Metric discrepancy theory and Kolmogorov's distribution

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Abstract

Let $(n_k)_{k\geq 1}$ be a lacunary sequence of integers, satisfying certain number-theoretic conditions. We determine the limit distribution of $\sqrt{N} D_N^*(n_k x)$ as $N \to \infty$, where $D_N^*(n_k x)$ denotes the star discrepancy of the sequence $(n_k x)_{k\geq 1}$ mod 1.

1 Introduction and statement of results

An infinite sequence $(x_k)_{k\geq 1}$ of real numbers is called uniformly distributed mod 1 if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{[a,b)}(x_k) = b - a \tag{1}$$

for any $0 \leq a \leq b \leq 1$; here $\mathbb{1}_{[a,b)}$ denotes the indicator function of the interval $[a,b)$, extended with period 1. It is known that (1) is equivalent to the relations $D_N(x_k) \to 0$ or $D_N^*(x_k) \to 0$, where

$$D_N(x_k) := \sup_{0\leq a\leq b\leq 1} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{[a,b)}(x_k) - (b-a) \right|$$

and

$$D_N^*(x_k) := \sup_{0\leq a\leq 1} \left| \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}_{[0,a)}(x_k) - a \right|$$

denote the discrepancy, resp. star discrepancy of the first N terms of $(x_k)_{k\geq 1}$. By a classical result of Weyl [15], for any increasing sequence $(n_k)_{k\geq 1}$ of positive integers the sequence $(n_k x)_{k\geq 1}$ is uniformly distributed mod 1 for almost all x in the sense of Lebesgue measure. Baker [5] proved that

$$D_N(n_k x) = O\left(\frac{(\log N)^{3/2+\varepsilon}}{\sqrt{N}} \right) \qquad \text{a.e.}$$

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and Berkes and Philipp [6] constructed an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that for almost real x the relation

$$D_N(n_k x) \geq \frac{(\log N)^{1/2}}{\sqrt{N}}$$

holds for infinitely many N. These results describe quite precisely the extremal behavior of $D_N(n_k x)$, but determining the order of magnitude of $D_N(n_k x)$ for a given $(n_k)_{k\geq 1}$ is a very difficult problem, solved only in a few special cases. Philipp [11] proved that if $(n_k)_{k\geq 1}$ satisfies the Hadamard gap condition

$$n_{k+1}/n_k \geq q > 1, \qquad (k = 1, 2, \dots) \tag{2}$$

then

$$\frac{1}{4} \leq \limsup_{N\to\infty} \left(\frac{N}{2\log\log N}\right)^{1/2} D_N(n_k x) \leq C \qquad \text{a.e.} \tag{3}$$

with some constant $C = C(q)$. Note that if $(X_k)_{k\geq 1}$ is an sequence of independent random variables in (0, 1) with $\mathbb{P}(X_k \leq x) = x \quad (0 \leq x \leq 1)$, then

$$\limsup_{N\to\infty} \left(\frac{N}{2\log\log N}\right)^{1/2} D_N(X_k) = \frac{1}{2} \tag{4}$$

with probability 1, see e.g. Shorack and Wellner [13], p. 504. A comparison of (3) and (4) shows that the sequence $(n_k x)_{k\geq 1}$ mod 1 behaves like a sequence of i.i.d. random variables. The analogy, however, is not perfect. Fukuyama [9] determined the limsup Σ_a in (3) in the case $n_k = a^k$ for $a > 1$; in particular he proved that

$$\begin{aligned}
&\Sigma_a = \sqrt{42}/9 \quad \text{a.e.} && \text{if } a = 2,\\
&\Sigma_a = \frac{\sqrt{(a+1)a(a-2)}}{2\sqrt{(a-1)^3}} \quad \text{a.e.} && \text{if } a \geq 4 \text{ is an even integer,}\\
&\Sigma_a = \frac{\sqrt{a+1}}{2\sqrt{a-1}} \quad \text{a.e.} && \text{if } a \geq 3 \text{ is an odd integer.}
\end{aligned} \tag{5}$$

Thus the limsup in (3) is generally different from the value 1/2 obtained in the i.i.d. case (cf. [1, 2]).

