Sankhyā : The Indian Journal of Statistics
2010, Volume 72-A, Part 2, pp. 1-13

Convergence Properties of Kemp's q-Binomial Distribution

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Abstract

We consider Kemp's q-analogue of the binomial distribution. Several convergence results involving the classical binomial, the Heine, the discrete normal, and the Poisson distribution are established. Some of them are q-analogues of classical convergence properties. From the results about distributions, we deduce some new convergence results for (q-)Krawtchouk and q-Charlier polynomials. Besides elementary estimates, we apply Mellin transform asymptotics.

AMS (2000) *subject classification.* Primary 60F05; secondary 33D15.
Keywords and phrases. q-binomial distribution, discrete normal distribution, Heine distribution, q-Krawtchouk polynomials, q-Charlier polynomials, Mellin transform, limit theorems.

1. **Introduction**

Kemp and Kemp (1991) introduced a q-analogue $KB(n, \theta, q)$ of the binomial distribution. It is well known that for $n \to \infty$ (and fixed θ) Kemp's q-binomial distribution converges to a Heine distribution. We are now interested in sequences of random variables X_n with $X_n \sim KB(n, \theta_n, q)$, where (θ_n) is a sequence of positive real parameters. Our main results contain q-analogues to the convergence of the classical binomial distribution to the Poisson distribution and the normal distribution, and show that the limits $q \to 1$ and $n \to \infty$ can be exchanged. From the limit theorems about distributions we deduce limit relations for q-polynomials that are orthogonal w.r.t. these distributions.

The paper is organised as follows. In Section 2 we give all definitions of q-calculus, q-distributions, and q-polynomials we need in the following; afterwards we sum up some important properties of Kemp's q-binomial distribution in Section 3. Section 4 deals with the case of convergent parameter θ_n, in particular with the case of constant mean. The pertinent limit law is the Heine distribution, and the involved q-polynomials are the q-Krawtchouk and the q-Charlier polynomials. In Section 5 we investigate parameter sequences that tend to infinity. If they do so fast enough, then it turns out that $n - X_n$ is either degenerate in the limit or tends to a Heine distribution. The main result of the paper is concerned with parameter sequences of slower growth, where the law of the normalized X_n converges to a discrete normal distribution. From this property we deduce a limit relation for the q-Krawtchouk and the Stieltjes-Wigert polynomials.

2. Notation and Definitions

Throughout the paper we use the notation of Gasper and Rahman (1990). The q-shifted factorial $(z;q)_n$ and the q-binomial coefficient are defined by

$$(z;q)_n = \prod_{i=0}^{n-1}(1-zq^i) \qquad \text{and} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

In the limit $q \to 1$ the q-shifted factorial converges to $(1-z)^n$, and the q-binomial coefficient to the binomial coefficient $\binom{n}{k}$. The q-number $[x]_q$ is defined as

$$[x]_q := \frac{1-q^x}{1-q};$$

for $q \to 1$, we have $[x]_q \to x$. Moreover, we will need two analogues of the exponential function:

$$e_q(z) = \frac{1}{(z;q)_\infty}, \qquad z \in \mathbb{C} \setminus \{q^{-i} : \ i = 0, 1, 2, \dots\},$$

and $E_q(z) = (-z;q)_\infty$. Here the limit relations $e_q((1-q)z) \to e^z$ and $E_q((1-q)z) \to e^z$ hold, as $q \to 1$. The basic hypergeometric series ${}_r\phi_s$ is defined by

$$\begin{aligned} &{}_r\phi_s(a_1, a_2, \dots, a_r; b_1, \dots, b_s; q, z) \\ &\qquad = \sum_{n=0}^{\infty} \frac{(a_1;q)_n \cdots (a_r;q)_n}{(q;q)_n (b_1;q)_n \cdots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \end{aligned}$$

Our main object of interest is the following q-analogue $KB(n,\theta,q)$ of the binomial distribution, see Kemp and Kemp (1991):

$$\mathbb{P}(X_{KB} = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \frac{\theta^x q^{x(x-1)/2}}{(-\theta;q)_n}, \qquad 0 \le x \le n, \ 0 < \theta, \ 0 < q < 1.$$

The Heine distribution (Benkherouf and Bather, 1988, Kemp, 1992a, 1992b) $H(\theta)$ is defined by

$$\mathbb{P}(X_H = x) = \frac{q^{x(x-1)/2}\theta^x}{(q;q)_x} e_q(-\theta), \qquad x \geq 0,\ 0 < q < 1.$$

If $q \to 1$, then $H((1-q)\theta) \to P(\theta)$, where $P(\theta)$ denotes the Poisson distribution with parameter θ.

