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A note on the existence of solutions to a stochastic recurrence equation

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Dedicated to Sándor Csörgő on the occasion of his 60th birthday

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Abstract. We provide a characterization of strictly stationary solutions to the stochastic recurrence equation $z_k = c(\varepsilon_{k-1})z_{k-1} + g(\varepsilon_{k-1})$ with Borel-measurable functions c and g, and independent, identically distributed random variables $\{\varepsilon_k\}$. Strictly stationary solutions that are functions of the past, respectively, of the future exist if and only if the expected value $E\log|c(\varepsilon_0)|$ is negative, respectively, positive. The main result of the paper is to show that there is no solution that is a function of the past or the future if $E\log|c(\varepsilon_0)| = 0$.

1. Introduction and results

Let $c(x)$ and $g(x)$ denote real-valued Borel-measurable functions. In the following, we study the recursion

$$z_k = c(\varepsilon_{k-1})z_{k-1} + g(\varepsilon_{k-1}), \qquad -\infty < k < \infty, \tag{1.1}$$

where $\{\varepsilon_k\}$ is a sequence of independent, identically distributed random variables. Recursions of the form (1.1), known as stochastic recurrence equations

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(also stochastic difference equations), have been the subject of intensive study for several decades. See the seminal papers of Furstenberg and Kesten (1960), Vervaat (1979), Brandt (1986), Bougerol and Picard (1992a), and Goldie and Maller (2000) for background information and variations. As specifications, the defining equations (1.1) include the classical autoregressive processes and also the volatility sequences of various generalized autoregressive conditionally heteroskedastic models. Section 2 contains a broader discussion with details. An approach focusing on applications is offered in Diaconis and Freedman (1999), who show that stochastic recurrence equations have impact, for instance, in the field of fractal images.

Related recursions defining, for example, random coefficient autoregressive sequences which possess a second random input, have been studied by Aue et al. (2006b). Then, (1.1) becomes $z_k = b_k z_{k-1} + \varepsilon_{k-1}$, where $\{b_k\}$ is a sequence of independent, identically distributed random variables, independent of $\{\varepsilon_k\}$.

Here, we are interested in a full characterization of strictly stationary solutions of (1.1). To this end, for $-\infty < k < \infty$, let

$$\mathcal{F}_k = \sigma(\varepsilon_i, -\infty < i \le k) \qquad \text{and} \qquad \mathcal{G}_k = \sigma(\varepsilon_i, k \le i < \infty)$$

denote the σ-algebras generated by the past and the future, respectively. Let $\log_+ x = 0$ if $x \le e$, and $= \log x$ if $x > e$. We assume for the rest of the paper that

$$E \log_+ |c(\varepsilon_0)| < \infty \qquad \text{and} \qquad E \log_+ |g(\varepsilon_0)| < \infty.$$

Writing $E \log |c(\varepsilon_0)| < 0$, we allow the possibility $E \log |c(\varepsilon_0)| = -\infty$. Similarly, the statement $E \log |c(\varepsilon_0)| > 0$ includes $E \log |c(\varepsilon_0)| = \infty$. Finally, a sequence $\{y_k\}$ of random variables is called strictly stationary if, for all integers $k_1, \ldots, k_n$, h and all nonnegative integers n,

$$(y_{k_1}, \ldots, y_{k_n}) \stackrel{\mathcal{D}}{=} (y_{k_1+h}, \ldots, y_{k_n+h}),$$

where $\stackrel{\mathcal{D}}{=}$ stands for equality in distribution. The following theorem provides sufficient conditions for the strict stationarity of solutions $\{z_k\}$ to the recursion defined by (1.1).

Theorem 1.1.
(i) *If $E \log |c(\varepsilon_0)| < 0$, there is a unique strictly stationary solution $\{z_k\}$ of (1.1). This solution is adapted to $\{\mathcal{F}_{k-1}\}$, that is, for any k, z_k is measurable with respect to $\mathcal{F}_{k-1}$.*
(ii) *If $E \log |c(\varepsilon_0)| > 0$, there is a unique strictly stationary solution $\{z_k\}$ of (1.1). This solution is adapted to $\{\mathcal{G}_k\}$.*

Versions of the first part of Theorem 1.1 are well studied in the literature (see, for instance, Bougerol and Picard (1992b), and Aue et al. (2006a) for results concerning GARCH-type sequences). It is the desired approach for any application, since it determines those solutions that are functions of the past. Hence, part (ii) of the theorem is less studied, although the result can be obtained by imitating the methods used to prove part (i).

