Stochastics and Dynamics


# ASYMPTOTIC BEHAVIOR OF TRIMMED SUMS *

ISTVÀN BERKES

*Institute of Statistics*
*Graz University of Technology*
*8010 Graz, Austria*
*berkes@tugraz.at*

LAJOS HORVÀTH

*Department of Mathematics*
*University of Utah*
*Salt Lake City, UT 84112-0090 USA*
*horvath@math.utah.edu*

JOHANNES SCHAUER

*Institute of Statistics*
*Graz University of Technology*
*8010 Graz, Austria*
*johannes@schauer.com*

Trimming is a standard method to decrease the effect of large sample elements in statistical procedures, used, e.g., for constructing robust estimators. It is also a powerful tool in understanding deeper properties of partial sums of independent random variables. In this paper we review some basic results of the theory and discuss new results in the central limit theory of trimmed sums. In particular, we show that for random variables in the domain of attraction of a stable law with parameter $0 < \alpha < 2$, the asymptotic behavior of modulus trimmed sums depends sensitively on the number of elements eliminated from the sample. We also show that under moderate trimming, the central limit theorem always holds if we allow random centering factors. Finally, we give an application to change point problems.

*Keywords*: order statistic, trimming, asymptotic normality

AMS Subject Classification: 60F05, 62E20

## 1. Introduction

Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables in the domain of attraction of a stable law with index $0 < \alpha < 2$, i.e. assume there exist real-valued sequences

*This work was supported by FWF Grant No. S9603-N23 and OTKA grants K 67961 and K 81928 (István Berkes), NSF Grant DMS 00905400 (Lajos Horváth) and FWF Grant No. S9603-N23 (Johannes Schauer)

$\{a_n, n \in \mathbb{N}\}$ and $\{b_n, n \in \mathbb{N}\}$ such that

$$\frac{1}{b_n}\left(\sum_{j=1}^{n} X_j - a_n\right) \xrightarrow{\mathcal{D}} Z_\alpha \tag{1.1}$$

as $n \to \infty$, where $Z_\alpha$ is a stable r.v. with index $\alpha$. As is known, this is the case if and only if the distribution function $F$ of $X_1$ satisfies

$$1 - F(x) \sim pL(x)\,x^{-\alpha}, \qquad F(-x) \sim qL(x)\,x^{-\alpha} \qquad \text{as } x \to \infty, \tag{1.2}$$

where $p, q \geq 0$, $p + q = 1$ and $L$ is slowly varying at infinity. In the case $p = 0$ the first relation of (1.2) is meant as $1 - F(x) = o(L(x)x^{-\alpha})$ and a similar remark holds for $q = 0$. As usual, in the case when (1.1) or (1.2) holds, we say that $F$ is in the domain of attraction of the stable variable $Z_\alpha$ and write $F \in D(\alpha)$. Relation (1.1) is analogous to the central limit theorem, but there is a crucial difference: letting $X_n^{(j)}$ denote the element of the sample $(X_1, \ldots X_n)$ with the $j$-th largest absolute value, $X_n^{(j)}/b_n$ is known to have a nondegenerate limit distribution for any fixed $j \geq 1$ and deleting $X_n^{(j)}$ from $S_n$, the resulting sum has a different asymptotic behavior as $S_n$. This sensitive dependence of the partial sum behavior on extremal elements of the sample is not desired in many statistical problems and starting in the 1960's, several authors investigated the asymptotic behavior of trimmed sums, i.e. sums where one or more extremal elements are omitted from $S_n$. In general, two types of trimmed sums are considered. Let

$$X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n} \tag{1.3}$$

be the order statistics of $(X_1, \ldots, X_n)$ and

$$|X_n^{(1)}| \geq |X_n^{(2)}| \geq \ldots \geq |X_n^{(n)}| \tag{1.4}$$

be the sample elements arranged in decreasing order corresponding to their absolute values. (In both cases, ties are broken according to priority of index.) The classical trimmed sum, denoted by $S_n(r_n, s_n)$, is defined as

$$S_n(r_n, s_n) = \sum_{j=r_n+1}^{n-s_n} X_{j,n}$$

for two nonnegative integer sequences $\{r_n, n \in \mathbb{N}\}$, $\{s_n, n \in \mathbb{N}\}$ satisfying $r_n + s_n < n$ for all $n \in \mathbb{N}$. Thus the $r_n$ smallest and the $s_n$ largest elements are removed from the partial sum $S_n$ to obtain $S_n(r_n, s_n)$. In the simplest case (e.g. for symmetric distributions $F$), we choose $r_n = s_n$ and set $S_n(r_n) := S_n(r_n, r_n)$. The second approach is based on the elements $X_n^{(j)}$ and leads to the so-called modulus trimmed sum $\tilde{S}_n(r_n)$ defined by

$$\tilde{S}_n(r_n) = \sum_{j=r_n+1}^{n} X_n^{(j)},$$

where $\{r_n, n \in \mathbb{N}\}$ is a numerical sequence with $r_n \in \{0, 1, \ldots, n-1\}$ for $n \in \mathbb{N}$. Here the $r_n$ largest elements according to their absolute values are removed. We distinguish three different types of trimming:

(1) light trimming: $r_n \equiv r$ (or bounded $r_n$)
(2) moderate trimming: $r_n \to \infty$, $r_n/n \to 0$
(3) heavy trimming: $r_n/n \to c \in (0, 1)$.

