|^4 \le 8(|a|^4 + |b|^4)$ we obtain that the left hand side of (3.11) is bounded by

$$8 \cdot 6^4 + 8 \cdot 2^4 E \left\{ \left(\sum_{r=1}^{N} \frac{1}{r} \max_{M \le N} \left| \sum_{\nu=H+1}^{H+M} e(r n_\nu \cdot) \right| \right)^4 \right\}. \tag{3.12}$$

We observe that for any $H+1 \le k < l \le H+M$

$$\int_0^1 \left| \sum_{\nu=k}^{l} e(rn_\nu \omega) \right|^4 d\omega \le u(k,l),$$

where $u(k,l)$ denotes the number of solutions of the Diophantine equation

$$n_{\nu_1} - n_{\nu_2} + n_{\nu_3} - n_{\nu_4} = 0, \qquad k \le \nu_1, \nu_2, \nu_3, \nu_4 \le l. \tag{3.13}$$

Clearly $u(k,l)$ satisfies

$$u(k,l) \le u(k,l+1), \quad u(k,l) + u(l+1,m) \le u(k,m),$$

and thus using Lemma A1 and condition (C*) with $h_i = \pm 1$ we get

$$\int_0^1 \max_{M \le N} \left| \sum_{\nu=H+1}^{H+M} e(rn_\nu \omega) \right|^4 d\omega \ll u(H+1, H+N) \log^4 N, \tag{3.14}$$

where the constant implied by $\ll$ is absolute. The number of solutions of (3.13) for which no proper subsum vanishes is at most $C_0 l^{1+\beta}$ by condition (C*) with $h_i = \pm 1$. Clearly, no 3-term subsum in (3.13) can vanish. If a two-term subsum vanishes, then either $\nu_1 = \nu_4$, $\nu_2 = \nu_3$ or $\nu_1 = \nu_2$, $\nu_3 = \nu_4$; the number of such solutions is clearly $\le 2(l-k)^2$. Thus

$$u(k,l) \le C_0 l^{1+\beta} + 2(l-k)^2,$$

and consequently the right hand side of (3.14) is at most

$$C_0 (H+N)^{1+\beta} \log^4 N + 2N^2 \log^4 N.$$

Hence by Minkowski's inequality the right hand side of (3.12) is

$$\begin{aligned} &\ll 1 + C_0 (H+N)^{1+\beta} \log^8 N + 2N^2 \log^8 N \\ &\ll N^{(1+\beta)(1+\gamma)} \log^8 N + N^2 \log^8 N \ll N^2 \log^8 N \end{aligned}$$

since $(1+\beta)(1+\gamma) < 3/2$ by (3.3). □

A byproduct of the proof of Lemma 3.4 is the following corollary.

Corollary 3.1. *Assume (n_k) satisfies condition (C*) with $h_i = \pm 1$ only. Then for all $k < l$ we have*

$$\int_0^1 \left| \sum_{\nu=k}^{l} e^{2\pi i n_\nu \omega} \right|^4 d\omega \ll (l-k)^2 + l^{1+\beta},$$

where the constant implied by $\ll$ depends only on C_0.

4. Martingale approximation

Fix an integer τ with

$$\tau \geq \max(1+2/\gamma,\ 1/\eta,\ 8/(1-2\beta),\ 90), \tag{4.1}$$

where β, γ and η are from conditions (C*) and (G). We define blocks H_1, I_1, H_2, $I_2, \dots$ of consecutive integers as follows: First, for each $j = 1, 2 \dots$,

$$\operatorname{card} H_j = \operatorname{card} I_j := N_j = \tau j^{\tau-1}. \tag{4.2}$$

Second, the members of H_j are smaller than the members of I_j which, in turn, are smaller than the members of H_{j+1}. There are no gaps between consecutive blocks. It is enough to prove the theorem for the sequence $(n_\nu \omega)$ with $\nu \in \bigcup_{j\geq 1} H_j$, since the proof of the corresponding statement for $\nu \in \bigcup_{j \geq 1} I_j$ is identical and since at the end the triangle inequality implies the desired result. Denote by h_j the largest member of H_j. Then

$$h_j = 2 \sum_{l \leq j-1} \tau l^{\tau-1} + \tau j^{\tau-1} \approx j^\tau, \tag{4.3}$$

where $\approx$ means same order of magnitude. Moreover, the indices of $n_\nu,\ \nu \in \bigcup_{l \leq j-1} H_l$, are separated from the indices of $n_\mu,\ \mu \in H_j$, by $\tau(j-1)^{\tau-1}$ at least.

Let r_k be the largest integer r with

$$2^r \leq n_k k^{12}, \qquad k \geq 1, \tag{4.4}$$

and $\mathcal{H}_k$ the σ-field generated by the dyadic intervals

$$U_{lk} = [l \cdot 2^{-r_k}, (l+1)2^{-r_k}), \quad 0 \leq l < 2^{r_k}. \tag{4.5}$$

We set

$$\mathcal{F}_j = \mathcal{H}_{h_j},\ \xi_\nu = E(x_\nu | \mathcal{F}_j), \quad \nu \in H_j, \tag{4.6}$$

and

$$w_j = \sum_{\nu \in H_j} x_\nu, \quad y_j = E(w_j | \mathcal{F}_j) = \sum_{\nu \in H_j} \xi_\nu. \tag{4.7}$$

Here s and t are fixed, $0 \leq s < t \leq 1$. Let ϱ satisfy

$$0 < \varrho \leq \min\left(1/(40\tau),\ \frac{1}{16}\left(\frac{1}{2} - \beta\right)\right). \tag{4.8}$$

Note that this ϱ is different from the ϱ introduced solely for the proof of Lemma 3.2. For fixed n we truncate $y_j, j \leq n$, by setting

$$z_j = z_{j,n} = y_j \mathbf{1}\{|y_j| \leq h_n^{\frac{1}{2}-\varrho}\}, \quad j \leq n. \tag{4.9}$$

From now on, and until the end of Section 6, we assume the *standing hypothesis*

$$t - s > h_n^{-1/2}. \tag{4.10}$$

LEMMA 4.1. *We have*

$$P\left(\sum_{j\le n}|z_j-w_j|\ge h_n^{\frac12-\varrho}\right)\ll(t-s)^{2\varepsilon}n^{-3/4}+h_n^{-\frac12-\frac34(\frac12-\beta)},$$

where ε *is defined by* (3.4).

Proof. Recall that by [20, Lemma 4.2.3]

$$||x_\nu-\xi_\nu||_2\ll\nu^{-6}.$$

Thus by Minkowski's inequality

$$E|w_j-y_j|^2\le\left(\sum_{\nu\in H_j}||x_\nu-\xi_\nu||_2\right)^2\ll h_j^{-10}\ll j^{-10\tau}. \tag{4.11}$$

Thus

$$\begin{aligned}P\left(\sum_{n^{1/4}<j\le n}|w_j-y_j|\ge\frac13 h_n^{\frac12-\varrho}\right)&\\ \ll h_n^{-\frac12+\varrho}\sum_{n^{1/4}<j\le n}E|w_j-y_j|&\\ \ll h_n^{-\frac12+\varrho}\sum_{n^{1/4}<j\le n}j^{-5\tau}\ll h_n^{-1}.&\end{aligned} \tag{4.12}$$

Since trivially

$$|w_j|\le 2\,\mathrm{card}\,H_j=2N_j,\quad |y_j|\le 2N_j$$

we have

$$\sum_{j\le n^{1/4}}|w_j-y_j|\le 4\sum_{j\le n^{1/4}}N_j\ll\sum_{j\le n^{1/4}}j^{\tau-1}\ll h_n^{1/4}. \tag{4.13}$$