Given a sequence $(n_k)_{k\geq 1}$ of positive integers, define

$$L(N, d, \nu) = \#\{1 \leq a, b \leq d,\ 1 \leq k, l \leq N :\ a n_k - b n_l = \nu\},$$

where we exclude the trivial solutions $k = l$ in the case $a = b,\ \nu = 0$. Aistleitner [3] proved

Theorem A. *Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers satisfying (2) and*

$$L(N, d, \nu) = O(N/(\log N)^{1+\varepsilon}) \qquad N \to \infty \tag{6}$$

for all $d \geq 2, \nu \in \mathbb{Z}$ and some $\varepsilon > 0$. Then we have

$$\limsup_{N\to\infty} \left(\frac{N}{2\log\log N}\right)^{1/2} D_N(n_k x) = \frac{1}{2} \qquad \text{a.e.} \tag{7}$$

Thus, under the Diophantine condition (6), the discrepancy behavior of $(n_k x)_{k\geq 1}$ follows exactly the i.i.d. case. Condition (6) holds e.g. if $n_{k+1}/n_k \to \infty$ or if $n_{k+1}/n_k \to \alpha$ for some $\alpha > 1$ such that α^r is irrational for $r = 1, 2, \ldots$.

The purpose of this paper is to prove the following

Theorem 1. *Let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying (2) and*

$$L(N, d, \nu) = o(N) \qquad \text{as } N \to \infty \tag{8}$$

for any $d \geq 2$ and $\nu \in \mathbb{Z}$. Then

$$\sqrt{N} D_N^*(n_k x) \xrightarrow{\mathcal{D}} K$$

where

$$K(y) = 1 - 2 \sum_{j=1}^{\infty} (-1)^{j-1} e^{-2j^2 y^2}$$

is the Kolmogorov distribution function.

Note that Theorem 1 does not cover the case $n_k = a^k$, $a \in \mathbb{N}$, $a \geq 2$. In this case (8) holds for all $\nu \neq 0$, but not for $\nu = 0$: we have namely $n_{k+1} - a n_k = 0$ for all $k \geq 1$. Our next theorem determines the limit distribution of $\sqrt{N} D_N^*(n_k x)$ in this case. For $0 \leq t \leq 1$ and $x \in \mathbb{R}$, put

$$\mathbf{I}_t(x) = \mathbb{1}_{[0,t]}(x) - t.$$

Theorem 2. *Let $a \geq 2$ be an integer, and set*

$$\Gamma(s,t) = \int_0^1 \mathbf{I}_s(x) \mathbf{I}_t(x)\, dx + \sum_{k=1}^{\infty} \int_0^1 \Big(\mathbf{I}_s(x) \mathbf{I}_t(a^k x) + \mathbf{I}_s(a^k x) \mathbf{I}_t(x) \Big)\, dx. \tag{9}$$

Then

$$\sqrt{N} D_N^*(a^k x) \xrightarrow{\mathcal{D}} K_\Gamma$$

where K_Γ denotes the distribution of $\sup_{0\leq x\leq 1} G_\Gamma(x)$, where G_Γ is a Gaussian process over $[0,1]$ with mean 0 and covariance function Γ.

Remark: It is not difficult to show that the function $\Gamma(s,t)$ in (9) is bounded and continuous, and that the infinite series in (9) is absolutely convergent. We omit the proof.

As mentioned before, Fukuyama recently calculated the value of the limsup in Philipp's discrepancy LIL (3) for sequences of the form $n_k = a^k$, $k \geq 1$, see (5). With the notation from Theorem 2 the value Σ_a of the limsup equals

$$\sup_{0\leq s\leq 1} \sqrt{\Gamma(s,s)} \tag{10}$$

for a.e. x, and thus Theorem 2 is the distributional analogue of Fukuyama's LIL. Computing explicitly the limit distribution K_Γ in Theorem 2 appears to be an extremely difficult problem, which we do not further investigate in this paper.

