On the law of the iterated logarithm for trigonometric series with bounded gaps

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Abstract

Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers. Bobkov and Götze proved that if the distribution of

$$\frac{\cos 2\pi n_1 x + \cdots + \cos 2\pi n_N x}{\sqrt{N}} \tag{1}$$

converges to a Gaussian distribution, then the value of the variance is bounded from above by $1/2 - \limsup k/(2n_k)$. In particular it is impossible that for a sequence $(n_k)_{k\geq 1}$ with bounded gaps (i.e. $n_{k+1} - n_k \leq c$ for some constant c) the distribution of (1) converges to a Gaussian distribution with variance $\sigma^2 = 1/2$ or larger.

In this paper we show that the situation is considerably different in the case of the law of the iterated logarithm. We prove the existence of an increasing sequence of positive integers satisfying

$$n_{k+1} - n_k \leq 2$$

such that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{2N \log\log N}} = +\infty \quad \text{a.e.}$$

1 Introduction and statement of results

Let $(n_k)_{k\geq 1}$ be an increasing sequence of positive integers. It is well known that the system $(\cos 2\pi n_k x)_{k\geq 1}$ behaves similar to a system of independent, identically distributed (i.i.d.) random variables if $(n_k)_{k\geq 1}$ is growing fast. For example, Salem and Zygmund [23] proved that the distribution of

$$\frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{N}} \tag{2}$$

converges to the normal (0,1/2) distribution if $(n_k)_{k\geq 1}$ satisfies

$$\frac{n_{k+1}}{n_k} \geq q > 1, \qquad k \geq 1. \tag{3}$$

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Condition (3) is called "Hadamard's gap condition", and a sequence satisfying this condition is called a "lacunary" sequence. Erdős and Gál [10] proved that the system $(\cos 2\pi n_k x)_{k\geq 1}$, where $(n_k)_{k\geq 1}$ is lacunary, also satisfies the law of the iterated logarithm (LIL). They showed that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{2N \log\log N}} = \frac{1}{\sqrt{2}} \quad \text{a.e.},$$

which is in perfect accordance with the behavior of systems of i.i.d. random variables (cf. also [26]).

The situation gets much more involved if the function $\cos 2\pi x$ is replaced by a general 1-periodic function (cf. [4, 13, 21]), or if the sequence $(n_k)_{k\geq 1}$ is of sub-lacunary growth (cf. [1, 6, 7, 17, 22]). There are only very few precise results which hold for general sequences $(n_k)_{k\geq 1}$ without any growth conditions, except for the case $n_k = k,\ k \geq 1$, where the theory of continued fractions can be used to obtain precise estimates (cf. [18, 25]).

Though it is not possible to determine the precise asymptotic behavior of systems $(\cos 2\pi n_k x)_{k\geq 1}$ for slowly growing $(n_k)_{k\geq 1}$, it turns out that random constructions can be used to prove the existence of slowly growing sequences $(n_k)_{k\geq 1}$ for which $(\cos 2\pi n_k x)_{k\geq 1}$ exhibits a certain probabilistic behavior. For example, Salem and Zygmund [24] showed that if $X_k(\omega)$ are independent random variables taking the values ± 1 which probability 1/2 each, then for the sequence $(n_k(\omega))_{k\geq 1}$ which consists of all number k for which $X_k(\omega) = 1$ the distribution of (2) converges to the normal (0,1/4) distribution for almost all ω. Observe that the sequences constructed in such a way satisfy

$$\lim n_k/k \to 2 \qquad \text{and} \qquad \sup_{k\geq 1}(n_{k+1} - n_k) = \infty$$

for almost all ω. This is in accordance with a result of Bobkov and Götze [8], which states that whenever the system $(\cos 2\pi n_k x)_{k\geq 1}$ satisfies the CLT for a sequence $(n_k)_{k\geq 1}$, the variance of the limiting distribution can be at most $1/2 - \limsup_{k\to\infty} k/(2n_k)$. In particular this implies that no sequence $(n_k)_{k\geq 1}$, for which the distribution of (2) converges to a normal (0,1/2) distribution, can have bounded gaps. Recently, it was proved [11, 12, 14] that a CLT for $(\cos 2\pi n_k x)_{k\geq 1}$ with variance arbitrarily close to 1/2 is possible for sequences with bounded gaps. On the other hand, Berkes [5] proved that if we allow $n_{k+1} - n_k \to \infty$, then a variance of 1/2 of the limiting distribution is possible.