To investigate the transition case $E\log|c(\varepsilon_0)|=0$, we say first that a sequence $\{z_k\}$ is a nondegenerate solution if $P\{|z_k|=|z_{k+1}|\}<1$ for all k.

Theorem 1.2.

(i) *If the recursion* (1.1) *has a nondegenerate, strictly stationary solution which is adapted to* $\{\mathcal{F}_{k-1}\}$, *then* $E\log|c(\varepsilon_0)|<0$.

(ii) *If the recursion* (1.1) *has a nondegenerate, strictly stationary solution which is adapted to* $\{\mathcal{G}_k\}$, *then* $E\log|c(\varepsilon_0)|>0$.

In other words, the sufficient conditions imposed in parts (i) and (ii) of Theorem 1.1 are also necessary.

Corollary 1.3.

(i) *The recursion* (1.1) *has a unique nondegenerate, strictly stationary solution which is adapted to* $\{\mathcal{F}_{k-1}\}$ *if and only if* $E\log|c(\varepsilon_0)|<0$.

(ii) *The recursion* (1.1) *has a unique nondegenerate, strictly stationary solution which is adapted to* $\{\mathcal{G}_k\}$ *if and only if* $E\log|c(\varepsilon_0)|>0$.

(iii) *If* $E\log|c(\varepsilon_0)|=0$, *then there is no nondegenerate, strictly stationary solution which is adapted to* $\{\mathcal{F}_{k-1}\}$ *or* $\{\mathcal{G}_k\}$.

Remark 1.4. Note that the constant sequence $z_k=a$, $-\infty<k<\infty$, is a solution of (1.1) if and only if $a=ac(\varepsilon_0)+g(\varepsilon_0)$ a.s. In this case no restriction on $E\log|c(\varepsilon_0)|$ is needed.

Remark 1.5. (i) If $g(x)=0$ and $c(x)=1$, then the general solution of (1.1) is $z_k=\varepsilon$, $-\infty<k<\infty$, where ε is an arbitrary random variable.

(ii) If $g(x)=0$ and $c(x)=-1$, then the general solution of (1.1) is $z_k=(-1)^k\varepsilon$, $-\infty<k<\infty$, where ε is an arbitrary symmetric random variable.

The next section contains a collection of possible specifications of (1.1), while the proofs can be found in Section 3.

2. Examples

Specific choices of the Borel-measurable functions $c(x)$ and $g(x)$ in (1.1) lead to prominent and well-known recursions.

Example 2.1. (Autoregressive processes). Let $c(x) = \varphi$ be constant and let $g(x) = x$ be the identity mapping. Then (1.1) becomes

$$z_k = \varphi z_{k-1} + \varepsilon_{k-1}, \qquad -\infty < k < \infty, \tag{2.1}$$

where z_{k-1} is independent of ε_{k-1}. Hence, (2.1) defines an autoregressive process of order one. Note that, in the time series context, the recursion is usually given as $z_k = \varphi z_{k-1} + \varepsilon_k$. If $\varphi = 1$, we obtain the random walk case.

Necessary and sufficient conditions for autoregressive processes are long known and can be found, for example, in Brockwell and Davis (1991). Since $c(\varepsilon_0) = \varphi$ is nonrandom, the conditions of Theorem 1.1 reduce to $|\varphi| < 1$, $|\varphi| > 1$ and $|\varphi| = 1$ for the respective parts (i), (ii) and (iii).

Another class of random variables included in the framework of (1.1) are augmented GARCH(1,1) processes (see Duan (1997)), which are, for $-\infty < k < \infty$, defined by the relations

$$x_k = \sigma_k \varepsilon_k, \tag{2.2}$$

$$\Lambda(\sigma_k^2) = c(\varepsilon_{k-1})\Lambda(\sigma_{k-1}^2) + g(\varepsilon_{k-1}), \tag{2.3}$$

where $\Lambda(x)$ is invertible. On substituting $z_k = \Lambda(\sigma_k^2)$, the recursion in (2.2) becomes a special case of (1.1).

Example 2.2. (Polynomial GARCH processes). If $\Lambda(x) = x^\delta$ with some $\delta > 0$ in (2.2), then the processes defined by (2.2) and (2.2) are referred to as polynomial GARCH. On specifying $c(x)$ and $g(x)$, this setting includes Bollerslev's (1986) classical GARCH version and a large variety of other models frequently used in econometrics. For details we refer to Aue et al. (2006a).