Next, we note that

$$E|y_j-z_j|=E\left|y_j\cdot\mathbf{1}\left\{|y_j|\ge h_n^{\frac12-\varrho}\right\}\right|\le h_n^{-\frac32+3\varrho}E|y_j|^4 \tag{4.14}$$

and also, by the conditional version of Jensen's inequality,

$$Ey_j^4=E\left(E(w_j|\mathcal{F}_j)^4\right)\le E\left(E(w_j^4|\mathcal{F}_j)\right)=Ew_j^4.$$

Thus by Lemma 3.3, (4.1), (4.2), (4.3) and (4.8)

$$P\left(\sum_{j\le n}|y_j - z_j| \ge \frac{1}{2}h_n^{\frac{1}{2}-\varrho}\right) \ll h_n^{-\frac{1}{2}+\varrho}h_n^{-\frac{3}{2}+3\varrho}\sum_{j\le n}E|y_j|^4 \tag{4.15}$$

$$\ll h_n^{-2+4\varrho}\left[(t-s)^{2\varepsilon}\sum_{j\le n}N_j^2 + \sum_{j\le n}N_j^{1+\beta}\log^4 j\right]$$

$$\ll (t-s)^{2\varepsilon}\cdot n^{-3/4} + h_n^{-2+4\varrho}\cdot h_n^{1+\beta}$$

$$\ll (t-s)^{2\varepsilon}n^{-3/4} + h_n^{-\frac{1}{2}-\frac{3}{4}(\frac{1}{2}-\beta)}.$$

The lemma now follows from (4.12), (4.13) and (4.15). □

A part of estimate (4.15) above yields:

COROLLARY 4.1. *We have*

$$\sum_{j\le n}E|y_j|^4 \ll (t-s)^{2\varepsilon}\cdot h_n^2 n^{-1} + h_n^{1+\beta}\cdot n^{-\beta}\log^4 n.$$

LEMMA 4.2. *Write*

$$Y_j = z_j - E(z_j|\mathcal{F}_{j-1}), \quad 1\le j\le n. \tag{4.16}$$

Then $\{Y_j,\mathcal{F}_j, 1\le j\le n\}$ *is a martingale difference sequence that is bounded by* $2h_n^{\frac{1}{2}-\varrho}$*. Moreover,*

$$P\left(\sum_{j\le n}|E(z_j|\mathcal{F}_{j-1})| \ge h_n^{\frac{1}{2}-\varrho}\right) \ll (t-s)^{2\varepsilon}n^{-3/4} + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}.$$

Proof. We first show that we have with probability 1

$$\sum_{j\le n}|E(y_j|\mathcal{F}_{j-1})| \le C_1 \tag{4.17}$$

for some non-random constant C_1. For this purpose we note that by the proof of [21, Lemma 3.3] we have

$$|E(x_\nu|\mathcal{F}_{j-1})| \ll 2^{r_{h_{j-1}}}(t-s)n_\nu^{-1}, \qquad \nu\in H_j,$$

and thus by (4.6), (4.7), (4.2), (4.4) we get

$$v_j := |E(y_j|\mathcal{F}_{j-1})| = \left|\sum_{\nu\in H_j}E(\xi_\nu|\mathcal{F}_{j-1})\right| = \left|\sum_{\nu\in H_j}E(x_\nu|\mathcal{F}_{j-1})\right| \tag{4.18}$$

$$\ll (t-s)2^{r_{h_{j-1}}}N_j/n_{h_{j-1}+N_{j-1}}$$

$$\ll (t-s)n_{h_{j-1}}h_{j-1}^{12}N_j/n_{h_{j-1}+N_{j-1}}.$$

For simplicity we set (cf. (4.3))

$$k := h_{j-1} \sim 2(j-1)^\tau.$$

To estimate the quotient $n(k+\tau(j-1)^{\tau-1})/n(k)$ (where, again, we wrote $n(j)$ instead of n_j to avoid subscripts) we note that for $j \geq j_0$ we have $k < 3(j-1)^\tau$, i.e., $j-1 > (k/3)^{1/\tau}$. Thus by relation (4.1)

$$\tau(j-1)^{\tau-1} > \tau(k/3)^{1-1/\tau} > \frac{\tau}{3}k^{1-\eta}.$$

Hence using Lemma 3.1 with $A \sim \tau/6$ we get

$$n(k+\tau(j-1)^{\tau-1})/n(k) \geq k^A. \tag{4.19}$$

Since by (4.1) we have $\tau \geq 90$, we can continue the estimating procedure in (4.18) and obtain in view of (4.19)

$$v_j \ll (t-s)h_{j-1}^{12}\, N_j \cdot h_{j-1}^{-A} \ll (t-s)h_{j-1}^{-2} \ll (t-s)j^{-180} \quad \text{a.s.}$$

This implies (4.17).

To finish the proof of the lemma we note that by (4.14) and part of the estimate (4.15) we have

$$\begin{aligned} &P\left(\sum_{j\leq n} |E(z_j|\mathcal{F}_{j-1}) - E(y_j|\mathcal{F}_{j-1})| \geq \frac{1}{2}h_n^{\frac{1}{2}-\varrho}\right) \\ &\ll h_n^{-\frac{1}{2}+\varrho}\sum_{j\leq n} E|z_j - y_j| \ll h_n^{-2+4\varrho}\sum_{j\leq n} E|y_j|^4 \\ &\ll (t-s)^{2\varepsilon}n^{-3/4} + h_n^{-\frac{1}{2}-\frac{3}{4}(\frac{1}{2}-\beta)}. \end{aligned}$$

This together with (4.17) yields the result. □

5. Estimates of the conditional variances

Eventually we will apply the exponential bound provided in Lemma A2 to the martingale difference sequence $\{Y_j, \mathcal{F}_j\}$. For this purpose we need an upper bound on the sum of the conditional variances $E(Y_j^2|\mathcal{F}_{j-1})$. By the conditional version of Jensen's inequality and since $|z_j| \leq |y_j|$ we have in view of (4.16)

$$E(Y_j^2|\mathcal{F}_{j-1}) \leq E(z_j^2|\mathcal{F}_{j-1}) \leq E(y_j^2|\mathcal{F}_{j-1}). \tag{5.1}$$

The next three lemmas will provide an estimate for the sums of $E(y_j^2|\mathcal{F}_{j-1})$ by the sums of Ew_j^2. The latter ones can be dealt with using Lemma 4.2. From now on let δ be a number with

$$0 < \delta \leq \frac{1}{4}\min\left(\varepsilon - \frac{1}{2}, \frac{1}{2} - \beta\right). \tag{5.2}$$

LEMMA 5.1. *We have*

$$P\left(\max_{k\le n}\left|\sum_{j\le k}(y_j^2 - E(y_j^2|\mathcal{F}_{j-1})\right| \ge \frac{1}{100}(t-s)^\delta h_n\right)$$
$$\ll (t-s)^{\varepsilon+\frac{1}{2}}n^{-1} + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}.$$

Proof. By Doob's maximal inequality [10, p. 314] for martingales, the probability in question does not exceed by (4.10), (5.2) and Corollary 4.1

$$\begin{aligned} &h_n^{-2}(t-s)^{-2\delta}E\left|\sum_{j\le n}(y_j^2 - E(y_j^2|\mathcal{F}_{j-1}))\right|^2 \\ &\quad\ll h_n^{-2}(t-s)^{-2\delta}\sum_{j\le n}E|y_j|^4 \\ &\quad\ll (t-s)^{2\varepsilon-2\delta}n^{-1} + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}. \end{aligned}$$ □