For details about the properties of the following q-polynomials, we refer to the encyclopaedic report by Koekoek and Swarttouw (1998) and the references therein. The q-Krawtchouk polynomials are given by

$$K_n(q^{-x}; p, N; q) = {}_3\phi_2(q^{-n}, q^{-x}, -pq^n; q^{-N}, 0; q, q), \quad n = 0, \ldots, N.$$

The q-Charlier polynomials are defined as

$$C_n(q^{-x}; a; q) = {}_2\phi_1\left(q^{-n}, q^{-x}; 0; q, -\frac{q^{n+1}}{a}\right),$$

and the Stieltjes-Wigert polynomials as

$$S_n(x; q) = \frac{1}{(q;q)_n} {}_1\phi_1(q^{-n}; 0; q, -q^{n+1}x).$$

3. Basic Properties

In this section we recall some of the properties of Kemp's q-binomial distribution (see Johnson, Kemp, and Kotz, 2005, Kemp, 2002, 2003, Kemp and Newton, 1990). In the limit $q \to 1$, the Kemp distribution $KB(n, \theta, q)$ converges to a binomial distribution:

$$KB(n, \theta, q) \to B\left(n, \frac{\theta}{1+\theta}\right),$$

whereas for $n \to \infty$ we obtain a Heine distribution $H(\theta)$. The random variable $X_{KB} \sim KB(n, \theta_n, q)$ can be written as the sum of independent Bernoulli random variables (Kemp and Newton, 1990), which leads to the expressions

$$\mu = \sum_{i=0}^{n-1} \frac{\theta q^i}{1+\theta q^i} \qquad \text{and} \qquad \sigma^2 = \sum_{i=0}^{n-1} \frac{\theta q^i}{(1+\theta q^i)^2} \tag{3.1}$$

for the mean and variance. Furthermore, the random variable $n - X_{KB}$ has the law $KB(n, \theta^{-1}q^{1-n}, q)$. We note in passing that Kemp (2003) deduced a characterization result for the KB distribution from a theorem of Rao and Shanbhag (1994).

4. **Convergent Parameter**

Our first result is a mild generalization of the convergence to the Heine distribution mentioned in the preceding section.

Proposition 4.1. *Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with limit $\theta \geq 0$. Then the sequence of Kemp's q-binomial distributions $KB(n, \theta_n, q)$ converges for $n \to \infty$ to a Heine distribution $H(\theta)$.*

Proof. Note that

$$\mathbb{P}(X_n = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \theta_n^x \frac{q^{x(x-1)/2}}{\prod_{i=0}^{n-1}(1+\theta_n q^i)}.$$

The q-binomial coefficient tends to $1/(q;q)_x$. As for the product in the denominator, apply the dominated convergence theorem to its logarithm to see that it tends to $E_q(\theta)$. □

Example 4.1. Let λ be a real number with $0 < \lambda < n$, and put $\theta_n(q) = \lambda/[n-\lambda]_q$. Then the sequence of Kemp's q-binomial distributions

$$\begin{array}{ccc} KB(n, \theta_n(q), q) & \xrightarrow{n\to\infty} & H((1-q)\lambda) \\ {\scriptstyle q\to 1}\big\downarrow & & \big\downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\lambda}{n}\right) & \xrightarrow[n\to\infty]{} & P(\lambda) \end{array}$$

The two preceding results yield limit relations for orthogonal polynomials. The orthogonal polynomials for Kemp's q-binomial, the Heine, and the binomial distribution, are, respectively, the q-Krawtchouk, the q-Charlier, and the Krawtchouk polynomials.

