MOMENT CONVERGENCE AND THE LAW OF ITERATED LOGARITHM FOR ADDITIVE FUNCTIONS

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Abstract: *Let $f(n)$ be a strongly additive arithmetic function and put $A_n = \sum_{p\le n} \frac{f(p)}{p}$, $B_n^2 = \sum_{p\le n} \frac{f^2(p)}{p}$. We prove a law of the iterated logarithm showing that the set*

$$\{n : |f(n) - A_n| \ge t(2B_n^2 \log\log B_n)^{1/2}\}$$

is 'small' for $t > 1$ and is 'large' for $t < 1$. The proof depends on asymptotic estimates for high moments of $(f(n) - A_n)/B_n$.

1. Introduction

Let f be a strongly additive arithmetic function and set

$$A_n = \sum_{p\le n} \frac{f(p)}{p}, \qquad B_n = \left(\sum_{p\le n} \frac{f^2(p)}{p}\right)^{1/2}. \tag{1}$$

By a classical result of Erdős and Kac [7], if $|f(p)| = O(1)$ and $B_n \to \infty$, then we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N : f(n) \le A_N + xB_N\} = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-u^2/2} du \tag{2}$$

for all $x \in \mathbb{R}$. The same conclusion holds for unbounded $f(p)$, provided f satisfies

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{p\le n, |f(p)|\ge \varepsilon B_n} \frac{f^2(p)}{p} = 0 \qquad \text{for any } \varepsilon > 0. \tag{3}$$

(See Kubilius [12], Shapiro [15].) Condition (3) is the analogue of the Lindeberg condition for the central limit theorem in probabability theory and the previous results show that the distributional behavior of additive functions is similar to that of sums of independent random variables. For extensions and further related results on the distribution of arithmetic functions see e.g. Kubilius [12], Elliott [4] and the references therein.

The standard proofs of the central limit theorem (2) (and in fact of most results on the distributional behavior of additive functions) depend on asymptotic estimates for the cardinality of the set

$$\{m \le N : \alpha_{p_i}(m) = \alpha_i,\ i = 1, 2, \dots, s\}$$

where

$$m = \prod_p p^{\alpha_p(m)}$$

is the prime factorization of m and $2 = p_1 < \cdots < p_s$ are the primes not exceeding r, where $r = r(N)$ satisfies $\log r / \log N \to 0$. Such estimates can be deduced using sieve methods and they show that 'not too many' of the arithmetic functions α_p are almost statistically independent with respect to the normalized counting measure on $\{1, 2, \dots, N\}$. A more elementary (although rather technical) proof was given by Halberstam [9] and simplified substantially by Billingsley [2], using the method of moments. They proved that letting

$$F_N(t) = \frac{1}{N} \#\{n \le N : f(n) < A_N + tB_N\}$$

we have

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} t^r dF_N(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} t^r e^{-t^2/2} dt \qquad (r = 1, 2, \dots). \tag{4}$$

From (4), the central limit theorem (2) follows immediately. The purpose of this paper is to show (see Theorem 2 below) that the r-th moment on the left hand side of (4) is asymptotically equal to the r-th moment of the standard Gaussian distribution not only for fixed r, but also if $r = r(N)$ tends to infinity not faster than $\log\log B_N$. Just as the validity of (4) for all fixed r implies the central limit theorem (2), this generalized moment behavior will lead, via a simple analysis, to a law of the iterated logarithm for $f(n)$. In view of (2), it is natural to expect that under conditions similar to (3) the set

$$H_t = \{n : |f(n) - A_n| \ge t(2B_n^2 \log\log B_n)^{1/2}\} \tag{5}$$

is "large" for $t < 1$ and "small" for $t > 1$. However, no such result seems to exist in the literature. The reason is that ordinary asymptotic density of sequences of integers, used in the central limit theorem (2), is too crude to measure the set H_t: the asymptotic density of H_t equals 0 for any $t > 0$, regardless whether $t > 1$ or $t < 1$. In this paper we will show, however, that using a finer measure of subsets of $\mathbb{N}$, depending on the growth of the variance function B_n, there is a sharp difference between the cases $t > 1$ and $t < 1$ in (5). Let μ denote the measure on subsets of $\mathbb{N}$ defined by

$$\mu(\{1, 2, \dots, N\}) = \log^* B_N, \qquad N = 1, 2, \dots \tag{6}$$

where the * means that we interpolate $\log B_N$ linearly between the points 2^k, $k = 0, 1, \dots$. We will prove the following

