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BAIRE RESULTS OF MULTISEQUENCES

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ABSTRACT. We extend Baire results about $n\alpha$-sequences in different ways, in particular we investigate sequences with multidimensional indices.

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Dedicated to the memory of Professor Edmund Hlawka

1. Introduction

A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers is called uniformly distributed modulo 1, if for every pair a, b of real numbers with $0 \leq a < b \leq 1$ the following condition holds:

$$\lim_{n\to\infty} \frac{A([a,b),n,\mathbf{x})}{n} = b - a,$$

where $A(I, n, \mathbf{x})$ is the number of elements x_i, $i \leq n$ with $x_i \in I$ for an interval I. For a general theory of uniform distribution we refer to Kuipers and Niederreiter [11] and Drmota and Tichy [4].

In [7] Goldstern, Schmeling and Winkler studied the size (in the sense of Baire) of the set

$$\mathcal{U} := \{\alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is uniformly distributed mod } 1\}$$

for a given sequence $\mathbf{n} = (n_j)_{j\in\mathbb{N}}$ of natural numbers; the size of this set depends on the growth rate of the sequence $\mathbf{n}$. In particular they showed that $\mathcal{U}$ is meager

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if $\mathbf{n}$ grows exponentially (for theory about Baire categories see Oxtoby [15]). By Ajtai, Havas, Komlós [2] this condition cannot be weakened.

Moreover, it was proven in [7] that the set

$$\mathcal{V} := \{\alpha \in \mathbb{R}/\mathbb{Z} : \mathbf{n}\alpha \text{ is maldistributed}\}$$

is residual if $\mathbf{n}$ grows very fast (for the precise statement we refer to [7]).

The aim of this paper (which is closely related to the very recent work of Winkler [19]) is to generalize these results in different ways. In Section 2 we consider for a given sequence $(\mathbf{n}_j)_{j\in\mathbb{N}}$ of r-dimensional vectors of nonnegative integers and an r-dimensional vector $\boldsymbol{\alpha}$ of real numbers the sequence $(\mathbf{n}_j\boldsymbol{\alpha})_{j\in\mathbb{N}}$, where $\mathbf{n}_j\boldsymbol{\alpha}$ means the scalar product of two vectors. Afterwards we investigate in Section 3 uniform distribution in $\mathbb{R}^d$, i.e., for a d-dimensional sequence $(\mathbf{n}_j)$ and a d-dimensional vector $\boldsymbol{\alpha}$ we consider the sequence $(\mathbf{n}_j\boldsymbol{\alpha})_{j\in\mathbb{N}}$, where $\mathbf{n}_j\boldsymbol{\alpha}$ means the Hadamard product of two vectors.

Section 4 is devoted to the generalization of elementary properties of uniform distribution of sequences to uniform distribution of nets. Afterwards we extend in Section 5 the characterization of the set of limit measures of a sequence (see Winkler [18]) to a special kind of nets over $\mathbb{N}^d$. Finally, we turn in Section 6 to $\mathbf{n}\alpha$-sequences with multidimensional indices. Besides the classical notion of uniform distribution of such sequences (see Kuipers and Niederreiter [11]) we study the $(s_1,\dots,s_d)$-uniform distribution (see Kirschenhofer and Tichy [10]) and introduce a new concept of uniform distribution modulo 1, which is inspired by Aistleitner [1]. In most of these cases it turns out that the known results for the classical case remain true in these generalized settings.

2. Vectors

In this section let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of r-dimensional vectors of nonnegative integers, i.e.,

$$\mathbf{n}_j = (n_{j,1},\dots,n_{j,r}) \quad \text{with} \quad n_{j,i} \in \mathbb{N},$$

and let $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_r)$ denote an r-dimensional vector of real numbers $0 \le \alpha_i \le 1$, $i = 1,\dots,r$. We are now interested in the distribution of the sequence

$$\mathbf{n}\boldsymbol{\alpha} := (\mathbf{n}_j\boldsymbol{\alpha})_{j\in\mathbb{N}} \qquad \text{with} \quad \mathbf{n}_j\boldsymbol{\alpha} = \sum_{i=1}^{r} n_{j,i}\alpha_i.$$

Note that $\mathbf{n}\boldsymbol{\alpha}$ is a one-dimensional sequence of real numbers. To study the size in the sense of Baire of the set

$$\mathcal{U} = \{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^r : \ (\mathbf{n}_j\boldsymbol{\alpha})_{j\in\mathbb{N}} \text{ is uniformly distributed mod } 1\}$$

we follow Goldstern, Schmeling, Winkler [7]. For this purpose we generalize the definition of ε-mixing sequences of functions:

Definition 2.1. A sequence of functions $f_i : [0,1)^r \to [0,1)$ is called ε-mixing in $(\delta_1, \dots, \delta_r)$ if for all sequences of intervals $J_1, J_2, \dots$ of length ε and for all cuboids J' of size $\delta_1 \times \dots \times \delta_r$ and for all $k \geq 0$

$$J' \cap \bigcap_{i=1}^{k} f_i^{-1}(J_i)$$

contains an inner point.

To proceed further we need a criterion when a sequence of functions is ε-mixing in $(\delta_1, \dots, \delta_r)$:

Lemma 2.2. *Let $f_j : [0,1)^r \to [0,1)$ be the function mapping $\boldsymbol{\alpha}$ to $\mathbf{n}_j \boldsymbol{\alpha}$ modulo 1, where $(\mathbf{n}_j)_{j\in\mathbb{N}}$ is a sequence of r-dimensional vectors of nonnegative integers satisfying*

(1) $n_{j+1,s} > \frac{4}{\varepsilon} n_{j,s}$ *for all j and*

(2) $n_{0,s} > \frac{\varepsilon}{2\delta_s}$

for a fixed $s \in \{1, \dots, r\}$. Then $(f_1, f_2, \dots)$ is ε-mixing in $(\delta_1, \dots, \delta_r)$.