Corollary 4.1. *(i) Let θ_n be as in Proposition 4.1. Then, as $n \to \infty$, the q-Krawtchouk polynomial $K_k(q^{-x}; q^{-n}\theta_n^{-1}, n; q)$ converges to the q-Charlier polynomial $C_k(q^{-x}; \theta; q)$.*

(ii) For the special parameter sequence $\theta_n(q) = \lambda/[n-\lambda]_q$, as $q \to 1$, the q-Krawtchouk polynomial $K_k(q^{-x}; q^{-n}\theta_n(q)^{-1}, n; q)$ converges to the Krawtchouk polynomial $K_k(x; \lambda/n, n)$.

The classical convergence of the binomial distribution with constant mean to the Poisson distribution has the following q-analogue.

Theorem 4.1. *Fix $\mu > 0$ and choose the parameter $\theta_n = \theta_n(\mu, q)$ of Kemp's q-binomial distribution such that $\mu_n = \mu$. Then we have:*

(i) The sequence $KB(n,\theta_n,q)$ converges for $n\to\infty$ to a Heine distribution $H(\theta)$, where $\theta=\lim_{n\to\infty}\theta_n$.

(ii) For fixed n, $KB(n,\theta_n,q)$ tends to a binomial distribution $B\left(n,\frac{\mu}{n}\right)$ in the limit $q\to 1$.

(iii) For $q\to 1$, the Heine distribution $H(\theta)$ converges to a Poisson distribution with parameter μ.

So we obtain the following commutative diagram:

$$\begin{tikzcd} KB(n,\theta_n(\mu,q),q) \arrow[r, "n\to\infty"] \arrow[d, "q\to 1"'] & H(\theta(\mu,q)) \arrow[d, "q\to 1"] \\ B\left(n,\frac{\mu}{n}\right) \arrow[r, "n\to\infty"'] & P(\mu) \end{tikzcd}$$

Proof. First we check that for given μ, q and large n, there is a unique θ_n such that $\mu_n(\theta_n,q)=\mu$. The function $\mu_n(\theta,q)$ is strictly increasing in θ, and $\mu_n(0,q)=0$. Since

$$\mu_n(q^{-n+1},q)\geq\sum_{i=0}^{n-1}\frac{q^{i-n+1}}{2q^{i-n+1}}=\frac{n}{2},$$

and $\mu_n(\theta,q)$ is continuous in θ, there exists a unique solution θ_n of $\mu_n(\theta,q)=\mu$ for each $n\geq 2\mu$. An easy continuity argument (de Haan and Ferreira, 2006, Lemma 1.1.1) shows that $\lim_{n\to\infty}\theta_n=\theta$, with θ the unique solution of $\mu_\infty(\theta,q)=\mu$. Thus $KB(n,\theta_n,q)\to H(\theta)$ by Proposition 4.1.

Again by Lemma 1.1.1 of de Haan and Ferreira (2006), we get $\theta_n\to\frac{\mu}{n-\mu}$ for $q\to 1$. Hence $KB(n,\theta_n,q)\to B\left(n,\frac{\mu}{n}\right)$.

It remains to check that $\theta/(1-q)$ converges to μ for $q\to 1$, which yields $H(\theta)\to P(\mu)$. The value $\theta/(1-q)$ is the unique solution of $\mu_\infty((1-q)\theta,q)=\mu$. Moreover, $\lim_{q\to 1}\mu_\infty((1-q)\theta,q)=\theta$, because $H((1-q)\theta)\to P(\theta)$. $\square$

Analogously to Corollary 4.1, Theorem 4.1 implies the following result about (q-)Krawtchouk polynomials.

Corollary 4.2. *Let θ_n and θ be as in Theorem 4.1. Then the q-Krawtchouk polynomial $K_k(q^{-x};q^{-n}\theta_n^{-1},n;q)$ converges to the Krawtchouk polynomial $K_k(x;\mu/n,n)$ for $q\to 1$.*

5. Increasing Parameter

If we consider fast growing parameter sequences, in the sense that $\theta_n = q^{-n-g(n)}$ with $g(n) \to \infty$ or convergent, we obtain the corresponding limit distribution easily.

Corollary 5.1. *Let $X_n \sim KB(n, q^{-n-g(n)}, q)$.*

(i) If $g(n)$ converges to a limit g_0, then the distribution of $n - X_n$ tends to the Heine distribution $H(q^{1+g_0})$ as $n \to \infty$.

(ii) If $g(n) \to \infty$ for $n \to \infty$, then the distribution of $n - X_n$ tends to the point measure δ_0 as $n \to \infty$.

