On permutations of Hardy-Littlewood-Pólya sequences

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Dedicated to the memory of Walter Philipp

Abstract: Let $\mathcal{H} = (q_1, \dots q_r)$ be a finite set of coprime integers and let $n_1, n_2, \dots$ denote the multiplicative semigroup generated by $\mathcal{H}$ and arranged in increasing order. The distribution of such sequences has been studied intensively in number theory and they have remarkable probabilistic and ergodic properties. For example, the asymptotic properties of the sequence $\{n_k x\}$ are very similar to those of independent, identically distributed random variables; here $\{\cdot\}$ denotes fractional part. However, the behavior of this sequence depends sensitively on the generating elements of (n_k) and the combination of probabilistic and number-theoretic effects results in a unique, highly interesting asymptotic behavior, see e.g. [6], [8]. In particular, the properties of $\{n_k x\}$ are not permutation invariant, in contrast to i.i.d. behavior. The purpose of this paper is to show that $\{n_k x\}$ satisfies a strong independence property ("interlaced mixing"), enabling one to determine the precise asymptotic behavior of permuted sums $S_N(\sigma) = \sum_{k=1}^N f(n_{\sigma(k)} x)$. As we will see, the behavior of $S_N(\sigma)$ still follows that of sums of independent random variables, but its growth speed (depending on σ) is given by the classical Gál function of Diophantine approximation theory. Some examples describing the class of possible growth functions are given.

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1 Introduction

Let $q_1, \ldots, q_r$ be a fixed set of coprime integers and let (n_k) be the set of numbers $q_1^{\alpha_1} \cdots q_r^{\alpha_r}$, $\alpha_i \geq 0$ integers, arranged in increasing order. Such sequences are called (sometimes) Hardy-Littlewood-Pólya sequences and their distribution has been investigated extensively in number theory. Thue [23] showed that $n_{k+1} - n_k \to \infty$ and this result was improved gradually until Tijdeman [24] proved that

$$n_{k+1} - n_k \geq \frac{n_k}{(\log n_k)^\alpha}$$

for some $\alpha > 0$, i.e. the growth of (n_k) is almost exponential. Except the value of the constant α, this result is best possible. Hardy-Littlewood-Pólya sequences also have remarkable probabilistic and ergodic properties. In his celebrated paper on the Khinchin conjecture, Marstrand [14] proved that if f is a bounded measurable function with period 1, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(n_k x) = \int_0^1 f(t)dt \quad \text{a.e.}$$

and Nair [15] showed (cf. Baker [2]) that this remains valid if instead of boundedness of f we assume only $f \in L^1(0,1)$. Letting $\{\cdot\}$ denote fractional part, it follows that $\{n_k x\}$ is not only uniformly distributed mod 1 for almost all x in the sense of Weyl [25], but satisfies the "strong uniform distribution" property of Khinchin [12]. Letting

$$D_N = D_N(x_1, \ldots, x_N) := \sup_{0\leq a<b<1} \left| \frac{1}{N} \#\{k \leq N : a \leq x_k < b\} - (b-a) \right|$$

denote the discrepancy of a sequence $(x_k)_{1\leq k\leq N}$ in $(0,1)$, Philipp [18] proved, verifying a conjecture of R.C. Baker, that

$$1/8 \leq \limsup_{N\to\infty} \sqrt{\frac{N}{2\log\log N}} D_N(\{n_k x\}_{1\leq k\leq N}) \leq C \qquad \text{a.e.,} \tag{1.1}$$

with a constant C depending on the generating elements of (n_k), establishing the law of the iterated logarithm for the discrepancies of $\{n_k x\}$. Note that if (ξ_k) is a sequence of independent random variables with uniform distribution over $(0,1)$, then

$$\limsup_{N\to\infty} \sqrt{\frac{N}{2\log\log N}} D_N(\xi_1, \ldots, \xi_N) = \frac{1}{2} \tag{1.2}$$

with probability one by the Chung-Smirnov LIL (see e.g. [22], p. 504). A comparison of (1.1) and (1.2) shows that the sequence $\{n_k x\}$ behaves like a sequence of independent random variables. In the same direction, Fukuyama and Petit [9] showed that under mild assumptions on the periodic function f, $\sum_{k\le N} f(n_k x)$ obeys the central limit theorem, another remarkable probabilistic property of Hardy-Littlewood-Pólya sequences. Surprisingly, however, the limsup in (1.1) is different from the constant 1/2 in (1.2) and, as Fukuyama [6] and Fukuyama and Nakata [8] showed, it depends sensitively on the generating elements $q_1, \dots, q_r$. For example, for $n_k = a^k$, $a \ge 2$ the limsup Σ_a in (1.1) equals

$$\begin{aligned}
\Sigma_a &= \sqrt{42}/9 && \text{if } a = 2 \\
\Sigma_a &= \frac{\sqrt{(a+1)a(a-2)}}{2\sqrt{(a-1)^3}} && \text{if } a \ge 4 \text{ is an even integer,} \\
\Sigma_a &= \frac{\sqrt{a+1}}{2\sqrt{a-1}} && \text{if } a \ge 3 \text{ is an odd integer,}
\end{aligned}$$

and if all the generating elements q_i of (n_k) are odd, then the limsup in (1.1) equals

$$\frac{1}{2}\left(\prod_{i=1}^{r} \frac{q_i+1}{q_i-1}\right)^{1/2}.$$

Even more surprisingly, Fukuyama [7] showed that the limsup Σ in (1.1) is not permutation-invariant: changing the order of the (n_k) generally changes the value of Σ. This is quite unexpected, since $\{n_k x\}$ are identically distributed in the sense of probability theory and the asymptotic properties of i.i.d. random variables are permutation invariant. The purpose of this paper is to give a detailed study of the structure of $\{n_k x\}$ in order to explain the role of arithmetic effects and the above surprising deviations from i.i.d. behavior. Specifically, we will establish an "interlaced" mixing condition for normed sums of $\{n_k x\}$, expressed by Lemmas 4 and 6, implying that the sequence $\{n_k x\}$ has mixing properties after any permutation of its terms. This property is considerably stronger than usual mixing properties of lacunary sequences, which are always directed, i.e. are valid only in the "natural" order of elements. In particular, we will see that for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ of the positive integers, $\sum_{k\le N} f(n_{\sigma(k)} x)$ still behaves like sums of independent random variables and the observed pathological properties of these sums are due to the unusual behavior of their L^2 norms which, as we will see, is a purely number theoretic effect. For example, in the case $f(x) = \{x\}$ the growth speed of the above sums is determined by $G(n_{\sigma(1)}, \dots, n_{\sigma(N)})$, where

$$G(m_1, \dots, m_N) = \sum_{1\le i\le j\le N} \frac{(m_i, m_j)}{[m_i, m_j]} \tag{1.3}$$

is the Gál function in Diophantine approximation theory; here (a, b) and $[a, b]$ denote the greatest common divisor, resp. least common multiple of a and b. While this function is completely explicit, the computation of its precise asymptotics for a specific permutation σ is a challenging problem and we will illustrate the situation only by a few examples.

As noted, the basic structural information on $\{n_k x\}$ is given by Lemmas 4 and 6, which are rather technical. The following result, which is a simple consequence of them, describes the situation more explicitly.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying the condition*

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\, dx = 0, \quad \mathrm{Var}_{[0,1]} f < +\infty \tag{1.4}$$

and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$. Assume that

$$A_{N,M}^2 := \int_0^1 \left(\sum_{k=M+1}^{M+N} f(n_{\sigma(k)} x) \right)^2 dx \geq CN, \qquad N \geq N_0,\ M \geq 1 \tag{1.5}$$

for some constant $C > 0$. Then letting $A_N = A_{N,0}$ we have

$$A_N^{-1} \sum_{k=1}^{N} f(n_{\sigma(k)} x) \to_d N(0,1) \tag{1.6}$$

and

$$\limsup_{N \to \infty} \frac{1}{(2A_N^2 \log\log A_N^2)^{1/2}} \sum_{k=1}^{N} f(n_{\sigma(k)} x) = 1 \quad \text{a.e.} \tag{1.7}$$

As the example $f(x) = \cos 2\pi x - \cos 4\pi x$, $n_k = 2^k$ shows, assumption (1.5) cannot be omitted in Theorem 1. It is satisfied, e.g., if all Fourier coefficients of f are nonnegative.

Theorem 1 shows that the growth speed of $\sum_{k=1}^N f(n_{\sigma(k)} x)$ is determined by the quantity

$$A_N^2 = A_n^2(\sigma) = \int_0^2 \left(\sum_{k=1}^{N} f(n_{\sigma(k)} x) \right)^2 dx.$$

In the harmonic case $f(x) = \sin 2\pi x$ we have $A_N(\sigma) = \sqrt{N/2}$ for any σ and thus the partial sum behavior is permutation-invariant. For trigonometric polynomials f containing at least two terms the situation is different: for example, in the case $f(x) = \cos 2\pi x + \cos 4\pi x$ the limits $\lim_{N \to \infty} A_N(\sigma)/\sqrt{N}$ for all permutations σ fill an interval. In the case $f(x) = \{x\} - 1/2$ we have, by a well known identity of Landau (see [13], p. 170)

$$\int_0^1 f(ax) f(bx) dx = \frac{1}{12} (a,b)/[a,b]$$

Hence in this case

$$A_N^2 = \frac{1}{12} G(n_{\sigma(1)}, \ldots, n_{\sigma(N)})$$

where G is the Gál function defined by (1.3). The function G plays an important role in the metric theory of Diophantine approximation and it is generally very difficult to estimate; see the profound paper of Gál [10] for more information on this point. Clearly, $G(n_{\sigma(1)}, \ldots, n_{\sigma(N)}) \geq N$ and from the proof of Lemma 2.2 of Philipp [18] it is easily seen that

$$G(n_{\sigma(1)}, \ldots, n_{\sigma(N)}) \ll N.$$

Here, and in the sequel, $\ll$ means the same as the O notation. In the case of the identity permutation σ the value of $\lim_{N\to\infty} N^{-1}G(n_1, \ldots, n_N)$ was computed by Fukuyama and Nakata [8], but to determine the precise asumptotics of $G(n_{\sigma(1)}, \ldots, n_{\sigma(N)})$ for general σ seems to be a very difficult problem. Again, in Section 3 we will see that in the case of $n_k = 2^k$ the class of limits $\lim_{N\to\infty} N^{-1}G(n_{\sigma(1)}, \ldots, n_{\sigma(N)})$ for all σ fills an interval.