Theorem 1. *Assume that $B_n \to \infty$ and*

$$|f(p)| = O(B_p^{1-\delta}) \qquad \text{for some } \delta > 0. \tag{7}$$

Then $\mu(H_t) < \infty$ for $t > 1$ and $\mu(H_t) = \infty$ for $t < 1$.

To clarify the meaning of Theorem 1 and in particular of the measure μ, let X_p, $p = 2, 3, 5, \ldots$ be independent random variables, defined on some probability space, such that X_p takes the values $f(p)$ and 0 with probabilities $1/p$ and $1 - 1/p$, respectively. Let $S_n = \sum_{p \le n} X_p$. By the classical arithmetic theory (see e.g. Kubilius [12]), the sequence $\{S_n,\, n \le N\}$ is an almost exact probabilistic replica of the sequence $\{f(n),\, n \le N\}$, where the latter sequence is meant with respect to the normalized counting measure on $\{1, 2, \ldots, N\}$. Since under (7) the sequence X_p trivially satisfies the central limit theorem

$$(S_n - A_n)/B_n \xrightarrow{\mathcal{D}} N(0,1),$$

this argument proves (2) and leads to a whole class of further interesting distribution results for additive functions. In contrast to this nice behavior, the probabilistic properties of the *infinite* sequences

$$\{f(n), n \ge 1\}, \qquad \{S_n, n \ge 1\}$$

are in general quite different. For example, the central limit theorem (2) implies that the asymptotic density of the set $G = \{n : f(n) > A_n\}$ is $1/2$; on the other hand, the sequence X_p satisfies the Lindeberg condition expressed by (3) and thus also the arc sine law (see e.g. Prohorov [14]), i.e.

$$\frac{1}{N} \sum_{k \le N} I(S_k > A_k) \xrightarrow{\mathcal{D}} H$$

where H is a nondegenerate distribution. The last relation obviously implies that the set $\{n : S_n > A_n\}$ has no asymptotic density; actually, its lower density is 0 and upper density is 1 a.s. To remedy this trouble, introduce the logarithmic density

$$\mu^*(A) = \lim_{N \to \infty} \frac{1}{\log B_N} \sum_{k \le N,\, k \in A} \log(B_k / B_{k-1}), \qquad A \subset \mathbb{N},$$

and note that by the so called almost sure central limit theorem (for a suitable version see Atlagh [1] or Ibragimov and Lifshits [10]) we have $\mu^*(n : S_n > A_n) = 1/2$ a.s. This suggests that logarithmic measure is the natural one in studying probabilistic statements of "almost sure" type and Theorem 1 shows that it works for the law of the iterated logarithm.

A law of the iterated logarithm for additive arithmetic functions was proved by Erdős (see [5], Theorem VI) and extended later by Kubilius (see [12], Theorem 7.2) and in several papers by Manstavičius (see [13] and the references therein). Specialized to the case $f(p) = 1$, the result of Erdős states that for any $\varepsilon > 0$ the asymptotic density of integers m which have at least one divisor d with

$$\omega(d) > \log\log d + (1 - \varepsilon)\sqrt{2 \log\log d \log_4 d}$$

is 1 and for every $\varepsilon > 0$ the density of integers m having at least one divisor $d > A$ with

$$\omega(d) > \log\log d + (1+\varepsilon)\sqrt{2\log\log d \log_4 d}$$

is tending to 0 if $A \to \infty$. Here $\omega(n)$ denotes the number of different prime divisors of n and $\log_r$ denotes r times iterated logarithm. While this formulation (and that of the results of Kubilius and Manstavičius) is very much in the spirit of the classical LIL, note that the objects for which the LIL is formulated is not $f(n)$ itself, and no information on the set H_t is obtained.