Less investigations have been made concerning the law of the iterated logarithm in the case of slowly growing $(n_k)_{k\geq 1}$. Salem and Zygmund [24] showed that if $X_k(\omega)$ are independent random variables taking the values ± 1 which probability 1/2 each, then, writing $(n_k(\omega))_{k\geq 1}$ for the sequence which consists of all the numbers k for which $X_k = 1$, for almost all ω

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{2N \log\log N}} = \frac{1}{2}$$

for almost all x (in fact they proved a result for more general trigonometric series, which contains the above result as a special case). As mentioned before, the sequences constructed by Salem and Zygmund have (almost surely) an asymptotic density of 1/2 and contain arbitrarily large gaps.

Very recently, it was shown that there exist sequences $(n_k)_{k\geq 1}$ with bounded gaps for which the lim sup in the LIL for the discrepancy D_N of $(\langle n_k x\rangle)_{k\geq 1}$ (where $\langle\cdot\rangle$ denotes the fractional part) is not a constant almost everywhere (see [15]), and that for any $\varepsilon > 0$ there exists a sequence $(n_k)_{k\geq 1}$ with bounded gaps such that the value of the lim sup in the LIL for $D_N(\langle n_k x\rangle)$ is larger than $1/2-\varepsilon$ for almost all x (see [14]). The same methods could be used to show that there exist sequences $(n_k)_{k\geq 1}$ with bounded gaps such that the value of

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{2N\log\log N}} \tag{4}$$

is not a constant a.e., or larger than $1/\sqrt{2}-\varepsilon$ a.e.

The purpose of this paper is to show that there even exist sequences $(n_k)_{k\geq 1}$ with bounded gaps for which (4) equals $+\infty$ for almost all x. We can limit the maximal distance between two consecutive elements of $(n_k)_{k\geq 1}$ to the smallest possible value, i.e. we can require that

$$n_{k+1}-n_k \leq 2, \qquad k\geq 1.$$

Theorem 1 *There exists a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers satisfying*

$$n_{k+1}-n_k \leq 2$$

such that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{2N\log\log N}} = \infty \quad \text{a.e.}$$

Combining this theorem with Koksma's inequality (see [9] or [19]) we get the following corollary:

Corollary 1 *There exists a strictly increasing sequence $(n_k)_{k\geq 1}$ of positive integers satisfying*

$$n_{k+1}-n_k \leq 2$$

such that

$$\limsup_{N\to\infty} \frac{N D_N(\langle n_k x\rangle)}{\sqrt{2N\log\log N}} = \infty \quad \text{a.e.}$$

The main idea in the proof is based on the observation that for arbitrary l there exists a slowly growing sequence $(n_k)_{k\geq 1}$ such that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{2N\log\log N}} \geq l \tag{5}$$

holds for all x contained in a set A_l of positive measure (the measure depending on l), and that it is possible to find slightly modified versions $\left(n_k^{(l,m)}\right)_{k\geq 1}$ of this sequence for which the sets $A_l^{(1)}, A_l^{(2)}, \dots, A_l^{(m)}, \dots$, where (5) holds, fill (almost) the whole interval (0,1). Mixing a sequence $(n_k)_{k\geq 1}$ out of all these sequences $(n_k^{(l,m)})_{k\geq 1}$ for $l, m \geq 1$, which contains "large" subblocks of elements of each of these sequences, will yield

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \cos 2\pi n_k x}{\sqrt{2N\log\log N}} = \infty$$

for almost all x. We believe that this idea of obtaining a sequence with a globally large value of the lim sup in the LIL by mixing sequences having locally a large value of the lim sup could also be used in other circumstances to construct sequences having a large value of the lim sup in the LIL, e.g. in the case of lacunary sequences. This technique might be particularly useful if there already exist constructions of sequences $(n_k)_{k\geq 1}$ for which the lim sup in (4) or in the LIL for the discrepancy is non-constant (cf. [2, 3, 16]).