Theorem 1.1 states that, in case $E\log|c(\varepsilon_0)| < 0$, there is a unique strictly stationary solution and it is adapted to $\{\mathcal{F}_{k-1}\}$. Usually, $c(x)$ and $g(x)$ are non-negative. Then, no $\{\mathcal{F}_{k-1}\}$-adapted solution exists if $E\log|c(\varepsilon_0)| \geq 0$. This follows from the fact that, on applying (2.2) repeatedly (see (3.1)) and using the fact that an iid random walk with zero mean is recurrent, we get

$$z_0 = \Lambda(\sigma_0^2) \geq \limsup_{N\to\infty} \sum_{i=1}^{N} g(\varepsilon_{-i}) \prod_{j=1}^{i-1} c(\varepsilon_{-j}) = \infty \qquad \text{a.s.}$$

But part (ii) of Theorem 1.1 implies the existence of a $\{\mathcal{G}_k\}$-adapted strictly stationary solution if $E\log|c(\varepsilon_0)| > 0$.

Example 2.3. (Exponential GARCH processes). If $\Lambda(x) = \log x$, then processes satisfying (2.2) and (2.3) are called exponential GARCH. The most prominent members of this subclass are the multiplicative GARCH model of Geweke (1986) and the exponential GARCH model introduced by Nelson (1991), whose respective specifications are given by

(2.4)
$$\log\sigma_k^2 = \omega + (\alpha+\beta)\log\sigma_{k-1}^2 + \alpha\log\varepsilon_{k-1}^2,$$
(2.5) ~~(2.3)~~
$$\log\sigma_k^2 = \omega + \beta\log\sigma_{k-1}^2 + \alpha_1\varepsilon_{k-1} + \alpha_2|\varepsilon_{k-1}|.$$

Again, Theorem 1.1 implies, for (2.4) (for (2.5)), the existence of a strictly stationary solution which (i) is ~~a~~ adapted to the past if $|\alpha+\beta| < 1$ (if $|\beta| < 1$), (ii) is adapted to the future if $|\alpha+\beta| > 1$ (if $|\beta| > 1$). If $|\alpha+\beta| = 1$ (if $|\beta| = 1$) there is no solution adapted to the past or the future.

Similar statements referring to strictly stationary solutions of the recursions defining random coefficient autoregressive processes could be obtained in the same fashion. Details are omitted here.

3. Proofs

First observe that, using recursion (1.1) repeatedly, we obtain, for any integer k and any $N \geq 1$,

$$z_k = z_{k-N}\prod_{i=1}^{N} c(\varepsilon_{k-i}) + \sum_{i=1}^{N} g(\varepsilon_{k-i})\prod_{j=1}^{i-1} c(\varepsilon_{k-j}). \tag{3.1}$$

On the other hand, rewriting (1.1) as

$$z_{k-1} = \frac{1}{c(\varepsilon_{k-1})}z_k - \frac{g(\varepsilon_{k-1})}{c(\varepsilon_{k-1})}$$

yields similarly that

$$z_k = z_{k+N}\prod_{i=0}^{N-1}\frac{1}{c(\varepsilon_{k+i})} - \sum_{i=0}^{N-1} g(\varepsilon_{k+i})\prod_{j=0}^{i}\frac{1}{c(\varepsilon_{k+j})}. \tag{3.2}$$

On letting $N \to \infty$, equations (3.1) and (3.2) provide natural candidates for strictly stationary solutions to (1.1) which are measurable with respect to the past and to the future, respectively. To proceed with the proof, we need to establish two auxiliary results. The first follows immediately from the strong law of large numbers and concerns properties of the product

$$\pi_N = \prod_{\ell=1}^{N} c(\varepsilon_{-\ell}), \qquad N \geq 1, \tag{3.3}$$

which will play a decisive role in the following.