In most cases light trimming does not improve the weak convergence behavior. For example, Mori [27] showed for general distributions that there are numerical sequences $\{c_n, n \in \mathbb{N}\}$, $\{d_n, n \in \mathbb{N}\}$ such that $(\tilde{S}_n(r) - c_n)/d_n \xrightarrow{\mathcal{D}} N$ if and only if $(S_n - c_n)/d_n \xrightarrow{\mathcal{D}} N$, where $N \sim N(0,1)$ denotes a standard normal random variable throughout the paper. Kesten [24] extended this result by showing that there exist $\{c_n\}$, $\{d_n\}$ such that $(\tilde{S}_n(r) - c_n)/d_n$ converges in distribution if and only if $(S_n - c_n)/d_n$ converges in distribution (to a possibly degenerate stable r.v.). The same result holds for $S_n(r)$. The complete solution of the asymptotic distribution problem for $(S_n(r) - c_n)/d_n$ was given by Csörgő et al. [7] (see also [6]) who described precisely the class of possible limit distributions and gave exact convergence criteria along the whole sequence of integers and along subsequences. Other important results concerning light trimming are due to Darling [12], Arov and Bobrov [1], Hall [21], Maller [26]. A classical result in case of heavy trimming with $r_n = \lfloor an \rfloor$, $s_n = \lfloor bn \rfloor$, $a, b > 0$, $a + b < 1$ is due to Stigler [29], who found the limit law of $S_n(r_n, s_n)$. He showed that to have a normal limit law, it is necessary and sufficient that the $a$-th and $b$-th quantiles of the underlying distribution are uniquely defined.

In contrast to weak convergence behavior, light trimming can improve almost sure behavior of partial sums. Feller [14] showed that removing the largest element from an i.i.d. sample $(X_1, \ldots, X_n)$ has a crucial effect on the LIL behavior of the partial sums $S_n$ in the case $EX^2 = +\infty$. Csörgő and Simons [11] found a similar effect for the strong law of large numbers for i.i.d. random variables in the domain of partial attraction of a semistable law. The same phenomenon occurs in analysis, for the partial sums of continued fraction digits, see Diamond and Vaaler [13].

The purpose of this paper is to review some basic results in the central limit theory of moderately trimmed sums and to formulate new results. For several further important contributions in trimming theory not discussed in the present paper, we refer to the book [20] and the references therein. In particular, [20] discusses a number of further variants of trimming like censoring and 'winsorized' trimming, see e.g. Hahn et al. [19] for an asymptotic theory.

## 2. Moderate trimming

In the rest of the paper we restrict our attention to moderate trimming, and thus we assume that $r_n \to \infty$ and $r_n/n \to 0$ (as well as $s_n \to \infty$ and $s_n/n \to 0$). Our main interest will be the weak limit theory of trimmed sums, i.e. the problem of finding sequences $\{c_n\}$ and $\{d_n\}$ such that

$$(S_n(r_n) - c_n)/d_n \xrightarrow{\mathcal{D}} G \tag{2.1}$$

or

$$(\tilde{S}_n(r_n) - c_n)/d_n \xrightarrow{\mathcal{D}} G \tag{2.2}$$

with a nondegenerate limit $G$ and characterization of the possible limit distributions $G$, together with precise criteria for convergence to a specific $G$. Another natural problem is partial attraction, i.e. finding criteria for (2.1), (2.2) to hold along subsequences of integers and characterizing the class of subsequential limits. For laws of the iterated logarithm for trimmed sequences we refer to Haeusler and Mason [22], [23].

In the case of i.i.d. random variables in the domain of attraction of a stable law with parameter $0 < \alpha < 2$, Teugels [30] studied the limiting behavior of the ratio $T_n(r_n) = \tilde{S}_n(r_n)/|X_n^{(r_n)}|$ under moderate trimming. He found sufficient conditions (stronger than (1.2)) for the asymptotic normality of $T_n(r_n)$. In particular, he showed asymptotic normality when $F$ is continuous and symmetric. In the nonsymmetric case he proved the following result.

**Theorem 2.1.** *Assume that as $x \to \infty$*

$$1 - F(x) = px^{-\alpha} + bx^{-\alpha-\beta} + o(x^{-\alpha-\beta})$$

$$F(-x) = qx^{-\alpha} + b'x^{-\alpha-\beta} + o(x^{-\alpha-\beta})$$

*where $0 < \alpha < 2$, $\alpha \neq 1$, $p, q \geq 0$, $p + q = 1$, $0 < \beta \leq \alpha$ and $b, b'$ are real constants. Then $T_n(r_n)$ is asymptotically normal provided $r_n = o(n^\gamma)$ with $\gamma$ chosen as*

$$\gamma = \begin{cases} \min\left\{\frac{2\beta}{\alpha+2\beta}, \frac{2(1-\alpha)}{2-\alpha}\right\}, & 0 < \alpha < 1, \\ \min\left\{\frac{2\beta}{\alpha+2\beta}, \frac{2}{2+\alpha}\right\}, & 1 < \alpha < 2,\ EX_1 \neq 0, \\ \frac{2\beta}{\alpha+2\beta}, & 1 < \alpha < 2,\ EX_1 = 0. \end{cases}$$

The surprising feature of this theorem is the assumption $r_n = o(n^\gamma)$ in the nonsymmetric case: this restricts $r_n$ from above, while one would expect that trimming more terms from the sum $S_n$ (i.e. increasing $r_n$) improves its behavior. In Section 3 we will see that this paradoxical restriction on $r_n$ is necessary: choosing $r_n$ too large, the CLT becomes generally false and we will give fairly precise bounds on $r_n$ such that the CLT holds.