Proof. For simplicity we just prove the case $r = 2$. Let $\mathbf{n}_j = (m_j, n_j)$ and assume that the conditions (1) and (2) hold for the sequence of n_j. We will show (by induction on k) that each set

$$J' \cap \bigcap_{i=1}^{k} f_i^{-1}(J_i)$$

contains a cuboid of size $c_k \times \frac{\varepsilon}{2n_k}$ with $c_k > 0$. This is true for $k = 0$, since $\delta_2 > \frac{\varepsilon}{2n_0}$ and $c_0 := \delta_1 > 0$.

Consider $k > 0$. Note that $f_k^{-1}(J_k)$ is a union of stripes of height $\frac{\varepsilon}{n_k}$ and distance $\frac{1-\varepsilon}{n_k}$ (see Figure 1). By induction hypothesis, the set $J' \cap \bigcap_{i=1}^{k-1} f_i^{-1}(J_i)$ contains a cuboid I of size $c_{k-1} \times \frac{\varepsilon}{2n_{k-1}}$. Since $\frac{\varepsilon}{2n_{k-1}} > \frac{2}{n_k}$, I crosses one stripe — say S — of height $\frac{\varepsilon}{n_k}$. Thus $I \cap S$ contains a cuboid I' of size $c_k \times \frac{\varepsilon}{2n_k}$ for some $c_k \leq c_{k-1}$ (see Figure 2). $\square$

To be able to state the theorem, we need the following definitions.

Definition 2.3. For a sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ of real numbers we define the measures $\mu_{\mathbf{x},n}$ by

$$\mu_{\mathbf{x},n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i},$$

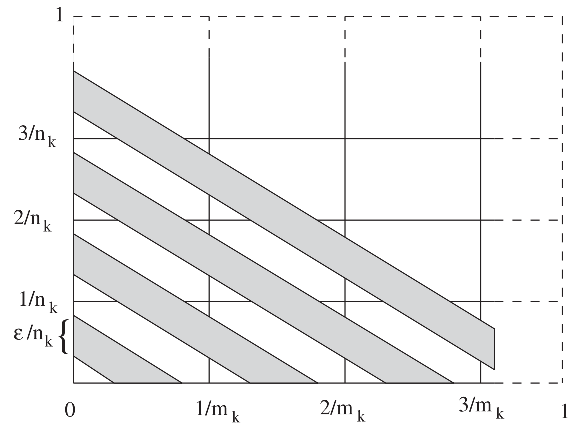

FIGURE 1.

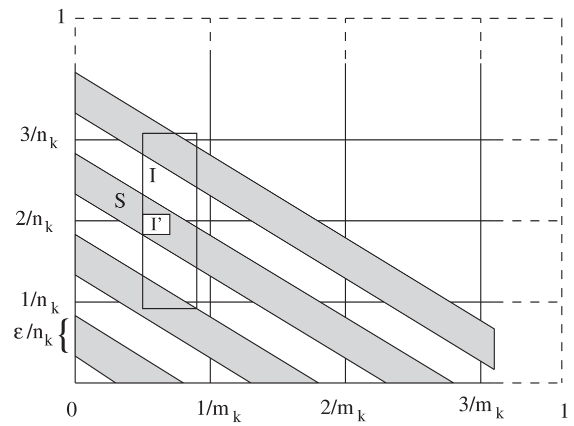

FIGURE 2.

where δ_x denotes the point measure in x. The set of accumulation points of the sequence $(\mu_{\mathbf{x},n})_{n\in\mathbb{N}}$ is denoted by $M(\mathbf{x})$ and is called the set of limit measures of the sequence $\mathbf{x}$.

DEFINITION 2.4. For any sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and any interval I we define $\overline{\mu}_{\mathbf{x}}(I)$ by

$$\overline{\mu}_{\mathbf{x}}(I) := \sup\{\mu(I) : \mu \in M(\mathbf{x})\}.$$

Now we can establish the theorem, which shows that the set of r-dimensional real vectors $\boldsymbol{\alpha}$, such that $\mathbf{n}\boldsymbol{\alpha}$ is uniformly distributed mod 1 is meager, if at least one component of the $\mathbf{n}_j$ grows exponentially.

Theorem 2.5. *Let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of r-dimensional vectors of nonnegative integers and assume $q := \liminf_j (n_{j,s+1}/n_{j,s}) > 1$ for an $s \in \{1,\dots,r\}$. Then the set*

$$\mathcal{U} := \{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^r :\ (\mathbf{n}_j\boldsymbol{\alpha})_{j\in\mathbb{N}} \text{ is uniformly distributed} \bmod 1\}$$

is meager.

Moreover: There is a number $Q > 0$ such that for all intervals I the set

$$\left\{\boldsymbol{\alpha} :\ \overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I) > \frac{Q}{-\log\lambda(I)}\right\}$$

is residual.

Before proving this theorem we state the following fact. It is completely analogous to the one-dimensional case. For details we refer to [7].

Definition 2.6. For an open cuboid I and a Borel set B we write $I \Vdash B$ for "$B \cap I$ is residual in I" or equivalently "$I \setminus B$ is meager".

Fact 2.7. *Let I be an open cuboid.*

(1) *If B_n is a Borel set for every $n \in \{0,1,2,\dots\}$ and $I \cap \bigcup_n B_n$ is residual in I, then there is some open nonempty cuboid $J \subseteq I$ and some n such that B_n is residual in J, i.e.,*

$$I \Vdash \bigcup_{n\in\mathbb{N}} B_n \Rightarrow \exists J \subseteq I\ \exists n \in \mathbb{N} : J \Vdash B_n.$$

(2) *If B_n is a Borel set for every $n \in \{0,1,2,\dots\}$, then $I \cap \bigcap_n B_n$ is residual in I iff each $I \cap B_n$ is residual in B_n:*

$$I \Vdash \bigcap_{n\in\mathbb{N}} B_n \Leftrightarrow \forall n \in \mathbb{N} : I \Vdash B_n.$$

(3) *If B is a Borel set, then $B \cap I$ is not residual in I iff there is some open cuboid $J \subseteq I$ such that B is meager in J:*

$$I \nVdash B \Leftrightarrow \exists J \subseteq I : J \Vdash B^C,$$

where B^C denotes the complement of B.

Proof of the theorem. The proof is completely analogous to the proof of Theorem 2.4 in [7], we have just to adapt the choice of Q and notation.