Lemma 3.1.
(i) *If* $E \log |c(\varepsilon_0)| > 0$, *then*

$$\sum_{i=1}^{N} \log |c(\varepsilon_{-i})| \to \infty \qquad a.s. \qquad (N \to \infty).$$

(ii) *If* $E \log |c(\varepsilon_0)| < 0$, *then*

$$\sum_{i=1}^{N} \log |c(\varepsilon_{-i})| \to -\infty \qquad a.s. \qquad (N \to \infty).$$

Proof. It suffices to prove (i), since symmetry then implies (ii). If $E \log |c(\varepsilon_0)| < \infty$ the assertion follows from the strong law of large numbers. So assume $E \log |c(\varepsilon_0)| = \infty$. It clearly holds, for any $K > 0$,

$$\sum_{i=1}^{N} \log |c(\varepsilon_{-i})| \geq \sum_{i=1}^{N} \min\{\log |c(\varepsilon_{-i})|, K\}.$$

As $K \to \infty$, $E \min\{\log |c(\varepsilon_0)|, K\} \to \infty$, hence $E \min\{\log |c(\varepsilon_0)|, K\} > 0$ for K large enough. Using the strong law of large numbers, (i) is readily proved. ∎

For $k \geq 1$, let $\mathcal{T}_k$ be the σ-algebra generated by the random variables $\varepsilon_{-1}, \ldots, \varepsilon_{-k}$. Recall that a random variable M is called a stopping time with respect to the filtration $\{\mathcal{T}_k\}$ if $\{M \leq k\} \in \mathcal{T}_k$ for all $k \geq 1$. Let

$$\mathcal{T}_M = \{A \in \mathcal{B} : A \cap \{M \leq k\} \in \mathcal{T}_k \text{ for all } k \geq 1\},$$

where $\mathcal{B}$ is the σ-algebra of the underlying probability space, and denote by $\{z_k\}$ a strictly stationary, $\{\mathcal{F}_{k-1}\}$-adapted solution of (1.1). Note that if M is a stopping time with respect to $\{\mathcal{T}_k\}$, then

(3.4) $$z_{-M} \text{ and } \mathcal{T}_M \text{ are independent;}$$

(3.5) ~~(3.4)~~ $$\pi_M \text{ and } \sum_{i=1}^{M} g(\varepsilon_{-i})\pi_{i-1} \text{ are measurable with respect to } \mathcal{T}_M;$$

(3.6) $$P\{z_{-M} \le x\} = F(x), \qquad \text{where } F(x) = P\{z_0 \le x\}.$$

Relation (3.4) follows from the strong Markov property, while (3.4) is evident from the definition of $\mathcal{T}_M$. Finally, (3.4) follows from (3.4) and the stationarity of $\{z_k\}$. δ5 P6

For a set A, let $I\{A\}$ be its set indicator function.

Lemma 3.2. *Let M be a stopping time with respect to $\{\mathcal{T}_k\}$. Then*

$$\begin{aligned} F(x) = E\Bigg[& I\Big\{\sum_{i=1}^{M} g(\varepsilon_{-i})\pi_{i-1} \le x\Big\} I\{\pi_M = 0\} + \\ & + F\Big(\pi_M^{-1}\Big[x - \sum_{i=1}^{M} g(\varepsilon_{-i})\pi_{i-1}\Big]\Big) I\{\pi_M > 0\} + \\ & + \Big[1 - F_-\Big(\pi_M^{-1}\Big[x - \sum_{i=1}^{M} g(\varepsilon_{-i})\pi_{i-1}\Big]\Big)\Big] I\{\pi_M < 0\}\Bigg], \end{aligned}$$

where $F_-(t) = \lim_{s \nearrow t} F(s)$ denotes the left limit.

Proof. Put $k = 0$ and $N = M$ in formula (3.1) and use the fact that $P\{z_0 \le x\} = E[P\{z_0 \le x | \mathcal{T}_M\}]$. By (3.4)–(3.4) and well-known properties of conditional expectations, Lemma 3.2 follows immediately. ∎ P6 ⅃- shorter

Proof of Theorem 1.1. (i) We assume that

(3.7) ~~(3.5)~~ $$E \log |c(\varepsilon_0)| < 0.$$

It follows from Brandt (1986) that the series

$$z_k^* = \sum_{i=1}^{\infty} g(\varepsilon_{k-i}) \prod_{j=1}^{i-1} c(\varepsilon_{k-j}), \qquad -\infty < k < \infty,$$

are absolutely convergent with probability one. It is also clear that z_k^* is measurable with respect to $\mathcal{F}_{k-1}$ and that it satisfies the recursion (1.1). Moreover, $\{z_k^*\}$ is strictly stationary. To finish the proof we need to show that it is the unique solution. To this end, let $\{z_k\}$ be another strictly stationary solution. Using equation (3.1), we obtain that, for any k and $N \geq 1$,

$$|z_k - z_k^*| \leq |z_{k-N}| \prod_{i=1}^{N} |c(\varepsilon_{k-i})| + \sum_{i=N+1}^{\infty} |g(\varepsilon_{k-i})| \prod_{j=1}^{i-1} |c(\varepsilon_{k-j})|.$$