The connection of the arithmetic central limit theorem (2) with almost sure central limit theory reveals a paradoxical property of additive functions from the probabilistic point of view. By the almost sure central limit theorem quoted above, the sequence X_p satisfies

$$\lim_{N\to\infty} \frac{1}{\log B_N} \sum_{k\le N} \log(B_k/B_{k-1}) I\left\{\frac{S_k - A_k}{B_k} \le x\right\} = \Phi(x) \qquad \text{a.s.},$$

and this relation fails if we replace logarithmic averages by ordinary averages. In contrast, for additive functions $f(n)$ we have by (2)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\le N} I\left\{\frac{f(k) - A_k}{B_k} \le x\right\} = \Phi(x)$$

and thus in this case the a.s. central limit theorem holds with ordinary averages. This shows that while the probabilistic behavior of additive functions is well understood in the case of distributional properties like the central limit theorem, much remains to be done in the case of "almost sure" type limit theorems.

Condition (7) obviously implies the Lindeberg condition (3). In analogy with Kolmogorov's classical condition (see [11]) for the LIL for independent random variables, it is natural to expect that the LIL of our paper remains valid under

$$f(p) = o(B_p/(\log\log B_p)^{1/2}).$$

However, the methods of our paper are not strong enough to decide the validity of this conjecture.

We finally note that using deeper tools from probabilistic number theory based on sieve methods, Theorem 1 can be sharpened in the same way as so called upper-lower class tests in probability theory improve the law of the iterated logarithm. (See e.g. Feller [8].) However, as our main interest in the present paper is the elementary moment approach, we do not investigate such improvements of Theorem 1 here.

2. Proofs

The first step of the argument is a truncation of the function f. Clearly $f = \sum_p f(p)\delta_p$, where the the function δ_p is defined by

$$\delta_p(m) = \chi(p|m).$$

Let the function f_n be defined by

$$f_n = \sum_{p \le \alpha_n} f(p)\delta_p \tag{8}$$

where

$$\alpha_n = n^{1/(\log\log B_n)^2}.$$

Set further

$$a_n = \sum_{p \le \alpha_n} \frac{f(p)}{p}, \qquad b_n = \left(\sum_{p \le \alpha_n} \frac{f^2(p)}{p} \right)^{1/2}. \tag{9}$$

Lemma 1. Let X_p, $p = 2, 3, 5, \dots$ be independent random variables, defined on some probability space, such that X_p takes the values $f(p)$ and 0 with probabilities $1/p$ and $1 - 1/p$, respectively. Let $S_n = \sum_{p \le \alpha_n} X_p$. Then we have

$$E\left\{ \left(\frac{S_n - a_n}{b_n} \right)^{2r} \right\} \sim \mu_{2r} \quad \text{as } n \to \infty, \text{ uniformly for } \ 1 \le r \le 4 \log\log b_n.$$

where $\mu_{2r} = 1 \cdot 3 \cdot \ldots \cdot (2r-1)$ is the $2r$-th moment of the standard normal law.

Proof. Let

$$X'_p = X_p - \frac{f(p)}{p}, \quad S'_n = \sum_{p \le \alpha_n} X'_p = S_n - a_n \quad s_n^2 = ES_n'^2 = \sum_{p \le \alpha_n} \frac{f^2(p)}{p} \left(1 - \frac{1}{p} \right).$$