Proof. As remarked in Section 3, $n - X_n \sim KB(n, \tau, q)$ with $\tau = q^{g(n)+1}$. Applying Proposition 4.1 yields the result. □

It follows from Corollary 5.1(i) that the q-Krawtchouk polynomials converge to the alternative q-Charlier polynomials, which is a known result, see (4.15.1) of Koekoek and Swarttouw (1998).

Now we turn to the main result of the paper, where we assume that

$$\theta_n = q^{-f(n)}, \quad \text{with} \quad f(n) \to \infty \quad \text{and} \quad n - f(n) \to \infty \quad \text{for} \quad n \to \infty. \tag{5.1}$$

This assumption on θ_n will be in force throughout the section. Since the sequence of means tends to infinity, we will normalize our random variables to $(X_n - \mu_n)/\sigma_n$. Still, this sequence does not converge in distribution without further assumptions on $f(n)$. It turns out that the fractional part $\{f(n)\} = \{-\log \theta_n / \log q\}$ has to be constant, which induces convergence to discrete normal distributions.

Theorem 5.1. *Suppose that $X_n \sim KB(n, \theta_n, q)$, such that the sequence θ_n satisfies (5.1) and $\{f(n)\} = \beta$ is constant. Then $(X_n - \mu_n)/\sigma_n$ converges for $n \to \infty$ to a limit X, with, for $x \in \mathbb{Z}$,*

$$\mathbb{P}\left(X = -(\beta + c)\frac{1}{\sigma} + \frac{1}{\sigma}x\right) = e_q(q)e_q(-q^\beta)e_q(-q^{1-\beta})q^{(x-1)(x-2\beta)/2}, \tag{5.2}$$

where $c = c(\beta, q)$ is a constant and $\sigma = \lim_{k\to\infty} \sigma_{n_k}$.

The discrete normal distribution is defined by

$$\mathbb{P}(X = x) = \frac{q^{-x\alpha}q^{x^2/2}}{\sum_{k=-\infty}^{\infty} q^{-k\alpha}q^{k^2/2}} \qquad \alpha \in \mathbb{R}, x \in \mathbb{Z}.$$

So the limit distributions in the preceding theorem are (scaled and shifted) discrete normal distributions with parameters

$$\begin{array}{ll} \alpha = \frac{1}{2} + \beta & \text{if } \beta < \frac{1}{2} \\ \alpha = -\frac{1}{2} + \beta & \text{if } \beta > \frac{1}{2} \\ \alpha = 0 & \text{if } \beta = \frac{1}{2} \end{array}.$$

For $q \to 1$, they converge to the standard normal distribution, see Szablowski (2001). Therefore, as in Proposition 4.1 and Theorem 4.1, the limits $q \to 1$ and $n \to \infty$ can be exchanged. Indeed, for $q \to 1$, the distribution of X_n in Theorem 5.1 tends to the binomial distribution $B(n, \frac{1}{2})$. The latter converges to the standard normal distribution after normalization.

To prepare the proof of Theorem 5.1, we will carry out a rather fine analysis of the asymptotics of the sequence of means μ_n. The following two propositions clarify its asymptotics up to order o(1).

PROPOSITION 5.1. *Let $X_n \sim KB(n, \theta_n, q)$ with θ_n as in* (5.1). *Then there is a function c such that, for $n \to \infty$,*

$$\mu_n = f(n) + c(\{f(n)\}, q) + \mathrm{o}(1). \tag{5.3}$$

In other words, the O(1) term of μ_n is constant if $\{f(n)\}$ is constant.

PROOF. We start from

$$\mu_n = \sum_{i=0}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} = \sum_{i=0}^{n-1} \frac{1}{1+q^{f(n)-i}} \tag{5.4}$$

and split the sum into two parts (w.l.o.g. $f(n) < n$): By expanding the denominator as a geometric series and changing the order of summation we find

$$\begin{aligned} \sum_{i=0}^{\lfloor f(n) \rfloor -1} \frac{1}{1+q^{f(n)-i}} &= \sum_{\ell \geq 0} (-1)^{\ell} q^{\ell f(n)} \sum_{i=0}^{\lfloor f(n) \rfloor -1} q^{-\ell i} \\ &= \lfloor f(n) \rfloor - \sum_{\ell \geq 1} \frac{(-1)^{\ell} q^{\ell \{f(n)\}}}{1-q^{-\ell}} + \mathrm{O}\left(q^{f(n)}\right). \end{aligned}$$