2 Preliminaries

The following Lemma 1 can be found in [20, Theorem 5.4, p. 149]. Lemma 2 is standard.

Lemma 1 (Esseen's inequality) *Let $X_1, \dots, X_N$ be independent random variables satisfying*

$$\mathbb{E}X_k = 0, \qquad \mathbb{E}|X_k|^3 < +\infty, \qquad k = 1, \dots, N.$$

Set

$$B_N = \sum_{k=1}^{N} \mathbb{V}X_k, \qquad F_N(y) = \mathbb{P}\left(B_N^{-1/2} \sum_{k=1}^{N} X_k < y\right).$$

Then

$$\sup_y |F_N(y) - \Phi(y)| \leq c B_N^{-3/2} \sum_{k=1}^{N} \mathbb{E}|X_k|^3,$$

where c is an absolute constant, and Φ is the normal (0,1) distribution function.

Lemma 2 *For any integers A and $B \neq 0$,*

$$\sum_{k=1}^{N} \cos 2\pi(A + Bk)x = \frac{\cos \pi(2A + NB)x \sin \pi(N+1)Bx}{\sin \pi Bx} = \mathcal{O}(1)$$

except for finitely many real numbers x. Especially

$$\left|\sum_{k=1}^{N} \cos 2\pi kx\right| = \mathcal{O}(1) \quad \textit{and} \quad \left|\sum_{1\leq k\leq N:\ k \text{ odd}} \cos 2\pi kx\right| = \mathcal{O}(1), \quad \text{a.e. } x$$

3 Proof of Theorem 1

Lemma 3 *There exists a sequence $(n_k)_{k\geq 1}$ having an asymptotic density of $1/2$ such that*

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \cos 2\pi n_k x}{\sqrt{2N \log\log N}} = \infty \quad \text{a.e.}$$

Theorem 1 is an easy consequence of Lemma 2 and Lemma 3: Let $(n_k)_{k\geq 1}$ be the sequence in Lemma 3. Define a new sequence $(m_k)_{k\geq 1}$, which consists of all odd positive integers and all even numbers j for which $j/2 \in (n_k)_{k\geq 1}$, sorted in increasing order. Since the sequence $(m_k)_{k\geq 1}$ contains all odd integers, we obviously have

$$m_{k+1} - m_k \leq 2, \qquad k \geq 1.$$

Let

$$M(N) = \#\{j \leq N : \ m_j/2 \in (n_k)_{k \geq 1}\}.$$

Then

$$\{m_k : \ 1 \leq k \leq N\} = \{2n_k : \ 1 \leq k \leq M(N)\} \cup \{1 \leq k \leq 2(N - M(N)) : \ k \text{ odd}\}.$$

Therefore

$$\sum_{k=1}^{N} \cos 2\pi m_k x = \sum_{k=1}^{M(N)} \cos 4\pi n_k x + \sum_{\substack{1 \leq k \leq 2(N-M(N)) \\ k \text{ odd}}} \cos 2\pi k x,$$

and

$$\begin{aligned} & \limsup_{N \to \infty} \frac{\sum_{k=1}^{N} \cos 2\pi m_k x}{\sqrt{2N \log \log N}} \\ \geq \ & \limsup_{N \to \infty} \frac{\sum_{k=1}^{M(N)} \cos 4\pi n_k x}{\sqrt{2N \log \log N}} - \limsup_{N \to \infty} \frac{\left| \sum_{1 \leq k \leq 2(N-M(N)): \ k \text{ odd}} \cos 2\pi k x \right|}{\sqrt{2N \log \log N}}, \end{aligned}$$

