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ON THE ASYMPTOTIC BEHAVIOR OF WEAKLY LACUNARY SERIES

C. AISTLEITNER, I. BERKES, AND R. TICHY

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Abstract. Let f be a measurable function satisfying

$$f(x+1)=f(x), \quad \int_0^1 f(x)\,dx=0, \quad \mathrm{Var}_{[0,1]} f<+\infty,$$

and let $(n_k)_{k\geq 1}$ be a sequence of integers satisfying $n_{k+1}/n_k \geq q>1$ ($k=1,2,\ldots$). By the classical theory of lacunary series, under suitable Diophantine conditions on n_k, $(f(n_k x))_{k\geq 1}$ satisfies the central limit theorem and the law of the iterated logarithm. These results extend for a class of subexponentially growing sequences $(n_k)_{k\geq 1}$ as well, but as Fukuyama showed, the behavior of $f(n_k x)$ is generally not permutation-invariant; e.g. a rearrangement of the sequence can ruin the CLT and LIL. In this paper we construct an infinite order Diophantine condition implying the permutation-invariant CLT and LIL without any growth conditions on $(n_k)_{k\geq 1}$ and show that the known finite order Diophantine conditions in the theory do not imply permutation-invariance even if $f(x)=\sin 2\pi x$ and $(n_k)_{k\geq 1}$ grows almost exponentially. Finally, we prove that in a suitable statistical sense, for almost all sequences $(n_k)_{k\geq 1}$ growing faster than polynomially, $(f(n_k x))_{k\geq 1}$ has permutation-invariant behavior.

1. Introduction

Let f be a measurable function satisfying

$$f(x+1)=f(x), \quad \int_0^1 f(x)\,dx=0, \quad \mathrm{Var}_{[0,1]} f<+\infty \tag{1}$$

and let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \geq q>1 \qquad (k=1,2,\ldots). \tag{2}$$

In the case $n_k=2^k$, Kac [14] proved that $f(n_k x)$ satisfies the central limit theorem

$$N^{-1/2}\sum_{k=1}^{N} f(n_k x) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma^2) \tag{3}$$

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with respect to the probability space $[0,1]$ equipped with the Lebesgue measure, where

$$\sigma^2 = \int_0^1 f^2(x)\,dx + 2\sum_{k=1}^{\infty}\int_0^1 f(x)f(2^k x)\,dx.$$

Gaposhkin [11] extended (3) to the case when the fractions n_{k+1}/n_k are all integers or if $n_{k+1}/n_k \to \alpha$, where α^r is irrational for $r = 1, 2, \ldots$. On the other hand, an example of Erdős and Fortet (see [15], p. 646) shows that the CLT (3) fails if $n_k = 2^k - 1$. Gaposhkin also showed (see [12]) that the asymptotic behavior of $\sum_{k=1}^{N} f(n_k x)$ is intimately connected with the number of solutions of the Diophantine equation

$$an_k + bn_l = c, \qquad 1 \le k, l \le N.$$

Improving these results, Aistleitner and Berkes [1] gave a necessary and sufficient condition for the CLT (3). For related laws of the iterated logarithm, see [5], [11], [13], and [17].

The previous results show that for arithmetically "nice" sequences $(n_k)_{k\ge1}$, the system $f(n_k x)$ behaves like a sequence of independent random variables. However, as an example of Fukuyama [9] shows, this result is not permutation-invariant: a rearrangement of $(n_k)_{k\ge1}$ can change the variance of the limiting Gaussian law or ruin the CLT altogether. A complete characterization of the permutation-invariant CLT and LIL for $f(n_k x)$ under the Hadamard gap condition (2) is given in our forthcoming paper [3]. In particular, it is shown there that in the harmonic case $f(x) = \cos 2\pi x$, $f(x) = \sin 2\pi x$ the CLT and LIL for $f(n_k x)$ hold after any permutation of $(n_k)_{k\ge1}$.

For subexponentially growing $(n_k)_{k\ge1}$ the situation changes radically. Note that in the case $f(x) = \cos 2\pi x$, $f(x) = \sin 2\pi x$, the unpermuted CLT and LIL remain valid under the weaker gap condition

$$n_{k+1}/n_k \ge 1 + ck^{-\alpha}, \qquad 0 < \alpha < 1/2;$$

see Erdős [8], Takahashi [18], [19]. However, as the following theorem shows, the slightest weakening of the Hadamard gap condition (2) can ruin the permutation-invariant CLT and LIL.

Theorem 1. *For any positive sequence $(\varepsilon_k)_{k\ge1}$ tending to 0, there exists a sequence $(n_k)_{k\ge1}$ of positive integers satisfying*

$$n_{k+1}/n_k \ge 1 + \varepsilon_k, \qquad k \ge k_0 \tag{4}$$

and a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ of the positive integers such that

$$N^{-1/2}\sum_{k=1}^{N} \cos 2\pi n_{\sigma(k)}x - b_N \xrightarrow{\mathcal{D}} G, \tag{5}$$

where G is a non-Gaussian distribution with characteristic function given by (14)-(16) *and $(b_N)_{N\ge1}$ is a numerical sequence with $b_N = O(1)$. Moreover, there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ of the positive integers such that*

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \cos 2\pi n_{\sigma(k)}x}{\sqrt{2N\log\log N}} = +\infty \quad \text{a.e.} \tag{6}$$