Corollary. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying (1.4) and assume that the Fourier coefficients of f are nonnegative. Let σ be a permutation of $\mathbb{N}$. Then $N^{-1/2}\sum_{k=1}^{N} f(n_{\sigma(k)}x)$ has a nondegenerate limit distribution iff*

$$\gamma^2 = \lim_{N\to\infty} N^{-1} \int_0^1 \left(\sum_{k=1}^{n} f(n_{\sigma(k)}x) \right)^2 dx > 0 \tag{1.8}$$

exists, and then

$$N^{-1/2} \sum_{k=1}^{N} f(n_{\sigma(k)}x) \to_d N(0, \gamma^2). \tag{1.9}$$

Also, if condition (1.8) is satisfied, then

$$\limsup_{N\to\infty} \frac{1}{\sqrt{2N \log\log N}} \sum_{k=1}^{N} f(n_{\sigma(k)}x) = \gamma \quad \text{a.e.} \tag{1.10}$$

As mentioned, for the original, unpermuted sequence (n_k), the value of $\gamma = \gamma_f$ in (1.8) was computed in [8]. Given an f satisfying condition (1.4), let Γ_f denote the set of limiting variances in (1.8) belonging to all permutations σ. (Note that the limit does not always exist.) Despite the simple description of Γ_f above, it seems a difficult problem to determine this set explicitly. In analogy with the theory of permuted function series (see e.g. Nikishin [16]), it is natural to expect that Γ_f is always a (possibly degenerate) interval. In Section 3 we will prove that for $n_k = 2^k$ and functions f with nonnegative Fourier coefficients, Γ_f is identical with the interval determined by $\|f\|^2$ and γ_f^2. For f with negative coefficients this is false, as an example in Section 3 will show.

2 An interlaced mixing condition

The crucial tool in proving Theorem 1 is a recent deep bound for the number of solutions $(k_1, \ldots, k_p)$ of the Diophantine equation

$$a_1 n_{k_1} + \ldots + a_p n_{k_p} = b. \tag{2.1}$$

Call a solution of (2.1) nondegenerate if no subsum of the sum on the left hand side equals 0. Amoroso and Viada [1] proved the following result, improving the quatitative subspace theorem of Schmidt [20] (cf. also Evertse et al. [5]).

Lemma 1. *For any nonzero integers $a_1, \ldots, a_p, b$ the number of nondegenerate solutions of (2.1) is at most $\exp(cp^6)$, where c is a constant depending only on the number of generators of (n_k).*

For the rest of the paper, C will denote positive constants, possibly different at different places, depending (at most) on f and (n_k). Similarly, the constants implied by O and by the equivalent relation $\ll$ will depend (at most) on f and (n_k).

Most results of this paper are probabilistic statements on the sequence $\{f(n_k x),\ k = 1, 2, \ldots\}$ and we will use probabilistic terminology. The underlying probability space for our sequence is the interval $[0, 1]$, equipped with Borel sets and the Lebesgue measure; we will denote probability and expectation in this space by $\mathbb{P}$ and $\mathbb{E}$.

Given any finite set I of positive integers, set

$$S_I = \sum_{k \in I} f(n_k x), \quad \sigma_I = (\mathbb{E} S_I^2)^{1/2}.$$

From Lemma 1 we deduce

Lemma 2. *Assume the conditions of Theorem 1 and let I be a set of positive integers with cardinality N. Then we have for any integer $p \geq 3$*

$$\mathbb{E} S_I^p = \begin{cases} \frac{p!}{(p/2)!} 2^{-p/2} \sigma_I^p + O(T_N) & \textit{if } p \textit{ is even} \\ O(T_N) & \textit{if } p \textit{ is odd} \end{cases}$$

where

$$T_N = \exp(Cp^7) N^{(p-1)/2} (\log N)^p.$$

Proof. Let $C_p = \exp(cp^6)$ be the constant in Lemma 1. We first note that

$$\sigma_I \leq K \|f\|^{1/2} |I|^{1/2}, \tag{2.2}$$

where K is a constant depending only on the generating elements of (n_k). This relation is implicit in the proof of Lemma 2.2 of Philipp [18]. Next we observe that for any fixed

$p \geq 3$ and any fixed nonzero coefficients $a_1, \ldots, a_p$, the number of nondegenerate solutions of (2.1) such that $b = 0$ and $k_1, \ldots, k_p \in I$ is at most $C_{p-1}N$. Indeed, the number of choices for k_p is at most N, and thus taking $a_p n_{k_p}$ to the right hand side and applying Lemma 1, our claim follows.

Without loss of generality we may assume that f is an even function and that $\|f\|_\infty \leq 1$, $\mathrm{Var}_{[0,1]} f \leq 1$; the proof in the general case is the same. (Here, and in the sequel, $\|\cdot\|_p$ denotes the L_p norm; for $p = 2$ we simply write $\|\cdot\|$.) Let

$$f \sim \sum_{j=1}^{\infty} a_j \cos 2\pi jx$$

be the Fourier series of f. $\mathrm{Var}_{[0,1]} f \leq 1$ implies (see Zygmund [26, p. 48])

$$|a_j| \leq j^{-1}, \tag{2.3}$$

and, writing

$$g(x) = \sum_{j=1}^{N^{2p}} a_j \cos 2\pi jx, \qquad r(x) = f(x) - g(x),$$

we have

$$\|g\|_\infty \leq \mathrm{Var}_{[0,1]} f + \|f\|_\infty \leq 2, \qquad \|r\|_\infty \leq \|f\|_\infty + \|g\|_\infty \leq 3.$$

For any positive integer n, (2.3) yields

$$\|(r(nx)\|^2 = \|r(x)\|^2 = \frac{1}{2} \sum_{j=N^{2p}+1}^{\infty} a_j^2 \leq N^{-2p}.$$

By Minkowski's inequality,

$$\|S_I\|_p \leq \left\| \sum_{k\in I} g(n_k x) \right\|_p + \left\| \sum_{k\in I} r(n_k x) \right\|_p,$$

and

$$\left\| \sum_{k\in I} r(n_k x) \right\|_p \leq 3 \sum_{k\in I} \|r(n_k x)/3\|_p \leq 3 \sum_{k\in I} \|r(n_k x)/3\|^{2/p} \leq 3 \sum_{k\in I} N^{-2} \leq 3. \tag{2.4}$$

By expanding and using elementary properties of the trigonometric functions we get

$$\begin{aligned} &\mathbb{E}\left(\sum_{k\in I} g(n_k x) \right)^p \\ &= 2^{-p} \sum_{1\leq j_1,\ldots,j_p \leq N^{2p}} a_{j_1} \cdots a_{j_p} \sum_{k_1,\ldots,k_p\in I} \mathbb{I}\{\pm j_1 n_{k_1} \pm \ldots \pm j_p n_{k_p} = 0\}, \end{aligned} \tag{2.5}$$

with all possibilities of the signs $\pm$ within the indicator function. Assume that $j_1, \dots, j_p$ and the signs $\pm$ are fixed, and consider a solution of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$. Then the set $\{1, 2, \dots, p\}$ can be split into disjoint sets $A_1, \dots, A_l$ such that for each such set A we have $\sum_{i \in A} \pm j_i n_{k_i} = 0$ and no further subsums of these sums are equal to 0. By the monotonicity of C_p and the remark at the beginning of the proof, for each A with $|A| \geq 3$ the number of solutions is $\leq C_{|A|-1} N \leq C_{p-1} N$; trivially, for $|A| = 2$ the number of solutions is at most N. Thus if $s_i = |A_i|$ $(1 \leq i \leq p)$ denotes the cardinality of A_i, the number of solutions of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$ admitting such a decomposition with fixed $A_1, \dots, A_l$ is at most

$$
\begin{aligned}
&\prod_{\{i : s_i \geq 3\}} C_{p-1} N \prod_{\{i : s_i = 2\}} N \leq (C_{p-1} N)^{\sum_{\{i : s_i \geq 3\}} 1 + \sum_{\{i : s_i = 2\}} 1} \\
&\leq (C_{p-1} N)^{\frac{1}{3} \sum_{\{i : s_i \geq 3\}} s_i + \frac{1}{2} \sum_{\{i : s_i = 2\}} s_i} = (C_{p-1} N)^{\frac{1}{3} \sum_{\{i : s_i \geq 3\}} s_i + \frac{1}{2} (p - \sum_{\{i : s_i \geq 3\}} s_i)} \\
&= (C_{p-1} N)^{\frac{p}{2} - \frac{1}{6} \sum_{\{i : s_i \geq 3\}} s_i}.
\end{aligned}
$$