By a recent result of Cuny and Weber on the speed of convergence of moments in the central limit theorem (see [3], Theorem 1.3) we have

$$\left| E\left(\frac{|S'_n|}{s_n} \right)^{2r} - \mu_{2r} \right| \le \left(C_1 \frac{r}{\log r} \right)^{2r} \max_{h \in \{1, \frac{1}{2r-2}\}} \left(\frac{\sum_{p \le \alpha_n} E|X'_p|^{2r}}{s_n^{2r}} \right)^h \tag{10}$$

where C_1 is an absolute constant. Here $E|X_p'|^{2r} = E|X_p - EX_p|^{2r} \le 2^{2r}E|X_p|^{2r}$ by Minkowski's inequality and thus we get, using $|f(p)| \le CB_p^{1-\delta}$

$$\begin{aligned}(11)\qquad &\sum_{p\le\alpha_n} E|X_p'|^{2r} \le 2^{2r}\sum_{p\le\alpha_n}\frac{|f(p)|^{2r}}{p} \le (2C)^{2r-2}B_n^{(2r-2)(1-\delta)}4\sum_{p\le\alpha_n}\frac{f^2(p)}{p}\\ &\le 4(2C)^{2r-2}B_n^{2r-\delta(2r-2)}.\end{aligned}$$

On the other hand, the well known relation

$$\sum_{p\le n}\frac{1}{p} = \log\log n + O(1)$$

implies

$$(12)\qquad \sum_{\alpha_n<p\le n}\frac{1}{p} = \log\log n - \log\log\alpha_n + O(1) \le 3\log\log\log B_n \qquad (n\ge n_0)$$

whence

$$(13)\qquad B_n^2 - b_n^2 = \sum_{\alpha_n<p\le n}\frac{f^2(p)}{p} \ll B_n^{2(1-\delta)}\sum_{\alpha_n<p\le n}\frac{1}{p} \ll B_n^{2(1-\delta)}\log\log\log B_n$$

and thus $s_n^2 \sim b_n^2 \sim B_n^2$. The statement of the lemma now follows from (10), (11) and the fact that

$$\left(\frac{r}{\log r}\right)^{2r} \le (4\log\log b_n)^{8\log\log b_n} \le \exp\left\{(\log\log b_n)^2\right\} \le b_n^{\delta/4} \le B_n^{\delta/4}$$

for $1\le r\le 4\log\log b_n$, $n\ge n_0$.

In what follows, P_n denotes normalized counting measure on $\{1,2,\dots,n\}$ and E_n denotes the corresponding expectation.

Lemma 2. We have

$$E_n\left\{\left(\frac{f_n-a_n}{b_n}\right)^r\right\} - E\left\{\left(\frac{S_n-a_n}{b_n}\right)^r\right\} \to 0 \quad \text{uniformly for}\ \ 1\le r\le 8\log\log b_n.$$

Proof. We follow Billingsley [2]. Clearly

$$(14)\qquad E(S_n^r) = \sum_{u=1}^{r}{\sum}'\frac{r!}{r_1!\cdots r_u!}{\sum}''E(X_{p_1}^{r_1}\cdots X_{p_u}^{r_u})$$

and by (8)

$$E_n(f_n^r) = \sum_{u=1}^{r} {\sum}' \frac{r!}{r_1! \cdots r_u!} {\sum}'' E(Y_{p_1}^{r_1} \cdots Y_{p_u}^{r_u}) \tag{15}$$

where $Y_p = f(p)\delta_p$, $\sum'$ extends over those u-tuples $(r_1, \cdots r_u)$ of positive integers satisfying $r_1 + \cdots + r_u = r$ and $\sum''$ extends over those u-tuples $(p_1, \cdots p_u)$ of primes satisfying $p_1 < \cdots < p_u \le \alpha_n$. Clearly,

$$E(X_{p_1}^{r_1} \cdots X_{p_u}^{r_u}) = \frac{f(p_1)^{r_1} \cdots f(p_u)^{r_u}}{p_1 \cdots p_u} \tag{16}$$

and

$$E_n(Y_{p_1}^{r_1} \cdots Y_{p_u}^{r_u}) = \frac{1}{n} \left[\frac{n}{p_1 \cdots p_u} \right] f(p_1)^{r_1} \cdots f(p_u)^{r_u}. \tag{17}$$