As we will see, with a slight change of the norming sequence $N^{-1/2}$ in (5) the limit distribution G can also be chosen as the Cauchy distribution with density $\pi^{-1}(1+x^2)^{-1}$. The proof of Theorem 1 will also show that (6) can be improved to

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_{\sigma(k)}x}{\sqrt{N\log N}} > 0 \quad \text{a.e.} \tag{7}$$

It should be noted that the subexponential gap condition (4) does not imply the permutation-invariant behavior of $f(n_k x)$ even for arithmetically "nice" sequences $(n_k)_{k\ge 1}$. Indeed, the sequence $(n_k)_{k\ge 1}$ in Theorem 1 can be chosen so that it satisfies conditions **B**, **C**, **G** in our paper [7] implying very strong independence properties of $\cos 2\pi n_k x$, $\sin 2\pi n_k x$, including the CLT and LIL. In fact, it is not easy to construct subexponential sequences $(n_k)_{k\ge 1}$ satisfying the permutation-invariant CLT and LIL: the only known example (see [2]) is the Hardy-Littlewood-Pólya sequence, i.e. the sequence generated by finitely many primes and arranged in increasing order; the proof uses deep number-theoretic tools. The purpose of this paper is to introduce a new, infinite order Diophantine condition $\mathbf{A}_{\boldsymbol{\omega}}$ which implies the permutation-invariant CLT and LIL for $f(n_k x)$ and then to show that, in a suitable statistical sense, almost all sequences $(n_k)_{\ge 1}$ growing faster than polynomially satisfy $\boldsymbol{A}_{\boldsymbol{\omega}}$. Thus, despite the difficulties to construct explicit examples, the permutation-invariant CLT and LIL are rather the rule than the exception.

Given a nondecreasing sequence $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots)$ of positive numbers tending to $+\infty$, let us say that a sequence $(n_k)_{k\ge 1}$ of different positive integers satisfies

Condition $\boldsymbol{A}_{\boldsymbol{\omega}}$ if for any $N \ge N_0$ the Diophantine equation

$$a_1 n_{k_1} + \ldots + a_r n_{k_r} = 0, \qquad 2 \le r \le \omega_N,\ 0 < |a_1|, \ldots, |a_r| \le N^{\omega_N} \tag{8}$$

with different indices k_j and nonzero integer coefficients a_j has only such solutions, where all n_{k_j} belong to the smallest N elements of the sequence $(n_k)_{k\ge 1}$.

Clearly, this property is permutation-invariant and it implies that for any fixed nonzero integer coefficients a_j the number of solutions of (8) with different indices k_j is at most N^r.

Theorem 2. *Let $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots)$ be a nondecreasing sequence tending to $+\infty$ and let $(n_k)_{k\ge 1}$ be a sequence of different positive integers satisfying condition $\boldsymbol{A}_{\boldsymbol{\omega}}$. Then for any f satisfying* (1) *we have*

$$N^{-1/2} \sum_{k=1}^N f(n_k x) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \|f\|^2), \tag{9}$$

where $\|f\|$ denotes the $L_2(0,1)$ norm of f. If $\omega_k \ge (\log k)^\alpha$ for some $\alpha > 0$ and $k \ge k_0$, then we also have

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N f(n_k x)}{(2N\log\log N)^{1/2}} = \|f\| \quad \text{a.e.} \tag{10}$$

Condition $\boldsymbol{A}_{\boldsymbol{\omega}}$ is different from the usual Diophantine conditions in lacunarity theory, which typically involve 4 or fewer terms. In contrast, $\boldsymbol{A}_{\boldsymbol{\omega}}$ is an "infinite order" condition; namely, it involves equations with arbitrary large order. As noted, the usual Diophantine conditions do not suffice in Theorem 2. Given any $\omega_k \uparrow \infty$, it is not hard to see that any sufficiently rapidly growing sequence $(n_k)_{k\ge 1}$ satisfies $\boldsymbol{A}_{\boldsymbol{\omega}}$; on the other hand, we do not have any "concrete" subexponential examples for

$\boldsymbol{A_\omega}$. However, we will show that, in a suitable statistical sense, almost all sequences growing faster than polynomially satisfy condition $\boldsymbol{A_\omega}$ for some appropriate $\boldsymbol{\omega}$. To make this precise requires defining a probability measure over the set of such sequences, or, equivalently, a natural random procedure to generate such sequences. A simple procedure is to choose n_k independently and uniformly from the integers in the interval

$$I_k = [a(k-1)^{\omega_{k-1}}, ak^{\omega_k}), \qquad k = 1, 2, \ldots . \tag{11}$$

Note that the length of I_k is at least $a\omega_k(k-1)^{\omega_k - 1} \geq a\omega_1$ for $k = 2, 3, \ldots$ and equals a for $k = 1$, and thus choosing a large enough, each I_k contains at least one integer. Let $\mu_{\boldsymbol{\omega}}$ be the distribution of the random sequence $(n_k)_{k\geq 1}$ in the product space $I_1 \times I_2 \times \ldots$.