Expanding the denominator as a geometric series and changing the order of summation again yields

$$\sum_{i=0}^{\lfloor f(n) \rfloor -1} \frac{1}{1+q^{f(n)-i}} = \lfloor f(n) \rfloor - \sum_{j \geq 0} \frac{1}{1+q^{-j-1-\{f(n)\}}} + \mathrm{O}\left(q^{f(n)}\right). \tag{5.5}$$

For the upper portion of the sum, we find

$$\begin{aligned}\sum_{i=\lfloor f(n)\rfloor+1}^{n-1}\frac{1}{1+q^{f(n)-i}} &= \sum_{i=\lfloor f(n)\rfloor+1}^{\infty}\frac{1}{1+q^{f(n)-i}}+\mathrm{O}\left(q^{n-f(n)}\right)\\ &= \sum_{i=0}^{\infty}\frac{1}{1+q^{\{f(n)\}-i-1}}+\mathrm{O}\left(q^{n-f(n)}\right). \qquad (5.6)\end{aligned}$$

The result now follows from (5.5) and (5.6). □

We next determine the Fourier series of the O(1)-term from Proposition 5.1, which shows that it is a $\frac{1}{2}$-periodic function of $\{f(n)\}$.

PROPOSITION 5.2. *Let $X_n \sim KB(n, \theta_n, q)$ with $\theta_n = q^{-f(n)}$. Then, as $n \to \infty$,*

$$\mu_n = f(n) + \frac{1}{2} + \sum_{k>0}\frac{2\pi\sin(2kf(n)\pi)}{\sinh\left(\frac{2k\pi^2}{\log q}\right)\log q} + \mathrm{o}(1). \qquad (5.7)$$

PROOF. We write

$$\mu_n = \sum_{i=0}^{n-1}\frac{1}{1+q^{f(n)-i}} = \sum_{i=0}^{\infty}\frac{1}{1+q^{f(n)-i}} + \mathrm{O}\left(q^{n-f(n)}\right)$$

and apply the Mellin transformation (see Flajolet, Gourdon and Dumas, 1995) to the function

$$h(t) = \sum_{i=0}^{\infty}\frac{1}{1+tq^{-i}}.$$

The linearity of the Mellin transformation $\mathcal{M}$ and its properties $\mathcal{M}\left(\frac{1}{1+t}\right) = \frac{\pi}{\sin \pi s}$ and $\mathcal{M}h(\alpha t)(s) = \alpha^{-s}\mathcal{M}(h)(s)$ give

$$\begin{aligned}\mathcal{M}(h)(s) &= \int_0^{\infty} x^{-s}h(x)\mathrm{d}x\\ &= \sum_{i=0}^{\infty}\left(q^{-i}\right)^{-s}\frac{\pi}{\sin\pi s} = \frac{1}{1-q^s}\frac{\pi}{\sin\pi s}.\end{aligned}$$

From the inverse transformation formula we get

$$h\left(q^{f(n)}\right) = \frac{1}{2\pi\mathrm{i}}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} q^{-f(n)s}\frac{1}{1-q^s}\frac{\pi}{\sin\pi s}\mathrm{d}s. \qquad (5.8)$$

Upon pushing the line of integration to the left, each pole of the integrand yields a term in the asymptotic expansion of h at infinity, see Flajolet, Gourdon and Dumas (1995). If k is a non-zero integer, then $1/(1-q^s)$ has a simple pole at $2\pi \mathrm{i}k/\log q$ with residue

$$\frac{\mathrm{i}\pi e^{-2\mathrm{i}f(n)k\pi}}{\sinh\left(\frac{2k\pi^2}{\log q}\right)\log q}. \tag{5.9}$$

The residue at the double pole at zero is easily computed, too, and equals $f(n)+\frac{1}{2}$. Merging the terms (5.9) corresponding to k and $-k$ gives the result. □

The following two lemmas complete our analysis of the means μ_n and prepare the proof of the main result of this section, viz. Theorem 5.1. Recall that (5.1) is assumed to hold throughout the present section.