But the right hand sides of (16) and (17) differ at most by $(1/n)|f(p_1)|^{r_1} \cdots |f(p_u)|^{r_u}$, and hence $E(S_n^r)$ and $E_n(f_n^r)$ cannot differ by more than the sum (14) with the inner summand replaced by $(1/n)|f(p_1)|^{r_1} \cdots |f(p_u)|^{r_u}$. It now follows by the multinomial theorem and the Cauchy-Schwarz inequality that

$$\begin{aligned} |E(S_n^r) - E_n(f_n^r)| &\le \frac{1}{n} \left(\sum_{p \le \alpha_n} |f(p)| \right)^r \le \frac{1}{n} \left(\sum_{p \le \alpha_n} \frac{f^2(p)}{p} \right)^{r/2} \left(\sum_{p \le \alpha_n} p \right)^{r/2} \\ &\le \frac{1}{n} b_n^r \alpha_n^r. \end{aligned} \tag{18}$$

Now

$$E((S_n - a_n)^r) = \sum_{k=0}^{r} \binom{r}{k} E(S_n^k)(-a_n)^{r-k}$$

and $E_n(f_n - a_n)^r$ has an analogous expansion. Comparing the two expansions term by term and applying (18) we get that

$$|E(S_n - a_n)^r - E_n(f_n - a_n)^r| \le \sum_{k=0}^{r} \binom{r}{k} \frac{\alpha_n^k b_n^k}{n} |a_n|^{r-k} = \frac{1}{n}(\alpha_n b_n + |a_n|)^r \le \frac{1}{n}(2\alpha_n b_n)^r,$$

where we used

$$|a_n| \le \sum_{p \le \alpha_n} \frac{|f(p)|}{p} \le \left(\sum_{p \le \alpha_n} \frac{f^2(p)}{p^2} \right)^{1/2} \alpha_n^{1/2} \le b_n \alpha_n.$$

Now

$$B_{2n}^2 - B_n^2 = \sum_{n < p \le 2n} \frac{f^2(p)}{p} \ll B_{2n}^{2(1-\delta)} \sum_{n < p \le 2n} \frac{1}{p} \ll B_{2n}^{2(1-\delta)}$$

which shows that $B_{2n}/B_n \to 1$ and thus B_n is slowly varying in the Karamata sense, which implies $B_n \ll n^\varepsilon$ for any $\varepsilon > 0$. Thus for $1 \le r \le 8 \log\log b_n$ we have

$$(2\alpha_n)^r \le 2^{8\log\log B_n} n^{8/(\log\log B_n)} \ll (\log B_n)^8 n^{1/2} = o(n)$$

and Lemma 2 is proved.

We can now easily get

Theorem 2. We have

$$E_n\left\{\left(\frac{f - A_n}{B_n}\right)^{2r}\right\} \sim \mu_{2r} \quad \text{as } n \to \infty, \text{ uniformly for } \; 1 \le r \le 4\log\log B_n.$$

Proof. By (13) we have $b_n^2/B_n^2 = 1 + O(B_n^{-\delta})$ and thus

$$b_n^r/B_n^r = (1 + O(B_n^{-\delta}))^{r/2} = 1 + o(1) \qquad \text{uniformly for } 1 \le r \le 8\log\log B_n.$$

Thus from Lemmas 1 and 2 it follows that

$$E_n\left\{\left(\frac{f_n - a_n}{B_n}\right)^{r}\right\} \sim E_n\left\{\left(\frac{f_n - a_n}{b_n}\right)^{r}\right\} \sim \mu_r \quad \text{as } n \to \infty, \tag{19}$$

uniformly for all even r with $1 \le r \le 8\log\log B_n$. Now

$$|f(m) - f_n(m)| \le \sum_{\alpha_n < p \le n} |f(p)|\delta_p(m) \qquad 1 \le m \le n$$