Since the sum defining z_k^* is absolutely convergent with probability one, the second term on the right-hand side of the latter inequality converges to zero almost surely. Also, by (3.5) and part (ii) of Lemma 3.1, we have that, for any fixed k,

$$\prod_{i=1}^{N} |c(\varepsilon_{k-i})| = \exp\Big(\sum_{i=1}^{N} \log|c(\varepsilon_{k-i})|\Big) \to 0 \qquad \text{a.s.} \qquad (N \to \infty).$$

Since $\{z_k\}$ is a strictly stationary solution, for any $\varepsilon > 0$, there is a constant $L > 0$ such that $P\{|z_{k-N}| \geq L\} \leq \varepsilon$ for all N. Hence, for any $\varepsilon > 0$ and $\delta > 0$,

$$\lim_{N\to\infty} P\Big\{|z_{k-N}| \prod_{i=1}^{N} |c(\varepsilon_{k-i})| + \sum_{i=N+1}^{\infty} |g(\varepsilon_{k-i})| \prod_{j=1}^{i-1} |c(\varepsilon_{k-j})| > \delta\Big\} \leq \varepsilon,$$

implying that

$$P\{z_k = z_k^*\} = 1 \qquad \text{for all } \; -\infty < k < \infty,$$

finishing the first part of the proof.

(ii) We repeat the arguments used in part (i) of the proof. Since

$$E \log|c(\varepsilon_0)| > 0,$$

the series

$$\tilde{z}_k = -\sum_{i=0}^{\infty} g(\varepsilon_{k+i}) \prod_{j=0}^{i} \frac{1}{c(\varepsilon_{k+j})}, \qquad -\infty < k < \infty,$$

are absolutely convergent with probability one. Clearly, $\tilde{z}_k$ is $\mathcal{G}_k$-measurable and $\{\tilde{z}_k\}$ is a strictly stationary sequence. Let $\{z_k\}$ denote another strictly stationary solution of (1.1). Then, by (3.2) and the definition of $\tilde{z}_k$, we have, for all k and $N \geq 1$,

$$|z_k - \tilde{z}_k| \leq |z_{k+N}| \prod_{i=0}^{N-1} \frac{1}{|c(\varepsilon_{k+i})|} + \sum_{i=N}^{\infty} |g(\varepsilon_{k+i})| \prod_{j=0}^{i} \frac{1}{|c(\varepsilon_{k-j})|}.$$

Since the sum defining $\tilde{z}_k$ is absolutely convergent with probability one,

$$\sum_{i=N}^{\infty} |g(\varepsilon_{k+i})| \prod_{j=0}^{i} \frac{1}{|c(\varepsilon_{k-j})|} \to 0 \qquad \text{a.s.} \qquad (N \to \infty),$$

and, by part (i) of Lemma 3.1,

$$\prod_{i=0}^{N-1} \frac{1}{|c(\varepsilon_{k+i})|} = \exp\Big(-\sum_{i=0}^{N-1} \log|c(\varepsilon_{k+i})|\Big) \to 0 \qquad \text{a.s.} \qquad (N \to \infty),$$

so that it follows as above that $P\{z_k = \tilde{z}_k\} = 1$. ∎

Proof of Theorem 1.2. We shall only give a proof of (i), since the proof of (ii) is essentially the same. Recall the definition of π_N in (3.3) and F in (3.4), and fix a real x. We assume that (1.1) has a nondegenerate, strictly stationary solution $\{z_k\}$ which is adapted to $\{\mathcal{F}_{k-1}\}$, but $E\log|c(\varepsilon_0)| \geq 0$. We show in the following that this leads to a contradiction. For $N \geq 1$, let 16

$$\begin{aligned}\eta_N =& I\Big\{\sum_{i=1}^{N} g(\varepsilon_{-i})\pi_{i-1} \leq x\Big\} I\{\pi_N = 0\} + \\ &+ F\Big(\pi_N^{-1}\Big[x - \sum_{i=1}^{N} g(\varepsilon_{-i})\pi_{i-1}\Big]\Big) I\{\pi_N > 0\} + \\ &+ \Big[1 - F_-\Big(\pi_N^{-1}\Big[x - \sum_{i=1}^{N} g(\varepsilon_{-i})\pi_{i-1}\Big]\Big)\Big] I\{\pi_N < 0\}.\end{aligned}$$