Theorem 3. *Let $\omega_k \uparrow \infty$ and let f be a function satisfying* (1). *Then with probability one with respect to $\mu_{\boldsymbol{\omega}}$ the sequence $(f(n_k x))_{k\geq 1}$ satisfies the CLT* (9) *after any permutation of its terms, and if $\omega_k \geq (\log k)^\alpha$ for some $\alpha > 0$ and $k \geq k_0$, $(f(n_k x))_{k\geq 1}$ also satisfies the LIL* (10) *after any permutation of its terms.*

The sequences $(n_k)_{k\geq 1}$ provided by $\mu_{\boldsymbol{\omega}}$ satisfy $n_k = O(k^{\omega_k})$; for slowly increasing ω_k, the so-obtained sequences grow much slower than exponentially, in fact they grow barely faster than polynomial speed. If ω_k grows so slowly that $\omega_k - \omega_{k-1} = o((\log k)^{-1})$, then the so-obtained sequence $(n_k)_{k\geq 1}$ has the precise speed $n_k \sim k^{\omega_k}$. We do not know if there exist polynomially growing sequences $(n_k)_{k\geq 1}$ satisfying the permutation-invariant CLT or LIL. The proof of Theorem 3 will also show that with probability 1, the sequences provided by $\mu_{\boldsymbol{\omega}}$ satisfy $\boldsymbol{A_{\omega^*}}$ with $\boldsymbol{\omega}^* = (c\omega_1^{1/2}, c\omega_2^{1/2}, \ldots)$.

2. Proofs

2.1. Proof of Theorem 1.

We begin with the CLT part. Let $(\varepsilon_k)_{k\geq 1}$ be a positive sequence tending to 0. Let $m_1 < m_2 < \ldots$ be positive integers such that $m_{k+1}/m_k \geq 2^{k^2}$, $k = 1, 2, \ldots$ and all the m_k are powers of 2; let $r_1 \leq r_2 \leq \ldots$ be positive integers satisfying $1 \leq r_k \leq k^2$. Put $I_k = \{m_k, 2m_k, \ldots, r_k m_k\}$; clearly the sets I_k, $k = 1, 2, \ldots$ are disjoint. Define the sequence $(n_k)_{k\geq 1}$ by

$$(n_k)_{k\geq 1} = \bigcup_{j=1}^{\infty} I_j. \tag{12}$$

Clearly, if $n_k, n_{k+1} \in I_j$, then $n_{k+1}/n_k \geq 1 + 1/r_j$ and thus if r_j grows sufficiently slowly, the sequence $(n_k)_{k\geq 1}$ satisfies the gap condition (4). Also, if r_j grows sufficiently slowly, there exists a subsequence $(n_{k_\ell})_{\ell\geq 1}$ of $(n_k)_{k\geq 1}$ which has exactly the same structure as the sequence in (12), just with $r_k \sim k$. By the proof of Theorem 1 in [4], $(\cos 2\pi n_{k_\ell} x)_{\ell\geq 1}$ satisfies

$$\frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} \cos 2\pi n_{k_\ell} x - b_N \xrightarrow{\mathcal{D}} G, \tag{13}$$

where $(b_N)_{N\geq 1}$ is a numerical sequence with $b_N = O(1)$ and G is a non-Gaussian infinitely divisible distribution with characteristic function

$$\exp\left\{\int\limits_{R\setminus\{0\}}\left(e^{itx}-1-\frac{itx}{1+x^2}\right)dL(x)\right\}, \tag{14}$$

where

$$L(x)=\begin{cases}-\frac{1}{\pi}\int_x^1\frac{F(t)}{t}dt & \text{if} \quad 0<x\leq 1,\\ \frac{1}{\pi}\int_{-x}^1\frac{G(t)}{t}dt & \text{if} \quad -1\leq x<0,\\ 0 & \text{if} \quad \mid x\mid>1\end{cases} \tag{15}$$

and

$$F(t)=\lambda\{x>0:\sin x/x\geq t\},\quad G(t)=\lambda\{x>0:\sin x/x\leq -t\}\quad (t>0), \tag{16}$$

where λ is the Lebesgue measure. Define a permutation σ in the following way:

- for $k\notin\{1,2,4,\dots,2^m,\dots\}$, $\sigma(k)$ takes the values of the set $\{k_1,k_2,\dots\}$ in consecutive order;
- for $k\in\{1,2,4,\dots,2^m,\dots\}$, $\sigma(k)$ takes the values of the set $\mathbb{N}\setminus\{k_1,k_2,\dots\}$ in consecutive order.

Then σ is a permutation of $\mathbb{N}$ and the sums

$$\sum_{k=1}^N\cos 2\pi n_{\sigma(k)}x \qquad \text{and} \qquad \sum_{l=1}^N\cos 2\pi n_{k_l}x$$

differ at most in $2\log_2 N$ terms. Therefore, (13) implies (5), proving the first part of Theorem 1.