Choose $Q > 0$ so small that $(\frac{1}{4Q} - 1) - \log 4 > 1$.

Without loss of generality we may assume $\frac{n_{j+1,s}}{n_{j,s}} > q$ for all k.

Let $\varepsilon := \lambda(I)$. Since $\overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}} > \frac{Q}{-\log\lambda(I)}$ will be trivially true for large intervals if we choose Q small enough, we may assume $\varepsilon < \frac{1}{q}$, so $(-\log\varepsilon) > 1$; here log always denotes the logarithm to base q. Hence $(-\log\varepsilon)(\frac{1}{4Q}-1)-\log 4>1$, thus the interval

$$\left(\log 4-\log\varepsilon,-\frac{1}{4Q}\log\varepsilon\right)$$

has length >1. Let c be an integer in this interval. Then

- $q^c>\frac{4}{\varepsilon}$
- $\frac{1}{2c}>\frac{2Q}{-\log\varepsilon}$.

Now suppose that the theorem is false. Since the set $\{\boldsymbol{\alpha}:\ \overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I)>\frac{Q}{-\log\varepsilon}\}$ is a Borel set and not residual, its complement is residual in I', for some open cuboid I':

$$I'\Vdash\left\{\boldsymbol{\alpha}:\ \overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I)\le\frac{Q}{-\log\varepsilon}\right\}.$$

Since $\overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I)\ge\limsup_{n\to\infty}\mu_{\mathbf{n}\boldsymbol{\alpha},n}$ the set $\{\boldsymbol{\alpha}:\overline{\mu}_{\mathbf{n}\boldsymbol{\alpha}}(I)\le\frac{Q}{-\log\varepsilon}\}$ is contained in

$$\left\{\boldsymbol{\alpha}:\ \exists m\ \forall N\ge m:\ \mu_{\mathbf{n}\boldsymbol{\alpha},N}(I)\le\frac{2Q}{-\log\varepsilon}\right\}.$$

Denote the set $\{j<N:\mathbf{n}_j\boldsymbol{\alpha}\in I\}$ by $Z_N(\boldsymbol{\alpha})$. So $\mu_{\mathbf{n}\boldsymbol{\alpha},N}(I)=\frac{\#Z_N(\boldsymbol{\alpha})}{N}$. Therefore

$$I'\Vdash\bigcup_m\bigcap_{N\ge m}\left\{\boldsymbol{\alpha}:\ \frac{\#Z_N(\boldsymbol{\alpha})}{N}\le\frac{2Q}{-\log\varepsilon}\right\}.$$

So, by Fact 2.7, we can find an open cuboid $J\subseteq I'$ and a k^* such that

$$J\Vdash\bigcap_{N\ge k^*}\left\{\boldsymbol{\alpha}:\ \frac{\#Z_N(\boldsymbol{\alpha})}{N}\le\frac{2Q}{-\log\varepsilon}\right\},$$

or equivalently, for all $N\ge k^*$:

$$J\Vdash\left\{\boldsymbol{\alpha}:\ \frac{\#Z_N(\boldsymbol{\alpha})}{N}\le\frac{2Q}{-\log\varepsilon}\right\},\tag{1}$$

Let $\delta_i:=\lambda_i(J)$, where $\lambda_i(J)$ is the length of the edge in the ith dimension. Without loss of generality we assume $n_{k^*c,s}>\frac{\varepsilon}{2\delta_s}$ (otherwise we just increase k^*). Now consider the functions $f_{k^*c},f_{(k^*+1)c},\dots,f_{(2k^*-1)c}$, defined as in Lemma 2.2. Since

$$\frac{n_{(k^*+i+1)c,s}}{n_{(k^*+i)c,s}}\ge q^c>\frac{4}{\varepsilon}$$

and $n_{k^*c,s}>\frac{\varepsilon}{\delta_s}$, these functions are ε-mixing in $(\delta_1,\dots,\delta_r)$ by Lemma 2.2. So there is an open cuboid $K\subseteq J$ such that for all $\boldsymbol{\alpha}\in K$ and all $i\in\{0,\dots,k^*\}$:

$$\boldsymbol{\alpha}\in f^{-1}_{(k^*+i)c}(I)\quad\text{i.e.,}\quad\mathbf{n}_{(k^*+i)c}\boldsymbol{\alpha}\in I.$$

Hence for all $\boldsymbol{\alpha} \in K$

$$\#Z_{2k^*c}(\boldsymbol{\alpha}) = \#\{i < 2k^*c : \ \mathbf{n}_i\boldsymbol{\alpha} \in I\} \geq \#\{k^*c, (k^*+1)c, \ldots, (2k^*-1)c\} = k^*.$$

So for all $\boldsymbol{\alpha} \in K$

$$\frac{\#Z_{2k^*c}(\boldsymbol{\alpha})}{2k^*c} \geq \frac{1}{2c}. \tag{2}$$

Since $\frac{1}{2c} > \frac{2Q}{-\log\varepsilon}$ and $K \subseteq J$, (1) with $N := 2k^*c$ implies

$$K \Vdash \left\{\boldsymbol{\alpha} : \ \frac{\#Z_{2k^*c}(\boldsymbol{\alpha})}{2k^*c} \leq \frac{1}{2c}\right\}. \tag{3}$$

Now consider the set $\{\boldsymbol{\alpha} : \ \frac{\#Z_{2k^*c}(\boldsymbol{\alpha})}{2k^*c} < \frac{1}{2c}\} \cap K$. By (2), this set is empty, but by (3) it is residual in K, which is a contradiction. $\square$

Note that the above theorem still remains true, if we require instead of $q := \liminf_j(n_{j,s+1}/n_{j,s}) > 1$ for an $s \in \{1,\ldots,r\}$ only that there exists $s \in \{1,\ldots,r\}$ and a constant C with

$$\left|\left\{j : \ 2^r \leq n_{j,s} < 2^{r+1}\right\}\right| \leq C \quad \forall r.$$

Then you can choose each $c := 2C\lceil 2 - \log_2\varepsilon\rceil$th term to obtain a growth of factor $4/\varepsilon$, and Q has to be chosen so small, that $\frac{1}{2c} > \frac{2Q}{-\log\varepsilon}$. Indeed, one can use instead of the base 2 in the above condition any number $K > 1$, but we will state all theorems in terms of the base 2 throughout this paper.