If there is at least one i with $s_i \geq 3$, then the last exponent is at most $(p-1)/2$ and since the number of partitions of the set $\{1, \dots, p\}$ into disjoint subsets is at most $p!\, 2^p$, we see that the number of solutions of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$ where at least one of the sets A_i has cardinality ≥ 3 is at most $p!\, 2^p (C_{p-1} N)^{(p-1)/2}$. If p is odd, there are no other solutions and thus using (2.3) the inner sum in (2.5) is at most $p!\, 2^p (C_{p-1} N)^{(p-1)/2}$ and consequently, taking into account the 2^p choices for the signs ± 1,

$$
\begin{aligned}
&\left| \mathbb{E} \left(\sum_{k \in I} g(n_k x) \right)^p \right| \\
&\leq p!\, 2^p (C_{p-1} N)^{(p-1)/2} 2^p \sum_{1 \leq j_1, \dots, j_p \leq N^{2p}} |a_{j_1} \cdots a_{j_p}| \ll \exp(C p^7) N^{(p-1)/2} (\log N)^p.
\end{aligned}
$$

If p is even, there are also solutions where each A has cardinality 2. Clearly, the contribution of the terms in (2.5) where $A_1 = \{1, 2\}, A_2 = \{3, 4\}, \dots$ is

$$
\begin{aligned}
&\left(\frac{1}{4} \sum_{1 \leq i, j \leq N^{2p}} \sum_{k, \ell \in I} a_i a_j \mathbb{I}\{\pm i n_k \pm j n_\ell = 0\} \right)^{p/2} = \left(\mathbb{E} \left(\sum_{k \in I} g(n_k x) \right)^2 \right)^{p/2} \\
&= \left\| \sum_{k \in I} g(n_k x) \right\|^p = \left\| S_I - \sum_{k \in I} r(n_k x) \right\|^p = (\sigma_I + O(1))^p \\
&= \sigma_I^p + p \left(\sigma_I + O(1) \right)^{p-1} O(1) \\
&= \sigma_I^p + O\left(p 2^{p-2} \right) \left(\sigma_I^{p-1} + O(1)^{p-1} \right) \\
&= \sigma_I^p + O(2^{p^2}) N^{(p-1)/2}, \qquad (2.6)
\end{aligned}
$$

using the mean value theorem and the relation

$$\left(\sum_{j=1}^{m} x_j\right)^{\alpha} \le \max(1, m^{\alpha-1}) \sum_{j=1}^{m} x_j^{\alpha}, \qquad (\alpha > 0,\ x_i \ge 0). \tag{2.7}$$

Here the constants implied by the O are absolute. Since the splitting of $\{1, 2, \ldots, p\}$ into pairs can be done in $\frac{p!}{(p/2)!}2^{-p/2}$ different ways, we proved that

$$\mathbb{E}\left(\sum_{k\in I} g(n_k x)\right)^{p} = \begin{cases} \frac{p!}{(p/2)!}2^{-\frac{p}{2}}\sigma_I^p + O(T_N^*) \\ O(T_N^*) \end{cases} \tag{2.8}$$

according as p is even or odd; here

$$T_N^* = \exp(Cp^7)N^{(p-1)/2}(\log N)^p.$$

Now, letting $G_I = \sum_{k\in I} g(n_k x)$, $R_I = \sum_{k\in I} r(n_k x)$, we get, using the mean value theorem, Hölder's inequality and (2.7),

$$\begin{aligned} &|\mathbb{E}S_I^p - \mathbb{E}G_I^p| \\ &\le \mathbb{E}|(G_I + R_I)^p - G_I^p| = \mathbb{E}|pR_I(G_I + \theta R_I)^{p-1}| \\ &\le p\|R_I\|_p\|(G_I + \theta R_I)^{p-1}\|_{p/(p-1)} = p\|R_I\|_p\|G_I + \theta R_I\|_p^{p-1} \\ &\le 3p(\|G_I\|_p + 3)^{p-1} \le 3p2^{p-2}(\|G_I\|_p^{p-1} + 3^{p-1}), \end{aligned} \tag{2.9}$$

for some $0 \le \theta = \theta(x) \le 1$. For even p we get from (2.8), together with (2.7) with $\alpha = 1/p$, that

$$\|G_I\|_p \ll p\sigma_I + \exp(Cp^6)\sqrt{N}\log N.$$

For p odd, we get the same bound, since $\|G_I\|_p \le \|G_I\|_{p+1}$. Thus for any $p \ge 3$ we get from (2.9)

$$\begin{aligned} \mathbb{E}S_I^p &\le \mathbb{E}G_I^p + 3p2^{p-2}(\|G_I\|_p^{p-1} + 3^{p-1}) \\ &\le \mathbb{E}G_I^p + 3p2^{p-2}\left[(p\sigma_I\exp(Cp^6))^{p-1} + 3^{p-1}\right]O(1)^{p-1} \\ &\le EG_I^p + \exp(Cp^7)\left(C\sqrt{N}\right)^{p-1}, \end{aligned}$$

completing the proof of Lemma 2.

Lemma 3. *Let*

$$f = \sum_{k=1}^{d}(a_k \cos 2\pi kx + b_k \sin 2\pi kx)$$

be a trigonometric polynomial and let I, J be disjoint sets of of positive integers with cardinality M and N, respectively, where $M/N \le C$ with a sufficiently small constant $0 < C < 1$. Assume $\sigma_I \gg |I|^{1/2}$, $\sigma_J \gg |J|^{1/2}$. Then for any integers $p \ge 2$, $q \ge 2$ we have

$$\mathbb{E}(S_I/\sigma_I)^p(S_J/\sigma_J)^q = \tag{2.10}$$

$$= \begin{cases} \dfrac{p!}{(p/2)!2^{p/2}} \dfrac{q!}{(q/2)!2^{q/2}} + O(T_{M,N}) & \textit{if } \ p, q \textit{ are even} \\ O(T_{M,N}) & \textit{otherwise} \end{cases}$$

where

$$T_{M,N} = C_{p+q}^{p+q} \big(M^{-1/2} + (M/N)^{1/2} \big),$$

and $C_p = \exp(cp^6)$ *is the constant in Lemma 1.*

Proof. To simplify the formulas, we assume again that f is a cosine polynomial, i.e.

$$f(x) = \sum_{j=1}^{d} a_j \cos 2\pi j x.$$

The general case requires only trivial changes. Clearly

$$S_I^p S_J^q = \frac{1}{2^{p+q}} \sum_{1 \le j_1, \dots, j_{p+q} \le d} a_{j_1} \dots a_{j_{p+q}} \times$$
$$\times \sum_{\substack{k_1, \dots, k_p \in I \\ k_{p+1}, \dots, k_{p+q} \in J}} \cos 2\pi \big(\pm j_1 n_{k_1} \pm \dots \pm j_{p+q} n_{k_{p+q}} \big) x$$

and thus

$$\mathbb{E} S_I^p S_J^q = \tag{2.11}$$
$$= \frac{1}{2^{p+q}} \sum_{1 \le j_1, \dots, j_{p+q} \le d} a_{j_1} \dots a_{j_{p+q}} \sum_{\substack{k_1, \dots, k_p \in I \\ k_{p+1}, \dots, k_{p+q} \in J}} I \big\{ \pm j_1 n_{k_1} \pm \dots \pm j_{p+q} n_{k_{p+q}} = 0 \big\}.$$

Assume that $j_1, \dots, j_{p+q}$ and the signs $\pm$ are fixed and consider a solution of

$$\pm j_1 n_{k_1} \pm \dots \pm j_{p+q} n_{k_{p+q}} = 0. \tag{2.12}$$

Clearly, the set $\{1, 2, \dots, p+q\}$ can be split into disjoint sets $A_1, \dots, A_\ell$ such that for each such set A we have $\sum\limits_{i \in A} \pm j_i n_{k_i} = 0$ and no further subsums of these sums are equal to 0. Call a set A type 1 or type 2 according as A intersects $\{1, 2, \dots, p\}$ or $A \subseteq \{p+1, \dots, p+q\}$. Similarly as in the proof of Lemma 2, the number of solutions of the equation $\sum_{i \in A} \pm j_i n_{k_i} = 0$ is at most $C_{p+q-1} M$ or $C_{p+q-1} N$ according as A is of type 1 or type 2. Thus the number of solutions of (2.12) belonging to a fixed decomposition $\{A_1, \dots, A_\ell\}$ is at most

$$(C_{p+q-1} M)^R (C_{p+q-1} N)^S \tag{2.13}$$

where R and S denote, respectively, the number of A_i's with type 1 and type 2. Let R^* and S^* denote the total cardinality of sets of type 1 and type 2. Then $R = R^*/2$ or

$R \leq (R^* - 1)/2$ according as all sets of type 1 have cardinality 2 or at least one of them has cardinality ≥ 3. A similar statement holds for sets of type 2 and thus if there exists at least one set A_i with $|A_i| \geq 3$, the expression in (2.13) can be estimated as follows, using also $R^* + S^* = p + q$, $S^* \leq q$,

$$\begin{aligned}
(C_{p+q-1}M)^R (C_{p+q-1}N)^S &\leq (C_{p+q-1}M)^{R^*/2} (C_{p+q-1}N)^{S^*/2} (C_{p+q-1}M)^{-1/2} \\
&= (C_{p+q-1}M)^{(p+q-S^*)/2} (C_{p+q-1}N)^{S^*/2} (C_{p+q-1}M)^{-1/2} \\
&= (C_{p+q-1}M)^{(p+q)/2} (N/M)^{S^*/2} (C_{p+q-1}M)^{-1/2} \\
&\leq (C_{p+q-1}M)^{(p+q)/2} (N/M)^{q/2} (C_{p+q-1}M)^{-1/2} \\
&\leq C_{p+q}^{(p+q)/2} M^{p/2} N^{q/2} M^{-1/2} \\
&\ll C^{p+q} C_{p+q}^{(p+q)/2} \sigma_I^p \sigma_J^q M^{-1/2},
\end{aligned}$$

where in the last step we used (1.5). Since the total number of decompositions of the set $\{1, 2, \ldots, p+q\}$ into subsets is $\leq (p+q)! 2^{p+q} \ll 2^{(p+q)^2}$, it follows that the contribution of those solutions of (2.12) in (2.11) where $|A_i| \geq 3$ for at least one set A_i is

$$\ll 2^{(p+q)^2} (\log d)^{p+q} C_{p+q}^{(p+q)/2} M^{-1/2} \sigma_I^p \sigma_J^q.$$