The proof of the LIL part of Theorem 1 is modeled after the proof of Theorem 1 in Berkes and Philipp [6]. Similarly as above, we construct a sequence $(n_k)_{k\geq1}$ satisfying (4) that contains a subsequence $(\mu_k)_{k\geq1}$ of the form (12) with $I_k=\{m_k,2m_k,\dots,r_km_k\}$, where $r_k\sim k\log k$ and $(m_k)_{k\geq1}$ is growing fast; specifically we choose m_k in such a way that it is a power of 2 and $m_{k+1}\geq r_k2^{2k}m_k$. Let $\mathcal{F}_i$ denote the σ-field generated of the dyadic intervals

$$[\nu 2^{-(\log_2 m_i)-i},(\nu+1)2^{-(\log_2 m_i)-i}),\quad 0\leq\nu<2^{(\log_2 m_i)+i}.$$

Write

$$X_i=\cos 2\pi m_ix+\cdots+\cos 2\pi r_im_ix$$

and

$$Z_i=\mathbb{E}(X_i|\mathcal{F}_i).$$

Then for all $x\in(0,1)$,

$$|X_i(x)-Z_i(x)|\ll r_i^2m_i2^{-(\log_2 m_i)-i}$$

and thus

$$\sum_{i\geq1}|X_i(x)-Z_i(x)|<\infty\qquad\text{for all } x\in(0,1).$$

As in [6, Lemma 2.1], the random variables $Z_1,Z_2,\dots$ are independent, and as in [6, Lemma 2.2], for almost every $x\in(0,1)$ we have

$$\limsup_{i\to\infty}X_i/r_i\geq 2/\pi.$$

Assume that for a fixed x and some $i \geq 1$ we have $X_i/r_i \geq 1/\pi$. Then either

$$|X_1 + \cdots + X_{i-1}| \geq r_i/2\pi$$

or

$$|X_1 + \cdots + X_i| \geq r_i/2\pi.$$

Since the total number of summands in $X_1, \ldots, X_i$ is $\ll i^2 \log i$, and since

$$r_i \gg (i^2 \log i)^{1/2} (\log(i^2 \log i))^{1/2},$$

we have

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \cos 2\pi \mu_k x\right|}{\sqrt{N \log N}} > 0 \qquad \text{a.e.},$$

and, in particular,

$$\limsup_{N\to\infty} \frac{\left|\sum_{k=1}^N \cos 2\pi \mu_k x\right|}{\sqrt{2N \log\log N}} = +\infty \qquad \text{a.e.}$$

Thus we constructed a subsequence of $(n_k)_{k\geq 1}$ failing the LIL and similarly as above, we can construct a permutation $(n_{\sigma(k)})_{k\geq 1}$ of (n_k) failing the LIL as well.

Note that the just completed proof of Theorem 1 provides the stronger relation (7) instead of (6). Also, if relation $r_k \sim k$ just preceding relation (13) is replaced by $r_k \sim k \log\log k$, then the proof of Theorem 2 in [4] shows that (5) will be replaced by

$$a_N^{-1} \sum_{k=1}^N \cos 2\pi n_{\sigma(k)} x - b_N \xrightarrow{\mathcal{D}} G,$$

where G is the Cauchy distribution with density $\pi^{-1}(1+x^2)^{-1}$ and $a_N \sim c\sqrt{N/\log\log N}$ with some $c > 0$ and $b_N = O(1)$.

2.2. **Proof of Theorem 2.**

Lemma 1. *Let $\omega_k \uparrow \infty$ and let $(n_k)_{k\geq 1}$ be a sequence of different positive integers satisfying condition $\boldsymbol{A_\omega}$. Let f satisfy (1) and put $S_N = \sum_{k=1}^N f(n_k x)$, $\sigma_N = (\mathbb{E}S_N^2)^{1/2}$. Then for any $p \geq 3$ we have*

$$\mathbb{E}S_N^p = \begin{cases} \frac{p!}{(p/2)!} 2^{-p/2} \sigma_N^p + O(T_N) & \text{if } p \text{ is even,} \\ O(T_N) & \text{if } p \text{ is odd,} \end{cases}$$

where

$$T_N = \exp(p^2) N^{(p-1)/2} (\log N)^p$$

and the constants implied by the O are absolute.

Proof. Fix $p \geq 2$ and choose the integer N so large that $\omega_{[N^{1/4}]} \geq 8p$. Without loss of generality we may assume that f is an even function and that $\|f\|_\infty \leq 1$, $\mathrm{Var}_{[0,1]} f \leq 1$; the proof in the general case is similar. Let

$$f \sim \sum_{j=1}^\infty a_j \cos 2\pi j x \tag{17}$$

be the Fourier series of f. $\mathrm{Var}_{[0,1]} f \leq 1$ implies that

$$|a_j| \leq j^{-1} \tag{18}$$

(see Zygmund [20, p. 48]), and writing

$$g(x) = \sum_{j=1}^{N^p} a_j \cos 2\pi jx, \qquad r(x) = f(x) - g(x),$$

we have

$$\|g\|_\infty \le \mathrm{Var}_{[0,1]} f + \|f\|_\infty \le 2, \qquad \|r\|_\infty \le \|f\|_\infty + \|g\|_\infty \le 3$$

by (4.12) of Chapter II and (1.25) and (3.5) of Chapter III of Zygmund [20]. Letting $\|\cdot\|$ and $\|\cdot\|_p$ denote the $L_2(0,1)$, resp. $L_p(0,1)$, norms, (18) yields for any positive integer n,

$$\|r(nx)\|^2 = \|r(x)\|^2 = \frac{1}{2}\sum_{j=N^p+1}^{\infty} a_j^2 \le N^{-p}. \tag{19}$$

By Minkowski's inequality,

$$\|S_N\|_p \le \|\sum_{k=1}^N g(n_k x)\|_p + \|\sum_{k=1}^N r(n_k x)\|_p$$

and

$$\|\sum_{k=1}^N r(n_k x)\|_p \le 3\sum_{k=1}^N \|r(n_k x)/3\|_p \le 3\sum_{k=1}^N \|r(n_k x)/3\|^{2/p} \le 3\sum_{k=1}^N N^{-1} \le 3. \tag{20}$$