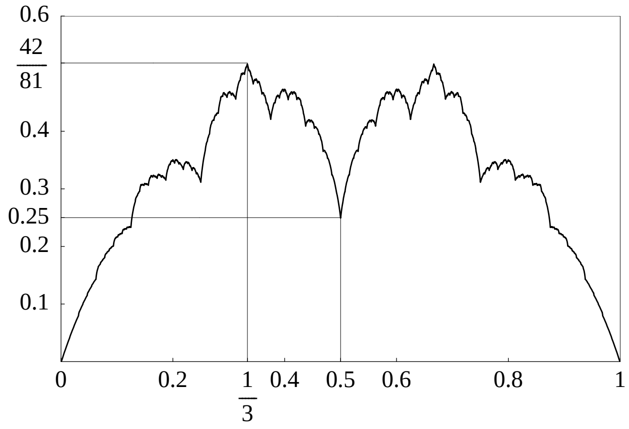

Figure 1: Covariance function $\Gamma(s,s)$ for $n_k = 2^k,\ k \geq 1$. The maximum of the function is $\Gamma(1/3, 1/3) = 42/81$, which causes the value $\sqrt{42}/9$ in Fukuyama's result (5). The functions $\mathbf{I}_{[0,1/2)}(2^k x)$ are independent for $k \geq 1$ (similar to the Rademacher functions), and thus $\Gamma(1/2, 1/2) = \|\mathbf{I}_{[0,1/2)}\|^2 = 1/4$.

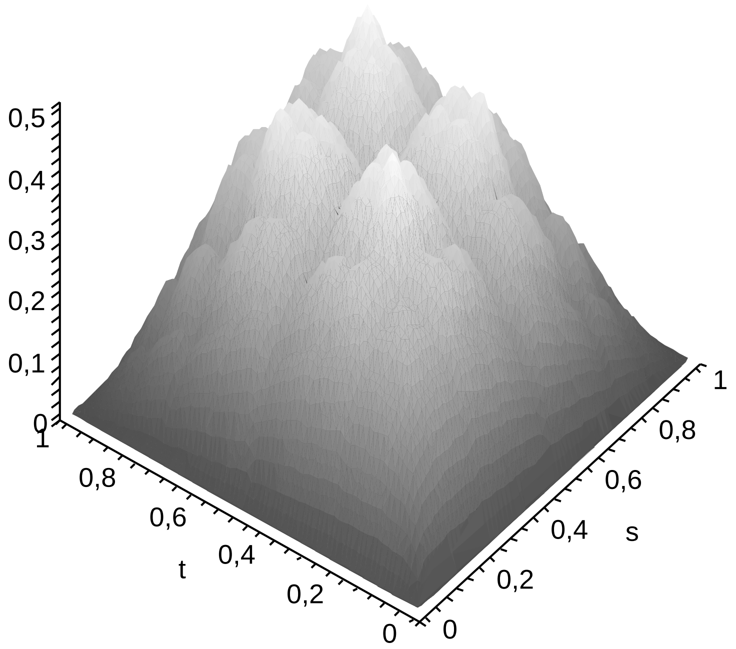

Figure 2: Covariance function $\Gamma(s,t)$ for $n_k = 2^k,\ k \geq 1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$f(x+1) = f(x), \qquad \int_0^1 f(x)\ dx = 0, \qquad \mathrm{Var}_{[0,1]} f < \infty. \tag{11}$$

In Aistleitner and Berkes [4] it is proved that under the conditions of Theorem 1 the central limit theorem for $(f(n_k x))_{k \geq 1}$ holds. More precisely, we have the following

Theorem B. *Let f be a function satisfying (11), and let $(n_k)_{k\geq 1}$ be a sequence of positive integers satisfying (2) and (8) for any $d \geq 2$ and $\nu \in \mathbb{Z}$. Then for all $t \in \mathbb{R}$*

$$\lim_{N\to\infty} \mathbb{P} \left\{ x \in (0,1) :\ \sum_{k=1}^N f(n_k x) \leq t \|f\| \sqrt{N} \right\} = \Phi(t),$$

where Φ is the standard normal distribution function.