LEMMA 5.1. *If the fractional part $\{f(n)\}=\beta$ is constant, then:*

(i) $c(0,q)=c(1/2,q)=1/2$

(ii) $c(\beta,q)+c(-\beta,q)=1$

(iii) $\lfloor c(\beta,q)+\beta\rfloor = \begin{cases} 0 & \textit{if} \quad 0\le\beta<1/2 \\ 1 & \textit{if} \quad 1/2\le\beta<1 \end{cases}$

PROOF. For (i) and (ii), use (5.7) and simple properties of sin. Part (iii) follows easily from (i), (ii), and the fact that the quantity $c(\{f(n)\},q)-1+\{f(n)\}$ increases w.r.t. $\{f(n)\}$, which in turn is readily seen from the proof of Proposition 5.1. □

We can now evaluate the integer part of the means μ_n.

LEMMA 5.2. *Suppose again that the fractional part $\{f(n)\}=\beta$ is constant.*

(i) If $\beta\neq\frac{1}{2}$, then $f(n)+c(\beta,q)\notin\mathbb{Z}$. Thus

$$\lfloor\mu_n\rfloor=\lfloor f(n)+c(\beta,q)\rfloor=\lfloor f(n)\rfloor+\lfloor\beta+c(\beta,q)\rfloor.$$

(ii) For $\beta=\frac{1}{2}$,

$$\mu_n>f(n)+\frac{1}{2},\ \text{if } 2f(n)\le n-1 \quad\text{and}\quad \mu_n<f(n)+\frac{1}{2},\ \text{if } 2f(n)\ge n.$$

Thus

$$\lfloor \mu_n \rfloor = f(n) + \frac{1}{2}, \qquad \text{if } 2f(n), \leq n-1$$

and

$$\lceil \mu_n \rceil = f(n) + \frac{1}{2}, \qquad \text{if } 2f(n) \geq n.$$

Proof. Part (i) is proved similarly to the preceding lemma. As for part (ii), assume $2f(n) \leq n-1$ first. Then μ_n equals

$$\begin{aligned}
\sum_{i=0}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} &= \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} + \sum_{i=f(n)+\frac{1}{2}}^{2f(n)} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} \\
&\quad + \sum_{2f(n)+1}^{n-1} \frac{q^{i-f(n)}}{1+q^{i-f(n)}} \\
&= \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{-i-\frac{1}{2}}}{1+q^{-i-\frac{1}{2}}} + \sum_{i=0}^{f(n)-\frac{1}{2}} \frac{q^{i+\frac{1}{2}}}{1+q^{i+\frac{1}{2}}} + \mathrm{o}(1) \\
&= f(n) + \frac{1}{2} + \mathrm{o}(1).
\end{aligned}$$

The o(1)-term is non-negative (and vanishes only for $2f(n) = n-1$). If $2f(n) \geq n$, then the third sum vanishes and the second sum just runs up to $n-1 < 2f(n)$, so $\mu_n < f(n) + \frac{1}{2}$. □

Now we are in a position to establish the announced convergence of the normalized X_n to a discrete normal random variable.

Proof of Theorem 5.1. Note that σ_n converges, which follows from the identity

$$\sigma_n^2 = \sum_{i=0}^{n} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2} = \sum_{i=0}^{\lfloor f(n) \rfloor} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2} + \sum_{i=\lfloor f(n) \rfloor+1}^{n} \frac{q^{i-f(n)}}{(1+q^{i-f(n)})^2}.$$

First we consider the case $\beta \neq 1/2$. To evaluate the product in the denominator of

$$\mathbb{P}(X_n = \lfloor \mu_n \rfloor + x) = \begin{bmatrix} n \\ \lfloor \mu_n \rfloor + x \end{bmatrix}_q \frac{q^{-(\lfloor \mu_n \rfloor + x)f(n) + (\lfloor \mu_n \rfloor + x)(\lfloor \mu_n \rfloor + x - 1)/2}}{\prod_{i=0}^{n-1} \left(1 + \frac{q^i}{q^{f(n)}}\right)}, \tag{5.10}$$

we split it into two parts:

$$\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right)=\prod_{i=0}^{\lfloor f(n)\rfloor}\left(1+\frac{q^i}{q^{f(n)}}\right)\prod_{i=\lfloor f(n)\rfloor+1}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right).$$