and thus similarly to the proof of Lemma 2, we have

$$E_n|f - f_n|^r \le \sum_{u=1}^{r} {\sum}' \frac{r!}{r_1! \cdots r_u!} {\sum}'' \frac{1}{n}\left[\frac{n}{p_1 \cdots p_u}\right] |f(p_1)|^{r_1} \cdots |f(p_u)|^{r_u},$$

where $\sum'$ extends over those u-tuples $(r_1, \cdots r_u)$ of positive integers satisfying $r_1 + \cdots + r_u = r$ and $\sum''$ extends over those u-tuples $(p_1, \cdots p_u)$ of primes satisfying $\alpha_n < p_1 < \cdots < p_u \le n$. Thus using (7) and (12) we get for $n \ge n_0$

$$E_n|f - f_n|^r \le C^r B_n^{r(1-\delta)} \sum_{u=1}^{r} {\sum}' \frac{r!}{r_1! \cdots r_u!} {\sum}'' \frac{1}{p_1 \cdots p_u} = C^r B_n^{r(1-\delta)} \left(\sum_{\alpha_n < p \le n} \frac{1}{p}\right)^r$$
$$\le (3C)^r B_n^{r(1-\delta)} (\log\log\log B_n)^r \le (3C)^r B_n^{r(1-\delta/2)},$$

where C is the constant implied by the O in (7). Thus letting $\|g\|_{r,n} = E_n(|g|^r)^{1/r}$ for any arithmetic function g, we get by Minkowski's inequality,

$$|\,\|(f - a_n)/B_n\|_{r,n} - \|(f_n - a_n)/B_n\|_{r,n}| \le 3CB_n^{-\delta/2}. \tag{20}$$

Further by (7) and (12) we have

$$|A_n - a_n| \le CB_n^{1-\delta} \sum_{\alpha_n < p \le n} \frac{1}{p} \le CB_n^{1-\delta/2} \qquad \text{for } n \ge n_0$$

and thus replacing $f - a_n$ by $f - A_n$ in the first term on the left hand side of (20) results in a change $\le CB_n^{-\delta/2}$ of the norm. Let now $\varepsilon > 0$. Relation (19) shows that for even r and $n \ge n_0(\varepsilon)$ the second term on the left hand side of (20) lies in the interval $[((1-\varepsilon)\mu_r)^{1/r}, ((1+\varepsilon)\mu_r)^{1/r}]$ and thus

$$\|(f - A_n)/B_n\|_{r,n} \le ((1+\varepsilon)\mu_r)^{1/r} + 4CB_n^{-\delta/2} \le ((1+2\varepsilon)\mu_r)^{1/r}$$

observing that $\mu_r \ge 1$ and

$$(1+2\varepsilon)^{1/r} - (1+\varepsilon)^{1/r} \ge 4CB_n^{-\delta/2}$$

for $1 \le r \le 8 \log\log B_n$ by the mean value theorem. A similar argument yields

$$\|(f - A_n)/B_n\|_{r,n} \ge ((1-2\varepsilon)\mu_r)^{1/r}$$

and Theorem 2 is proved.

Using Theorem 2 we can now get upper and lower tail estimates for $|f - A_n|$ using a method going back to Kolmogorov [11] in the context of the moment generating functions and to Erdős and Gál [6] in the case of moment convergence.

Lemma 3. We have

$$P_n\{|f - A_n| \ge (2(1+\varepsilon)B_n^2 \log\log B_n)^{1/2}\} \ll \exp(-(1+\varepsilon)\log\log B_n).$$

Proof. Let

$$G(t) = P_n\{|f - A_n| \ge (2tB_n^2 \log\log B_n)^{1/2}\}, \qquad t > 0$$

and

$$Z_n = (f - A_n)^2/(2B_n^2 \log\log B_n).$$

Since

$$\mu_{2r} = \frac{(2r)!}{2^r r!} \sim \sqrt{2}(2r/e)^r \qquad \text{as } r \to \infty,$$

we get by Lemmas 1 and 2 for $1 \le r \le 4\log\log B_n$, $n \ge n_0$

$$(r/e)^r (\log\log B_n)^{-r} \ll E_n Z_n^r \ll (r/e)^r (\log\log B_n)^{-r} \tag{21}$$

where the constants implied by $\ll$ are absolute. By (21) and the Markov inequality

$$G(t) = P_n(Z_n \ge t) \le t^{-r} E_n Z_n^r \ll t^{-r}(r/e)^r (\log\log B_n)^{-r}. \tag{22}$$