LEMMA 5.2. *We have*

$$P\left(\sum_{j\le n}|y_j^2 - w_j^2| \ge \frac{1}{100}(t-s)^\delta h_n\right)$$
$$\ll h_n^{-1}(t-s)^{1/4}.$$

Proof. By (4.11) and Lemma 3.2 and since $Ey_j^2 \le Ew_j^2$ by the conditional version of Jensen's inequality, we have

$$\begin{aligned} E|y_j^2 - w_j^2| &\le (E|y_j - w_j|^2 E|y_j + w_j|^2)^{1/2} \\ &\le (E|y_j - w_j|^2 \cdot 4Ew_j^2)^{1/2} \\ &\ll j^{-5\tau}(t-s)^{\frac{1}{2}\varepsilon}N_j^{1/2}, \end{aligned}$$

and thus by (4.1), (4.2), (5.2) the probability in question does not exceed

$$\ll h_n^{-1}(t-s)^{\frac{1}{2}\varepsilon-\delta}\sum_{j\le n}j^{-4\tau} \ll h_n^{-1}(t-s)^{1/4}.$$ □

LEMMA 5.3. *We have*

$$P\left(\max_{k\le n}\left|\sum_{j\le k}(w_j^2 - Ew_j^2)\right| \ge \frac{1}{100}(t-s)^\delta h_n\right)$$
$$\ll (t-s)^{\varepsilon+\frac{1}{2}}\cdot n^{-1}\log^2 n + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}.$$

Proof. Using the notation introduced in the proof of Lemma 3.3 we write

$$w_j = \sum_{\nu \in H_j} x_\nu^* + \sum_{\nu \in H_j} x_\nu^{**} = u_j + w_j^{**} \quad \text{say,}$$

where the truncation of the corresponding Fourier series is, for each $j \leq n$, adapted to the index set $H_j = (H, H+N]$; i.e., the cutoff index is $N_j^{3(1+\gamma)}$. First we estimate the terms $|w_j^{**}|^2, |u_j||w_j^{**}|$ and their expectations. Their contributions to the probability bound in the statement of the lemma will turn out to be well below the bound claimed. At the end we shall apply Lemma A1 to the sequence $\{u_j^2 - Eu_j^2,\, j \leq n\}$.

By (3.8) and Minkowski's inequality we have

$$E|w_j^{**}|^2 \leq N_j^{-(1+3\gamma)} \ll j^{-(1+3\gamma)(\tau-1)}, \tag{5.3}$$

and thus, as $\tau \geq 90$,

$$\sum_{j \geq 1} E|w_j^{**}|^2 < \infty. \tag{5.4}$$

Since by (3.7)

$$|w_j^{**}| \ll N_j \log j,$$

we have

$$\sum_{j \leq n^{1/4}} |w_j^{**}|^2 \ll \sum_{j \leq n^{1/4}} N_j^2 \log^2 j \ll h_n^{1/2}. \tag{5.5}$$

Moreover, by (5.3)

$$\begin{aligned} &P\left(\sum_{n^{1/4}<j\leq n} |w_j^{**}|^2 \geq \frac{1}{200}(t-s)^\delta h_n\right) \\ &\leq \sum_{n^{1/4}<j\leq n} P\left(|w_j^{**}|^2 \geq \frac{1}{200n}(t-s)^\delta h_n\right) \\ &\ll h_n^{-1}(t-s)^{-\delta} n \sum_{j>n^{1/4}} j^{-(\tau-1)(1+3\gamma)} \ll h_n^{-5/4}(t-s)^{-\delta} \\ &\ll h_n^{-1} \end{aligned} \tag{5.6}$$

in view of (4.10), (5.2) and (4.1) as $\delta < 1/8$. Since $\|u_j\|_2 \leq \|w_j\|_2 + \|w_j^{**}\|_2$, applying (5.3) and Lemma 3.2 we obtain

$$\begin{aligned} E|u_j||w_j^{**}| &\leq (E|w_j^{**}|^2)^{1/2}(E|u_j|^2)^{1/2} \ll j^{-(1+3\gamma)(\tau-1)/2}\left(N_j(t-s)^\varepsilon\right)^{1/2} \\ &\ll (t-s)^{\frac{1}{2}\varepsilon} j^{-3/2}. \end{aligned}$$

Hence we obtain

$$\sum_{j\geq 1} E|u_j||w_j^{**}| < \infty, \tag{5.7}$$

and similar to (5.6)

$$\begin{aligned} P\left(\sum_{j\leq n} |u_j||w_j^{**}| \geq \frac{1}{200}(t-s)^{\delta} h_n\right) &\\ \leq \sum_{j\leq n} P\left(|u_j||w_j^{**}| \geq \frac{1}{200n}(t-s)^{\delta} h_n\right) &\\ \ll (t-s)^{\frac{1}{2}\varepsilon-\delta} h_n^{-1} n \sum_{j\geq 1} j^{-3/2} \ll h_n^{-1+1/\tau}. & \end{aligned} \tag{5.8}$$

The relations $w_j = u_j + w_j^{**}$, (5.4), (5.5), (5.6), (5.7) and (5.8) show that the contributions of $|w_j^{**}|^2, |u_j||w_j^{**}|$ and their expectations of the claimed probability bound are negligible, i.e., are within the bound on the right hand side of the Lemma.

We can now prepare for the application of Lemma A1. For fixed $1 \leq j < k \leq n$ we shall estimate

$$E\left|\sum_{j<p\leq k}(u_p^2 - Eu_p^2)\right|^2 = \sum_{j<p,q\leq k} E(u_p^2 - Eu_p^2)(\bar{u}_q^2 - E\bar{u}_q^2).$$

By Lemma 3.3, $w_p = u_p + w_p^{**}$, (4.1), (3.9) and Minkowski's inequality

$$\begin{aligned} E|u_p^2 - Eu_p^2|^2 \leq E|u_p|^4 &\ll E|w_p|^4 + E|w_p^{**}|^4 \\ &\ll (t-s)^{2\varepsilon} N_p^2 + N_p^{1+\beta} \log^4 N_p + N_p \log^2 N_p \\ &\ll (t-s)^{2\varepsilon} p^{2\tau-2} + p^{(\tau-1)(1+\beta)} \log^4 p. \end{aligned} \tag{5.9}$$

Next we obtain for $p < q$

$$\begin{aligned} E(u_p^2 - Eu_p^2)(\bar{u}_q^2 - E\bar{u}_q^2) &= Eu_p^2\bar{u}_q^2 - Eu_p^2 E\bar{u}_q^2 \\ &= \sum_{\substack{0<|h_1|,|h_2|\leq N_p^{3(1+\gamma)} \\ 0<|h_3|,|h_4|\leq N_q^{3(1+\gamma)}}} c_{h_1} c_{h_2} \bar{c}_{h_3} \bar{c}_{h_4} \times \\ &\quad \times \sum_{\substack{\nu_1,\nu_2 \in H_p \\ \nu_3,\nu_4 \in H_q}} \mathbf{1}\{h_1 n_{\nu_1} + h_2 n_{\nu_2} - h_3 n_{\nu_3} - h_4 n_{\nu_4} = 0\} \\ &\quad - \sum_{\substack{0<|h_1|,|h_2|\leq N_p^{3(1+\gamma)} \\ 0<|h_3|,|h_4|\leq N_q^{3(1+\gamma)}}} c_{h_1} c_{h_2} \bar{c}_{h_3} \bar{c}_{h_4} \times \\ &\quad \times \sum_{\substack{\nu_1,\nu_2 \in H_p \\ \nu_3,\nu_4 \in H_q}} \mathbf{1}\{h_1 n_{\nu_1} + h_2 n_{\nu_2} = 0\} \cdot \mathbf{1}\{h_3 n_{\nu_3} + h_4 n_{\nu_4} = 0\} \\ &= J_1 - J_2, \quad \text{say.} \end{aligned}$$