By assumption

$$\frac{n_k}{k} \to 2,$$

which implies

$$\frac{M(N)}{N} \to \frac{1}{3}.$$

Since by Lemma 3

$$\limsup_{N \to \infty} \frac{\sum_{k=1}^{M(N)} \cos 4\pi n_k x}{\sqrt{2N \log \log N}} = \limsup_{N \to \infty} \frac{\sum_{k=1}^{M(N)} \cos 4\pi n_k x}{\sqrt{6M(N) \log \log M(N)}} = \infty \quad \text{a.e.},$$

and since

$$\limsup_{N \to \infty} \frac{\left| \sum_{1 \leq k \leq 2(N-M(N)): \ k \text{ odd}} \cos 2\pi k x \right|}{\sqrt{2N \log \log N}} = 0 \quad \text{a.e.}$$

by Lemma 2, we conclude

$$\limsup_{N \to \infty} \frac{\sum_{k=1}^{N} \cos 2\pi m_k x}{\sqrt{2N \log \log N}} = \infty \quad \text{a.e.},$$

which is Theorem 1. $\square$

Proof of Lemma 3:

We write I_h for the set of integers in the interval $\left[2^{(h^2)} + 1, 2^{((h+1)^2)}\right]$. Then $|I_h| = 2^{(h^2+2h+1)} - 2^{(h^2)}$, where $|\cdot|$ denotes the number of elements of a set.

We write J_i for the set of integers in the interval

$$\left[\frac{(i-1)i(2i-1)}{6} + 1, \frac{i(i+1)(2i+1)}{6}\right].$$

Then $|J_i| = i^2$, and every positive integer h is contained in exactly one set J_i for appropriate i. If $h \in J_i$, then h has a unique representation of the form

$$h = \frac{(i-1)i(2i-1)}{6} + 1 \ + li + m, \qquad \text{where} \qquad 0 \le l, m \le i-1.$$

We call the pair (l,m), which is defined in this way, the "type" of h and of the interval I_h. If h is of type (l,m), then necessarily

$$l, m \le (4h)^{1/3}. \tag{6}$$

We define a sequence of *independent* random variables $X_1, X_2, \dots$ over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ which have the following properties:

For any number $k \ge 1$ there exists a uniquely defined number h such that $k \in I_h$. Assume that I_h is of type (l,m). If $(k \mod 2^{l+m}) \in \{1, \dots, 2^m\}$, then

$$X_k = X_k(x,\omega) = \begin{cases} \frac{1}{2}\sum_{j=0}^{2^l-1} \cos 2\pi \left(k + j2^m\right) x & \text{with probability } \mathbb{P} = 1/2 \\ -\frac{1}{2}\sum_{j=0}^{2^l-1} \cos 2\pi \left(k + j2^m\right) x & \text{with probability } \mathbb{P} = 1/2 \end{cases} \tag{7}$$

If $(k \mod 2^{l+m}) \notin \{1, \dots, 2^m\}$, then

$$X_k \equiv 0.$$

If $k \in I_h$ for some h and the interval I_h is of type (l,m), then way say that the random variable X_k is of type (l,m). We will use the symbols $\mathbb{E}, \mathbb{V}, \mathbb{P}$ with respect to the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and write λ for the Lebesgue measure on $(0,1)$. The notation "a.s" will always be used with respect to $\mathbb{P}$, while "a.e." is meant with respect to λ.
Observe that for any $j \in I_h$ the function $\cos 2\pi jx$ appears in the definition of exactly one random variable $X_k,\ k \in I_h$, and that

$$\sum_{k \in I_h} X_k = \frac{1}{2} \sum_{k \in I_h} \pm \cos 2\pi kx.$$

By construction

$$\mathbb{E} X_k = 0 \qquad k \ge 1.$$

We write Δ^* for the set of indices k for which $X_k \equiv 0$, and $\Delta = \mathbb{N} \backslash \Delta^*$. For $k \in \Delta^*$