We now turn to the contribution of those solutions of (2.12) where all sets $A_1, \ldots, A_\ell$ have cardinality 2. This can happen only if $p+q$ is even and then $\ell = (p+q)/2$. Fixing $A_1, \ldots A_\ell$, the sum of the corresponding terms in (2.11) can be written as

$$2^{-(p+q)} \sum_{1 \leq j_1, \ldots, j_{p+q} \leq d} a_{j_1} \ldots a_{j_{p+q}} I\Big\{ \sum_{i \in A_1} \pm j_i n_{k_i} = 0 \Big\} \ldots I\Big\{ \sum_{i \in A_{(p+q)/2}} \pm j_i n_{k_i} = 0 \Big\}$$

and this is the product of $(p+q)/2$ such sums belonging to $A_1, \ldots, A_{(p+q)/2}$. For an $A_i \subseteq \{1, \ldots, p\}$ we get

$$\frac{1}{4} \sum_{\substack{1 \leq i, j \leq d \\ k_i, k_j \in I}} a_i a_j I\{ \pm i n_{k_i} \pm j n_{k_j} = 0 \} = ES_I^2 = \sigma_I^2.$$

Similarly, for any $A_i \subseteq \{p+1, \ldots, p+q\}$ the corresponding sum equals $ES_J^2 = \sigma_J^2$. Finally, if a set A_i is "mixed", i.e. if one of its elements is in $\{1, \ldots, p\}$, the other in $\{p+1, \ldots, p+q\}$, then we get $ES_I S_J := \sigma_{I,J}$ (cf. (2.11) with $p = q = 1$). Thus, if we have t_1 sets $A_i \subseteq \{1, \ldots, p\}$, t_2 sets $A_i \subseteq \{p+1, \ldots, p+q\}$ and t_3 "mixed" sets, we get $\sigma_I^{2t_1} \sigma_J^{2t_2} \sigma_{I,J}^{t_3}$. Clearly $t_3 = 0$ can occur only if p and q are both even and then $t_1 = p/2$, $t_2 = q/2$, i.e. we get $\sigma_I^p \sigma_J^q$ which, taking into account the fact that $\{1, 2, \ldots, p\}$ can be split into 2-element subsets in $\frac{p!}{(p/2)!} 2^{-p/2}$ different ways, gives the contribution

$$\frac{p!}{(p/2)! 2^{p/2}} \frac{q!}{(q/2)! 2^{q/2}} \sigma_I^p \sigma_J^q.$$

Assume now that $t_3 = s$, $1 \leq s \leq p \wedge q$. Then $t_1 = (p-s)/2$, $t_2 = (q-s)/2$; clearly if p and q are both even, then s can be $0, 2, 4, \dots$ and if p and q are both odd, then s can be $1, 3, 5, \dots$. Thus the contribution in this case is

$$\sigma_I^{p-s} \sigma_J^{q-s} \sigma_{I,J}^s. \tag{2.14}$$

From

$$\sigma_{I,J} = \frac{1}{4} \sum_{\substack{1 \leq i,j \leq d \\ k \in I, \ell \in J}} a_i a_j I\{\pm i n_k \pm j n_\ell = 0\}$$

we see that $\sigma_{I,J} \ll (|I| \wedge |J|) = M$ and thus dividing with $\sigma_I^p \sigma_J^q$ and summing for s, (2.14) yields, using again (1.5),

$$\begin{aligned} \sum_{s \geq 1} \sigma_I^{-s} \sigma_J^{-s} \sigma_{I,J}^s \leq \sum_{s \geq 1} C^{2s} (MN)^{-s/2} M^s = \\ = \sum_{s \geq 1} C^{2s} (M/N)^{s/2} \ll (M/N)^{1/2}, \end{aligned}$$

provided C is small enough. □

Lemma 4. *Under the conditions of Lemma 3 we have for any* $0 < \delta < 1$

$$\begin{aligned} \left| \mathbb{E} \left(\exp \left(it S_I / \sigma_I + is S_J / \sigma_J \right) \right) - e^{-(t^2+s^2)/2} \right| \ll \\ \ll e^{-C(\log M)^\delta} + e^{C(\log M)^{7\delta}} \left(M^{-1/2} + \sqrt{M/N} \right) \end{aligned}$$

for $|t|, |s| \leq \frac{1}{4} (\log M)^{\delta/2}$.

Lemma 4 (and also Lemma 6 below) show that the random variables S_I/σ_I and S_J/σ_J are asymptotically independent if $|I| \to \infty$, $|J| \to \infty$, $|I|/|J| \to 0$. Note that I and J are arbitrary disjoint subsets of $\mathbb{N}$: they do not have to be intervals, or being separated by some number $x \in \mathbb{R}$, they can be also "interlaced". Thus $\{n_k x\}$ obeys an "interlaced" mixing condition, an unusually strong near independence property introduced by Bradley [3]. Note that this property is permutation-invariant, explaining the permutation-invariance of the CLT and LIL in Theorem 1.

It is easy to extend Lemma 4 for the joint characteristic function of normed sums $S_{I_1}/\sigma_{I_1}, \dots S_{I_d}/\sigma_{I_d}$ of d disjoint blocks $I_1, \dots I_d$, $d \geq 3$. Since, however, the standard mixing conditions like α-mixing, β-mixing, etc. involve pairs of events and the present formulation will suffice for the CLT and LIL for $f(n_{\sigma(k)} x)$, we will consider only the case $d = 2$.

Proof. Using $\left| e^{ix} - \sum_{p=0}^{k-1} \frac{(ix)^p}{p!} \right| \leq \frac{|x|^k}{k!}$, valid for any $x \in \mathbb{R}$, $k \geq 1$ we get for any $L \geq 1$

$$\exp(it S_I / \sigma_I) = \sum_{p=0}^{L-1} \frac{(it)^p}{p!} (S_I/\sigma_I)^p +$$

$$+\theta_L(t,x,I)\frac{|t|^L}{L!}|S_I/\sigma_I|^L =$$
$$=: U_L(t,x,I)+\theta_L(t,x,I)\frac{|t|^L}{L!}|S_I/\sigma_I|^L$$

where $|\theta_L(t,x,I)|\le 1$. Writing a similar expansion for $\exp(isS_J/\sigma_J)$ and multiplying, we get

$$\begin{aligned}
&\mathbb{E}\left(\exp(itS_I/\sigma_I+isS_J/\sigma_J)\right)=\\
&=\mathbb{E}\big(U_L(t,x,I)U_L(s,x,J)\big)+\mathbb{E}\left(U_L(t,x,I)\theta_L(s,x,J)\frac{|s|^L}{L!}|S_J/\sigma_J|^L\right)+\\
&\quad+\mathbb{E}\left(U_L(s,x,J)\theta_L(t,x,I)\frac{|t|^L}{L!}|S_I/\sigma_I|^L\right)+\\
&\quad+\mathbb{E}\left(\frac{|t|^L}{L!}\frac{|s|^L}{L!}|S_I/\sigma_I|^L|S_J/\sigma_J|^L\theta_L(t,x,I)\theta_L(s,x,J)\right)=\\
&=I_1+I_2+I_3+I_4.
\end{aligned}$$

We estimate I_1, I_2, I_3, I_4 separately. We choose $L=2\big[(\log M)^\delta\big]$ and use Lemma 3 to get

$$\begin{aligned}
I_1 &= \sum_{\substack{p,q=0\\ p,q \text{ even}}}^{L-1}\frac{(it)^p}{(p/2)!2^{p/2}}\frac{(is)^q}{(q/2)!2^{q/2}}+\\
&\quad+O(1)\sum_{p,q=0}^{L-1}\frac{|t|^p}{p!}\frac{|s|^q}{q!}C_{2L}^{2L}\big(M^{-1/2}+(M/N)^{1/2}\big)\\
&=: I_{1,1}+I_{1,2}.
\end{aligned}$$

Here

$$\begin{aligned}
&e^{-(t^2+s^2)/2}-I_{1,1}=\\
&=\Bigg(\sum_{\substack{p=0\\ p\text{ even}}}^{L-1}\frac{(it)^p}{(p/2)!2^{p/2}}\Bigg)\Bigg(\sum_{\substack{q=L\\ q\text{ even}}}^{\infty}\frac{(is)^q}{(q/2)!2^{q/2}}\Bigg)+\Bigg(\sum_{\substack{p=L\\ p\text{ even}}}^{\infty}\frac{(it)^p}{(p/2)!2^{p/2}}\Bigg)\Bigg(\sum_{\substack{q=0\\ q\text{ even}}}^{\infty}\frac{(is)^q}{(q/2)!2^{s/2}}\Bigg).
\end{aligned}$$