Note that if $E\log|c(\varepsilon_0)| = 0$ and $P\{\log|c(\varepsilon_0)| = 0\} < 1$, then the Chung–Fuchs law of large numbers (cf. Chow and Teicher (1988)) implies that

(3.8) ~~(3.6)~~

$$\limsup_{N\to\infty} |\pi_N| = \limsup_{N\to\infty} \prod_{i=1}^{N} |c(\varepsilon_{-i})| = \limsup_{N\to\infty} \exp\Big(\sum_{i=1}^{N} \log|c(\varepsilon_{-i})|\Big) = \infty \qquad \text{a.s.}$$

Hence, with probability one,

(3.9) ~~(3.7)~~ $(\alpha)\ \limsup_{N\to\infty} \pi_N = +\infty$ or $(\beta)\ \liminf_{N\to\infty} \pi_N = -\infty$ or (γ) both.

Assume now that $P\{\log|c(\varepsilon_0)| = 0\} = 1$. If $P\{g(\varepsilon_0) = 0\} = 1$, then the solution of (1.1) satisfies $|z_k| = 1$ a.s. and it is thus degenerate. In the following, we study the case when $|c(\varepsilon_0)| = 1$ a.s. and $P\{g(\varepsilon_0) = 0\} < 1$. First observe that, by the latter assumption on $g(\varepsilon_0)$ and the Borel–Cantelli lemma, there is a $\delta > 0$ such that $P\{|g(\varepsilon_k)| \ge \delta \text{ infinitely often}\} = 1$. Hence, $\sum_{i=1}^{N} g(\varepsilon_{-i})\pi_{i-1}$ cannot converge a.s., since

$$P\{|g(\varepsilon_{-i})\pi_{i-1}| = |g(\varepsilon_{-i})| \ge \delta \text{ infinitely often}\} = 1.$$

Consequently, we have either that

$$\limsup_{N\to\infty} \sum_{i=1}^{N} g(\varepsilon_{-i})\pi_{i-1} = c_1 \ne c_2 = \liminf_{N\to\infty} \sum_{i=1}^{N} g(\varepsilon_{-i})\pi_{i-1} \qquad \text{a.s.} \tag{$\cancel{3.8}$ (3.10)}$$

or at least one of the next two options holds true:

$$\limsup_{N\to\infty} \sum_{i=1}^{N} g(\varepsilon_{-i})\pi_{i-1} = +\infty \qquad \text{a.s.,} \tag{3.11}$$

$$\liminf_{N\to\infty} \sum_{i=1}^{N} g(\varepsilon_{-i})\pi_{i-1} = -\infty \qquad \text{a.s.} \tag{$\cancel{3.9}$ (3.12)}$$

10 Consider the case (3.8) and assume that there is a subsequence $N(k)$ along which $\pi_{N(k)} = 1$ and

$$\sum_{i=1}^{N(k)} g(\varepsilon_{-i})\pi_{i-1} \to c_1 \qquad \text{a.s.}$$

We can and shall assume that $\sum_{i=1}^{N(k)} g(\varepsilon_{-i})\pi_{i-1} \le c_1$. Hence, $\eta_{N(k)} \to F(x - c_1)$ a.s., and therefore $F(x) = F(x - c_1)$ for all x. If $c_1 \ne 0$, then this is a contradiction to the fact that F is the distribution function of a nondegenerate solution. If $c_1 = 0$, then $c_2 \ne 0$ and similar arguments lead to the same contradiction. The case $\pi_{N(k)} = -1$ can be treated in an analogous fashion.

11 Consider now the case (3.9) and assume that there is a subsequence $N(\ell)$ along which $\pi_{N(\ell)} = 1$ and

$$\lim_{\ell\to\infty} \sum_{i=1}^{N(\ell)} g(\varepsilon_{-i})\pi_{i-1} = +\infty \qquad \text{a.s.}$$

Then, $\eta_{N(\ell)} \to 0$ and, therefore, for all x, $F(x) = 0$, a contradiction. Similar arguments apply if $\pi_{N(\ell)} = -1$. They provide the same contradiction moreover in

12 the case (3.9).