Note that the greatest element of the set H_q is at most $2\tau q^\tau \le 2N_q^{\tau/(\tau-1)} \le (2N_q)^{1+\gamma}$ by (4.1), and thus by condition (C*) the inner sum in J_1 contains at most $C_0(2N_q)^{1+\beta}$ nonzero terms for fixed h_i, $i=1,2,3,4$, provided that no proper subsum vanishes. Thus in view of (3.5) the contribution of these terms in the double sums J_1 and J_2 is $\ll N_q^{1+\beta}\log^4 N_q$. Suppose now that the fourfold Diophantine equation holds. Then a threefold subsum of $h_1 n_{\nu_1} + h_2 n_{\nu_2} - h_3 n_{\nu_3} - h_4 n_{\nu_4}$ clearly cannot vanish, since if, e.g., $h_1 n_{\nu_1} + h_2 n_{\nu_2} - h_3 n_{\nu_3} = 0$, then we would have $h_4 n_{\nu_4} = 0$, which is impossible. It remains to consider the terms for which a twofold subsum vanishes. If $h_1 n_{\nu_1} + h_2 n_{\nu_2} = 0$, then automatically $h_3 n_{\nu_3} + h_4 n_{\nu_4} = 0$, and thus the inner sums in J_1 and J_2 are equal, hence their contribution in $J_1 - J_2$ is 0. On the other hand, if $h_1 n_{\nu_1} - h_3 n_{\nu_3} = 0$, then the fourfold Diophantine equation can hold only if $h_2 n_{\nu_2} - h_4 n_{\nu_4} = 0$. The contribution of these terms in J_1 equals $(Eu_p\bar{u}_q)^2$, while their contribution in J_2 is at most N_q, since if ν_1 is chosen, then the equations $h_1 n_{\nu_1} - h_3 n_{\nu_3} = 0$, $h_2 n_{\nu_2} - h_4 n_{\nu_4} = 0$, $h_1 n_{\nu_1} + h_2 n_{\nu_2} = 0$, $h_3 n_{\nu_3} + h_4 n_{\nu_4} = 0$ determine ν_2, ν_3, ν_4 uniquely. To estimate $(Eu_p\bar{u}_q)^2$, we note that by (5.3), $u_j = w_j - w_j^{**}$ and Lemma 3.2

$$|Eu_p\bar{u}_q - Ew_p w_q| \le \|w_p^{**}\|_2(\|w_q\|_2 + \|w_q^{**}\|_2) + \|w_p\|_2\|w_q^{**}\|_2 \ll N_q^{1/2}.$$

Similarly, by Lemma 3.2, (4.7), (4.11) and the conditional Jensen inequality

$$|Ew_p w_q - Ey_p y_q| \le \|w_p - y_p\|_2\|y_q\|_2 + \|w_p\|_2\|w_q - y_q\|_2 \ll N_q^{1/2}.$$

Since $p<q$, we have by the proof of Lemma 4.2

$$|E(y_p y_q)| = |E(E(y_p y_q|\mathcal{F}_{q-1}))| = |E(y_p v_q)| \le \|y_p\|_2\|v_q\|_2 \ll p^{1/2}q^{-1800} \ll 1.$$

Adding these estimates we see that the contribution of the considered terms to J_1 and J_2 is $\ll N_q$.

Collecting the above estimates we obtain in view of (5.9)

$$\begin{aligned} E\left|\sum_{j<p\le k}(u_p^2 - Eu_p^2)\right|^2 &\ll \sum_{j<p\le q\le k} N_q^{1+\beta}\log^4 N_q \\ &\quad + (t-s)^{2\varepsilon}\sum_{j<p\le k} p^{2\tau-2} + \sum_{j<p\le k} p^{(\tau-1)(\beta+1)}\log^4 p \\ &\ll (t-s)^{2\varepsilon}\sum_{j<p\le k} p^{2\tau-2} + \sum_{j<q\le k} q\cdot q^{(\tau-1)(\beta+1)}\log^4 q. \end{aligned} \tag{5.10}$$

Thus by Lemma A1 with $\gamma = 2$

$$P\left(\max_{k\le n}\left|\sum_{j\le k}(u_j^2 - Eu_j^2)\right| \ge \frac{1}{100}(t-s)^\delta h_n\right)$$
$$\ll h_n^{-2}\log^2 n\ (t-s)^{-2\delta}\left[(t-s)^{2\varepsilon}n^{2\tau-1} + \log^4 n\cdot n^{\tau(1+\beta)+1-\beta}\right]$$
$$\ll (t-s)^{2(\varepsilon-\delta)}n^{-1}\log^2 n + (t-s)^{-2\delta}h_n^{-1+\beta}n^{1-\beta}\log^6 n$$
$$\ll (t-s)^{\varepsilon+\frac{1}{2}}n^{-1}\log^2 n + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}$$

by (4.1), (5.2) and (4.10). □

In conclusion we note that if we replace the definition of u_j by $u_j = \sum_{\nu\in H_j} e^{2\pi i n_\nu \omega}$ or its real part $u_j = \sum_{\nu\in H_j}\cos(2\pi n_\nu\omega)$ (i.e., from the Fourier series $\sum_{h\ne 0} c_h e^{2\pi i h n_\nu\omega}$ of $\mathbf{1}_{(s,t]}(n_\nu\omega) - (t-s)$ we keep only the term corresponding to $h = 1$), then the estimate (5.10) remains valid under the assumption of condition (C) for $h_i = \pm 1$ only. This observation will be needed for the proof of Theorem 2 in Section 8.

6. The exponential bound

Let Δ be the constant implied by $\ll$ in Lemma 3.2. Then

$$\sum_{j\le n} Ew_j^2 \le \Delta(t-s)^\varepsilon \sum_{j\le n} N_j \le \Delta(t-s)^\delta h_n\ .$$

Hence by (4.10), (5.1) and Lemmas 5.1, 5.2 and 5.3 we obtain, noting that $\varepsilon < 1$ by (3.4) and $\tau \ge 90$ by (4.1),

$$P\left(\sum_{j\le n} E\left(Y_j^2|\mathcal{F}_{j-1}\right) \ge (\Delta+1)(t-s)^\delta h_n\right) \ll (t-s)^{\varepsilon+\frac{1}{2}}n^{-1}\log^2 n + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}\ . \tag{6.1}$$

Recall that δ and ϱ were arbitrary, but subject to (5.2) and (4.8). From now on we choose, in addition, that

$$\delta = \frac{3}{2}\varrho. \tag{6.2}$$

Lemma 6.1. *We have for any constant $A \ge \Delta + 1$*

$$P\left(\max_{k\le n}\left|\sum_{j\le k} w_j\right| \ge 2A\cdot(t-s)^{\varrho/2}(h_n\log\log h_n)^{\frac{1}{2}}\right)$$
$$\ll \exp\left(-\frac{1}{4}A(t-s)^{-\varrho/2}\log\log h_n\right) + (t-s)^{\varepsilon+\frac{1}{2}}n^{-\frac{3}{4}} + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}.$$

Proof. We apply the exponential bound of Lemma *A*2 to the martingale difference sequences $\{Y_j,\, \mathcal{F}_j,\, j \le n\}$ and $\{-Y_j,\, \mathcal{F}_j,\, j \le n\}$ and add the probability bounds of Lemmas 4.1 and 4.2 at the end. We set

$$\begin{aligned} U_k &= \sum_{j \le k} Y_j && \text{for } k \le n, \\ &= U_n && \text{for } k > n, \\ s_k^2 &= \sum_{j \le k} E(Y_j^2 | \mathcal{F}_{j-1}) && \text{for } k \le n, \\ &= s_n^2 && \text{for } k > n, \end{aligned}$$

and

$$c = 2h_n^{\frac{1}{2}-\varrho}, \quad \lambda = \left(\frac{\log\log h_n}{h_n}\right)^{1/2} (t-s)^{-\varrho}, \quad K = A \cdot (t-s)^{\frac{3}{2}\varrho} h_n.$$