Similarly,

$$\left|\|S_N\| - \|\sum_{k=1}^N g(n_k x)\|\right| \le \|\sum_{k=1}^N r(n_k x)\| \le N^{-\frac{p}{2}+1},$$

and therefore

$$\begin{aligned}
&\left|\|S_N\|^p - \|\sum_{k=1}^N g(n_k x)\|^p\right| \\
&\qquad \le p\ \max\left(\|S_N\|^{p-1}, \|\sum_{k=1}^N g(n_k x)\|^{p-1}\right)\cdot\left|\|S_N\| - \|\sum_{k=1}^N g(n_k x)\|\right| \\
&\qquad \ll p\left(N(\log\log N)^2\right)^{\frac{p-1}{2}} N^{-\frac{p}{2}+1} \\
&\qquad \ll p(\log\log N)^{p-1} N^{1/2} \qquad (21)
\end{aligned}$$

since by a result of Gál [10] and Koksma [16],

$$\|S_N\|^2 \ll N(\log\log N)^2 \quad \text{and} \quad \|\sum_{k=1}^N g(n_k x)\|^2 \ll N(\log\log N)^2, \tag{22}$$

where the implied constants are absolute.

By expanding and using elementary properties of the trigonometric functions we get

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=1}^N g(n_k x)\right)^p \\
&\quad = 2^{-p}\sum_{1\le j_1,\dots,j_p\le N^p} a_{j_1}\cdots a_{j_p}\sum_{1\le k_1,\dots,k_p\le N} \mathbb{I}\{\pm j_1 n_{k_1} \pm \ldots \pm j_p n_{k_p} = 0\}, \qquad (23)
\end{aligned}$$

with all possibilities of the signs $\pm$ within the indicator function. Assume that $j_1, \dots, j_p$ and the signs $\pm$ are fixed, and consider a solution of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$. Then the set $\{1, 2, \dots, p\}$ can be split into disjoint sets $A_1, \dots, A_l$ such that for each such set A we have $\sum_{i \in A} \pm j_i n_{k_i} = 0$ and no further subsums of these sums are equal to 0. Group the terms of $\sum_{i \in A} \pm j_i n_{k_i}$ with equal k_i. If after grouping there are at least two terms, then by the restriction on subsums, the sum of the coefficients j_i in each group will be different from 0 and will not exceed

$$pN^p \le \omega_{[N^{1/4}]} N^{\frac{1}{8}\omega_{[N^{1/4}]}} \le 2^{\frac{1}{8}\omega_{[N^{1/4}]}} N^{\frac{1}{8}\omega_{[N^{1/4}]}} \le N^{\frac{1}{4}\omega_{[N^{1/4}]}} \qquad (N \ge N_0).$$

Also the number of terms after grouping will be at most $p \le \omega_{[N^{1/4}]}$ and thus applying condition $\boldsymbol{A_\omega}$ with the index $[N^{1/4}]$ shows that within a block A the n_{k_i} belong to the smallest $[N^{1/4}]$ terms of the sequence. Thus letting $|A| = m$, the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ is at most $N^{m/4}$. If after grouping there is only one term, then all the k_i are equal and thus the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ is at most N. Thus if $m \ge 3$, then the number of solutions of $\sum_{i \in A} \pm j_i n_{k_i} = 0$ in a block is at most $N^{m/3}$. If $m = 2$, then the number of solutions is clearly at most N. Thus if $s_i = |A_i|$ $(1 \le i \le l)$ denotes the cardinality of A_i, the number of solutions of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$ admitting such a decomposition with fixed $A_1, \dots, A_l$ is at most

$$\begin{aligned} \prod_{\{i: s_i \ge 3\}} N^{s_i/3} \prod_{\{i: s_i = 2\}} N &= N^{\frac{1}{3}\sum_{\{i:s_i \ge 3\}} s_i + \sum_{\{i:s_i=2\}} 1} \\ &= N^{\frac{1}{3}\sum_{\{i:s_i \ge 3\}} s_i + \frac{1}{2}\sum_{\{i:s_i=2\}} s_i} = N^{\frac{1}{3}\sum_{\{i:s_i \ge 3\}} s_i + \frac{1}{2}(p - \sum_{\{i:s_i \ge 3\}} s_i)} \\ &= N^{\frac{p}{2} - \frac{1}{6}\sum_{\{i:s_i \ge 3\}} s_i}. \end{aligned}$$

If there is at least one i with $s_i \ge 3$, then the last exponent is at most $(p-1)/2$ and since the number of partitions of the set $\{1, \dots, p\}$ into disjoint subsets is at most $p!\,2^p$, we see that the number of solutions of $\pm j_1 n_{k_1} \pm \dots \pm j_p n_{k_p} = 0$, where at least one of the sets A_i has cardinality ≥ 3, is at most $p!\,2^p N^{(p-1)/2}$. If p is odd, there are no other solutions and thus using (18) the inner sum in (23) is at most $p!\,2^p N^{(p-1)/2}$ and consequently, taking into account the 2^p choices for the signs ± 1,

$$\begin{aligned} &\left| \mathbb{E} \left(\sum_{k \le N} g(n_k x) \right)^p \right| \\ &\quad \le p!\,2^p N^{(p-1)/2} \sum_{1 \le j_1, \dots, j_p \le N^p} |a_{j_1} \cdots a_{j_p}| \ll \exp(p^2) N^{(p-1)/2} (\log N)^p. \end{aligned}$$