Using $n!\ge (n/3)^n$ and $t^2\le L/24\le p/24$ we get

$$\begin{aligned}
\Bigg|\sum_{\substack{p=L\\ p\text{ even}}}^{\infty}\frac{(it)^p}{(p/2)!2^{p/2}}\Bigg| &\le \sum_{p=L}^{\infty}\frac{|t|^p}{(p/3)^{p/2}}=\sum_{p=L}^{\infty}\left(\frac{t^2}{p/3}\right)^{p/2}\le\\
&\le\sum_{p=L}^{\infty}\left(\frac{1}{4}\right)^{p/2}\le 2\cdot 2^{-L}\le 8e^{-(\log M)^\delta}
\end{aligned}$$

and similarly
$$\left|\sum_{\substack{q=L\\ q \text{ even}}}^{\infty} \frac{(is)^q}{(q/2)!2^{q/2}}\right| \ll e^{-(\log M)^\delta}.$$
Thus
$$\left|\sum_{\substack{p=0\\ p \text{ even}}}^{L-1} \frac{(it)^p}{(p/2)!2^{p/2}}\right| \le e^{-t^2/2} + \left|\sum_{\substack{p=L\\ p \text{ even}}}^{\infty} \frac{(it)^p}{(p/2)!2^{p/2}}\right| \le 9,$$
and a similar estimate holds for
$$\sum_{\substack{q=0\\ q \text{ even}}}^{L-1} \frac{(is)^q}{(q/2)!2^{q/2}}.$$
Consequently
$$\left|I_{1,1} - e^{-(t^2+s^2)/2}\right| \ll e^{-(\log M)^\delta}.$$
On the other hand,
$$\begin{aligned}|I_{1,2}| &\ll \left(\sum_{p=0}^{\infty} \frac{|t|^p}{p!}\right)\left(\sum_{q=0}^{\infty} \frac{|s|^q}{q!}\right) C_{2L}^{2L}\big(M^{-1/2} + (M/N)^{1/2}\big)\\ &\ll e^{|t|+|s|} e^{C(2L)^7}\big(M^{-1/2} + (M/N)^{1/2}\big)\\ &\ll e^{C(\log M)^{7\delta}}\big(M^{-1/2} + (M/N)^{1/2}\big).\end{aligned}$$
Thus we proved
$$\left|I_1 - e^{-(t^2+s^2)/2}\right| \ll e^{-C(\log M)^\delta} + e^{C(\log M)^{7\delta}}\big(M^{-1/2} + (M/N)^{1/2}\big).$$
Next we estimate I_4. Using Lemma 3 and $t^2 \le L/24$ we get, since L is even,
$$\begin{aligned}I_4 &\le \frac{|t|^L}{L!}\frac{|s|^L}{L!}\mathbb{E}|S_I/\sigma_I|^L|S_J/\sigma_J|^L\\ &\ll \frac{|t|^L}{L!}\frac{|s|^L}{L!}\left[\left(\frac{L!}{(L/2)!2^{L/2}}\right)^2 + C_{2L}^{2L}\big(M^{-1/2} + (M/N)^{1/2}\big)\right]\\ &\ll \frac{|t|^L|s|^L}{((L/2)!)^2} + \frac{|t|^L|s|^L}{(L!)^2}C_{2L}^{2L}\big(M^{-1/2} + (M/N)^{1/2}\big)\\ &\ll \left(\frac{t^2}{L/6}\right)^{L/2}\left(\frac{s^2}{L/6}\right)^{L/2}\\ &\quad + \left(\frac{t^2}{L/6}\right)^{L/2}\left(\frac{s^2}{L/6}\right)^{L/2} e^{C(2L)^7}\big(M^{-1/2} + (M/N)^{1/2}\big)\\ &\ll 4^{-L} + 4^{-L}e^{C(2L)^7}\big(M^{-1/2} + (M/N)^{1/2}\big)\end{aligned}$$

$$\ll e^{-C(\log M)^{\delta}} + e^{C(\log M)^{7\delta}}\big(M^{-1/2} + (M/N)^{1/2}\big).$$

Finally we estimate I_2 and I_3. Clearly

$$\begin{aligned}|U_L(t,x,I)| &\le |\exp(itS_I/\sigma_I)| + \frac{|t|^L}{L!}|S_I/\sigma_I|^L \\ &\le 1 + \frac{|t|^L}{L!}|S_I/\sigma_I|^L\end{aligned}$$

and thus

$$|I_2| \le \mathbb{E}\left(\frac{|s|^L}{L!}|S_J/\sigma_J|^L\right) + \mathbb{E}\left(\frac{|t|^L}{L!}\frac{|s|^L}{L!}|S_I/\sigma_I|^L|S_J/\sigma_J|^L\right).$$

Here the second summand can be estimated exactly in the same way as I_4 and the first one can be estimated by using Lemma 2. Thus we get

$$|I_2| \ll e^{-C(\log M)^{\delta}} + e^{C(\log M)^{7\delta}}\big(M^{-1/2} + (M/N)^{1/2}\big).$$

A similar bound holds for I_3 and this completes the proof of Lemma 4. □

Lemma 5. *Let F and G be probability distributions on $\mathbb{R}^2$ with characteristic functions φ and γ, respectively and let $T > 0$. Then there exists a probability distribution H on $\mathbb{R}^2$ such that $H\big(|x| \ge T^{-1/2}\log T\big) \ll e^{-T/2}$ and for any Borel set $B \subset [-T,T]^2$*

$$\big|(F*H)(B) - (G*H)(B)\big| \ll T^2 \int\limits_{[-T,T]^2} |\varphi(u) - \gamma(u)|du + e^{-(\log T)^2/4}.$$

The constants implied by $\ll$ are absolute.

Proof. Let ζ_0 be a standard $N(0,\mathbf{I})$ random variable in $\mathbb{R}^2$ and $\zeta = \frac{\log T}{T}\zeta_0$. Clearly we have

$$P\left(|\zeta| \ge \frac{\log T}{\sqrt{T}}\right) = P\big(|\zeta_0| \ge \sqrt{T}\big) \ll e^{-T/2}.$$

Letting ψ and H denote, respectively, the characteristic function and distribution of ζ, we get

$$\begin{aligned}\big|f_{F*H}(x) - f_{G*H}(x)\big| &\le (2\pi)^{-1}\int\limits_{\mathbb{R}^2} |\varphi(u) - \gamma(u)|\,|\psi(u)|du \le \\ &\le \int\limits_{[-T,T]^2} |\varphi(u) - \gamma(u)|du + 2\int\limits_{u\notin[-T,T]^2} |\psi(u)|du,\end{aligned}$$

where f_{F*H}, f_{G*H} denote the density functions corresponding to the distributions $F*H$ and $G*H$, respectively. Letting $\tau = T^{-1}\log T$, we clearly have $\psi(u) = e^{-\tau^2|u|^2/2}$ for $u \in \mathbb{R}^2$ and a simple calculation shows

$$\int_{u \notin [-T,T]} |\psi(u)| du \ll e^{-(\log T)^2/3}.$$

Thus

$$\left| f_{F*H}(x) - f_{G*H}(x) \right| \ll \int\limits_{[-T,T]^2} |\varphi(u) - \gamma(u)| du + e^{-(\log T)^2/3} \quad \text{for all } x \in \mathbb{R}^2$$

whence for $B \subseteq [-T,T]^2$ we get

$$|(F*H)(B) - (G*H)(B)| \ll T^2 \int\limits_{[-T,T]} |\varphi(u) - \gamma(u)| du + T^2 e^{-(\log T)^2/3},$$

proving Lemma 5. $\square$

Lemma 6. *Under the conditions of Lemma 3 we have for any $0 < \delta < 1$ and for $|x|, |y| \le \frac{1}{8}(\log M)^{\delta/2}$,*

$$\begin{aligned} &|P\left(S_I/\sigma_I \le x,\, S_J/\sigma_J \le y\right) - \Phi(\hat{x})\Phi(\hat{y})| \ll \\ &\ll e^{-C(\log\log M)^2} + e^{C(\log M)^{7\delta}} \left(M^{-1/2} + (M/N)^{1/2}\right) \end{aligned} \tag{2.15}$$

where Φ is the standard normal distribution function and $\hat{x}$, $\hat{y}$ are suitable numbers with $|\hat{x} - x| \le C(\log M)^{-\delta/8}$, $|\hat{y} - y| \le C(\log M)^{-\delta/8}$.

Proof. Let

$$F = \operatorname{dist}\left(S_I/\sigma_I, S_J/\sigma_J\right), \quad G = N(0, I), \quad T = (\log M)^{\delta/2}.$$

By Lemmas 4 and 5 we have for any Borel set $B \subseteq [-T,T]^2$

$$\begin{aligned} &\left|(F*H)(B) - (G*H)(B)\right| \\ &\ll T^2 \int\limits_{[-T,T]^2} |\varphi(u) - \gamma(u)| du + e^{-(\log T)^2/4} \\ &\ll (\log M)^{2\delta} \left[e^{-C(\log M)^{\delta}} + e^{C(\log M)^{7\delta}} \left(M^{-1/2} + (M/N)^{1/2}\right)\right] + e^{-c(\log\log M)^2} \\ &\ll e^{-C(\log M)^{\delta}} + e^{C(\log M)^{7\delta}} \left(M^{-1/2} + (M/N)^{1/2}\right) + e^{-C(\log\log M)^2} \end{aligned}$$

where H is a distribution on $\mathbb{R}^2$ such that

$$H\left(x : |x| \ge C(\log M)^{-\delta/8}\right) \le e^{-C(\log M)^{\delta/2}}.$$