Then $Y_j \le c$, $j \le n$, and by (4.10) we have $\lambda c < 1$ for $n \ge n_0$. Then $\{U_k,\, k \ge 1\}$ is a martingale and by Lemma A2 we have for $n \ge n_0$, defining T_n as in Lemma A2,

$$\begin{aligned} &P\left(\max_{k \le n} \sum_{j \le k} Y_j > A \cdot (t-s)^{\varrho/2} (h_n \log\log h_n)^{1/2}\right) \\ &= P\left(\sup_{m \ge 0} U_m > \lambda K\right) \\ &= P\left(\sup_{m \ge 0} \exp(\lambda U_m) > \exp(\lambda^2 K)\right) \\ &\le P\left(\sup_{m \ge 0} T_m > \exp\left(\lambda^2 K - \frac{1}{2}\lambda^2 \left(1 + \frac{1}{2}\lambda c\right) s_n^2\right)\right) \\ &\le P\left(\sup_{m \ge 0} T_m > \exp\left(\lambda^2 K - \frac{3}{4}\lambda^2 s_n^2\right),\ s_n^2 \le (\Delta+1)(t-s)^{\delta} h_n\right) \\ &\qquad\qquad + P(s_n^2 > (\Delta+1)(t-s)^{\delta} h_n) := I + II, \text{ say.} \end{aligned}$$

The first term does not exceed

$$I \le P\left(\sup_{m \ge 0} T_m > \exp(\lambda^2 K - \frac{3}{4}\lambda^2 (\Delta+1)(t-s)^{\delta} h_n\right).$$

Now

$$\begin{aligned} \lambda^2 \left(K - \frac{3}{4}(\Delta+1)(t-s)^{\delta} h_n\right) &\ge \frac{\log\log h_n}{h_n} (t-s)^{-2\varrho} A h_n \frac{1}{4} \cdot (t-s)^{3\varrho/2} \\ &= \frac{1}{4} A (t-s)^{-\rho/2} \log\log h_n. \end{aligned}$$

Thus by Lemma A2

$$I \ll \exp\left(-\frac{1}{4}A(t-s)^{-\varrho/2}\log\log h_n\right).$$

By (6.1)

$$II \ll (t-s)^{\varepsilon+\frac{1}{2}}n^{-1}\log^2 n + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}\ .$$

Since $\{-Y_j,\, j \le n\}$ is also a martingale difference sequence, we obtain a similar bound and as a consequence

$$P\left(\max_{k\le n}\left|\sum_{j\le k}Y_j\right| \ge A(t-s)^{\varrho/2}(h_n\log\log h_n)^{1/2}\right)$$
$$\ll \exp\left(-\frac{1}{4}A(t-s)^{-\varrho/2}\log\log h_n\right) + (t-s)^{\varepsilon+\frac{1}{2}}n^{-1}\log^2 n + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}.$$

We add the probability bounds of Lemmas 4.1 and 4.2 and obtain the result in view of (3.4). □

7. Conclusion of the proof of Theorem 3

Let us recall that all the lemmas in Sections 4–6 were proved under the standing hypothesis (4.10). For the rest of the paper we lift this hypothesis and subsequent results and arguments will involve all values $0 \le s < t \le 1$.

Let

$$A = \max(80,\, \Delta + 1). \tag{7.1}$$

LEMMA 7.1. *With probability 1 there exists an $n_0 = n_0(\omega)$ such that for all $n \ge n_0$, and all s, t with $0 \le s < t \le 1$*

$$\max_{k\le n}\left|\sum_{j\le k}w_j\right| \le 2^{\tau}A(t-s)^{\varrho/2}(h_n\log\log h_n)^{1/2} + \frac{1}{2}h_n^{1/2}\ .$$

Proof. We follow the proof of [21, Lemma 3.10]. We set

$$Z(s,t) = Z(k;\, s,t) = \left|\sum_{j\le k}w_j(s,t)\right|$$

and

$$\phi(x) = 2A(x\log\log x)^{1/2},$$

$$m = m(n) = \left[\frac{1}{16}A\log\log h_n\right], \quad M = M(n) = \left[\frac{1}{2\log 2}\log h_n\right] + 4.$$

Then, as in [21, (3.37)], we have the chaining relation

$$
\begin{aligned}
(7.2)\quad Z(s,t) \le\ & Z(a2^{-m}, b2^{-m}) + \sum_{i=m+1}^{M} Z(a_i 2^{-i}, (a_i+1)2^{-i}) \\
& + \sum_{i=m+1}^{M} Z(b_i 2^{-i}, (b_i+1)2^{-i}) + Z(a_{M+1}2^{-M}, (a_{M+1}+1)2^{-M}) \\
& + Z(b_{M+1}2^{-M}, (b_{M+1}+1)2^{-M}) + 2h_n 2^{-M},
\end{aligned}
$$

where a, b, a_i, b_i $(m < i \le M+1)$ are integers with $0 \le a, b \le 2^m$, $0 \le a_i$, $b_i < 2^i$ $(m < i \le M+1)$. Define

$$
\begin{aligned}
E_n(a,b) &= \left\{ \max_{k\le n} Z(k;\ a2^{-m}, b2^{-m}) \ge ((b-a)2^{-m})^{\varrho/2}\phi(h_n) \right\}, \\
E_n &= \bigcup_{0\le a,b\le 2^m} E_n(a,b), \\
F_n(i,a) &= \left\{ \max_{k\le n} Z(k;\ a2^{-i}, (a+1)2^{-i}) \ge 2^{-\varrho i/2}\phi(h_n) \right\}, \\
F_n &= \bigcup_{m<i\le M} \ \bigcup_{0\le a<2^i} E_n(a,b).
\end{aligned}
$$

Then by Lemma 6.1

$$
P(E_n(a,b)) \ll \exp\left(-\frac{1}{4}A\log\log h_n\right) + n^{-3/4}
$$

and so

$$
\begin{aligned}
(7.3)\qquad P(E_n) &\ll 2^{2m}\exp\left(-\frac{1}{4}A\log\log h_n\right) + 2^{2m}n^{-3/4} \\
&\ll \exp\left(-\frac{1}{8}A\log\log h_n\right) + n^{-1/2} \\
&\ll (\log n)^{-A/8} \ll (\log n)^{-10}\ .
\end{aligned}
$$

Similarly,

$$
P(F_n(i,a)) \ll \exp\left(-\frac{1}{4}A\cdot 2^{i\varrho/2}\log\log h_n\right) + 2^{-i(\varepsilon+\frac{1}{2})}n^{-3/4} + h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)}
$$

and so

$$
\begin{aligned}
(7.4) \qquad P(F_n) &\ll \sum_{m<i\leq M} \exp\left(-\frac{1}{4}A\cdot 2^{i\varrho/2}\log\log h_n + i\right) \\
&\qquad + n^{-3/4}\sum_{m<i\leq M} 2^{-i(\varepsilon-\frac{1}{2})} + 2^M h_n^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\beta)} \\
&\ll \exp\left(-\frac{1}{8}A\cdot 2^{m\varrho/2}\log\log h_n\right) \\
&\qquad + n^{-3/4}\cdot 2^{-m(\varepsilon-\frac{1}{2})} + h_n^{-\frac{1}{2}(\frac{1}{2}-\beta)} \\
&\ll (\log n)^{-10}.
\end{aligned}
$$