In the case $F \in D(\alpha)$, Csörgő et al. [10] proved the following central limit theorem for the symmetrically trimmed sums $S_n(r_n)$.

**Theorem 2.2.** *Assume (1.2). Then there exists a numerical sequence $\{c_n\}$ and a slowly varying function $L$ (at infinity) such that*

$$\frac{S_n(r_n) - c_n}{n^{1/\alpha} r_n^{(\alpha-2)/2\alpha} L(n/r_n)} \xrightarrow{\mathcal{D}} N, \tag{2.3}$$

*where $N$ is a standard normal random variable.*

In other words, in the case $F \in D(\alpha)$ the trimmed sum $S_n(r_n)$ is asymptotically normal under moderate trimming without any additional conditions on $F$. Note that the analogous statement for $S_n(r_n, s_n)$ is not always valid, see Griffin and Pruitt [16], p. 1206. The authors are indebted to David Mason for pointing out this fact, as well as for pointing out a number of further relevant references to the considered limit problem.

A complete solution of the asymptotic distribution problem for $S_n(r_n, s_n)$ was given by Csörgő et al. [8]; for the formulation we need some definitions. Let $Q$ be the quantile function of $F$ defined as

$$Q(y) = \inf\{x : F(x) \geq y\}$$

for $0 < y \leq 1$ with $Q(0) = \lim_{x\to 0+} Q(x)$. Put further

$$c_n^* = n \int_{r_n/n}^{1-s_n/n} Q(y)dy,$$

$$(d_n^*)^2 = \int_{r_n/n}^{1-s_n/n} \int_{r_n/n}^{1-s_n/n} (y \wedge z - yz)dQ(y)dQ(z),$$

where $y \wedge z = \min(y, z)$, $y \vee z = \max(y, z)$, and with this for all $\gamma \in \mathbb{R}$

$$\Psi_{1,n}(\gamma) = \begin{cases} \left(\frac{r_n}{n}\right)^{1/2} \frac{\left(Q\left(\frac{r_n}{n}+\gamma\frac{r_n^{1/2}}{n}\right)-Q\left(\frac{r_n}{n}\right)\right)}{d_n^*}, & -\frac{1}{2}r_n^{1/2} \leq \gamma \leq \frac{1}{2}r_n^{1/2}, \\ \Psi_{1,n}\left(-\frac{r_n^{1/2}}{2}\right), & -\infty < \gamma < -\frac{1}{2}r_n^{1/2}, \\ \Psi_{1,n}\left(\frac{r_n^{1/2}}{2}\right), & \frac{1}{2}r_n^{1/2} < \gamma < \infty, \end{cases}$$

$$\Psi_{2,n}(\gamma) = \begin{cases} \left(\frac{s_n}{n}\right)^{1/2} \frac{\left(Q\left(1-\frac{s_n}{n}+\gamma\frac{s_n^{1/2}}{n}\right)-Q\left(1-\frac{s_n}{n}\right)\right)}{d_n^*}, & -\frac{1}{2}s_n^{1/2} \leq \gamma \leq \frac{1}{2}s_n^{1/2}, \\ \Psi_{2,n}\left(-\frac{s_n^{1/2}}{2}\right), & -\infty < \gamma < -\frac{1}{2}s_n^{1/2}, \\ \Psi_{2,n}\left(\frac{s_n^{1/2}}{2}\right), & \frac{1}{2}s_n^{1/2} < \gamma < \infty, \end{cases}$$

$$u_{1,n} = -\left(\frac{r_n}{n}\right)^{1/2} (d_n^*)^{-1} \int_{r_n/n}^{1-s_n/n} (1-y)dQ(y),$$

$$u_{2,n} = -\left(\frac{s_n}{n}\right)^{1/2} (d_n^*)^{-1} \int_{r_n/n}^{1-s_n/n} ydQ(y).$$

**Theorem 2.3.** *There exist numerical sequences $\{c_n\}$ and $\{d_n\}$ such that*

$$(S_n(r_n, s_n) - c_n)/d_n \xrightarrow{\mathcal{D}} N$$

*if and only if for all $\gamma \in \mathbb{R}$*

$$\lim_{n\to\infty} \Psi_{1,n}(\gamma) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \Psi_{2,n}(\gamma) = 0. \tag{2.4}$$

*In this case one can set* $c_n = c_n^*$ *and* $d_n = n^{1/2} d_n^*$.