Remark 2.8. With the same argument as above, Theorem 2.4 in [7] holds also for sequences $(n_j)_{j\in\mathbb{N}}$ with

$$\left|\left\{j : \ 2^r \leq n_j < 2^{r+1}\right\}\right| \leq C \quad \forall r.$$

So far we gave sufficient conditions that $\mathbf{n}\boldsymbol{\alpha}$ is u.d. mod 1 only for $\boldsymbol{\alpha}$ in a set of first category. If we weaken the growth condition in the following way, there will be sequences $\mathbf{n}$, such that $\mathbf{n}\boldsymbol{\alpha}$ is u.d. mod 1 for $\boldsymbol{\alpha}$ in a set of second category. Therefor we start with an extension of a result due to Ajtai, Havas, Komlós [2].

Lemma 2.9. *Given any r sequences*

$$(\varepsilon_{j,k})_{j\in\mathbb{N}}, \quad 1 \leq k \leq r, \quad \varepsilon_{j,k} \geq 0, \quad \lim_{j\to\infty} \varepsilon_{j,k} = 0 \quad \textit{for all} \ \ k,$$

there is a sequence of r-dimensional vectors of nonnegative integers $(\mathbf{n}_j)_{j\in\mathbb{N}}$ with

$$\frac{n_{j+1,k}}{n_{j,k}} > 1 + \varepsilon_{j,k}, \quad 1 \leq k \leq r$$

such that for all $\boldsymbol{\alpha}$ with $\sum_{i=1}^r \alpha_i \notin \mathbb{Q}$ the sequence $\mathbf{n}\boldsymbol{\alpha}$ is u.d. mod 1.

Proof. Set $\varepsilon_j := \max\{\varepsilon_{j,k} : \ 1 \le k \le r\}$. Then, by [2, Lemma 1], there exists a sequence $(n_j)_{j\in\mathbb{N}}$ with $n_{j+1}/n_j > 1 + \varepsilon_j$ such that $n_j\alpha$ is u.d. mod 1 for all irrational α. Define $\mathbf{n}_j := (n_j, \ldots, n_j)$. Then

$$\mathbf{n}_j\boldsymbol{\alpha} = \sum_{i=1}^{r} n_j\alpha_i = n_j \sum_{i=1}^{r} \alpha_i = n_j\alpha'$$

with irrational α'. □

To get a statement in Baire's categories, we need a lemma which tells us that the set of d-dimensional real vectors, whose entries are linearly independent over $\mathbb{Q}$, is residual in $\mathbb{R}^d$.

Lemma 2.10. *The set*

$$\mathcal{I} := \{\boldsymbol{\alpha} : \ 1, \alpha_1, \ldots, \alpha_d \textit{ are linearly independent over } \mathbb{Q}\}$$

is residual, and hence of second category, in $\mathbb{R}^d$.

Proof. Note that

$$\mathcal{I} = \mathbb{R}^d \ \setminus \bigcup_{\substack{a_0,\ldots,a_d\in\mathbb{Q} \\ (a_0,\ldots,a_d)\neq(0,\ldots,0)}} S(a_1, \ldots a_d),$$

where

$$S(a_1, \ldots a_d) = \{\alpha : a_0 + a_1\alpha_1 + \cdots + a_d\alpha_d = 0\} .$$

Since $S(a_1, \ldots a_d)$ is a subspace of dimension smaller than d, all these sets $S(a_1, \ldots a_d)$ are nowhere dense. Therefore,

$$\bigcup_{\substack{a_0,\ldots,a_d\in\mathbb{Q} \\ (a_0,\ldots,a_d)\neq(0,\ldots,0)}} S(a_1, \ldots a_d)$$

is of first category. □

Consequently we have

Theorem 2.11. *Given any r sequences $(\varepsilon_{j,k})_{j\in\mathbb{N}}$, $1 \le k \le r$, $\varepsilon_{j,k} \ge 0$, $\lim_{j\to\infty} \varepsilon_{j,k} = 0$ for all k, there is a sequence of r-dimensional vectors of non-negative integers $(\mathbf{n}_j)_{j\in\mathbb{N}}$ with*

$$\frac{n_{j+1,k}}{n_{j,k}} > 1 + \varepsilon_{j,k}, \quad 1 \le k \le r$$

such that the set

$$\{\boldsymbol{\alpha} : \ \mathbf{n}\boldsymbol{\alpha} \textit{ is u.d.} \bmod 1\}$$

is residual.

In [7] Goldstern, Schmeling and Winkler also proved, that if the sequence $(n_j)_{j\in\mathbb{N}}$ grows very fast (i.e., if $\lim_{j\to\infty} n_{j+1}/n_j = \infty$), then the set of α, for which $\mathbf{n}\alpha$ is maldistributed, is residual. A sequence $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is called maldistributed, iff the set $M(\mathbf{x})$ is the whole set of Borel probability measures on $[0,1]$. It is as easy as the modification of the proof of [7, Theorem 2.4] to the proof of Theorem 2.5 to obtain a generalization of [7, Theorem 2.6]:

Theorem 2.12. *Let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of r-dimensional vectors of nonnegative integers and assume that there is an $s \in \{1,\dots,r\}$ such that*

$$\lim_{k\to\infty} n_{s,k+1}/n_{s,k} = \infty.$$

Then the set

$$\{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^r :\ \mathbf{n}\boldsymbol{\alpha} \text{ is maldistributed}\}$$

is residual.

3. $\mathbf{n}\boldsymbol{\alpha}$-sequences in $\mathbb{R}^d$

In this section we investigate uniform distribution in $\mathbb{R}^d$. For a sequence $(\mathbf{n}_j)_{j\in\mathbb{N}}$ of d-dimensional vectors of nonnegative integers and a d-dimensional vector $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d)$ of real numbers we are interested in the sequence

$$\mathbf{n}\boldsymbol{\alpha} := (n_{j,1}\alpha_1,\dots,n_{j,d}\alpha_d)_{j\in\mathbb{N}}.$$

To obtain results for such sequences we use the connection between uniform distribution modulo 1 in $[0,1]^d$ and uniform distribution in $[0,1]$. As in the previous section our first theorem shows that the set of $\boldsymbol{\alpha}$ such that $\mathbf{n}\boldsymbol{\alpha}$ is u.d. is meager if at least one component of the $\mathbf{n}_j$ grows exponentially.