$$\mathbb{V} X_k = 0,$$

and for $k \in \Delta$

$$\begin{aligned} \mathbb{V} X_k &= \frac{1}{4} \left(\sum_{j=0}^{2^l-1} \cos 2\pi \left(k + j2^m\right) x \right)^2 \\ &= \frac{1}{4} \left(\frac{\sin(\pi 2^l 2^m x) \cos(\pi(2k + (2^l-1)2^m)x)}{\sin \pi 2^m x} \right)^2, \end{aligned} \tag{8}$$

if X_k is of type (l, m).

Write

$$B_N = B_N(x) = \sum_{k=1}^{N} \mathbb{V} X_k.$$

If X_k is of type (l, m), then

$$|X_k| \le 2^l.$$

By (6) this implies

$$|X_k| \le 2^{(4h)^{1/3}} \tag{9}$$

and

$$B_k \le 2^{(h+1)^2 + (4h)^{1/3}} = o(2^{(h+2)^2}) \tag{10}$$

for $k \in I_h$.

We define

$$A_{l,m} = \bigcup_{j=0}^{2^m-1} \left[\frac{j}{2^m} - \frac{1}{2^{l+m+1}}, \frac{j}{2^m} + \frac{1}{2^{l+m+1}} \right], \qquad l, m \ge 0.$$

For $x \in A_{l,m}$ we have

$$\frac{\sin\left(\pi 2^l 2^m x\right)}{\sin \pi 2^m x} \ge 2^{l-1}. \tag{11}$$

For any h, we write its type by $(l(h), m(h))$. By the orthogonality of the trigonometric system,

$$\int_0^1 \sum_{k \in I_h} \left(\cos(\pi(2k + (2^{l(h)} - 1) 2^{m(h)} x) \right)^2 dx = \frac{|I_h|}{2},$$

and

$$\begin{aligned}
&\int_0^1 \left(\left(\sum_{k \in I_h} (\cos(\pi(2k + (2^{l(h)} - 1) 2^{m(h)}) x)^2 \right) - |I_h|/2 \right)^2 dx \\
= \quad &\int_0^1 \left(\sum_{k \in I_h} \cos(2\pi(2k + (2^{l(h)} - 1) 2^{m(h)}) x) \right)^2 dx \\
= \quad &|I_h|/2.
\end{aligned}$$

Therefore, by Chebyshev's inequality,

$$\lambda \left(\left\{ \sum_{k \in I_h} (\cos(\pi(2k + (2^{l(h)} - 1) 2^{m(h)}) x)^2 < |I_h|/4 \right\} \right) \le \frac{8}{|I_h|}.$$

Obviously we have

$$\sum_{h=1}^{\infty} \frac{8}{|I_h|} < +\infty,$$

so by the Borel-Cantelli lemma there exist a set A with $\lambda(A) = 1$ such that for all $x \in A$ there exists an $H = H(x)$ such that for all $h \geq H$

$$\sum_{k \in I_h} (\cos(\pi(2k + (2^{l(k)} - 1)2^{m(k)})x)^2 \geq |I_h|/4. \tag{12}$$

Combining (8), (11) and (12) we obtain for all $x \in A \cap A_{l,m}$ and $h \geq H$

$$\sum_{k \in I_h} \mathbb{V} X_k(x) > 2^{2l(h)-4} |I_h|. \tag{13}$$

We write

$$B^{(h)} = \sum_{k \in I_h} \mathbb{V} X_k. \tag{14}$$

For $N \in I_h$ and $x \in A$, by (13) we have

$$B_N \geq B^{(h-1)} \geq 2^{-4} |I_{h-1}| \geq 2^{-5} 2^{\left(h^2\right)}.$$

Combining this together with (9), we can verify the condition

$$B_N \to \infty \quad \text{and} \quad \|X_N\|_\infty = o\left(\sqrt{\frac{B_N}{\log\log B_N}}\right) \quad \text{as} \quad N \to \infty.,$$

of Kolmogorov's law of the iterated logarithm, and have

$$\limsup_{N \to \infty} \frac{\left|\sum_{k=1}^N X_k\right|}{\sqrt{2 B_N \log\log B_N}} = 1 \quad \text{a.s.}$$