Theorem 2.3 characterizes the domain of attraction of the normal law for trimmed sums. The results of [8] also give, in terms of the quantities $\Psi_{1,n}$, $\Psi_{2,n}$, $u_{1,n}$, $u_{2,n}$, necessary and sufficient conditions for weak convergence of centered and normed trimmed sums along any fixed subsequence of integers, together with a complete description of the possible limit distributions, thereby characterizing domains of partial attraction for the trimmed sums $S_n(r_n, s_n)$. The method of Csörgő et al. [8] depends on the quantile transform and weighted approximation of the uniform empirical process. Using a different approach, Griffin and Pruitt [16] (see also [17]) gave later equivalent convergence criteria for $(S_n(r_n, s_n) - c_n)/d_n$ in terms of the truncated moments of $X$. For comparison, we formulate here their CLT characterization. Put for all $\gamma, \delta \in \mathbb{R}$ and sufficiently large $n$,

$$\begin{aligned} a_n(\gamma) &= \inf\{x : F(-x) \leq n^{-1}(r_n - \gamma r_n^{1/2})\}, \\ b_n(\delta) &= \inf\{x : 1 - F(x) < n^{-1}(s_n - \delta s_n^{1/2})\}. \end{aligned}$$

Then a necessary and sufficient condition for the asymptotic normality of $S_n(r_n, s_n)$ is

$$\frac{E[(X \wedge b_n(\delta)) \vee (-a_n(\gamma))]^2}{E[(X \wedge b_n(0)) \vee (-a_n(0))]^2} \to 1 \qquad \text{for all } \gamma, \delta \in \mathbb{R}. \tag{2.5}$$

Until now, we discussed the weak convergence problem for the trimmed sums $S_n(r_n, s_n)$. The analogous problem for modulus trimmed sums $\tilde{S}_n(r_n)$ has a different character. In case of symmetric distributions, Griffin and Pruitt [15] gave the following characterization of asymptotic normality.

**Theorem 2.4.** *Let the distribution of* $X$ *be symmetric and continuous and let* $\tilde{a}_n(\gamma)$ *be any solution of* $P(|X| > x) = n^{-1}(r_n - \gamma r_n^{1/2})$. *Then there exists a numerical sequence* $\{d_n\}$ *such that*

$$\tilde{S}_n(r_n)/d_n \xrightarrow{\mathcal{D}} N$$

*if and only if for all* $\gamma \in \mathbb{R}$

$$\frac{EX^2 I(|X| \leq \tilde{a}_n(\gamma))}{EX^2 I(|X| \leq \tilde{a}_n(0))} \to 1, \tag{2.6}$$

*where* $I$ *denotes the indicator function. In this case one can take*

$$d_n = [n\, EX^2 I(|X| \leq \tilde{a}_n(0))]^{1/2}.$$

Note that we assumed here that $X$ is continuous, in which case the criterion is slightly simpler. In [15], a complete description of subsequential limits of $(\tilde{S}_n(r_n) - c_n)/d_n$ is also given in the symmetric case, together with the corresponding convergence criteria. The class of sequential limits was later determined by Griffin and Qazi [18] (see also [17]). Alternative proofs for the convergence criteria using the quantile-empirical process approach were given by Csörgő et al. [9].

Note that symmetry is essential in Theorem 2.4, as is in most existing results on $\tilde{S}_n(r_n)$. In fact, in [15], pp. 346–349, Griffin and Pruitt gave, for any sequence $\{r_n\}$, $r_n \to \infty$, $r_n/n \to 0$ an example of an i.i.d. sequence in the domain of attraction of a symmetric stable law with index $\alpha$, $0 < \alpha < 2$, such that $(\tilde{S}_n(r_n) - c_n)/d_n$ is not asymptotically normal for any $\{c_n\}$, $\{d_n\}$. Thus not even the symmetry of the limiting stable law generally suffices for modulus trimmed sums to satisfy the CLT in the moderately trimmed case.

## 3. Recent results and applications

The results of the previous section give a complete description of the weak limit behavior of trimmed sums $S_n(r_n, s_n)$: we have convergence criteria for normed sums $(S_n(r_n, s_n) - c_n)/d_n$ along $\mathbb{N}$ and subsequences, together with a characterization of the class of possible limits and subsequential limits. In contrast, the theory for modulus trimmed sums $\tilde{S}_n(r_n)$ is less complete: there exist exact convergence criteria for symmetric distributions, but the nonsymmetric case remains open. Even in the case $F \in D(\alpha)$, no necessary and sufficient conditions have been found for the asymptotic normality of $\tilde{S}_n(r_n)$ in the moderate trimming case. Recall that by Csörgő et. al [10], the trimmed sum $S_n(r_n)$ is asymptotically normal for $F \in D(\alpha)$ without any additional conditions, but by the earlier mentioned counterexample of Griffin and Pruitt [15], the corresponding result for $\tilde{S}_n(r_n)$ generally fails, even if the limiting stable law is symmetric. The only positive result in the nonsymmetric case appears to be the theorem of Teugels [30] giving normal convergence criteria for $\tilde{S}_n(r_n)/|X_n^{(r_n)}|$ under moderate trimming, provided $r_n$ is not too large. The purpose of the present section is to give a detailed discussion of the nonsymmetric case and formulate new results, together with an application to change point problems.