If p is even, there are also solutions where each A has cardinality 2. Clearly, the contribution of the terms in (23), where $A_1 = \{1,2\}, A_2 = \{3,4\}, \ldots$, is

$$\left(\frac{1}{4} \sum_{1 \le i,j \le N^{2p}} \sum_{1 \le k,\ell \le N} a_i a_j \mathbb{I}\{\pm i n_k \pm j n_\ell = 0\}\right)^{p/2} = \left(\mathbb{E}\left(\sum_{k \le N} g(n_k x)\right)^2\right)^{p/2}$$
$$= \left\|\sum_{k \le N} g(n_k x)\right\|^p$$
$$= \|S_N\|^p + \mathcal{O}\left(p (\log\log N)^{p-1} N^{1/2}\right)$$

by (21).

Since the splitting of $\{1,2,\ldots,p\}$ into pairs can be done in $\frac{p!}{(p/2)!} 2^{-p/2}$ different ways, we proved that

$$\mathbb{E}\left(\sum_{k \le N} g(n_k x)\right)^p = \begin{cases} \frac{p!}{(p/2)!} 2^{-\frac{p}{2}} \sigma_N^p + O(T_N), \\ O(T_N) \end{cases} \tag{24}$$

according as p is even or odd; here

$$T_N = \exp(p^2) N^{(p-1)/2} (\log N)^p.$$

Now, letting $G_N = \sum_{k \le N} g(n_k x)$ we get, using (20), (22) and (24),

$$\begin{aligned} &|\mathbb{E} S_N^p - \mathbb{E} G_N^p| \\ &\quad \le p \ \max\left(\|S_N\|_p^{p-1}, \|G_N\|_p^{p-1}\right) \cdot \left|\|S_N\|_p - \|G_N\|_p\right| \\ &\quad \ll p \left(\frac{p!}{(p/2)!} 2^{-\frac{p}{2}} \sigma_N^p\right)^{\frac{p-1}{p}} \\ &\quad \ll T_N, \end{aligned}$$

where in the last step we used the fact that $\sigma_N \ll \sqrt{N} \log\log N$ by (22). This completes the proof of Lemma 1. □

Lemma 2. *Let $\omega_k \uparrow \infty$ and let $(n_k)_{k \ge 1}$ be a sequence of different positive integers satisfying condition $\boldsymbol{A_\omega}$. Then for any f satisfying* (1) *we have*

$$\int_0^1 \left(\sum_{k=1}^N f(n_k x)\right)^2 dx \sim \|f\|^2 N \quad \text{as } N \to \infty. \tag{25}$$

Proof. Clearly, $\omega_{[N^{1/4}]} \ge 4$ for sufficiently large N and thus applying Condition $\boldsymbol{A_\omega}$ for the index $[N^{1/4}]$ it follows that for $N \ge N_0$ the Diophantine equation

$$j_1 n_{i_1} + j_2 n_{i_2} = 0, \qquad i_1 \ne i_2,\ 0 < |j_1|, |j_2| \le N \tag{26}$$

has only such solutions, where n_{i_1}, n_{i_2} belong to the set J_N of $[N^{1/4}]$ smallest elements of the sequence $(n_k)_{k \ge 1}$. Write $p_N(x)$ for the N-th partial sum of the Fourier series of f, and r_N for the N-th remainder term. Then we have for any f

satisfying (1),
(27)
$$\left\|\sum_{k=1}^{N} f(n_k x)\right\| \geq \left\|\sum_{k\in[1,N]\setminus J_N} p_N(n_k x)\right\| - \left\|\sum_{k\in J_N} f(n_k x)\right\| - \left\|\sum_{k\in[1,N]\setminus J_N} r_N(n_k x)\right\|.$$

Using the previous remark on the number of solutions of (26) we get, as in (23),

$$\left\|\sum_{k\in[1,N]\setminus J_N} p_N(n_k x)\right\| = (N-[N^{1/4}])^{1/2}\ \|p_N\| \sim N^{1/2}\|f\|,$$

since $\|p_N\| \to \|f\|$. Further, $\|r_N\| \ll N^{-1/2}$ by (18) and thus using Minkowski's inequality and the results of Gál and Koksma mentioned in (22), we get

$$\left\|\sum_{k\in J_N} f(n_k x)\right\| \ll N^{1/4}, \qquad \left\|\sum_{k\in[1,N]\setminus J_N} r_N(n_k x)\right\| \ll \sqrt{N}\log\log N\|r_N\| \ll \log\log N.$$

These estimates, together with (27), prove Lemma 2.

Lemma 1 and Lemma 2 imply that for any fixed $p \geq 2$, the p-th moment of S_N/σ_N converges to $\frac{p!}{(p/2)!}2^{-p/2}$ if p is even and to 0 if p is odd; in other words, the moments of S_N/σ_N converge to the moments of the standard normal distribution. By $\sigma_N \sim \|f\|\sqrt{N}$ and a well-known result in probability theory, this proves the CLT part of Theorem 2. The proof of the LIL part of Theorem 2 is more involved, and we will give just a sketch of the proof. The details can be modeled after the proof of [2, Theorem 1]. The crucial ingredient is Lemma 3 below, which yields the LIL part of Theorem 2, just as [2, Theorem 1] follows from [2, Lemma 3].