If $t \geq 3$, we choose $r = [e \log\log B_n]$ to get

$$G(t) \ll t^{-2\log\log B_n}, \qquad t \geq 3. \tag{23}$$

For $0 < t < 3$ we choose $r = [t \log\log B_n]$ to get

$$G(t) \ll \exp(-t \log\log B_n) \qquad 0 < t < 3, \tag{24}$$

and Lemma 3 is proved.

Lemma 4. We have

$$P_n\{|f - A_n| \geq (2(1-\varepsilon)B_n^2 \log\log B_n)^{1/2}\} \gg \exp(-(1-\varepsilon^2/16)\log\log B_n).$$

Proof. Let

$$D_1 = \{1-\varepsilon \leq Z_n \leq 1\},\ D_2 = \{0 \leq Z_n < 1-\varepsilon\},\ D_3 = \{1 < Z_n \leq 3\},\ D_4 = \{Z_n > 3\}.$$

Then by (21) we have for $1 \leq r \leq 4\log\log B_n$, $n \geq n_0$

$$\begin{aligned} G(1-\varepsilon) = P_n(Z_n \geq 1-\varepsilon) \geq P_n(D_1) \geq \int_{D_1} Z_n^r \, dP_n \\ \geq A(r/e)^r(\log\log B_n)^{-r} - (I_2 + I_3 + I_4) \end{aligned} \tag{25}$$

where A is an absolute constant and

$$I_k = \int_{D_k} Z_n^r \, dP_n, \qquad k = 2, 3, 4.$$

We choose $r = [(1-\varepsilon/2)\log\log B_n]$ and estimate I_2, I_3 and I_4 from above. First we get, using (24) and $G(t) = P_n(Z_n \geq t)$,

$$\begin{aligned} I_2 &= -\int_0^{1-\varepsilon} t^r dG(t) \leq 2r \int_0^{1-\varepsilon} t^{r-1} G(t) dt \ll 2r \int_0^{1-\varepsilon} t^{r-1} \exp(-t \log\log B_n) dt \\ &= 2r(\log\log B_n)^{-r} \int_0^{(1-\varepsilon)\log\log B_n} u^{r-1} e^{-u} du. \end{aligned}$$

Since $u^{r-1}e^{-u}$ reaches its maximum at $u = r-1$ which exceeds the upper limit of the last integral by the choice of r, we get

$$\begin{aligned} I_2 &\leq 2r(1-\varepsilon)^r e^{-(1-\varepsilon)\log\log B_n} \leq 4\log\log B_n \cdot (1-\varepsilon)^{(1-\varepsilon/2)\log\log B_n} (\log B_n)^{-(1-\varepsilon)} \\ &= 4(\log\log B_n)\,(\log B_n)^{-\gamma} \end{aligned} \tag{26}$$

where

$$\gamma = 1 - \varepsilon - (1-\varepsilon/2)\log(1-\varepsilon).$$

Similarly as above, we get

$$I_3 \le 2r(\log\log B_n)^{-r} \int_{\log\log B_n}^{3\log\log B_n} u^{r-1}e^{-u}du.$$

Now the maximum of the integrand is reached at a point which is smaller than the lower limit of the integral and we get

$$I_3 \le 4(\log\log B_n)\,(\log B_n)^{-1}. \tag{27}$$

Finally, to estimate I_4 we proceed as with I_2, but instead of (24) we use (23) to get

$$\begin{aligned} I_4 &\le 2r\int_3^\infty t^{r-1}G(t)dt \ll 2r\int_3^\infty t^{r-1}t^{-2\log\log B_n}dt \\ &\ll e^{-\log\log B_n} = (\log B_n)^{-1}. \end{aligned}$$