Let $X_1, X_2, \ldots$ be i.i.d. random variables belonging to the domain of attraction of a stable law with parameter $0 < \alpha < 2$. To avoid minor complications, we assume in this section that $X$ has a continuous distribution and thus, with probability one, all the elements $|X_j|$, $j = 1, 2, \ldots$ are different. Consider the modulus trimmed sum

$$\tilde{S}_n(r_n) = \sum_{j=1}^{n} X_j I\{|X_j| \le |X_n^{(r_n)}|\}, \tag{3.1}$$

where $X_n^{(j)}$ denotes, as in Section 1, the element of the sample $(X_1, \ldots X_n)$ with the $j$-th largest absolute value. (Note that in (3.1) the number of elements eliminated from the total sum is $r_n - 1$ and thus by the notation of Section 1 this sum is $\tilde{S}_n(r_n - 1)$. However, using the present notation will lighten the formulas.) It was proved by Kiefer [25] (see also Shorack and Wellner [28]) that $|X_n^{(r_n)}|$ is close to $H^{-1}(r_n/n)$, where $H(x) = P(|X| > x)$ and $H^{-1}(x)$ is its generalized inverse. Thus it is natural to compare the trimmed sum $\tilde{S}_n(r_n)$ in (3.1) with the truncated sum

$$\tilde{T}_n(r_n) = \sum_{j=1}^{n} X_j I\{|X_j| \le H^{-1}(r_n/n)\}. \tag{3.2}$$

We still assume that $r_n \to \infty$ and $r_n/n \to 0$. Clearly, $\tilde{T}_n(r_n)$ is a sum of independent, identically distributed random variables and using Ljapunov's condition it follows immediately that

$$(\tilde{T}_n(r_n) - E\tilde{T}_n(r_n))/D_n \xrightarrow{\mathcal{D}} N, \tag{3.3}$$

where

$$D_n^2 = \frac{\alpha}{2-\alpha} r_n \left(H^{-1}(r_n/n)\right)^2. \tag{3.4}$$

Hence if we could prove that

$$(\tilde{S}_n(r_n) - \tilde{T}_n(r_n))/D_n \xrightarrow{P} 0, \tag{3.5}$$

then the asymptotic normality of $\tilde{S}_n(r_n)$ would follow. Relation (3.5), however, is generally false for nonsymmetric distributions, as the following example shows.

**Example 3.1.** Assume that $X$ is concentrated on $(0, \infty)$ with $P(X > x) = cx^{-\alpha}$ for $x \geq x_0$. Then the left hand side of (3.5) has a nondegenerate Gaussian limit distribution.

We thus see that the difference between the trimmed sum $\tilde{S}_n(r_n)$ and the truncated sum $\tilde{T}_n(r_n)$ is generally not $o_P(D_n)$, and in fact, the asymptotic behavior of trimmed and truncated sums can be different. To explore this further, it will be convenient to work with the functional CLT. Put, for $0 \leq t \leq 1$,

$$\tilde{S}_{n,r_n}(t) = \sum_{j=1}^{\lfloor nt \rfloor} X_j I\{|X_j| \leq |X_n^{(r_n)}|\}, \ \tilde{T}_{n,r_n}(t) = \sum_{j=1}^{\lfloor nt \rfloor} X_j I\{|X_j| \leq H^{-1}(r_n/n)\}.$$

By the functional version of (3.3) we have

$$(\tilde{T}_{n,r_n}(t) - c_n(t))/D_n \xrightarrow{\mathcal{D}[0,1]} W(t), \tag{3.6}$$

where

$$c_n(t) = \lfloor nt \rfloor E(XI\{|X| \leq H^{-1}(r_n/n)\})$$

and $\{W(t),\ 0 \leq t \leq 1\}$ is a standard Wiener process. However, the analogue of (3.6) for $\tilde{S}_{n,r_n}(t)$ is generally false. To get the correct result, put

$$m(t) = EXI\{|X| \leq t\}, \quad t \geq 0.$$

Then we have

**Theorem 3.1.** *Let $X_1, X_2, \ldots$ be i.i.d. random variables with a continuous distribution function $F$ satisfying (1.2) for $0 < \alpha < 2$ and in the case $\alpha = 1$ assume that $X_1$ is symmetric. Assume that $r_n/n \to 0$, $r_n/(\log n)^{7+\varepsilon} \to \infty$ for some $\varepsilon > 0$. Then we have*

$$\frac{1}{D_n} \sum_{j=1}^{\lfloor nt \rfloor} \left( X_j I\{|X_j| \leq |X_n^{(r_n)}|\} - c_n \right) \xrightarrow{\mathcal{D}[0,1]} W(t), \tag{3.7}$$

*where $D_n$ is defined by (3.4), further*

$$c_n = m(|X_n^{(r_n)}|) \tag{3.8}$$

*and $\{W(t),\ 0 \le t \le 1\}$ is a standard Wiener process.*

For the proof of Theorem 3.1 and Example 3.1, we refer to Berkes et al. [5]. The case $\alpha = 1$ remains unsolved in the nonsymmetric case. Relation (3.7) is a functional CLT for the modulus trimmed sums $\tilde{S}_n(r_n)$, but the centering factor $c_n$ in (3.8) is random. For symmetric $F$ we have $c_n = 0$, leading to a standard (nonrandom) functional CLT for trimmed sums, but in the case of Example 3.1 above, $nc_n/D_n$ has, after a nonrandom translation, a nondegenerate Gaussian limit distribution and thus $c_n$ cannot be replaced by a nonrandom centering factor. Thus we see that under $F \in D(\alpha)$ the center of the modulus trimmed sample is generally a nondegenerate random variable, explaining why the ordinary CLT fails in general for such sums. This also means that modulus trimming is generally unsuitable for statistical purposes under $F \in D(\alpha)$.