Applying Lemma 2 with $p = 2[\log\log M]$ and using the Markov inequality, we get

$$
\begin{aligned}
P(|S_I/\sigma_I| \geq T) &\leq T^{-p}\mathbb{E}(|S_I/\sigma_I|^p) \ll T^{-p}\frac{p!}{(p/2)!}2^{-p/2} \\
&\ll 4^p(\log M)^{-\delta p/2}p^p = 4^p \exp\left(p\log p - \frac{\delta p}{2}\log\log M\right) \\
&\ll \exp(-C(\log\log M)^2)
\end{aligned}
$$

and a similar inequality holds for $P(|S_J/\sigma_J| \geq T)$. Convolution with H means adding an (independent) r.v. which is $< C(\log M)^{-\delta/8}$ with the exception of a set with probability $e^{-C(\log M)^{\delta/2}}$. Thus choosing $B = [-T, x] \times [-T, y]$ with $|x| \leq C(\log M)^{\delta/2}$, $|y| \leq C(\log M)^{\delta/2}$ we get

$$
\begin{aligned}
&|P(S_I/\sigma_I \leq x,\ S_J/\sigma_J \leq y) - \Phi(\hat{x})\Phi(\hat{y})| \ll \\
&\ll e^{-C(\log\log M)^2} + e^{C(\log M)^{7\delta}}\left(M^{-1/2} + (M/N)^{1/2}\right)
\end{aligned}
\tag{2.16}
$$

where $|\hat{x} - x| \leq C(\log M)^{-\delta/8}$, $|\hat{y} - y| \leq C(\log M)^{-\delta/8}$. $\square$

Remark. The one-dimensional analogue of Lemma 6 can be proved in the same way (in fact, the argument is much simpler):

$$|P(S_I/\sigma_I \leq x) - \Phi(\hat{x})| \ll e^{-C(\log\log M)^2}$$

for $|x| \leq \frac{1}{8}(\log M)^{\delta/2}$, where $|\hat{x} - x| \leq C(\log M)^{-\delta/8}$. Using this fact, the statement of Lemma 6 and simple algebra show that for $|x|, |y| \leq \frac{1}{8}(\log M)^{\delta/2}$ we have

$$
\begin{aligned}
&|P(S_I/\sigma_I > x,\ S_J/\sigma_J > y) - \Psi(\hat{x})\Psi(\hat{y})| \ll \\
&\ll e^{-C(\log\log M)^2} + e^{C(\log M)^{7\delta}}\left(M^{-1/2} + (M/N)^{1/2}\right)
\end{aligned}
$$

where $\hat{x}$, $\hat{y}$ are suitable numbers with $|\hat{x} - x| \leq C(\log M)^{-\delta/8}$, $|\hat{y} - y| \leq C(\log M)^{-\delta/8}$. Here $\Psi(x) = 1 - \Phi(x)$.

3 Proof of Theorem 1

The CLT (1.6) in Theorem 1 follows immediately from Lemma 6; see also the remark after Lemma 6. To prove the LIL (1.7), assume the conditions of Theorem 1 and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$. Clearly for $p = O(\log\log N)$ we have $\exp(Cp^7) \ll N^{1/4}$ and thus Lemma 2 implies

$$\int_0^1 \left(\sum_{k=M+1}^{M+N} f(n_{\sigma(k)}x)\right)^{2p} dx \sim \frac{(2p)!}{p!}2^{-p}(1 + O(N^{-1/4}))A_{N,M}^p \qquad \text{as } N \to \infty$$

uniformly for $p = O(\log\log N)$ and $M \ge 1$. Using this fact, the upper half of the LIL (1.7) can be proved by following the classical proof of Erdős and Gál [4] of the LIL for lacunary trigonometric series. (The observation that the upper half of the LIL follows from asymptotic moment estimates was already used by Philipp [17] to prove the LIL for mixing sequences.) To prove the lower half of the LIL we first observe that the upper half of the LIL and relation (2.2) imply

$$\limsup_{N\to\infty} (N \log\log N)^{-1/2} \sum_{k=1}^{N} f(n_{\sigma(k)}x) \le K\|f\|^{1/8} \qquad \text{a.e.} \tag{3.1}$$

where K is a constant depending on the generating elements of (n_k). Given any f satisfying (1.4) and $\varepsilon > 0$, f can be written as $f = f_1 + f_2$ where f_1 is a trigonometric polynomial and $\|f_2\| \le \varepsilon$, and thus applying (3.1) with $f = f_2$ it is immediately seen that it suffices to prove the lower half of the LIL for trigonometric polynomials f.

Let $\theta \ge 2$ be an integer and set

$$\eta_n = \frac{X_{\theta^n+1} + \cdots + X_{\theta^{n+1}}}{\gamma_n}$$

where $X_j = f(n_{\sigma(j)}x)$, $\gamma_n^2 = \operatorname{Var}\big(X_{\theta^n+1} + \cdots + X_{\theta^{n+1}}\big)$. Fix $\varepsilon > 0$ and put

$$A_n = \big\{\eta_n \ge (1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big\}.$$

We will prove that $P(A_n \text{ i.o.}) = 1$; we use here an idea of Révész [19] and the following generalization of the Borel-Cantelli lemma, see Spitzer [21], p. 317.

Lemma 7. *Let A_n, $n = 1, 2, \ldots$ be events satisfying $\sum_{n=1}^{\infty} P(A_n) = \infty$ and*

$$\lim_{N\to\infty} \frac{\sum_{1\le m,n\le N} \big|P(A_m \cap A_n) - P(A_m)P(A_n)\big|}{\Big(\sum_{n=1}^{N} P(A_n)\Big)^2} = 0.$$

Then $P(A_n$ i.o.$) = 1$.

By the one-dimensional version of Lemma 6 (see the remark at the end of Section 2) we have

$$P(A_n) = \Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{\frac{1}{2}} + z_n\big) + O(e^{-C(\log n)^2}) \tag{3.2}$$

where $|z_n| \le Cn^{-\delta/8}$. By the mean value theorem, $\Psi(x) \sim (2\pi)^{-1/2}x^{-1}\exp(-x^2/2)$ and $\theta^n \ll \gamma_n^2 \ll \theta^n$ we have

$$\begin{aligned} &\Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2} + z_n\big) \\ &= \Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big) + \exp\left(-\frac{1}{2}\left[(1-\varepsilon)(2\log\log\gamma_n)^{1/2} + O(1)n^{-\delta/8}\right]^2\right) O(n^{-\delta/8}) \end{aligned} \tag{3.3}$$

$$= \Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big) + \exp\big(-(1-\varepsilon)^2\log\log\gamma_n + O(1)\big)\, O(n^{-\delta/8})$$
$$= \Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big) + O(1)\Psi\left((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\right)(\log n)^{1/2} n^{-\delta/8}.$$

In particular,

$$\Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2} + z_n\big) \sim \Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big)$$

and thus (3.2) implies

$$P(A_n) \sim \Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big) \gg \frac{1}{n^{(1-\varepsilon)^2}(\log n)^{1/2}}. \tag{3.4}$$

Hence the estimates in (3.3) yield

$$\Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2} + z_n\big) = \Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big) + O(P(A_n)n^{-\delta/16}). \tag{3.5}$$

Now by Lemma 6 for $m \le n$ (see the Remark at the end of Section 2)

$$\begin{aligned} P(A_m\cap A_n) = {} & \Psi\big((1-\varepsilon)(2\log\log\gamma_m)^{1/2} + z_1\big)\Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2} + z_2\big) \\ & + O(1)\Big[e^{-C(\log m)^2} + e^{Cm^{7\delta}}(e^{-Cm} + e^{-C(n-m)})\Big], \end{aligned} \tag{3.6}$$

provided $\log n \le m^{\delta/2}$. The expression $\Psi(\dots)\Psi(\dots)$ in (3.6) equals by (3.4), (3.5),

$$\Psi\big((1-\varepsilon)(2\log\log\gamma_m)^{1/2}\big)\Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big) + O\left(P(A_m)P(A_n)m^{-\delta/16}\right).$$

Hence, assuming also $n-m \ge m^{8\delta}$ we get from (3.6),

$$\begin{aligned} P(A_m\cap A_n) = {} & \Psi\big((1-\varepsilon)(2\log\log\gamma_m)^{1/2}\big)\Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big) \\ & + O\left(P(A_m)P(A_n)m^{-\delta/16}\right) + O\left(e^{-C(\log m)^2}\right). \end{aligned} \tag{3.7}$$

Further, by (3.2) and the above estimates

$$\begin{aligned} P(A_m)P(A_n) = {} & \Psi\big((1-\varepsilon)(2\log\log\gamma_m)^{1/2}\big)\Psi\big((1-\varepsilon)(2\log\log\gamma_n)^{1/2}\big) \\ & + O\left(P(A_m)P(A_n)m^{-\delta/16}\right) + O\left(e^{-C(\log m)^2}\right), \end{aligned}$$

and thus we obtained

Lemma 8. *We have*

$$\big|P(A_m\cap A_n) - P(A_m)P(A_n)\big| \ll P(A_m)P(A_n)m^{-\delta/16} + e^{-C(\log m)^2}$$

provided $n-m \ge m^{8\delta}$ *and* $\log n \le m^\delta$.