Now using $r = [(1-\varepsilon/2)\log\log B_n]$ we see that the first term on the right hand side of (25) is

$$A(r/e)^r(\log\log B_n)^{-r} \gg (r/e)^r\left(\frac{r}{1-\varepsilon/2}\right)^{-r} \gg (\log B_n)^{-\gamma'} \tag{28}$$

where

$$\gamma' = (1-\varepsilon/2) - (1-\varepsilon/2)\log(1-\varepsilon/2)$$

and the constants implied by $\gg$ are absolute. Simple calculations show that for sufficiently small ε we have $\gamma' < \gamma$ and $\gamma' < 1-\varepsilon^2/16$ which imply that all of I_2, I_3 and I_4 are of smaller order of magnitude than the expression in (28). Thus we get

$$G(1-\varepsilon) \gg (\log B_n)^{-\gamma'} \gg (\log B_n)^{-(1-\varepsilon^2/16)}$$

and Lemma 4 is proved.

We can now easily prove Theorem 1. Let $0<\varepsilon<1$. By Lemma 3 we have

$$P_{2^k}\{|f-A_{2^k}| \ge (2(1+\varepsilon)B_{2^k}^2\log\log B_{2^k})^{1/2}\} \ll \exp(-(1+\varepsilon)\log\log B_{2^k}). \tag{29}$$

As we have seen in the proof of Lemma 2, we have $B_{2^k}/B_{2^{k-1}} \to 1$. Also, the fluctuation of A_n in the interval $[2^{k-1}, 2^k]$ is at most

$$\sum_{2^{k-1}<p\le 2^k} \frac{|f(p)|}{p} \ll B_{2^k}^{1-\delta} \sum_{2^{k-1}<p\le 2^k} \frac{1}{p} \ll B_{2^k}^{1-\delta}.$$

Thus the number of $j \in [2^{k-1}, 2^k]$ belonging to $H_{(1+2\varepsilon)^{1/2}}$ is

$$\ll 2^k\exp(-(1+\varepsilon)\log\log B_{2^k}) = \frac{2^k}{(\log B_{2^k})^{(1+\varepsilon)}}.$$

By the definition of $\log^* B_N$, the μ-measure of any point j with $2^{k-1} \le j < 2^k$ is

$$2^{-(k-1)} \log(B_{2^k}/B_{2^{k-1}}) \sim 2^{-(k-1)}(B_{2^k}/B_{2^{k-1}} - 1).$$

Thus

$$\mu(H_{(1+2\varepsilon)^{1/2}} \cap [2^{k-1}, 2^k]) \ll 2^{-k} \frac{B_{2^k} - B_{2^{k-1}}}{B_{2^{k-1}}} \frac{2^k}{(\log B_{2^k})^{(1+\varepsilon)}} \ll \int_{B_{2^{k-1}}}^{B_{2^k}} \frac{1}{x(\log x)^{(1+\varepsilon)}} dx.$$

Summing for k we get the first part of the theorem. The proof of the second part is similar, but instead of (29) we use

$$P^*_{2^k}\{|f - A_{2^k}| \ge (2(1-\varepsilon)B^2_{2^k} \log\log B_{2^k})^{1/2}\} \gg \exp(-(1-\varepsilon^2/16)\log\log B_{2^k}) \tag{30}$$

where $P^*_{2^k}$ denotes uniform probability on the set $\{2^{k-1}+1, \dots, 2^k\}$. Relation (30) is similar to our lower tail estimate

$$P_{2^k}\{|f - A_{2^k}| \ge (2(1-\varepsilon)B^2_{2^k} \log\log B_{2^k})^{1/2}\} \gg \exp(-(1-\varepsilon^2/16)\log\log B_{2^k})$$

in Lemma 4 and can be proved in the same way.

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