By noting (10), we conclude

$$\lim_{h \to \infty} \frac{\left|\sum_{k=1}^{2^{\left(h^2\right)}} X_k\right|}{\sqrt{2^{(h+1)^2} \log\log 2^{(h+1)^2}}} = 0 \tag{15}$$

Let l, m and $x \in A_{l,m} \cap A'_{l,m}$ be fixed, and assume that $h \geq H$ is of type (l, m). By (13) we can assume that

$$B^{(h)} \geq 2^{2l-4} |I_h|.$$

Then by (9)

$$\left(B^{(h)}\right)^{-3/2} \sum_{k \in I_h} \mathbb{E}|X_k|^3 \leq 2^{-3l+6} |I_h|^{-1/2} 2^{3(16 \log_2 k)^{1/6}}. \tag{16}$$

Set

$$U_h = \left\{ \sum_{k \in I_h} X_k > 2^{l-3} \sqrt{|I_h| \log\log |I_h|} \right\}.$$

By Lemma 1, (14) and (16), for some constant c,

$$\begin{aligned}
\mathbb{P}(U_h) &\geq \mathbb{P}\left(\sum_{k\in I_h} X_k > 2^{-1}\sqrt{B^{(h)}\log\log|I_h|}\right) \\
&\geq \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\log\log|I_h|/2}}^{\infty} e^{-t^2/2}dt - c2^6 2^{3(16\log_2 k)^{1/6}-3l}|I_h|^{-1/2} \\
&\geq \frac{e^{-\left(\sqrt{\log\log|I_h|/2}\right)^2/2}}{\sqrt{\log\log|I_h|}} - c2^6 2^{6(h+1)^{1/3}}\left(2^{(h^2+2h+1)} - 2^{(h^2)}\right)^{-1/2} \\
&\gg \frac{1}{h^{1/4}(\log h)^{1/2}}.
\end{aligned}$$

For every pair (l,m) and every sufficiently large i, there is a number h of type (l,m) in every interval J_i. Since the size of the integers in J_i is somewhere between $i^3/3$ and $(i+2)^3/3$, this implies

$$\sum_{\substack{h\geq 1\\ h \text{ is of type } (l,m)}} \frac{1}{h^{1/4}(\log h)^{1/2}} = \infty.$$

Since the sets U_h are independent this implies

$$\limsup_{h\to\infty} \frac{\sum_{k\in I_h} X_k}{2^{l-3}\sqrt{|I_h|\log\log|I_h|}} \geq 1 \qquad \text{a.s.}$$

and, since $|I_h|/2^{((h+1)^2)} \to 1$,

$$\limsup_{h\to\infty} \frac{\sum_{k\in I_h} X_k}{2^{l-3}\sqrt{2^{(h+1)^2}\log\log 2^{(h+1)^2}}} \geq 1 \qquad \text{a.s.,}$$

Finally, by (15)

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k}{2^{l-3}\sqrt{N\log\log N}} \geq \limsup_{h\to\infty} \frac{\sum_{k=1}^{2^{(h^2)}} X_k + \sum_{k\in I_h} X_k}{2^{l-3}\sqrt{2^{(h+1)^2}\log\log 2^{(h+1)^2}}} \geq 1 \qquad \text{a.s.,}$$

and therefore for every $x \in A_{l,m}\cap A$

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k}{\sqrt{2N\log\log N}} \geq 2^{l-4} \qquad \text{a.s.,}$$