Let $\theta > 1$ and define $\Delta'_M = \{k \in \mathbb{N} : \theta^M < k \leq \theta^{M+1}\}$ and $T'_M = \sum_{k\in\Delta'_M} f(n_k x)$. By the standard method of proof of the LIL, we need precise bounds for the tails of T'_M and also, a near independence relation for the T'_M for the application of the Borel-Cantelli lemma in the lower half of the LIL. From the set Δ'_M we remove its $[\theta^{M/4}]$ elements with the smallest value of n_k (recall that the sequence $(n_k)_{k\geq 1}$ is not assumed to be increasing) and denote the remaining set by Δ_M. Since the number of removed elements is $\ll |\Delta'_M|^{1/4}$, this operation does not influence the partial sum asymptotics of T'_M. As in the proof of Lemma 1, we assume that we have a representation of f in the form (17) and that (18) holds. Define

$$g_M(x) = \sum_{j=1}^{[\theta^M]^2} a_j \cos 2\pi j x, \qquad \sigma_M^2 = \int_0^1 \left(\sum_{k\in\Delta_M} g_M(n_k x)\right)^2 dx$$

and

$$T_M = \sum_{k\in\Delta_M} g_M(n_k x), \qquad Z_M = T_M/\sigma_M.$$

From Lemma 2 it follows easily that

$$\sigma_M \gg |\Delta_M|^{1/2}. \tag{28}$$

Assume that $(n_k)_{k\geq 1}$ satisfies Condition $\boldsymbol{A_\omega}$ for a sequence $(\omega_k)_{k\geq 1}$ with $\omega_k \geq (\log k)^\alpha$ for some $\alpha > 0$, $k \geq k_0$. Without loss of generality we may assume

$0 < \alpha < 1/2$. Choose $\delta > 0$ so small that for sufficiently large r,

$$\left(\log \theta^{\sqrt{r}/4}\right)^{\alpha} > 4\left(\log \theta^{r}\right)^{\delta}. \tag{29}$$ □

Lemma 3. *For sufficiently large M, N satisfying $N^{1-\alpha/2} \le M \le N$, and for positive integers p, q satisfying $p+q \le (\log \theta^N)^{\delta}$ we have*

$$\mathbb{E} Z_M^p Z_N^q = \begin{cases} \frac{p!}{(p/2)!2^{p/2}} \frac{q!}{(q/2)!2^{q/2}} + \mathcal{O}(R_{M,N}) & \text{if } p, q \text{ are even,} \\ \mathcal{O}(R_{M,N}) & \text{otherwise,} \end{cases}$$

where

$$R_{M,N} = 2^{p+q}(p+q)!\,(\log M)^{p+q}|\Delta_M|^{-1/2}.$$

Proof. Note that $N \le M^{1+3\alpha/4}$ and thus, setting $L = [\theta^{M/4}]$, relation $p+q \le (\log \theta^N)^{\delta}$ and (29) imply

$$p+q \le \frac{1}{4}(\log \theta^{\sqrt{N}/4})^{\alpha} \le \frac{1}{4}(\log \theta^{M/4})^{\alpha} \le \omega_L$$

and by a simple calculation,

$$(p+q)\left(\theta^N\right)^2 \le [\theta^{M/4}]^{(\log[\theta^{M/4}])^{\alpha}} \le L^{\omega_L}$$

provided N is large enough. Applying condition $\mathbf{A}_{\boldsymbol{\omega}}$ we get that for all solutions of the equation

$$\pm j_1 n_{k_1} \pm \cdots \pm j_p n_{k_p} \pm j_{p+1} n_{k_{p+1}} \pm \cdots \pm j_{p+q} n_{k_{p+q}} = 0 \tag{30}$$

with different indices $k_1, \ldots, k_{p+q}$, where

$$1 \le j_i \le (p+q)\left(\theta^N\right)^2, \tag{31}$$

the n_{k_j} belong to the $[\theta^{M/4}]$ smallest elements of $(n_k)_{k\ge1}$. By construction not a single one of these elements is contained in Δ_M or Δ_N. Thus the equation (30) subject to (31) has no solution $(k_1, \ldots, k_{p+q})$, where $k_1, \ldots, k_{p+q}$ are different and satisfy

$$k_1, \ldots, k_p \in \Delta_M,\ k_{p+1}, \ldots, k_{p+q} \in \Delta_N.$$

Now

$$\mathbb{E} Z_M^p Z_N^q = \frac{2^{-p-q}}{\sigma_M^p \sigma_N^q} \sum_{\substack{1\le j_q,\ldots,j_p \le [\theta^M]^2,\\ 1\le j_{p+1},\ldots,j_{p+q}\le[\theta^N]^2}}$$

$$\times \sum_{\substack{k_1,\ldots,k_p\in\Delta_M,\\ k_{p+1},\ldots,k_{p+q}\in\Delta_N}} a_{j_1}\ldots a_{j_{p+q}}\ \mathbf{1}\{\pm j_1 n_{k_1} \pm \cdots \pm j_{p+q} n_{k_{p+q}} = 0\}. \tag{32}$$