Note that Lemma 6.1, used above for the estimation of $P(E_n(a,b))$ and $P(F_n)$, was established under the standing hypothesis (4.10) which is not assumed in the present section. However, the special intervals $(a2^{-m}, b2^{-m})$ and $(a2^{-i}, (a+1)2^{-i})$ in the definition of $E_n(a,b)$ and $F_n(i,a)$ satisfy, as trivial calculations show, the hypothesis (4.10) and so our estimates are correct. Now by (7.3), (7.4)

$$
\sum_{p=1}^{\infty} P(E_{2^p}\cup F_{2^p}) < \infty\ .
$$

The Borel–Cantelli Lemma implies that only finitely many of the events E_{2^p} or F_{2^p} occur with probability 1. Let n be sufficiently large and define p by $2^{p-1}\leq n < 2^p$. Then by (7.2) we have with probability 1 for all $0\leq s<t\leq 1$ and $n\geq n_0$,

$$
\begin{aligned}
&\max_{k\leq n} Z(k;s,t) \\
&\quad\leq \left(\left((b-a)2^{-m(2^p)}\right)^{\varrho/2} + 2\sum_{m(2^p)\leq i\leq M(2^p)} 2^{-i\varrho/2}\right)\phi(h_{2^p}) + \frac{1}{4}h_n^{1/2} \\
&\quad\leq (t-s)^{\varrho/2}\phi(h_{2^p}) + o\left(h_{2^p}^{1/2}\right) + \frac{1}{4}h_n^{1/2} \\
&\quad\leq 2^\tau 4A\cdot(t-s)^{\varrho/2}(h_n\log\log h_n)^{1/2} + \frac{1}{2}h_n^{1/2}\ . \qquad \square
\end{aligned}
$$

Lemma 7.2. *With probability 1*

$$
\max_{q\leq h_n-h_{n-1}}\ \sup_{0\leq s<t\leq 1}\left|\sum_{\nu=h_{n-1}+1}^{h_{n-1}+q} x_\nu\right| \ll h_n^{(1/2)-1/(8\tau)}\ .
$$

Proof. Note that $h_n \ll n^\tau$, $h_n - h_{n-1} \geq n^{\tau-1}$, and thus by (4.1) we have $h_{n-1} \ll (h_n - h_{n-1})^{1+\gamma}$ for $n \geq n_0$. Hence applying Lemma 3.4 we obtain

$$P\left(\max_{q \leq h_n - h_{n-1}} \sup_{0 \leq s < t \leq 1} \left| \sum_{\nu = h_{n-1}+1}^{h_{n-1}+q} x_\nu \right| \geq h_n^{(1/2)-1/(8\tau)}\right)$$
$$\ll h_n^{-2+1/(2\tau)} (h_n - h_{n-1})^2 \log^8 n \ll n^{-3/2} \log^8 n.$$

The lemma follows now from the convergence part of the Borel–Cantelli lemma. □

We now can finish the proof of Theorem 3 by using Lemma 7.2 to break into the blocks. Let N be given and choose n so that $h_{n-1} \leq N < h_n$. By Lemma 7.1 and the analogous statement for the blocks I_j, there exists with probability 1 an index $N_0 = N_0(\omega)$ such that for all $N \geq N_0$ and all s, t with $0 \leq s < t \leq 1$

$$\begin{aligned}
\max_{j \leq N} j \,|F_j(t) - F_j(s) - (t-s)| &= \max_{j \leq N} \left| \sum_{\nu \leq j} x_\nu(s,t) \right| \\
&\leq \max_{k \leq n-1} \left| \sum_{\nu \leq h_k} x_\nu(s,t) \right| + \max_{q \leq h_n - h_{n-1}} \sup_{0 \leq s < t \leq 1} \left| \sum_{\nu = h_{n-1}+1}^{h_{n-1}+q} x_\nu \right| \\
&\leq 2^{\tau+1} A (t-s)^{\varrho/2} (h_n \log\log h_n)^{1/2} + h_n^{1/2} + h_n^{(1/2)-1/(8\tau)} \\
&\leq 2^{\tau+2} A (t-s)^{\varrho/2} (N \log\log N)^{1/2} + 2N^{1/2}\ .
\end{aligned}$$

This concludes the proof of Theorem 3. □

8. Proof of Theorem 2

We follow [21, Section 4], using basically the same notation. Define the blocks H_j, I_j as in Section 4 above, with the only difference that instead of (4.2) we now choose

$$\text{card } H_j := N_j = \tau j^{\tau-1}, \qquad \text{card } I_j = \tau j^{\tau-10}, \tag{8.1}$$

where

$$\tau > \max(10/\eta,\ 120). \tag{8.2}$$

Thus H_j are "long" blocks and I_j are "short" blocks. As we will see, the short blocks provide enough separation between the long blocks so that the martingale property of the long blocks used in the proof of Theorem 3 remains valid. Let h_j denote again the largest member of H_j and define h_j, r_k, $\mathcal{H}_k$, $\mathcal{F}_j$, w_j as in Section 4 above. As in [21, Section 4], we extend all r.v.'s and σ-fields to the product space $[0,1)^2$. Specifically, let $\mathcal{F}_j'$ be the set of all rectangles $A \times [0,1)$, where $A \in \mathcal{F}_j$, and define

$$\xi_\nu = \xi_\nu(\omega_1, \omega_2) = \sqrt{2}\, E(\cos 2\pi n_\nu \cdot |\mathcal{F}_j'), \qquad \nu \in H_j.$$

Then, as in [21, (4.1)], we have

$$|\xi_\nu(\omega_1,\omega_2) - \sqrt{2}\cos 2\pi n_\nu \omega_1| \ll h_j^{-12}, \qquad \nu \in H_j. \tag{8.3}$$

Set

$$y_j = \sum_{\nu \in H_j} \xi_\nu, \quad X_j = y_j - E(y_j|\mathcal{F}_{j-1}'). \tag{8.4}$$

Instead of [21, Lemma 4.1] we now have:

Lemma 8.1. *For almost all* $(\omega_1,\omega_2) \in [0,1)^2$

$$\sum_{\nu \le h_n} \sqrt{2}\, \cos 2\pi n_\nu \omega_1 - \sum_{j \le n} X_j \ll h_n^{\frac{1}{2} - \frac{1}{\tau}}.$$

Proof. We follow the proof of [21, Lemma 4.1], replacing the estimate [21, (4.5)] by (8.5) below. To prepare for it, we note that h_j satisfies now

$$h_j = \sum_{l \le j} \tau l^{\tau - 1} + O\left(j^{\tau - 9}\right) = j^\tau (1 + o(1))$$

and so

$$\begin{aligned} h_{j-1} + \tau(j-1)^{\tau-10} &\ge h_{j-1}\left(1 + \frac{\tau}{2}(j-1)^{-10}\right) \ge h_{j-1}\left(1 + \frac{\tau}{4} h_{j-1}^{-10/\tau}\right) \\ &\ge h_{j-1}\left(1 + \frac{\tau}{4} h_{j-1}^{-\eta}\right). \end{aligned}$$

Thus we get, using the second statement of Lemma 3.1 with $A = \tau/8$,

$$n\left(h_{j-1} + \tau(j-1)^{\tau-10}\right) / n\left(h_{j-1}\right) \ge h_{j-1}^{\tau/8}.$$

Hence, by part of [21, (4.5)] and (4.4), (8.2), we obtain for $\nu \in H_j$

$$\begin{aligned} |E(\xi_\nu|\, \mathcal{F}_{j-1}')| &\le 4 \cdot 2^{r_{h_{j-1}}} n_\nu^{-1} \\ &\ll n(h_{j-1}) h_{j-1}^{12} n\left(h_{j-1} + \tau(j-1)^{\tau-10}\right)^{-1} \\ &\ll h_{j-1}^{-\tau/8+12} \le h_{j-1}^{-2}. \end{aligned} \tag{8.5}$$