If, finally, $E \log |c(\varepsilon_0)| > 0$, the strong law of large numbers implies that (3.6) 8
and consequently relation (3.7) hold.

9

It is enough to deduce now a contradiction for case (α) of (3.8). The proof for (β) and (γ) is the same. 19

Assume that (α) in (3.8) holds and define a sequence $N(m)$ of integer-valued random variables by $N(1) = 1$ and $N(m) = \min\{j > N(m-1) : \pi_j \geq m\}$ for $m \geq 2$. It is obvious that $N(m) \to \infty$, $\pi_{N(m)} \to \infty$ and that, for each m, $N(m)$ is a stopping time with respect to $\{\mathcal{T}_k\}$. Let

$$K = \limsup_{m\to\infty} \pi_{N(m)}^{-1} \sum_{i=1}^{N(m)} g(\varepsilon_{-i})\pi_{i-1}.$$

Clearly, for each integer $L \geq 1$,

$$K = \limsup_{m\to\infty} \pi_{N(m)}^{-1} \sum_{i=L}^{N(m)} g(\varepsilon_{-i})\pi_{i-1}.$$

ρ- If $N(m) < L$, the sum on the right hand side is 0, and if $N(m) \geq L$, the expression $\pi_{N(m)}^{-1} \sum_{i=L}^{N(m)} g(\varepsilon_{-i})\pi_{i-1}$ is a function of $\varepsilon_{-(L-1)}, \varepsilon_{-L}, \ldots$. Hence by the $0-1$ law, K is a constant with probability one. Clearly, infinitely many of the numbers

$$\pi_{N(m)}^{-1} \sum_{i=1}^{N(m)} g(\varepsilon_{-i})\pi_{i-1} \qquad (3.13) \; \cancel{(3.10)}$$

lie on one side of K and keeping only these indices from the sequence $N(m)$, the stopping time property remains valid. We now construct an increasing sequence $N^*(m)$ of integers such that, for each m, $N^*(m)$ is a stopping time with respect to the filtration $\{\mathcal{T}_k\}$ and

$$\lim_{m\to\infty} \pi_{N^*(m)}^{-1} \sum_{i=1}^{N^*(m)} g(\varepsilon_{-i})\pi_{i-1} = K.$$

Assume first that K is finite and the numbers in (3.10) lie, e.g., on the right side of K. Let $N^*(1) = 1$ and if $N^*(1) < \cdots < N^*(m-1)$ are already defined, let $N^*(m)$ be the smallest integer $k > N^*(m-1)$ such that k belongs to the sequence $\{N(j)\}$ and γ3 (3.13)

$$\sum_{i=1}^{k} g(\varepsilon_{-i})\pi_{i-1} < K + 1/m.$$

The validity of the relation $\{N^*(m) \leq \ell\}$ depends only on the values of the random variables $\varepsilon_{-1}, \ldots, \varepsilon_{-\ell}, \pi_1, \ldots, \pi_\ell$ and thus $\{N^*(m) \leq \ell\} \in \mathcal{T}_\ell$, showing that $N^*(m)$

satisfies the requirement. A similar argument holds if $K = \pm\infty$. Clearly,

$$\frac{x - \sum_{i=1}^{N^*(m)} g(\varepsilon_{-i})\pi_i}{\pi_{N^*(m)}} \to -K.$$

If there is a subsequence $N^*(m(k))$ of $N^*(m)$ such that

$$\frac{x - \sum_{i=1}^{N^*(m(k))} g(\varepsilon_{-i})\pi_i}{\pi_{N^*(m(k))}} \geq -K,$$

then $\eta_{N^*(m(k))} \to F(-K)$ a.s. and therefore by Lemma 3.2 $F(x) = F(-K)$. If for some subsequence $N^*(m(k))$ we have that

$$\frac{x - \sum_{i=1}^{N^*(m(k))} g(\varepsilon_{-i})\pi_i}{\pi_{N^*(m(k))}} \leq -K,$$

then $\eta_{N^*(m(k))} \to F_{-}(-K)$ a.s. and therefore by Lemma 3.2 $F(x) = F_{-}(-K)$. , 2x
This means that $F(x)$ can take only two values at most. If F takes only one value, it cannot be a distribution function, so we have a contradiction. If F takes two values, then z_0 is a constant with probability one, contradicting that the solution is nondegenerate. This completes the proof of part (i) of Theorem 1.2. ■

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