Moreover, it is shown in [4] that condition (8) is optimal for the CLT: replacing (8) by

$$L(N,d,\nu) \leq \delta N \qquad N \geq 1$$

the CLT becomes generally false. Thus condition (8) is the precise condition for the CLT for $f(n_k x)$. One can show that (8) is also optimal in Theorems 1. However, the proof is very complicated and will not be given here.

The following central limit theorem for $(f(a^k x))_{k\geq 1}$ is due to Takahashi [14], who improved an earlier result of Kac [10].

Theorem C. *Let f be a function satisfying (11). Then for all $t \in \mathbb{R}$*

$$\lim_{N\to\infty} \mathbb{P} \left\{ x \in (0,1) :\ \sum_{k=1}^N f(n_k x) \leq t \sigma_f \sqrt{N} \right\} = \Phi(t),$$

where

$$\sigma_f^2 = \|f\|^2 + 2 \sum_{k=1}^\infty \int_0^1 f(x) f(a^k x)\ dx.$$

A functional LIL for the empirical process of $(n_k x)_{k\geq 1}$ was proved by Philipp [12]; this enables one to get laws of the iterated logarithm for various functionals of the empirical process. Theorems 1 and 2 will be deduced from a functional CLT for the empirical process, which has a number of further applications. However, in the present paper we will deal only with the asymptotics of the discrepancy of $(n_k x)_{k\geq 1}$.

2 Preliminaries

In the sequel, set

$$F_N(t) = F_N(x;t) = \frac{1}{\sqrt{N}} \sum_{k=1}^N \mathbf{I}_t(n_k x).$$

Lemma 1. *Under the assumptions of Theorem 1 we have for any $r \geq 1$ and $(t_1, \dots, t_r) \in [0,1]^r$*

$$(F_N(t_1), \dots, F_N(t_r)) \xrightarrow{\mathcal{D}} (B(t_1), \dots, B(t_r)) \qquad \textit{as} \quad N \to \infty.$$

Lemma 2. *Under the assumptions of Theorem 2 we have for any $r \geq 1$ and $(t_1, \dots, t_r) \in [0,1]^r$*

$$(F_N(t_1), \dots, F_N(t_r)) \xrightarrow{\mathcal{D}} (K_\Gamma(t_1), \dots, K_\Gamma(t_r)) \qquad \textit{as} \quad N \to \infty,$$

where K_Γ is defined like in Theorem 2.

Lemma 3. *For any $(n_k)_{k \geq 1}$ satisfying (2) there exists a constant c (depending only the growth factor q in (2)) such that for $N \geq 1$ and $t_1, t_2, t_3 \in [0,1]$, $t_1 \leq t_2 \leq t_3$,*

$$\mathbb{E}\left((F_N(t_1) - F_N(t_2))^2 (F_N(t_2) - F_N(t_3))^2 \right) \leq c(t_3 - t_1)^2.$$

Lemma 4. *Under the assumptions of Theorem 1 we have*

$$F_N(t) \Rightarrow B(t) \qquad \textit{as} \qquad N \to \infty,$$

where $\Rightarrow$ denotes weak convergence in the Skorokhod space $D([0,1])$.

Lemma 5. *Under the assumptions of Theorem 2 we have*

$$F_N(t) \Rightarrow K_\Gamma(t) \qquad \textit{as} \qquad N \to \infty.$$

Proof of Lemma 1: By the Cramér-Wold theorem (see [7, Theorem 29.4]) it suffices to show that for any $r \geq 1$, $(c_1, \dots, c_r) \in \mathbb{R}^r$ and $(t_1, \dots, t_r) \in [0,1]^r$, $t_1 < \dots < t_r$,

$$c_1 F_N(t_1) + \dots + c_r F_N(t_r) \xrightarrow{\mathcal{D}} c_1 B(t_1) + \dots + c_r B(t_r) \qquad \text{as} \quad N \to \infty. \tag{12}$$