Theorem 3.1. *Let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of d-dimensional vectors of nonnegative integers and assume that there exists $s \in \{1,\dots,d\}$ and a constant C with*

$$\left|\left\{j :\ 2^r \le n_{j,s} < 2^{r+1}\right\}\right| \le C \quad \forall r.$$

Then

$$\mathcal{A} := \left\{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^d :\ \mathbf{n}\boldsymbol{\alpha} \text{ is u.d.} \bmod 1 \text{ in } \mathbb{R}^d\right\}$$

is meager.

Proof. By [11, Theorem 6.3], uniform distribution of $\mathbf{n}\boldsymbol{\theta}$ implies that each component $\mathbf{n}_i\theta_i := (n_{j,i}\theta_i)_{j\in\mathbb{N}}$, $1 \le i \le d$ is u.d. mod 1, especially $\mathbf{n}_s\theta_s$ is u.d. mod 1. Therefore

$$\mathcal{A} \subseteq \mathbb{R} \times \cdots \times \mathbb{R} \times \mathcal{A}_s \times \mathbb{R} \times \cdots \times \mathbb{R},$$

where

$$\mathcal{A}_s := \{\theta :\ \mathbf{n}_s\theta \text{ u.d. mod } 1\}.$$

By Remark 2.8, the set $\mathcal{A}_s$ is meager. Hence, by [15, Th. 15.3.], $\mathcal{A}$ is meager. $\square$

As before the grow condition in the theorem above cannot be weakened:

Lemma 3.2. *Given any d sequences $(\varepsilon_{j,k})_{j\in\mathbb{N}}$, $1 \le k \le d$, $\varepsilon_{j,k} \ge 0$, $\lim_{j\to\infty} \varepsilon_{j,k} = 0$ for all k, there is a sequence of d-dimensional vectors of nonnegative integers $(\mathbf{n}_j)_{j\in\mathbb{N}}$ with*

$$\frac{n_{j+1,k}}{n_{j,k}} > 1 + \varepsilon_{j,k}, \quad 1 \le k \le d$$

such that for all $\boldsymbol{\alpha}$ with $1, \alpha_1, \ldots, \alpha_d$ linearly independent over $\mathbb{Q}$, the sequence $(n_{j,1}\alpha_1, \ldots, n_{j,d}\alpha_d)_{j\in\mathbb{N}}$ is u.d. $\bmod 1$ in $\mathbb{R}^d$.

Proof. Set $\varepsilon_j := \max\{\varepsilon_{j,k} : \ 1 \le k \le d\}$. Then, by [2, Lemma 1], there exists a sequence $(n_j)_{j\in\mathbb{N}}$ with $n_{j+1}/n_j > 1 + \varepsilon_j$ such that $n_j\alpha$ is u.d. $\bmod 1$ for all irrational α. Define $\mathbf{n}_j := (n_j, \ldots, n_j)$. By [11, Theorem 6.3] we have to show that for all $\mathbf{h} \in \mathbb{Z}^d$, $\mathbf{h} \ne 0$ the sequence $\langle \mathbf{h}, n_j\boldsymbol{\alpha}\rangle$ is u.d. $\bmod 1$ for all $\boldsymbol{\alpha}$ with $1, \alpha_1, \ldots, \alpha_d$ linearly independent over $\mathbb{Q}$. This is true since

$$\langle \mathbf{h}, n_j\boldsymbol{\alpha}\rangle = \sum_{i=1}^{d} h_i n_j \alpha_i = n_j \sum_{i=1}^{d} h_i\alpha_i = n_j\alpha'$$

with $\alpha' \in \mathbb{R} \setminus \mathbb{Q}$. □

Using Lemma 2.10 we get as an immediate consequence

Theorem 3.3. *Given any d sequences $(\varepsilon_{j,k})_{j\in\mathbb{N}}$, $1 \le k \le d$, $\varepsilon_{j,k} \ge 0$, $\lim_{j\to\infty} \varepsilon_{j,k} = 0$ for all k, there is a sequence of d-dimensional vectors of nonnegative integers $(\mathbf{n}_j)_{j\in\mathbb{N}}$ with*

$$\frac{n_{j+1,k}}{n_{j,k}} > 1 + \varepsilon_{j,k}, \quad 1 \le k \le d$$

such that the set

$$\left\{\boldsymbol{\alpha} : \ (n_{j,1}\alpha_1, \ldots, n_{j,d}\alpha_d)_{j\in\mathbb{N}} \text{ is u.d. } \bmod 1 \text{ in } \mathbb{R}^d\right\}$$

is residual.

Again using [15, Th. 15.3.] we obtain for fast growing sequences $(\mathbf{n}_j)$:

Theorem 3.4. *Let $(\mathbf{n}_j)_{j\in\mathbb{N}}$ be a sequence of d-dimensional vectors of nonnegative integers and assume $\lim_{k\to\infty} n_{t,k+1}/n_{t,k} = \infty$ for all $t \in \{1, \ldots, d\}$, then the set*

$$\{\boldsymbol{\alpha} \in (\mathbb{R}/\mathbb{Z})^d : \ \mathbf{n}\boldsymbol{\alpha} \text{ is maldistributed}\}$$

is residual.