Let

$$A_l = \bigcup_{m=1}^{\infty} A_{l,m}.$$

Then $\mathbb{P}(A_l) = 1$. In fact, $x \in A_{l,2}$ if and only if $\langle 2x\rangle \in A_{l,1}$ Similarly, $x \in A_{l,m}$ if and only if $\langle 2^{m-1}x\rangle \in A_{l,1}$, and generally

$$x \in \bigcup_{m=1}^{\infty} A_{l,m}$$

if and only if at least one element of the sequence $(\langle 2^{m-1}x\rangle)_{m\geq 1}$ is contained in $A_{l,1}$. Since the sequence $(2^{m-1}x)_{m\geq 1}$ is uniformly distributed modulo 1 for almost all real x, this implies

$$\mathbb{P}\left(\bigcup_{m=1}^{\infty} A_{l,m}\right) = 1, \qquad \text{and} \qquad \mathbb{P}(A_l) = 1.$$

Since there are only countably many choices for m we have for every $x \in A_l \cap A$,

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k}{\sqrt{2N \log\log N}} \geq 2^{l-4} \qquad \text{a.s.} \tag{17}$$

Write

$$A^* = \left(\bigcap_{l=1}^{\infty} A_l\right) \cap A$$

Then $\mathbb{P}(A^*) = 1$, and since there are only countable many values for l we have for almost all $x \in (0,1)$

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k}{\sqrt{2N \log\log N}} = \infty \qquad \text{a.s.}$$

By Fubini's theorem this also implies

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k}{\sqrt{2N \log\log N}} = \infty \qquad \text{a.e.,}$$

$\mathbb{P}$-almost surely.

We define sets $\Delta^+ = \Delta^+(\omega)$ and $\Delta^- = \Delta^-(\omega)$ in the following way:

$$\Delta^+ = \{k \in \Delta : \ X_k(\omega, 0) > 0\} \qquad \text{and} \qquad \Delta^- = \Delta \backslash \Delta^+.$$

In other words, Δ^+ contains those indices in Δ for which X_k has sign "+" in (7), and Δ^- the indices where X_k has sign "$-$" (depending on ω). We can choose an $\hat{\omega}$ for which

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N X_k(\hat{\omega}, x)}{\sqrt{2N \log\log N}} = \infty \qquad \text{a.e.}$$

and (by the strong law of large numbers)

$$\frac{\#\{k \in \Delta^+, \ k \leq N\}}{\#\{k \in \Delta^-, \ k \leq N\}} \to 1 \quad \text{as} \quad N \to \infty. \tag{18}$$

hold. In the sequel we will always assume that this particular $\hat{\omega}$ has been chosen, and write $X_k = X_k(\hat{\omega})$.

For all $k \in \mathbb{N}\backslash\Delta$ we have $X_k \equiv 0$. This means

$$\sum_{k \leq N} X_k = \sum_{k \in \Delta, k \leq N} X_k,$$

and

$$\sum_{k\leq N}\left(X_k+\frac{1}{2}\cos 2\pi kx\right)=2\sum_{k\in\Delta^+,\ k\leq N}X_k$$

The function $\sum_{k\in\Delta^+}$ is an infinite cosine-series. We write $(n_k)_{k\geq 1}$ for the sequence which consists of all frequencies which are contained in this series, sorted in increasing order. Then by (18) the sequence $(n_k)_{k\geq 1}$ has asymptotic density 1/2, and since

$$\limsup_{N\to\infty}\frac{2\sum_{k\in\Delta^+,\ k\leq N}X_k}{\sqrt{2N\log\log N}}\geq\limsup_{N\to\infty}\frac{\sum_{k\leq N}X_k}{\sqrt{2N\log\log N}}-\underbrace{\limsup_{N\to\infty}\frac{\frac{1}{2}\left|\sum_{k\leq N}\cos 2\pi kx\right|}{\sqrt{2N\log\log N}}}_{=0\text{ a.e. by Lemma 2}}$$

we have

$$\limsup_{N\to\infty}\frac{\sum_{k=1}^N\cos 2\pi n_kx}{\sqrt{2N\log\log N}}=\infty\qquad\text{a.e.},$$

which proves Lemma 3.

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