Set

$$f(x) = \sum_{m=1}^{r} c_m \mathbf{I}_{t_m}(x),$$

and

$$V = V(t_1, \dots, t_r) = \sum_{m=1}^{r} c_j^2 t_j (1 - t_j) + 2 \sum_{1 \leq m < n \leq r} c_m c_n t_m (1 - t_n).$$

We have

$$\begin{aligned}
\|f\|^2 &= \int_0^1 \left(\sum_{m=1}^{r} c_m \mathbf{I}_{t_m}(x) \right)^2 dx \\
&= \sum_{m=1}^{r} \int_0^1 c_m^2 I_{t_m}(x)\ dx + 2 \sum_{1 \leq m < n \leq r} \int_0^1 c_m c_n I_{t_m}(x) I_{t_n}(x)\ dx \\
&= \sum_{m=1}^{r} c_m^2 t_m (1 - t_m) + 2 \sum_{1 \leq m < n \leq r} c_m c_n t_m (1 - t_n) \\
&= V(t_1, \dots, t_r).
\end{aligned}$$

Thus by Theorem A

$$\frac{\sum_{k=1}^N f(n_k x)}{\sqrt{N}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,V),$$

which implies

$$c_1 F_N(t_1) + \cdots + c_r F_N(t_r) \xrightarrow{\mathcal{D}} \mathcal{N}(0,V).$$

On the other hand

$$\begin{aligned}
& \mathbb{E}\left(c_1 B(t_1) + \cdots + c_r B(t_r)\right)^2 \\
= & \sum_{m=1}^r \mathbb{E}\left(c_j B(t_j)\right) + 2 \sum_{1 \le m < n \le r} \mathbb{E}\left(c_m B(t_m) c_n B(t_n)\right) \\
= & \sum_{m=1}^r c_j^2 t_j (1-t_j) + 2 \sum_{1 \le m < n \le r} c_m c_n t_m (1-t_n) \\
= & V(t_1, \ldots, t_r),
\end{aligned}$$

and hence

$$c_1 B(t_1) + \cdots + c_r B(t_r) \sim \mathcal{N}(0,V).$$

Thus we have established (12), which proves the lemma. $\square$

Proof of Lemma 2: Again it suffices to show that for any $r \ge 1$, $(c_1, \ldots, c_r) \in \mathbb{R}^r$ and $(t_1, \ldots, t_r) \in [0,1]^r$, $t_1 < \cdots < t_r$,

$$c_1 F_N(t_1) + \cdots + c_r F_N(t_r) \xrightarrow{\mathcal{D}} c_1 K_\Gamma(t_1) + \cdots + c_r K_\Gamma(t_r) \qquad \text{as} \quad N \to \infty.$$

Setting again

$$f(x) = \sum_{m=1}^r c_m \mathbf{I}_{t_m}(x),$$

we have by Theorem C

$$\frac{\sum_{k=1}^N f(n_k x)}{N} \xrightarrow{\mathcal{D}} \mathcal{N}(0,\sigma_f^2),$$

where

$$\begin{aligned}
\sigma_f^2 &= \|f\|^2 + 2\sum_{k=1}^\infty \int_0^1 f(x) f(a^k x)\, dx \\
&= \int_0^1 \left(\sum_{m=1}^r c_m \mathbf{I}_{t_m}(x)\right)^2 dx \\
&\quad +2\sum_{k=1}^\infty \int_0^1 \left(\sum_{m=1}^r c_m \mathbf{I}_{t_m}(x)\right)\left(\sum_{n=1}^r c_n \mathbf{I}_{t_n}(a^k x)\right) dx \\
&= \sum_{m=1}^r \int_0^1 c_m^2 \mathbf{I}_{t_m}(x)^2 dx + 2 \sum_{1 \le m < n \le r} \int_0^1 c_m c_n \mathbf{I}_{t_m}(x) \mathbf{I}_{t_n}(x)\, dx \\
&\quad +2 \sum_{m=1}^r \sum_{k=1}^\infty \int_0^1 c_m^2 \mathbf{I}_{t_m}(x) \mathbf{I}_{t_m}(a^k x)
\end{aligned}$$