We can combine the ideas of this and the previous section: Consider a sequence of $d \times r$-matrices of nonnegative integers

$$(N_j)_{j\in\mathbb{N}} \quad \text{with} \quad N_j = \left(n_{ik}^{j}\right), \qquad i = 1, \ldots, d, \ k = 1, \ldots, r.$$

We are now interested in the distribution of the sequence $\mathbf{N}\boldsymbol{\alpha} := (N_j\boldsymbol{\alpha})_{j\in\mathbb{N}}$, where $N_j\boldsymbol{\alpha}$ means the classical matrix-vector-product. Same argumentation as in the proof of Theorem 3.1 yields

Theorem 3.5. *Let $(N_j)_{j\in\mathbb{N}}$ be a sequence of $d\times r$-matrices of nonnegative integers and assume that there exist $s\in\{1,\dots,d\}$, $t\in\{1,\dots,r\}$ and a constant C with*

$$\left|\left\{j : 2^r \le n_{st}^j < 2^{r+1}\right\}\right| \le C \quad \forall r.$$

Then the set

$$\mathcal{A} := \left\{\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_r) : \ \mathbf{N}\boldsymbol{\alpha} \text{ is u.d. } \bmod 1 \text{ in } \mathbb{R}^d\right\}$$

is meager.

4. Uniform distribution of nets

In this section we define uniform distribution of nets of elements of a locally compact Hausdorff space and give a list of some elementary properties which generalize the results for classical sequences given in Bauer [3], Helmberg [8], Kuipers and Niederreiter [11] and Winkler [18]. The proofs are analogous to the ones of the case of one-dimensional sequences, so we omit them and just state the theorems. As explained in the following such nets induce nets of certain discrete probability measures. Uniform distribution properties of nets of general probability measures on locally compact groups were studied in Gerl [5] and Maxones and Rindler [13, 14]. A special kind of nets are sequences indexed by d-dimensional vectors in $\mathbb{N}^d$. Such sequences of random variables also appear in probability theory, see e.g., Jacod and Shiryaev [9]. For an introduction to nets we refer to Willard [17].

Throughout this and the following section let $X \neq \emptyset$ be a locally compact Hausdorff space with countable topology base. Moreover, let $\mathcal{M}(X)$ be the compact sets of nonnegative finite Borel measures with $\mu(X) = 1$ if X is compact and $\mu(X) \le 1$ if X is not compact, equipped with the topology of weak convergence. On $\mathcal{M}(X)$ we use the metric given in [18]. Furthermore let Λ (equipped with two relations $(\le_1, \le_2)$) be a countable directed set (w.r.t. both relations) with the additional property that for all $\lambda\in\Lambda$ the sets $\mathcal{V}_i(\lambda) := \{\nu : \nu \le_i \lambda\}$ $(i = 1, 2)$ is finite. Moreover, assume

$$|\{\lambda : \ |\mathcal{V}_1(\lambda)|\}| = o(n^\alpha) \quad \text{as} \quad n\to\infty \quad \text{and} \quad \alpha\in\mathbb{R}.$$

For a net $\mathbf{x} = (x_\lambda)_{\lambda\in\Lambda}$ of elements in X and a function $f\in\mathcal{K}(X)$, the space of all continuous real-valued functions on X whose support is compact, we define the net $\boldsymbol{\mu}_f = (\mu_{\lambda,f})_{\lambda\in\Lambda}$ by

$$\mu_{\lambda,f} = \frac{1}{|\mathcal{V}_1(\lambda)|} \sum_{\ell \leq_1 \lambda} f(x_\ell). \tag{4}$$

If the nets $\boldsymbol{\mu}_f$ converges (w.r.t. the relation $\leq_2$) to the integral

$$\int_X f \, \mathrm{d}\mu$$

for all $f \in \mathcal{K}(X)$ then we say $\mathbf{x}$ is *μ-uniformly distributed* (μ-u.d.) in X.

Now we give some basic properties:

(i) If $\mathscr{V}$ is a class of functions from $\mathcal{K}(X)$ such that $\mathrm{sp}(\mathscr{V})$ is dense in $\mathcal{K}(X)$, then $\mathscr{V}$ is convergence-determining with respect to any μ in X.

(ii) If $\mathrm{sp}(\mathscr{V})$ is a subalgebra of $\mathcal{K}(X)$ that separates points and vanishes nowhere, then $\mathscr{V}$ is a convergence-determining class with respect to any μ in X.

(iii) The net $\mathbf{x} = (x_\lambda)_{\lambda \in \Lambda}$ is μ-u.d. in X iff the nets $\mathbf{y}^M = (y_\lambda^M)_{\lambda \in \Lambda}$ defined by

$$y_\lambda^M = \frac{A(M;\lambda)}{|\mathcal{V}_1(\lambda)|}$$

converge (w.r.t. $\leq_2$) to $\mu(M)$ for all compact μ-continuity sets $M \subseteq X$. Here $A(M;\lambda) = \sum_{\ell \leq_1 \lambda} \mathbf{1}_M(x_\ell)$.

(iv) In a locally compact Hausdorff space X with countable space, there exists a countable convergence-determining class of real-valued continuous functions with compact support with respect to any $\mu \in \mathcal{M}(X)$.

(v) Let S be the set of all μ-u.d. sequences in X, viewed as a subset of $X_\Lambda := \prod_{\lambda \in \Lambda}$. Then $\mu_\infty(S) = 1$.

(vi) If X contains more than one element, then the set S from the above theorem is a set of first category in X_Λ.

(vii) The set S is everywhere dense in X_Λ.

Generalizing the concept of uniform distribution we introduce the set $M(\mathbf{x})$, the set of limit measures of the net $\mathbf{x}$, as the set of cluster points of the net (w.r.t. $\leq_2$) $\boldsymbol{\mu} = (\mu_\lambda)_{\lambda \in \Lambda}$ of induced measures defined by

$$\mu_\lambda = \frac{1}{|\mathcal{V}_1(\lambda)|} \sum_{\ell \leq_1 \lambda} \delta_{x_\ell}. \tag{5}$$

If $M(\mathbf{x}) = \{\mu\}$ ($\mu \in \mathcal{M}(X)$), then this net is μ-u.d. in X. If $M(\mathbf{x}) = \mathcal{M}(X)$ we say $\mathbf{x}$ is *maldistributed* in X.

As in the classical case (see [18]) only very few (in a topological sense) nets are μ-u.d. Moreover, almost all nets are maldistributed. We have:

The typical situation in the sense of Baire is $M(\mathbf{x}) = \mathcal{M}(X)$, i.e., the set

$$Y = \{\mathbf{x} \in X_\Lambda |\ M(\mathbf{x}) = \mathcal{M}(X)\} \subseteq X_\Lambda$$

is residual.