If for some $k_1, \ldots, k_{p+q}$ we have (note that in (32) these indices need not be different)

$$\pm j_1 n_{k_1} \pm \cdots \pm j_{p+q} n_{k_{p+q}} = 0, \tag{33}$$

then grouping the terms of the equation according to identical indices, we get a new equation of the form

$$j_1' n_{l_1} + \ldots + j_s' n_{l_s} = 0, \qquad l_1 < \ldots < l_s,\ s \le p+q,\ j_i' \le (p+q)(\theta^N)^2$$

and using the above observation, all the coefficients $j_1', \ldots, j_s'$ must be equal to 0. In other words, in any solution of (33) the terms can be divided into groups such that in each group the n_{k_j} are equal and the sum of the coefficients is 0. Consider first the solutions where all groups have cardinality 2. This can happen only if both p and q are even, and similarly to the proof of Lemma 1, the contribution of such solutions in (32) is

$$\frac{p!}{(p/2)!2^{p/2}}\frac{q!}{(q/2)!2^{q/2}}.$$

Consider now the solutions of (33), where at least one group has cardinality ≥ 3. Clearly the sets $\{k_1,\ldots,k_p\}$ and $\{k_{p+1},\ldots k_{p+q}\}$ are disjoint; let us denote the number of groups within these two sets by R and S, respectively. Evidently $R \leq p/2$, $S \leq q/2$, and at least one of the inequalities is strict. Fixing $j_1,\ldots,j_{p+q}$ and the groups, the number of such solutions cannot exceed

$$|\Delta_M|^R|\Delta_N|^S \leq |\Delta_M|^{p/2}|\Delta_N|^{q/2}|\Delta_M|^{-1/2} \ll \sigma_M^p\sigma_N^q|\Delta_M|^{-1/2},$$

where we used (28) and the fact that $|\Delta_M| \leq |\Delta_N|$. Since the number of partitions of the set $\{1,2,\ldots,p+q\}$ into disjoint subsets is at most $(p+q)!2^{p+q}$ and since the number of choices for the signs $\pm$ in (33) is at most 2^{p+q}, we see, after summing over all possible values of $j_1,\ldots,j_{p+q}$, that the contribution of the solutions containing at least one group with cardinality ≥ 3 in (32) is at most $2^{p+q}(p+q)!|\Delta_M|^{-1/2}(\log[\theta^N])^{p+q}$. This completes the proof of Lemma 3. □

The rest of the proof of the LIL part of Theorem 2 can be modeled following the lines of Lemma 4, Lemma 5, Lemma 6 and the proof of Theorem 1 in [2].

2.3. **Proof of Theorem 3.** Let $\omega_k \uparrow \infty$ and set $\eta_k = \frac{1}{2}\omega_k^{1/2}$, $\boldsymbol{\eta} = (\eta_1,\eta_2,\ldots)$. Clearly

$$(2k)^{\eta_k^2+2\eta_k} \leq (2k)^{\omega_k/2} \leq k^{-2}|I_k| \qquad \text{for } k \geq k_0 \tag{34}$$

since, as we noted, $|I_k| \geq a\omega_k(k-1)^{\omega_k-1} \geq (k/2)^{\omega_k-1}$ for large k. We choose n_k, $k=1,2,\ldots$, independently and uniformly from the integers of the intervals I_k in (11). We claim that, with probability 1, the sequence $(n_k)_{k\geq 1}$ is increasing and satisfies condition $\boldsymbol{A_\eta}$. To see this, let $k \geq 1$ and consider the numbers of the form

$$(a_1n_{i_1}+\ldots+a_sn_{i_s})/d, \tag{35}$$

where $1 \leq s \leq \eta_k$, $1 \leq i_1,\ldots,i_s \leq k-1$, $a_1,\ldots,a_s,d$ are nonzero integers with $|a_1|,\ldots,|a_s|,|d| \leq k^{\eta_k}$. Since the number of values in (35) is at most $(2k)^{\eta_k^2+2\eta_k}$, and by (34), the probability that n_k equals any of these numbers is at most k^{-2}. Thus by the Borel-Cantelli lemma, with probability 1 for $k \geq k_1$, n_k will be different from all the numbers in (35) and thus the equation

$$a_1n_{i_1}+\ldots+a_sn_{i_s}+a_{s+1}n_k = 0$$

has no solution with $1 \leq s \leq \eta_k$, $1 \leq i_1 < \ldots < i_s \leq k-1$, $0 < |a_1|,\ldots,|a_{s+1}| \leq k^{\eta_k}$. By monotonicity, the equation

$$a_1n_{i_1}+\ldots+a_sn_{i_s} = 0$$

has no solutions provided the indices i_ν are all different, the maximal index is at least k, the number of terms is at most η_k, and $0 < |a_1|,\ldots,|a_s| \leq k^{\eta_k}$. In other words, (n_k) satisfies condition $\boldsymbol{A_\eta}$. Now using Theorem 2, we get Theorem 3.

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Institute of Mathematics A, Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria

E-mail address: aistleitner@math.tugraz.at

Institute of Statistics, Graz University of Technology, Münzgrabenstrasse 11, 8010 Graz, Austria

E-mail address: berkes@tugraz.at

Institute of Mathematics A, Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria

E-mail address: tichy@tugraz.at