$$
\begin{aligned}
& +2 \sum_{1 \le m < n \le r} \sum_{k=1}^{\infty} \int_0^1 c_m c_n \left(\mathbf{I}_{t_m}(x) \mathbf{I}_{t_n}(a^k x) + \mathbf{I}_{t_m}(a^k x) \mathbf{I}_{t_n}(x) \right) \, dx \\
= & \sum_{m=1}^{r} c_m^2 \Gamma(t_m, t_m) + 2 \sum_{1 \le m < n \le r} c_m c_n \Gamma(t_m, t_n).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \mathbb{E} \left(c_1 K_\Gamma(t_1) + \cdots + c_r K_\Gamma(t_r) \right)^2 \\
= & \sum_{m=1}^{r} \mathbb{E}(c_m^2 K_\Gamma(t_m)^2) + 2 \sum_{1 \le m < n \le r} \mathbb{E}(c_m c_n K_\Gamma(t_m) K_\Gamma(t_n)) \\
= & \sum_{m=1}^{r} c_m^2 \Gamma(t_m, t_m) + 2 \sum_{1 \le m < n \le r} c_m c_n \Gamma(t_m, t_n),
\end{aligned}
$$

which proves the lemma. $\square$

Proof of Lemma 3: Let $N \ge 1$ and $t_1, t_2, t_3 \in [0,1]$, $t_1 \le t_2 \le t_3$ be given. Let $Q \ge 1$ be a number for which

$$
q^Q > 4 \tag{13}
$$

(here q is the growth factor from (2)). To shorten formulas we assume that $\mathbf{I}_{t_1} - \mathbf{I}_{t_2}$ is an even function, i.e. that it can be expanded into a pure cosine-series (the proof in the general case is exactly the same). Write

$$
\mathbf{I}_{t_1}(x) - \mathbf{I}_{t_2}(x) \sim \sum_{j=1}^{\infty} a_j \cos 2\pi j x.
$$

for the Fourier series of $\mathbf{I}_{t_1} - \mathbf{I}_{t_2}$. Then

$$
\sum_{j=1}^{\infty} a_j^2 = \| \mathbf{I}_{t_1} - \mathbf{I}_{t_2} \|^2 \le |t_1 - t_2|.
$$

Let $k_1 \ne k_2 \ne k_3 \ne k_4$, such that $k_1 \equiv k_2 \equiv k_3 \equiv k_4 \mod Q$, and let $j_1, j_2, j_3, j_4 \in [2^n, 2^{n+1})$ for some $n \ge 0$. Then by (13)

$$
j_1 n_{k_1} \pm j_2 n_{k_2} \pm j_3 n_{k_3} \pm j_4 n_{k_4} \ne 0, \tag{14}
$$

no matter how the signs $\pm$ are chosen. Thus by Markov's inequality and the orthogonality of the trigonometric system

$$
\begin{aligned}
& \left(\mathbb{E} \left(F_N(t_1) - F_N(t_2) \right)^4 \right)^{1/4} \\
= & \left(\int_0^1 \left(\frac{1}{\sqrt{N}} \sum_{k=1}^{N} \mathbf{I}_{t_1}(n_k x) - \mathbf{I}_{t_2}(n_k x) \, dx \right)^4 \right)^{1/4} \\
\le & \frac{1}{\sqrt{N}} \sum_{m=0}^{Q-1} \left(\int_0^1 \left(\sum_{\substack{1 \le k \le N, \\ k \equiv m \mod Q}} \mathbf{I}_{t_1}(n_k x) - \mathbf{I}_{t_2}(n_k x) \, dx \right)^4 \right)^{1/4}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\sqrt{N}} \sum_{m=0}^{Q-1} \sum_{n=0}^{\infty} \left(\int_0^1 \left(\sum_{\substack{1 \leq k \leq N, \\ k \equiv m \mod Q}} \sum_{j=2^n}^{2^{n+1}-1} \cos 2\pi j n_k x \; dx \right)^4 \right)^{1/4} \\
&= \frac{1}{\sqrt{N}} \sum_{m=0}^{Q-1} \sum_{n=0}^{\infty} \left(\sum_{\substack{1 \leq k_1,k_2,k_3,k_4 \leq N \\ k_1,k_2,k_3,k_4 \equiv m \mod Q}} \sum_{2^n \leq j_1,j_2,j_3,j_4 < 2^{n+1}} \right. \\
&\qquad \left. \sum_{\pm} \frac{a_{j_1} a_{j_2} a_{j_3} a_{j_4}}{8} \; \mathbb{1}(j_1 n_{k_1} \pm j_2 n_{k_2} \pm j_3 n_{k_3} \pm j_4 n_{k_4} = 0) \right)^{1/4}, \qquad (15)
\end{aligned}
$$