We can now prove

Lemma 9. *We have*

$$\lim_{N\to\infty} \frac{\sum_{1\le m,n\le N} \big|P(A_m \cap A_n) - P(A_m)P(A_n)\big|}{\left(\sum_{n=1}^{N} P(A_n)\right)^2} = 0.$$

Proof. By Lemma 8 we have

$$\begin{aligned}
&\sum_{\substack{1\le m,n\le N\\ n-m\ge m^{8\delta}\\ m\ge CN^{\delta}}} \big|P(A_m \cap A_n) - P(A_m)P(A_n)\big| \\
&\ll \left(\sum_{m=1}^{N} P(A_m) m^{-\delta/16}\right)\left(\sum_{n=1}^{N} P(A_n)\right) + N^2 e^{-c(\log N)^2} \\
&= o_N(1)\left(\sum_{m=1}^{N} P(A_m)\right)\left(\sum_{n=1}^{N} P(A_n)\right) + O(1) = o_N(1)\left(\sum_{m=1}^{N} P(A_m)\right)^2,
\end{aligned}$$

since $\sum\limits_{n=1}^{N} P(A_n) = +\infty$ by (3.4). Further

$$\begin{aligned}
&\sum_{\substack{1\le m,n\le N\\ 0\le n-m\le m^{8\delta}}} \big|P(A_m \cap A_n) - P(A_m)P(A_n)\big| \\
&\le \sum_{\substack{1\le m,n\le N\\ m\le n\le m+m^{8\delta}}} 2P(A_m) \le 2\sum_{m=1}^{N} m^{8\delta} P(A_m) \le 2N^{8\delta} \sum_{m=1}^{N} P(A_m)
\end{aligned}$$

and

$$\sum_{\substack{1\le m\le n\le N\\ m\le CN^{\delta}}} \big|P(A_m \cap A_n) - P(A_m)P(A_n)\big| \le \sum_{\substack{1\le m\le n\le N\\ m\le CN^{\delta}}} 2P(A_n) \le 2N^{\delta} \sum_{n=1}^{N} P(A_n).$$

The previous estimates imply

$$\begin{aligned}
&\sum_{1\le m,n\le N} \big|P(A_m \cap A_n) - P(A_m)P(A_n)\big| \\
&\qquad \ll o_N(1)\left(\sum_{n=1}^{N} P(A_n)\right)^2 + N^{8\delta}\left(\sum_{n=1}^{N} P(A_n)\right).
\end{aligned}$$

Since

$$\sum_{n=1}^{N} P(A_n) \gg \sum_{n=1}^{N} \frac{1}{n^{(1-\varepsilon)^2} (\log n)^{1/2}} \gg \sum_{n=1}^{N} \frac{1}{n^{1-\varepsilon}} \gg N^{\varepsilon},$$

choosing $\delta < \varepsilon/8$ we get

$$\frac{\sum_{1\le m,n\le N}\left|P(A_m\cap A_n)-P(A_m)P(A_n)\right|}{\left(\sum_{n=1}^N P(A_n)\right)^2} \ll o_N(1) + \frac{N^{8\delta}}{\sum_{n=1}^N P(A_n)} \to 0.$$

□

We can now complete the proof of the lower half of the LIL. By Lemmas 7 and 9 and we have with probability 1

$$\left|X_{\theta^n+1}+\cdots+X_{\theta^{n+1}}\right| \ge (1-\varepsilon)(2\gamma_n^2\log\log\gamma_n)^{1/2} \qquad \text{i.o.} \tag{3.8}$$

where $\gamma_n = \left\|X_{\theta^n+1}+\cdots+X_{\theta^{n+1}}\right\|$. By the already proved upper half of the LIL we have

$$\left|X_1+\cdots+X_{\theta^n}\right| \le (1+\varepsilon)(2A_{\theta^n}^2\log\log A_{\theta^n})^{1/2} \qquad \text{a.s.} \tag{3.9}$$

and (2.2) and the assumptions of Theorem 1 imply

$$A_{\theta^{n+1}}/A_{\theta^n} \ge C\theta^{1/2}, \tag{3.10}$$

whence

$$\begin{aligned}\gamma_n &\ge \left\|X_1+\cdots+X_{\theta^{n+1}}\right\| - \left\|X_1+\cdots+X_{\theta^n}\right\| = A_{\theta^{n+1}} - A_{\theta^n} \\ &\ge A_{\theta^{n+1}}\left(1-O(\theta^{-1/2})\right).\end{aligned} \tag{3.11}$$

Thus using (3.8), (3.9), (3.10) and (3.11) we get with probability 1 for infinitely many n

$$\begin{aligned}&\left|X_1+\cdots+X_{\theta^{n+1}}\right| \\ &\ge (1-\varepsilon)(2\gamma_n^2\log\log\gamma_n)^{1/2} - (1+\varepsilon)(2A_{\theta^n}^2\log\log A_{\theta^n})^{1/2} \\ &\ge (1-2\varepsilon)(2A_{\theta^{n+1}}^2\log\log A_{\theta^{n+1}})^{1/2},\end{aligned}$$

provided we choose $\theta=\theta(\varepsilon)$ large enough. This completes the proof of the lower half of the LIL.

To prove the Corollary, assume that

$$N^{-1/2}\sum_{k=1}^N f(n_{\sigma(k)}x) \xrightarrow{d} G \tag{3.12}$$

with a nondegenerate distribution G. By Lemma 2 and (2.2) we have

$$\mathbb{E}\left(\sum_{k=1}^N f(n_{\sigma(k)}x)\right)^4 \ll N^2,$$

and thus the sequence

$$N^{-1}\left(\sum_{k=1}^{N} f(n_{\sigma(k)}x)\right)^2, \qquad N=1,2,\ldots$$

is bounded in L_2 norm and consequently uniformly integrable. Thus the second moment of the left hand side of (3.12) converges to the second moment γ^2 of G, which is nonzero, since G is nondegenerate. Thus we proved (1.8), and since the nonnegativity of the Fourier coefficients of f implies (1.5), Theorem 1 yields (1.9) and (1.10).

In conclusion, we prove the remark made at the end of the Introduction concerning the set Γ_f of limiting variances corresponding to all permutations σ. Let f be a function satisfying (1.4) with nonnegative Fourier coefficients. Assume that f is even, i.e. its Fourier series

$$f(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x$$

is a pure cosine series; the general case requires only trivial changes. Note that the Fourier coefficients of f satisfy (2.3) and by Kac [11] we have

$$\int_0^1 \left(\sum_{k=1}^{N} f\left(2^k x\right)\right)^2 dx \sim \gamma_f^2 N$$

where

$$\gamma_f^2 = \|f\|^2 + 2\sum_{r=1}^{\infty}\int_0^1 f(x)f(2^r x)dx \geq \|f\|^2.$$

We first note that for any permutation $\sigma:\mathbb{N}\to\mathbb{N}$ we have

$$\|f\|^2 \leq \frac{1}{N}\int_0^1 \left(\sum_{k=1}^{N} f(2^{\sigma(k)}x)\right)^2 dx \leq \gamma_f^2 \tag{3.13}$$

for any $N \geq 1$. To see this, we observe that

$$\begin{aligned} &\int_0^1 \left(\sum_{k=1}^{N} f(2^{\sigma(k)}x)\right)^2 dx \qquad (3.14)\\ &= N\|f\|^2 + \sum_{1\leq i\neq j\leq N}\int_0^1 f(2^{\sigma(i)}x)f(2^{\sigma(j)}x)\,dx\\ &= N\|f\|^2 + \sum_{1\leq i\neq j\leq N}\int_0^1 f(x)f(2^{|\sigma(j)-\sigma(i)|}x)\,dx\\ &= N\|f\|^2 + \sum_{r=1}^{\infty} a_r^{(N)}\int_0^1 f(x)f(2^r x)\,dx \end{aligned}$$

where

$$a_r^{(N)} = \#\{1 \leq i \neq j \leq N : |\sigma(j) - \sigma(i)| = r\}.$$

Fix $r \geq 1$. Clearly, for any $1 \leq i \leq N$ there exist at most two indices $1 \leq j \leq N$, $j \neq i$ such that $|\sigma(j) - \sigma(i)| = r$. Hence $a_r^{(N)} \leq 2N$ and by the nonnegativity of the Fourier coefficients of f, the integrals in the last line of (3.14) are nonnegative. Thus (3.13) is proved. Next we claim that for any $\rho \in [\|f\|, \gamma_f]$ we can find a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that

$$\int_0^1 \left(\sum_{k=1}^N f\left(2^{\sigma(k)} x\right) \right)^2 dx \sim \rho^2 N. \tag{3.15}$$

To this end, we will need

Lemma 10. *For some $J \geq 0$ let*

$$g(x) = \sum_{j=J+1}^{\infty} a_j \cos 2\pi j x.$$

Then for any set $\{m_1, \ldots, m_N\}$ of distinct positive integers we have

$$\left\| \sum_{k=1}^N g\left(2^{m_k} x\right) \right\| \leq \begin{cases} 2\sqrt{N} & \text{for } J = 0 \\ \sqrt{2N} J^{-1/2} & \text{for } J \geq 1. \end{cases}$$

Proof: Similarly to (2.5) we have

$$\begin{aligned}
& \int_0^1 \left(\sum_{k=1}^N g\left(2^{m_k} x\right) \right)^2 dx \\
= \quad & \frac{1}{2} \sum_{1 \leq k_1, k_2 \leq N} \sum_{j_1, j_2 \geq J+1} a_{j_1} a_{j_2} \cdot \mathbf{1}\left(j_1 2^{m_{k_1}} = j_2 2^{m_{k_2}}\right) \\
\leq \quad & \sum_{1 \leq k_1 \leq k_2 \leq N} \sum_{j_1, j_2 \geq J+1} \frac{1}{j_1 j_2} \cdot \mathbf{1}\left(j_1 2^{m_{k_1}} = j_2 2^{m_{k_2}}\right) \\
\leq \quad & \sum_{1 \leq k_1 \leq k_2 \leq N} \sum_{j \geq J+1} \frac{2^{m_{k_1}}}{j^2 2^{m_{k_2}}} \\
\leq \quad & N \sum_{v=0}^{\infty} 2^{-v} \sum_{j \geq J+1} \frac{1}{j^2} \\
\leq \quad & \begin{cases} 4N & \text{for } J = 0 \\ 2NJ^{-1} & \text{for } J \geq 1. \end{cases} \qquad \square
\end{aligned}$$