Theorem 3.1 yields a randomized functional CLT for $\tilde{S}_n(r_n)$ in the domain of attraction of a stable law, but it does not give any information when the ordinary (nonrandom) CLT holds. The following theorems give a partial answer to this question. We start with a simple special case.

**Theorem 3.2.** *Assume that*

$$P(X > x) = px^{-\alpha}L(x) \quad \text{and} \quad P(X \le -x) = qx^{-\alpha}L(x) \quad \text{for } x \ge x_0,$$

*where $p, q \ge 0$, $p + q = 1$, $0 < \alpha < 2$ and $L$ is slowly varying at infinity. Assume that $r_n \to \infty$, $r_n/n \to 0$. Then*

$$\frac{1}{r_n^{1/2} H^{-1}(r_n/n)} (\tilde{S}_n(r_n) - c_n') \xrightarrow{\mathcal{D}} \left( \frac{\alpha}{2-\alpha} + (p-q)^2 \right)^{1/2} N,$$

*where*

$$c_n' = n \left( \int_{-x_0}^{x_0} x dF(x) + (p-q) \int_{r_n/n}^{H(x_0)} H^{-1}(t) dt \right).$$

In Theorem 3.2 the tails of the distribution of $X$ are "perfectly balanced", i.e. $P(X \le -x)/P(X > x) = c$ for $x \ge x_0$ with some constant $c > 0$ or $X$ is one-sided, i.e. it is concentrated on a half line. In the case when $L$ is a constant (i.e. when $X$ is in the domain of normal attraction of a stable law), this condition can be substantially weakened, as our next theorem shows.

**Theorem 3.3.** *Let, as $x \to \infty$,*

$$P(X > x) = px^{-\alpha} + O(x^{-\beta}) \quad \text{and} \quad P(X \le -x) = qx^{-\alpha} + O(x^{-\beta}) \tag{3.9}$$

*where $p, q > 0$, $p + q = 1$, $0 < \alpha < 2$ and $\beta > \alpha$, and assume that*

$$r_n \to \infty, \quad r_n = o\big(n^{(\beta-\alpha)/\tau}\big) \quad \text{with} \quad \tau = \max(1 + 2\alpha, \beta - \alpha/2). \tag{3.10}$$

*Then there exist numerical sequences* $\{c_n\}$ *and* $\{d_n\}$ *such that*

$$(\tilde{S}_n(r_n) - c_n)/d_n \xrightarrow{\mathcal{D}} N. \tag{3.11}$$

*Replacing (3.10) by*

$$r_n \to \infty, \quad r_n = O\big(n^{(\beta-\alpha)/\tau}\big) \quad \textit{with} \quad \tau = \beta - 3\alpha/4,$$

*the CLT (3.11) becomes generally false.*

Note that, like in Theorem 3 of Teugels [30], we assumed here an upper bound for $r_n$ to guarantee the validity of the CLT for $\tilde{S}_n(r_n)$. This is rather surprising, since for larger $r_n$ we remove more elements from the sample and hence one would believe that the effectiveness of the trimming increases. However, as the last statement of Theorem 3.3 shows, an upper bound for $r_n$ for the validity of the CLT for $\tilde{S}_n(r_n)$ is needed as well. The critical order of magnitude is $r_n \sim n^\gamma$ for some $0 < \gamma < 1$ whose value remains unknown.

The remainder term in the tail condition (3.9) in Theorem 3.3 was $O(x^{-\beta})$ with $\beta > \alpha$. The following theorem describes the situation when the remainder term is $O(x^{-\alpha}/L(x))$ with a function $L(x) \to \infty$ growing slower than any power of $x$.

**Theorem 3.4.** *Assume that, as* $x \to \infty$,

$$P(X > x) = px^{-\alpha} + O(x^{-\alpha}/L(x)), \qquad P(X \le -x) = qx^{-\alpha} + O(x^{-\alpha}/L(x))$$

*where* $p, q > 0$, $p + q = 1$, $0 < \alpha < 2$ *and* $L$ *is a nondecreasing function satisfying*

$$\lim_{x\to\infty} L(x) = \infty \qquad \textit{and} \quad \limsup_{x\to\infty} L(x^2)/L(x) < \infty. \tag{3.12}$$

*Then under*

$$r_n \to \infty, \qquad r_n = o(L(n)^{\alpha/(2\alpha+1)}) \tag{3.13}$$

*we have the CLT (3.11) with suitable numerical sequences* $\{c_n\}$, $\{d_n\}$. *Replacing the second relation of (3.13) by* $r_n = O(L(n)^4)$, *the CLT (3.11) becomes generally false.*

Relation (3.12) and the monotonicity of $L$ imply that $L$ is slowly varying. Actually, instead of (3.12) it suffices to assume that $L(x) \to \infty$ and $L$ is slowly varying, but then condition (3.13) should be replaced by

$$r_n \to \infty, \quad r_n = o(L(n^{1/\alpha-\varepsilon}))^{\alpha/(2\alpha+1)}$$

for some $0 < \varepsilon < 1/\alpha$.