where the sum $\sum_{\pm}$ is meant as a sum over all possible choices of signs "+" and "−" in the indicator $\mathbb{1}(j_1 n_{k_1} \pm j_2 n_{k_2} \pm j_3 n_{k_3} \pm j_4 n_{k_4} = 0)$. Now by (14) the only solutions of $j_1 n_{k_1} \pm j_2 n_{k_2} \pm j_3 n_{k_3} \pm j_4 n_{k_4}$, subject to the given restrictions of the coefficients, are of the form

$$
\underbrace{j_1 n_{k_1} - j_1 n_{k_1}}_{=0} \pm \underbrace{j_2 n_{k_2} - j_2 n_{k_2}}_{=0}
$$

(where we have $\binom{4}{2}$ possible combinations of the pairs). Thus (15) is bounded by

$$
\begin{aligned}
& \frac{1}{\sqrt{N}} \sum_{m=0}^{Q-1} \sum_{n=0}^{\infty} \left(\sum_{\substack{1 \leq k_1,k_2 \leq N \\ k_1,k_2 \equiv m \mod Q}} \sum_{2^n \leq j_1,j_2 < 2^{n+1}} 2 \binom{4}{2} \frac{a_{j_1}^2 a_{j_2}^2}{8} \right)^{1/4} \\
\leq \;& 3^{1/4} Q \left(\sum_{j=1}^{\infty} a_j^2 \right)^{1/2} \\
\leq \;& 3^{1/4} Q (t_2 - t_1)^{1/2}.
\end{aligned}
$$

Hence

$$
\mathbb{E} \left(F_N(t_1) - F_N(t_2) \right)^4 \leq 3 Q^4 (t_2 - t_1)^2. \qquad (16)
$$

In the same way we obtain

$$
\mathbb{E} \left(F_N(t_2) - F_N(t_3) \right)^4 \leq 3 Q^4 (t_3 - t_2)^2. \qquad (17)
$$

By (16), (17) and Hölders inequality

$$
\begin{aligned}
& \mathbb{E} \left(\left(F_N(t_1) - F_N(t_2) \right)^2 \left(F_N(t_2) - F_N(t_3) \right)^2 \right) \\
\leq \;& \left(\mathbb{E} \left(F_N(t_1) - F_N(t_2) \right)^4 \right)^{1/2} \left(\mathbb{E} \left(F_N(t_2) - F_N(t_3) \right)^4 \right)^{1/2} \\
\leq \;& 3 Q^4 (t_2 - t_1)(t_3 - t_2) \\
\leq \;& 3 Q^4 (t_3 - t_1)^2,
\end{aligned}
$$

which proves Lemma 3. □

Proof of Lemma 4 and Lemma 5: Lemma 4 follows from Lemma 1, Lemma 3 and [8, Theorem 13.5]. Lemma 5 follows similarly from Lemma 2, Lemma 3 and [8, Theorem 13.5].

3 Proof of Theorem 1 and Theorem 2

By Lemma 4 the distribution of F_N converges weakly to the distribution of the Brownian bridge. In particular this implies

$$\sup_{t\in[0,1]} |F_N(t)| \xrightarrow{\mathcal{D}} \sup_{t\in[0,1]} |B(t)|,$$

and hence

$$\sqrt{N} D_N^*(n_k x) \xrightarrow{\mathcal{D}} K.$$

Theorem 2 follows in the same way from Lemma 5.

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