Thus

$$\sum_{j \ge 1} |X_j - y_j| \ll \sum_{j \ge 1} h_j \cdot h_{j-1}^{-2} < \infty. \tag{8.6}$$

Next, by Corollary 3.1, (8.1) and (8.2) we have

$$E\left|\sum_{\nu \in I_j} \cos 2\pi n_\nu \cdot\right|^4 \ll (\text{card } I_j)^2 + h_{j+1}^{1+\beta} \ll j^{2\tau - 20}.$$

Thus

$$\begin{aligned} P&\left(\sum_{j\le n}\left|\sum_{\nu\in I_j}\cos 2\pi n_\nu\cdot\right| \ge h_n^{\frac12-\frac1\tau}\right)\\ &\le \sum_{j\le n} P\left(\left|\sum_{\nu\in I_j}\cos 2\pi n_\nu\cdot\right| \ge \frac1n h_n^{\frac12-\frac1\tau}\right)\\ &\ll n^4 h_n^{-2+4/\tau}\sum_{j\le n} j^{2\tau-20} \ll n^4 n^{-2\tau+4} n^{2\tau-19} \ll n^{-11} \ . \end{aligned}$$

The conclusion of the lemma follows now from the last estimate, the convergence part of the Borel–Cantelli lemma and by (8.3), (8.6). □

Set

$$V_n := \sum_{j\le n} E(X_j^2|\mathcal{F}'_{j-1}). \tag{8.7}$$

Lemma 8.2. *We have*

$$P\left(\max_{k\le n}|V_k-h_k| \ge h_n n^{-1/8}\right) \ll n^{-3/4}\log^2 n.$$

Proof. Set

$$u_j = \sum_{\nu\in H_j}\sqrt{2}\cos 2\pi n_\nu\cdot\ . \tag{8.8}$$

By (8.1) and (8.7) we have

$$\begin{aligned} \max_{k\le n}|V_k-h_k| &\le \sum_{j\le n}\left|E(X_j^2|\mathcal{F}'_{j-1}) - E(y_j^2|\mathcal{F}'_{j-1})\right| \\ &\quad + \max_{k\le n}\left|\sum_{j\le k}\left(y_j^2 - E(y_j^2|\mathcal{F}'_{j-1})\right)\right| \\ &\quad + \sum_{j\le n}|y_j^2-u_j^2| + \max_{k\le n}\left|\sum_{j\le k}(u_j^2-N_j)\right| \\ &\quad + O(n^{\tau-9}). \end{aligned} \tag{8.9}$$

We estimate the terms separately. Using (8.4) and (8.5) we have with probability 1

$$\left|E(X_j^2|\mathcal{F}'_{j-1}) - E(y_j^2|\mathcal{F}'_{j-1})\right| = (E(y_j|\mathcal{F}'_{j-1}))^2 \ll (h_{j-1}^{-2}h_j)^2 \ll j^{-2}\ . \tag{8.10}$$

Thus the first sum on the right hand side of (8.9) is bounded with probability 1. Similarly, we obtain by (8.3), (8.4) and (8.8)

$$|u_j^2-y_j^2| \le 2\sqrt{2}N_j|y_j-u_j| \ll N_j^2 h_j^{-12} \ll j^{-2}.$$

Thus the third sum on the right hand side of (8.9) is also bounded. By Doob's maximal inequality [10, p. 314] for martingales we obtain for the second term on the right hand side of (8.9)

$$\tag{8.11} \begin{aligned} P\left(\max_{k\le n}\left|\sum_{j\le k}(y_j^2 - E(y_j^2|\mathcal{F}'_{j-1}))\right| \ge \frac{1}{4}h_n n^{-1/8}\right) \\ \ll h_n^{-2}n^{1/4}\sum_{j\le n}E|y_j|^4 \ll h_n^{-2}n^{1/4}\sum_{j\le n}E|u_j|^4 \\ \ll h_n^{-2}n^{1/4}\sum_{j\le n}(N_j^2 + h_j^{1+\beta}) \ll n^{-3/4} \end{aligned}$$

by (8.2), (8.4), (8.8), the conditional Jensen inequality and Corollary 3.1. To estimate the fourth sum on the right hand side of (8.9) we observed at the end of Section 5 that the estimate (5.10) remains valid (assuming condition (C) with $h_i = \pm 1$), with the present definition of u_p in (8.8). Thus using Lemma A1 with $\gamma = 2$ we get

$$\begin{aligned} &E\max_{k\le n}\left|\sum_{p\le k}(u_p^2 - Eu_p^2)\right|^2 \\ &\ll \left[\sum_{p\le n}p^{2\tau-2} + \sum_{q\le n}q^{(\tau-1)(\beta+1)+1}\log^4 q\right]\log^2 n \\ &\ll n^{2\tau-1}\log^2 n \ll h_n^2 n^{-1}\log^2 n. \end{aligned}$$

Since $Eu_p^2 = N_p$ we obtain by Chebyshev's inequality

$$P\left(\max_{k\le n}\left|\sum_{j\le k}(u_j^2 - N_j)\right| \ge \frac{1}{4}h_n n^{-1/8}\right) \ll n^{-3/4}\log^2 n.$$

The result follows now from (8.9) and the above estimates. □

LEMMA 8.3. *With probability 1*

$$V_k = h_k + O(h_k k^{-1/8}).$$

Proof. Lemma 8.2 and the Borel-Cantelli lemma imply

$$\max_{n^2<k\le(n+1)^2}|V_k - h_k| \ll n^{-1/4}h_{(n+1)^2} \ll n^{-1/4}h_{n^2} \quad \text{a.s.} \qquad \square$$

We now apply Lemma A3 to the martingale difference sequence $\{X_n, \mathcal{F}_n, n\ge 1\}$ with $f(x) = x^{1-\varrho}$ where, as before, $\varrho \le 1/(40\tau)$ (cf. (4.8).) We have to check the almost sure convergence of

$$\sum_{k\geq 1} V_k^{-1+\varrho} E\left(X_k^2 \mathbf{1}\left\{X_k^2 > V_k^{1-\varrho}\right\} | \mathcal{F}'_{k-1}\right). \tag{8.12}$$

In view of Lemma 8.3 and the inequality

$$E\left(X_k^2 \mathbf{1}\left\{X_k^2 > V_k^{1-\varrho}\right\} | \mathcal{F}'_{k-1}\right) \leq V_k^{-1+\varrho} E\left(X_k^4 | \mathcal{F}'_{k-1}\right)$$

it suffices to show the almost sure convergence of

$$\sum_{k\geq 1} h_k^{-2+2\varrho} E(X_k^4 | \mathcal{F}'_{k-1}).$$

Since by Corollary 3.1 and (8.2) we have

$$E\left(E\left(X_k^4 | \mathcal{F}'_{k-1}\right)\right) = E|X_k|^4 \ll E|y_k|^4 \leq E|u_k|^4 \ll N_k^2 + h_k^{1+\beta} \ll N_k^2$$

(cf. (8.11)) and since by $\varrho \leq 1/(40\tau)$ we have

$$\sum_{k\geq 1} h_k^{-2+2\varrho} N_k^2 \ll \sum_{k\geq 1} k^{-2\tau+2\varrho\tau} k^{2\tau-2} \ll \sum_{k\geq 1} k^{-7/4} < \infty,$$

the Beppo Levi theorem implies that the series in (8.12) converges with probability one.