For the proof of Theorems 3.2–3.4, see Berkes and Horváth [4]. Note that in Theorems 3.3 and 3.4 we assumed $p > 0$, $q > 0$, i.e. our results do not cover the extremely asymmetric cases when in (3.9) or in the corresponding relation in Theorem 3.4 we have $p = 0$ or $q = 0$. In these cases one tail of $F$ has a smaller order of magnitude than the other and need not even be regularly varying. This situation is rather pathological and we have only partial results in this case. For example, in [4] we proved that if

$$1-F(x)=px^{-\alpha}+O(x^{-\beta}), \qquad F(-x)=qx^{-\gamma}+o(x^{-\gamma}) \qquad (x\to\infty)$$

for $p,q>0$, $0<\alpha<2$, $\alpha<\beta<\gamma$ (note that in this case the smaller tail is actually regularly varying) then under the condition (3.10) the trimmed CLT (3.11) is still valid for sufficiently large $\gamma$. The CLT also holds in the one sided case, i.e. when one of the tail conditions in (3.9) is satisfied and the other tail is indentically 0. For a detailed discussion and more results we refer to [4].

Theorems 3.2–3.4 give a fairly clear picture on the central limit theorem for modulus trimmed sums, but their application for statistical purposes is made difficult by the fact that it is impossible to decide from a sample if the underlying distribution $F$ satisfies the technical conditions of Theorems 3.2–3.4. In contrast, Theorem 3.1 is very suitable for statistical purposes, despite the random centering factor $c_n$. Consider e.g. the location model

$$X_j=\mu+\delta I(j>m^*)+e_j \qquad \text{for } j=1,\dots,n,$$

where $1\le m^*\le n$, $\mu$ and $\delta=\delta_n$ are unknown parameters. We assume that $e_1,\dots,e_n$ are i.i.d. random variables. We want to test the hypothesis $H_0: m^*=n$ (no change) against $H_1: m^*<n$. In the case when $Ee_1=0$, $Ee_1^2=1$, the standard CUSUM test uses the supremum of the function

$$U_n(t)=\sum_{j=1}^{\lfloor nt\rfloor}\left(X_j-\overline{X}_n\right), \quad \text{where } \overline{X}_n=\frac{1}{n}\sum_{j=1}^{n}X_j,$$

which, suitably normalized, converges weakly to the supremum of the Brownian bridge over $[0,1]$. If the $e_j$ are in the domain of attraction of a stable law with parameter $1<\alpha<2$, the $e_j$ still have a finite mean and it is natural to use the CUSUM functional in this case, too. In Aue et. al [2] it was shown that $U_n(t)$, suitably normalized, converges weakly to a nongaussian process, whose distribution, however, is not known explicitly, and thus critical values for the test cannot be computed analytically. A natural solution of this difficulty is to use a bootstrap or permutation test, but, as it was shown in [2] and [3], the normalized CUSUM functional converges in this case to a continuous process containing random parameters and thus it is unsuitable for statistical purposes. Since the presence of the random parameters in the limit process is due to the large elements of the sample, it is natural to try trimming, i.e. using the trimmed CUSUM functional

$$U_{n,r_n}(t)=\frac{1}{D_n}\sum_{j=1}^{\lfloor nt\rfloor}\left(X_jI\{|X_j|\le|X_n^{(r_n)}|\})-\overline{X}_n\right), \qquad (0\le t\le 1) \tag{3.14}$$

where

$$\overline{X}_n=\frac{1}{n}\sum_{j=1}^{n}X_jI\{|X_j|\le|X_n^{(r_n)}|\}.$$

By Theorem 3.1 the trimmed sample $X_j I\{|X_j| \le |X_n^{(r_n)}|\}$, $j = 1, \ldots, n$, satisfies the functional CLT after a suitable random translation, and since a random translation does not change the value of the CUSUM functional, we get

**Theorem 3.5.** *Let $X_1, X_2, \ldots$ be i.i.d. random variables satisfying the conditions of Theorem 3.1 and let $U_{n,r_n}(t)$ denote the trimmed CUSUM process defined by (3.14). Then we have*

$$U_{n,r_n}(t)/D_n \xrightarrow{\mathcal{D}[0,1]} B(t),$$

*where $\{B(t),\, 0 \le t \le 1\}$ is a Brownian bridge.*

Using this result, the trimmed CUSUM test can be applied without any problem in the case of i.i.d. random variables with infinite variances.