At the end of this section we define two notions of uniform distribution on $\Lambda = \mathbb{N}^d$, which we will use in the following. First let $\mathbb{N}^d$ be equipped with the relations $(\leq_1, \leq_2) = (\leq, \leq)$ defined by $\mathbf{x} \leq \mathbf{y}$ iff $x_i \leq y_i$ $(1 \leq i \leq d)$. The second concept is to introduce the relation $\leq_s$ defined by $\mathbf{x} \leq_s \mathbf{y}$ iff $|\mathbf{x}| \leq |\mathbf{y}|$, where $|\mathbf{x}| := \prod_{i=1}^d x_i$ and to consider $(\mathbb{N}^d, \leq, \leq_s)$. A μ-u.d. net w.r.t. to this relation on $\mathbb{N}^d$ we will call strongly uniformly distributed (s.u.d.). The set of limit measure we denote by $M_s(\mathbf{x})$. The first concept is in accord with Kuipers and Niederreiter [11], the second concept is motivated by Aistleitner [1], who studied the discrepancy of sequences with multidimensional indices.

5. Characterization of $M(\mathbf{x})$ and distribution of subnets for a special kind of nets on $\mathbb{N}^d$

This section is devoted to the generalization of the characterization of the sets of limit measures given in Winkler [18, Theorem 3.1] to nets defined on $\Lambda = (\mathbb{N}^d, \leq, \leq)$ (see Section 4).

To simplify notation we introduce some operations on multidimensional indices. For an index $\mathbf{i} = (i_1, \ldots, i_d)$ we define

$$\mathbf{i} + c = (i_1 + c, \ldots, i_d + c),$$
$$\mathbf{i} \bmod c = (i_1 \bmod c, \ldots, i_d \bmod c).$$

Furthermore, we define the index-sets

$$I[\mathbf{i}, \mathbf{j}] := \{\mathbf{k} : \mathbf{k} \geq \mathbf{i} \text{ and } \exists \ell :\ k_\ell \leq j_\ell\},$$
$$I[\mathbf{i}, \mathbf{j}) := \{\mathbf{k} : \mathbf{k} \geq \mathbf{i} \text{ and } \exists \ell :\ k_\ell < j_\ell\} = I[\mathbf{i}, \mathbf{j} - 1],$$
$$I(\mathbf{i}, \mathbf{j}] := \{\mathbf{k} : \mathbf{k} > \mathbf{i} \text{ and } \exists \ell :\ k_\ell \leq j_\ell\} = I[\mathbf{i} + 1, \mathbf{j}].$$

A sequence of the form $\mathbf{x} = (x_\mathbf{i})_{\mathbf{i} \in I[\mathbf{1}, \mathbf{N}]}$ we call an angle-sequence, and by the periodic continuation of an angle-sequence by a finite sequence $\mathbf{y} = (y_\mathbf{i})_{\mathbf{1} \leq \mathbf{i} \leq \mathbf{N}_1}$ we mean the sequence $\mathbf{x}' = (x'_\mathbf{i})_{\mathbf{i} \in \mathbb{N}^d}$ defined by

$$x'_\mathbf{i} = \begin{cases} x_\mathbf{i} & \text{if } \mathbf{i} \in I[\mathbf{1}, \mathbf{N}], \\ y^k_{\mathbf{i} - N \bmod N_1} & \text{if } \mathbf{i} > \mathbf{N}. \end{cases}$$

Here we assume $\mathbf{N} = (N, \dots, N)$ and $\mathbf{N}_1 = (N_1, \dots, N_1)$. In fact, we could define the above construction for arbitrary indices $\mathbf{N}$ and $\mathbf{N}_1$, but in the following we will just need this definition.

Example 5.1. In two dimensions the periodic continuation of $\mathbf{x}$ with period $\mathbf{y}$ looks like the following:

$$\begin{array}{ccccc} \vdots & \vdots & \vdots & \vdots & \iddots \\ \vdots & \mathbf{y} & \mathbf{y} & \mathbf{y} & \cdots \\ \vdots & \mathbf{y} & \mathbf{y} & \mathbf{y} & \cdots \\ \vdots & \mathbf{y} & \mathbf{y} & \mathbf{y} & \cdots \\ \mathbf{x} & \cdots & \cdots & \cdots & \cdots \end{array}$$

The following lemma generalizes [18, Section 2] and gives some properties of the set $M(\mathbf{x})$.

Lemma 5.2. *For all sequences $\mathbf{x} \in X^{\omega \times \dots \times \omega}$ the set $M(\mathbf{x})$ has the following properties: $M(\mathbf{x})$ is*

(i) *nonempty,*
(ii) *contained in $M(X)$,*
(iii) *closed (hence compact),*
(iv) *connected.*

Proof. (*i*): $M(\mathbf{x}) \neq \emptyset$ since every net in a compact space has a convergent subnet and all $\lambda_{\mathbf{N}} \in \mathcal{M}(X)$.

(ii): The proof is completely analogous to the proof in [18], you just have to take the multidimensional limit.

(iii): $M(\mathbf{x})$ is the set of cluster points of the net $\mathbf{y} = (y_{\mathbf{N}})_{\mathbf{N} \in \mathbb{N}^k}$, and the set of cluster points of any net in any topological space is closed.