Using Relation I.3 of Gasper and Rahman (1990), we obtain

$$\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right)=q^{-f(n)(\lfloor f(n)\rfloor+1)+(\lfloor f(n)\rfloor+1)\lfloor f(n)\rfloor/2}\times$$
$$\left(-q^{\beta};q\right)_{\lfloor f(n)\rfloor+1}\left(-q^{-\beta+1};q\right)_{n-\lfloor f(n)\rfloor-2}. \tag{5.11}$$

The last two terms in (5.11) tend to $e_q\left(-q^{\beta}\right)$ and $e_q\left(-q^{-\beta+1}\right)$, respectively. The q-binomial coefficient in (5.10) tends to $e_q(q)$. By Lemma 5.2, we can simplify the exponent of q resulting from (5.10) and (5.11) to

$$\frac{1}{2}(x-1+\delta)(\delta-2\beta+x),$$

where $c=c(\beta,q)$ and

$$\delta=\lfloor\beta+c\rfloor=\begin{cases}0 & \beta<1/2\\ 1 & \beta>1/2\end{cases}$$

by Lemma 5.1 (iii). Putting things together, we obtain

$$\mathbb{P}(X_n=\lfloor\mu_n\rfloor+x)\to e_q(q)e_q\left(-q^{\beta}\right)e_q\left(-q^{-\beta+1}\right)q^{\frac{(\delta+x-1)(\delta+x-2\beta)}{2}}.$$

By normalizing X_n we get (5.2).

For $\beta=1/2$ define

$$G(\mu_n):=\begin{cases}\lfloor\mu_n\rfloor & \text{if } 2f(n)\le n-1\\ \lceil\mu_n\rceil & \text{if } 2f(n)\ge n.\end{cases}$$

Then

$$\mathbb{P}\left(X_n=G(\mu_n)+x\right)=\begin{bmatrix}n\\ G(\mu_n)+x\end{bmatrix}_q\frac{q^{-(G(\mu_n)+x)f(n)+(G(\mu_n)+x)(G(\mu_n)+x-1)/2}}{\prod_{i=0}^{n-1}\left(1+\frac{q^i}{q^{f(n)}}\right)}.$$

The q-binomial-coefficient tends to $e_q(q)$, and the product can be transformed as above. This time the exponent of q equals $\frac{x^2}{2}$. So we have

$$\mathbb{P}\left(X_n=G(\mu_n)+x\right)\to e_q(q)e_q\left(-q^{\frac{1}{2}}\right)^2q^{\frac{x^2}{2}},$$

from which (5.2) follows by normalizing X_n. $\square$

With little extra effort one can see that the limit distribution is symmetric if and only if $\beta = 0$ or $\beta = 1/2$.

Again, the convergence of the distributions in Theorem 5.1 yields a convergence property of the corresponding orthogonal polynomials. The orthogonal polynomials for the discrete normal distribution are the Stieltjes-Wigert polynomials $S_k(x; q)$, see Christiansen and Koelink (2006), Koekoek and Swarttouw (1998).

Corollary 5.2. *Let x be a real number, and $f(n)$ as in* (5.1)*, with $\{f(n)\}$ constant. Then, as $n \to \infty$, the q-Krawtchouk polynomial given by $K_k(q^{-x-f(n)+o(1)}; q^{f(n)-n}, n; q)$ tends to $(q; q)_k \times S_k(q^{-x}; q)$.*

Acknowledgement. The first author received financial support by Christian Doppler Laboratory for Portfolio Risk Management (PRisMa Lab).The fruitful collaboration and support by the Bank Austria Creditanstalt (BA-CA) and the Austrian Federal Financing Agency (ÖBFA) through CDG is gratefully acknowledged.

The second author was supported by the NAWI-Graz project and the Austrian Science Fund project S9611 of the National Research Network S9600 Analytic Combinatorics and Probabilistic Number Theory.

The authors thank Stephan Wagner for the idea of using Mellin transform asymptotics for the sequence of means in Section 5, Tom Koornwinder for the hint to think about our convergence results in terms of orthogonal polynomials, and the anonymous referee for some remarks that improved the presentation.

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Paper received June 2008; revised October 2009.