Let now $\rho \in [\|f\|, \gamma_f]$ be given and write

$$\alpha = \frac{\rho^2 - \|f\|^2}{\gamma_f^2 - \|f\|^2}.$$

Clearly

$$\alpha \in [0,1].$$

Postponing the extremal cases $\alpha = 0$ and $\alpha = 1$, assume $\alpha \in (0,1)$. Set

$$\Delta_i = \{i^2+1, \dots, (i+1)^2\}, \quad i \geq 0.$$

For every positive integer k there exists exactly one number $i = i(k)$ such that $k \in \Delta_i$. Now we set $n_1 = 1$ and define a sequence $(n_k)_{k \geq 1}$ recursively by

$$n_k \quad = \quad \begin{cases} n_{k-1} + i + 1 & \text{if} \quad k = i^2 + 1 \text{ for some } i \\ n_{k-1} + 1 & \text{if} \quad k \in \{i^2+2, i^2 + \lceil 2i\alpha \rceil\} \text{ for some } i \\ n_{k-1} + i + 1 & \quad \text{otherwise.} \end{cases}$$

For any $i \geq 0$, set

$$p^{(i)}(x) = \sum_{j=1}^{2^i} a_j \cos 2\pi j x, \qquad r^{(i)}(x) = f(x) - p^{(i)}(x) = \sum_{j=2^i+1}^{\infty} a_j \cos 2\pi j x.$$

We want to calculate

$$\int_0^1 \left(\sum_{k=1}^N f\left(2^{n_k} x\right) \right)^2 dx$$

asymptotically. There is an i such that $N \in \Delta_i$, and since $N - i^2 \leq (i+1)^2 - i^2 = 2i + 1 \leq 2\sqrt{N} + 1$, we have by Lemma 10

$$\left\| \sum_{k=1}^N f\left(2^{n_k} x\right) \right\| - \left\| \sum_{k=1}^{i^2} f\left(2^{n_k} x\right) \right\| \leq \left\| \sum_{k=i^2+1}^{N} f\left(2^{n_k} x\right) \right\| \leq 2 \left(2\sqrt{N} + 1 \right)^{1/2}. \tag{3.16}$$

Using Lemma 10 again, we get

$$\begin{aligned} & \left\| \sum_{k=1}^{i^2} f\left(2^{n_k} x\right) \right\| \\ = \quad & \left\| \sum_{h=0}^{i-1} \left(\sum_{k \in \Delta_h} p^{(h)}\left(2^{n_k} x\right) + \sum_{k \in \Delta_h} r^{(h)}\left(2^{n_k} x\right) \right) \right\|, \end{aligned}$$

and

$$\begin{aligned} & \left\| \sum_{k=1}^{i^2} f\left(2^{n_k} x\right) \right\| - \left\| \sum_{h=0}^{i-1} \left(\sum_{k \in \Delta_h} p^{(h)}\left(2^{n_k} x\right) \right) \right\| \\ \leq \quad & \sum_{h=0}^{i-1} \left\| \left(\sum_{k \in \Delta_h} r^{(h)}\left(2^{n_k} x\right) \right) \right\| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{h=0}^{i-1} \sqrt{2|\Delta_h|}2^{-h/2} \\ &\leq \sum_{h=0}^{i-1} \sqrt{2(2h+1)}2^{-h/2} \\ &\ll 1. \end{aligned} \tag{3.17}$$

Now we calculate

$$\left\| \sum_{h=0}^{i-1} \left(\sum_{k \in \Delta_h} p^{(h)}\left(2^{n_k} x\right) \right) \right\|.$$

By the construction of the sequence $(n_k)_{k\geq 1}$, the functions

$$\sum_{k \in \Delta_{h_1}} p^{(h_1)}\left(2^{n_k} x\right), \qquad \sum_{k \in \Delta_{h_2}} p^{(h_2)}\left(2^{n_k} x\right) \tag{3.18}$$

are orthogonal if $h_1 \neq h_2$. In fact, if $h_2 > h_1$, and $k_1 \in \Delta_{h_1},\ k_2 \in \Delta_{h_2}$, then $n_{k_2} \geq n_{k_1} + h_2 + 1$, which implies that the largest frequency of a trigonometric function in the Fourier series of $p^{(h_1)}(2^{n_{k_1}} x)$ is $2^{h_1} 2^{n_{k_1}} < 2^{n_{k_2}}$. Thus the functions in (3.18) are really orthogonal. A similar argument shows that for fixed h and $k_1, k_2 \in \Delta_h$ the functions $p^{(h)}(2^{n_{k_1}} x)$ and $p^{(h)}(2^{n_{k_2}} x)$ are orthogonal if not both k_1 and k_2 are in the set $\{h^2+1, h^2+\lceil 2h\alpha \rceil\}$. Thus

$$\int_0^1 \left(\sum_{h=0}^{i-1} \left(\sum_{k \in \Delta_h} p^{(h)}\left(2^{n_k} x\right) \right) \right)^2 \tag{3.19}$$

$$= \sum_{h=0}^{i-1} \left(\int_0^1 \left(\sum_{k \in \{h^2+1, h^2+\lceil 2h\alpha \rceil\}} p^{(h)}\left(2^{n_k} x\right) \right)^2 dx + \sum_{k \in \{h^2+\lceil 2h\alpha \rceil+1, (h+1)^2\}} \|p^{(h)}\|^2 \right).$$

For $h \to \infty$,

$$\int_0^1 \left(\sum_{k \in \{h^2+1, h^2+\lceil 2h\alpha \rceil\}} p^{(h)}\left(2^{n_k} x\right) \right)^2 dx \sim \gamma_f^2 \left(h^2 + \lceil 2h\alpha \rceil - (h^2+1)\right) \sim \gamma_f^2 2h\alpha,$$

and

$$\sum_{k \in \{h^2+\lceil 2h\alpha \rceil+1, (h+1)^2\}} \|p^{(h)}\|^2 \sim \|f\|^2 \left((h+1)^2 - (h^2 + \lceil 2h\alpha \rceil)\right) \sim \|f\|^2 2h(1-\alpha).$$

Thus by (3.19) for $i \to \infty$

$$\int_0^1 \left(\sum_{h=0}^{i-1} \left(\sum_{k \in \Delta_h} p^{(h)}\left(2^{n_k} x\right) \right) \right)^2$$

$$
\begin{aligned}
&\sim \sum_{h=0}^{i-1} \left(\gamma_f^2 2h\alpha + \|f\|^2 2h(1-\alpha)\right) \\
&\sim \left(\gamma_f^2\alpha + \|f\|^2(1-\alpha)\right) \sum_{h=0}^{i-1} 2h \\
&\sim \rho^2 i^2. \qquad (3.20)
\end{aligned}
$$

Combining (3.16), (3.17), (3.20) we finally obtain

$$
\int_0^1 \left(\sum_{k=1}^{N} f\left(2^{n_k}x\right)\right)^2 dx \sim \rho^2 N \qquad \text{as } N \to \infty. \qquad (3.21)
$$

Note that in our argument we assumed $\alpha \in (0,1)$, i.e. that ρ is an inner point of the interval $[\|f\|, \gamma_f]$. The case $\alpha = 1$ (i.e. $\rho = \gamma_f$) is trivial, with $n_k = k$. In the case $\alpha = 0$ we choose (n_k) growing very rapidly and the theory of lacunary series implies (3.21) with $\rho = \|f\|$.

Relation (3.21) is not identical with (3.15), since the sequence (n_k) is not a permutation of $\mathbb{N}$. However, from (n_k) we can easily construct a permutation σ such that (3.15) holds. Let H denote the set of positive integers not contained in (n_k) and insert the elements of H into the sequence $n_1, n_2, \dots$ by leaving very rapidly increasing gaps between them. The so obtained sequence is a permutation σ of $\mathbb{N}$ and if the gaps between the inserted elements grow sufficiently rapidly, then clearly the asyimptotics of the integrals in (3.15) and (3.21) are the same, i.e. (3.15) holds. This completes the proof of the fact that the class of limits

$$
\lim_{N\to\infty} \frac{1}{N} \int_0^1 \left(\sum_{k=1}^{N} f\left(2^{\sigma(k)}x\right)\right)^2 dx
$$

is identical with the interval $[\|f\|^2, \gamma_f^2]$.

In conclusion we show that without assuming the nonnegativity of the Fourier coefficients of f, the class Γ_f of limiting variances in Theorem 1 for permuted sequences $f(n_{\sigma(k)}x)$ is not necessarily the closed interval with endpoints $\|f\|^2$ and γ_f^2. Let

$$
f(x) = \cos 2\pi x - \cos 4\pi x + \cos 8\pi x
$$

and again $n_k = 2^k$. Then taking into account the cancellations in the sum $\sum_{k=1}^{N} f(n_k x)$ we get

$$
\sum_{k=1}^{N} f(n_k x) = \cos 4\pi x + \cos 16\pi x + \cos 32\pi x + \cos 64\pi x + \cos 128\pi x + \dots
$$

whence

$$
\int_0^1 \left(\sum_{k=1}^{N} f(n_k x)\right)^2 dx \sim N/2
$$

so that $\gamma_f^2 = 1/2$ and clearly $\|f\|^2 = 3/2$. (Note that in this case $\gamma_f < \|f\|$.) Now

$$\sum_{k=1}^{N} f(4^k x) = \cos 8\pi x - \cos 16\pi x$$
$$+ 2\cos 32\pi x - \cos 64\pi x + 2\cos 128\pi x - \cos 256x + 2\cos 512x - \dots$$

and thus

$$\int_0^1 \left(\sum_{k=1}^{N} f(4^k x) \right)^2 dx \sim 5N/2.$$

Similarly as above, we can get a permutation σ of $\mathbb{N}$ such that

$$\int_0^1 \left(\sum_{k=1}^{N} f(2^{\sigma(k)} x) \right)^2 dx \sim 5N/2.$$

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