## References


1. D. Z. Arov and A. A. Bobrov. The extreme terms of a sample and their role in the sum of independent variables, *Teor. Verojatnost. i Primenen.* **5** (1960) 415–435.
2. A. Aue, I. Berkes and L. Horváth, Selection from a stable box, *Bernoulli* **14** (2008) 125–139.
3. I. Berkes, L. Horváth and J. Schauer, Noncentral limit theorems for random selections, *Probab. Theory Rel. Fields* **147** (2010) 449–479.
4. I. Berkes and L. Horváth, The central limit theorem for sums of trimmed variables, *Preprint.*
5. I. Berkes, L. Horváth and J. Schauer, Asymptotics of trimmed CUSUM statistics, *Bernoulli*, to appear.
6. M. Csörgő, S. Csörgő, L. Horváth and D.M. Mason, Normal and stable convergence of integral functions of the empirical distribution function, *Ann. Probab.* **14** (1986) 86–118.
7. S. Csörgő, E. Haeusler and D.M. Mason, A probabilistic approach to the asymptotic distribution of sums of independent, identically distributed random variables, *Adv. Appl. Math.* **9** (1988) 259–333.
8. S. Csörgő, E. Haeusler and D.M. Mason, The asymptotic distribution of trimmed sums, *Ann. Probab.* **16** (1988) 672–699.
9. S. Csörgő, E. Haeusler and D.M. Mason, The quantile-transform approach to the asymptotic distribution of modulus trimmed sums, In: *Sums, Trimmed Sums and Extremes*, M.G. Hahn, D.M. Mason, D.C. Weiner (eds). (Birkhäuser, 1991) 337–353.
10. S. Csörgő, L. Horváth and D.M. Mason, What portion of the sample makes a partial sum asymptotically stable or normal? *Probab. Theory Related Fields* **72** (1986) 1–16.
11. S. Csörgő and G. Simons, A strong law of large numbers for trimmed sums, with applications to generalized St. Petersburg games, *Statistics and Probability Letters* **26** (1996) 65–73.
12. D. A. Darling, The influence of the maximum term in the addition of independent random variables, *Trans. Amer. Math. Soc.* **73** (1952) 95–107.
13. H. Diamond and J. Vaaler, Estimates for partial sums of continued fraction partial quotients, *Pacific J. Math.* **122** (1986) 73–82.
14. Feller, W, An extension of the law of the iterated logarithm to variables without variance, *J. Math. Mech.* **18** (1968) 343–355.

15. P. S. Griffin and W. E. Pruitt, The central limit problem for trimmed sums, *Math. Proc. Cambridge Philos. Soc.* **102** (1987) 329–349.
16. P. S. Griffin and W. E. Pruitt, Asymptotic normality and subsequential limits of trimmed sums, *Ann. Probab.* **17** (1989) 1186–1219.
17. P. S. Griffin and W. E. Pruitt, Weak convergence of trimmed sums, In: *Sums, Trimmed Sums and Extremes*, M.G. Hahn, D.M. Mason, D.C. Weiner (eds). (Birkhäuser, 1991) 55–80.
18. P. S. Griffin and F. S. Qazi, Limit laws of modulus trimmed sums, *Ann. Probab.* **30** (2002) 1466-1485.
19. M.G. Hahn, J. Kuelbs and D.C. Weiner, Asymptotic behavior of partial sums: a more robust approach via trimming and self-normalization. In: *Sums, Trimmed Sums and Extremes*, M.G. Hahn, D.M. Mason, D.C. Weiner (eds). (Birkhäuser, 1991) 1–53.
20. M.G. Hahn, D.M. Mason, D.C. Weiner (eds), *Sums, trimmed sums and extremes.* (Birkhäuser, 1991).
21. P. Hall, On the extreme terms of a sample from the domain of attraction of a stable law, *J. London Math. Soc.* **18** (1978) 181–191.
22. E. Haeusler and D. M. Mason, Laws of the iterated logarithm for sums of the middle portion of the sample, *Math. Proc. Cambridge Philos. Soc.* **101** (1987) 301–312.
23. E. Haeusler and D. M. Mason, A law of the iterated logarithm for modulus trimming, In: *Limit theorems in probability and statistics (Pécs, 1989)*, Coll. Math. Soc. János Bolyai 57. (North-Holland, Amsterdam, 1990) 241–254.
24. H. Kesten, Convergence in distribution of lightly trimmed and untrimmed sums are equivalent, *Math. Proc. Cambridge Philos. Soc.* **113** (1993) 615–638.
25. J. Kiefer, Iterated logarithm analogues for sample quantiles when $p_n \downarrow 0$, In: *Proc. Sixth Berkeley Symp. Math. Stat. Probab.,* Univ. California, Berkeley, 1970/1971, Vol. I. (Univ. California Press, Berkeley, 1972) 227–244.
26. R. A. Maller, Relative stability of trimmed sums, *Z. Wahrsch. Verw. Gebiete* **66** (1984) 61–80.
27. T. Mori, On the limit distributions of lightly trimmed sums, *Math. Proc. Cambridge Philos. Soc.* **96** (1984) 507–516.
28. G. Shorack and J. Wellner, *Empirical processes with applications to statistics.* (Wiley, 1986.)
29. S. M. Stigler, The asymptotic distribution of the trimmed mean, *Ann. Statist.* **1** (1973) 472–477.
30. J. L. Teugels, Limit theorems on order statistics, *Ann. Probab.* **9** (1981) 868–880.