Observe now that

$$U(\omega) = U(\omega_1, \omega_2) = \omega_2, \qquad \omega \in [0,1)^2,$$

is a random variable having uniform distribution over the unit square and independent of the sequence $\{X_k\}$, which depends only on the variable ω_1. Thus by Lemma A3 there exists a sequence $\{Y_n,\, n \geq 1\}$ of independent standard normal random variables defined on $[0,1)^2$ such that with probability 1

$$\sum_{k\geq 1} X_k \mathbf{1}\{V_k \leq t\} - \sum_{m\leq t} Y_m \ll (tf(t))^{1/4} \log t. \tag{8.13}$$

Let now $n \geq 1$ and

$$V_n \leq t < V_{n+1}. \tag{8.14}$$

By (8.13) we have almost surely

$$\sum_{k\leq n} X_k - \sum_{m\leq t} Y_m \ll t^{\frac{1}{2}-\varrho/5}. \tag{8.15}$$

By Lemma 8.3, relation (4.8), Markov's inequality and the Borel–Cantelli lemma there exists with probability 1 an $n_0(\omega)$ such that

$$|V_n - h_n| \leq \frac{1}{2} h_n n^{-1/9} \leq \frac{1}{2} h_n^{1-\varrho} \quad \text{for } n \geq n_0.$$

Thus we have for all t and n subject to (8.14) that

$$\begin{aligned} h_n - \frac{1}{2}h_n^{1-\varrho} \le V_n \le t < V_{n+1} \le h_{n+1} + \frac{1}{2}h_{n+1}^{1-\varrho} \\ \le h_n + O(n^{\tau-1}) + O(h_n^{1-\varrho}) \le h_n + h_n^{1-\varrho/40} \end{aligned}$$

for $n \ge n_0(\omega)$. Thus

$$p_n := h_n - h_n^{1-\varrho/40} \le t \le q_n := h_n + h_n^{1-\varrho/40}$$

for $n \ge n_0(\omega)$. For $a < b$ let

$$R(a,b) = \max_{a \le h < j \le b} \left| \sum_{m=h+1}^{j} Y_m \right|.$$

Then by (8.15) we have for all $n \ge n_0$

$$\left| \sum_{k \le n} X_k - \sum_{m \le h_n} Y_m \right| \le R(p_n, q_n) + O\left(h_n^{\frac{1}{2}-\varrho/5}\right) . \tag{8.16}$$

But for sufficiently large n we have by Lévy's maximal inequality

$$\begin{aligned} P\left(R(p_n,q_n) \ge h_n^{\frac{1}{2}-\varrho/160}\right) &\le 2P\left(N(0,1) \ge h_n^{\frac{1}{2}-\varrho/160}(q_n - p_n)^{-\frac{1}{2}}\right) \\ &\le 2P\left(N(0,1) \ge \frac{1}{2}n^{\tau\varrho/160}\right) \ll \exp\left(-cn^{\tau\varrho/80}\right) \end{aligned}$$

for some $c > 0$. Hence we obtain from the Borel–Cantelli Lemma and (8.16) that with probability 1 for all $n \ge n_0$

$$\left| \sum_{k \le n} X_k - \sum_{m \le h_n} Y_m \right| \ll h_n^{\frac{1}{2}-\varrho/160} . \tag{8.17}$$

By Corollary 3.1, Lemma A1 and (4.8), (8.2) we have

$$\begin{aligned} &P\left(\max_{q \le h_{n+1}-h_n} \left| \sum_{\nu=h_n+1}^{h_n+q} \cos 2\pi n_\nu \omega_1 \right| \ge h_n^{\frac{1}{2}-\varrho} \right) \\ &\quad \ll h_n^{-2+4\varrho}\left[(h_{n+1}-h_n)^2 + h_{n+1}^{1+\beta}\right] \log^4 n \ll n^{2\tau-2-\tau(2-4\varrho)} \ll n^{-3/2}, \end{aligned}$$

and thus by the Borel-Cantelli lemma

$$\max_{q \le h_{n+1}-h_n} \left| \sum_{\nu=h_n+1}^{h_n+q} \cos 2\pi n_\nu \omega_1 \right| \ll h_n^{\frac{1}{2}-\varrho} . \tag{8.18}$$

An analogous inequality holds for the $Y_m's$. The result follows now from Lemma 8.1, (8.17) and (8.18).

Appendix

LEMMA A.1 ([19, Corollary 3.1]). *Let $\gamma \ge 1$ be a given real. Suppose that there exists a nonnegative function $g(i,j)$, $1 \le i \le j \le n$, satisfying*

$$g(i,j) \le g(i,j+1) \quad \text{for } 1 \le i \le j \le n$$

and

$$g(i,j) + g(j+1,k) \le g(i,k) \quad \text{for } 1 \le i \le j < k \le n.$$

Let $X_1, X_2, \dots, X_n$ be a sequence of random variables with finite γth moments and assume that

$$E\left|\sum_{k=i}^{j} X_k\right|^{\gamma} \le g(i,j) \quad \text{for } 1 \le i \le j \le n.$$

Then

$$E\max_{k\le n}\left|\sum_{p\le k} X_p\right|^{\gamma} \le g(1,n)(1+\log n)^{\gamma}.$$

LEMMA A.2 ([23, p. 299]). *Let $(U_n,\, \mathcal{F}_n,\, n \ge 1)$ be a supermartingale with $EU_1 = 0$. Put*

$$U_0 = 0 \quad \text{and} \quad Y_j = U_j - U_{j-1},\ j \ge 1.$$

Suppose that

$$Y_j \le c \qquad \text{a.s.}$$

for some constant $c > 0$ and for all $j \ge 1$. For $\lambda > 0$ define

$$T_n = \exp\left(\lambda U_n - \frac{1}{2}\lambda^2\left(1+\frac{1}{2}\lambda c\right)\sum_{j\le n} E\left(Y_j^2|\mathcal{F}_{j-1}\right)\right),\ n \ge 1,$$

and $T_0 = 1$ a.s.. Then for each λ with $\lambda c \le 1$ the sequence $(T_n,\, \mathcal{F}_n \ge 0)$ is a non-negative supermartingale satisfying

$$P\left(\sup_{n\ge 0} T_n > \alpha\right) \le 1/\alpha$$

for each $\alpha > 0$.

LEMMA A.3 ([24, p. 334]). *Let $\{X_n,\, \mathcal{F}_n,\, n \ge 1\}$ be a real-valued square integrable martingale difference sequence defined on some probability space $(\Omega, \mathcal{F}, P)$. Let f be a positive non-decreasing function on $\mathbb{R}^+$ such that $f(x)/x$ is non-increasing. Suppose that*

$$V_n := \sum_{j\le n} E(X_j^2|\mathcal{F}_{j-1}) \longrightarrow \infty \qquad \text{a.s.}$$

and that

$$\sum_{k\geq 1} f(V_k)^{-1} E(X_k^2 \mathbf{1}\{X_k^2 > f(V_k)\}|\mathcal{F}_{k-1}) < \infty \qquad a.s.$$

Suppose that there exists a random variable U, uniformly distributed over $(0,1)$ and independent of the sequence $\{X_n\}$. Then there exists a sequence $\{Y_n,\, n \geq 1\}$ of independent standard normal random variables defined on $(\Omega, \mathcal{F}, P)$ such that with probability 1

$$\sum_{n\geq 1} X_n \mathbf{1}\{V_n \leq t\} - \sum_{n\leq t} Y_n \ll (tf(t))^{1/4} \log t.$$

Note added in proof: With great sadness, we inform the reader that Walter Philipp passed away on July 19, 2006, at the age of 69, near Graz, Austria. — I. Berkes and R.F. Tichy.

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