(iv): Assume that $M = M(\mathbf{x})$ is not connected. Therefore there are nonempty disjoint closed subsets M_1, $M_2 \subseteq M$ with $M = M_1 \cup M_2$. Since compact Hausdorff spaces are normal, we can find open sets O_i and V_i, $i = 1, 2$, in $M(X)$ satisfying

$$M_i \subseteq O_i \subseteq \overline{O}_i \subseteq V_i, \quad i = 1, 2, \quad \text{and} \quad V_1 \cap V_2 = \emptyset.$$

Thus the closures of the O_i are compact and disjoint. This yields that they have positive distance

$$d(\overline{O}_1, \overline{O}_2) = \inf_{\mu_i \in O_i} d(\mu_1, \mu_2) = \varepsilon > 0.$$

Now consider the compact set $L = \mathcal{M}(X) \setminus O_1 \setminus O_2$. Since both M_1 and M_2 contain cluster points of Λ, the net has to be infinitely many times in O_1 as well as in O_2 for all tails $(\lambda_{\mathbf{N}})_{\mathbf{N}\geq\mathbf{n}}$ with $\mathbf{n}\in\mathbb{N}^d$. Observe that $d(\lambda_{\mathbf{N}},\lambda_{\mathbf{N}+1})\leq c/(N+1)$, where $\mathbf{N}=(N,\dots,N)$. Thus the distance of subsequent members in the diagonal of Λ gets arbitrarily small, say less than ε. This means that Λ has to intersect L infinitely many times for all tails $(\lambda_{\mathbf{N}})_{\mathbf{N}\geq\mathbf{n}}$ with $\mathbf{n}\in\mathbb{N}^d$. Since L is compact, there must be a cluster point of Λ in L, but we also have

$$L\cap M=L\cap(M_1\cup M_2)\subseteq(L\cap O_1)\cup(L\cap O_2)=\emptyset,$$

which is a contradiction. □

The parts (i), (ii), and (iii) of the lemma above are valid for arbitrary nets as considered in Section 4, whereas part (iv) fails in general. We give the following example: Let $\mathbf{x}=(x_{n,m})_{(n,m)\in\mathbb{N}^2}$ be the net defined by

$$x_{(n,m)}=\begin{cases}0 & \text{if } m=0,\\ 1 & \text{if } m>0\end{cases}$$

and consider the pair of relations $(\leq,\leq_s)$. Then the set of limit measures is the set

$$M(\mathbf{x})=\{\lambda:\ \lambda(0)=\tfrac{1}{n},\ \lambda(1)=1-\tfrac{1}{n}\}\cup\{\delta_1\}.$$

Now we turn to the main result of this section. The proof uses two lemmas which we will present afterwards. With the definitions given in Section 4 we have

Theorem 5.3. *Let X be a locally compact Hausdorff space with countable topological base and $\mathbf{x}=(x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ a net. Then:*

(1) *Every $M(\mathbf{x})$ is a nonempty, closed (hence compact) and connected subset of $\mathcal{M}(X)$.*
(2) *Let $M\subseteq\mathcal{M}(X)$ be nonempty, compact and connected. Then there is a net $\mathbf{x}\in X^{\omega\times\cdots\times\omega}$ with $M(\mathbf{x})=M$.*

Proof. (1) see Lemma 5.2

(2) By Lemma 5.4 there exists a net $(\mu_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ in M whose set of cluster points equals M and with the additional property that $\lim_{\mathbf{k}\to\infty}\varepsilon_{\mathbf{k}}=0$ with a monotonically nonincreasing sequence of $\varepsilon_{\mathbf{k}}>d_{\mathbf{k}}$, where $d_{\mathbf{k}}$ is the maximum of the distances of $\mu_{\mathbf{k}}$ to its successors (see Lemma 5.4). Now we construct a sequence $\mathbf{x}=(x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ such that the induced sequence of the $\lambda_{\mathbf{N}}$ approximates the $\mu_{\mathbf{k}}$ in the following sense: There are indices $N_1<N_2<\cdots$ such that $d(\mu,\lambda_{\mathbf{N}}(\mathbf{x}))<2\varepsilon_{\mathbf{k}}$ for all $\mathbf{N}\in I[\mathbf{N}_k,\mathbf{N}_{k+1})$, where $\mathbf{N}_j=(N_j,\dots,N_j)$. Then the relation $M=M(\mathbf{x})$ is an immediate consequence.

To construct such a sequence we take a finite sequence $\mathbf{x}_0=(x^0_{\mathbf{i}})_{\mathbf{1}\leq\mathbf{i}\leq\mathbf{N}_0}$ such that $d(\lambda_{\mathbf{N}_0}(\mathbf{x}_0),\mu_1)<\varepsilon_1$ (the existence of $\mathbf{x}_0$ is guaranteed by Section 4). Consider the sequence $\mathbf{x}_1=(\mathbf{x}^1_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ with $x^1_{\mathbf{i}}=x_{\mathbf{i}\bmod N_0}$ such that there is a number N_1 with $d(\lambda_{\mathbf{N}}(\mathbf{x}_1), \mu_1) < \varepsilon_1$ for all $\mathbf{N} \geq \mathbf{N}_1$. Now we proceed by induction:

For arbitrary $k \geq 1$ assume that there is an angle-sequence $\mathbf{x}_k = (x_{\mathbf{i}}^k)_{\mathbf{i} \in I[\mathbf{1}, \mathbf{N}_k]}$ with the following properties:

(1) $d(\lambda_{\mathbf{N}_k}(\mathbf{x}_k), \mu_k) < \varepsilon_k$.

(2) There is a finite sequence $\mathbf{y}_k = (y_{\mathbf{i}}^k)_{\mathbf{1} \leq \mathbf{i} \leq (K,\ldots,K)}$ such that for the periodic continuation $\mathbf{x}_k^c$ of $\mathbf{x}_k$ with period $\mathbf{y}_k$ we have $d(\lambda_{\mathbf{N}}(\mathbf{x}_k^c), \mu_k) < \varepsilon_k$ for all $\mathbf{N} \geq \mathbf{N}_k$.

By Lemma 5.6 there is an angle-sequence $\mathbf{x}' = (x'_{\mathbf{i}})_{\mathbf{i} \in I(\mathbf{N}_k, \mathbf{N}_{k+1}]}$ such that for the angle-sequence $\mathbf{x}_{k+1} = (x_{\mathbf{i}}^{k+1})_{\mathbf{i} \in I[\mathbf{1}, \mathbf{N}_{k+1}]}$ defined by

$$x_{\mathbf{i}}^{k+1} = \begin{cases} x_{\mathbf{i}}^k & \text{if } \mathbf{i} \in I[1, \mathbf{N}_k], \\ x'_{\mathbf{i}} & \text{if } \mathbf{i} \in I(\mathbf{N}_k, \mathbf{N}_{k+1}] \end{cases}$$

the following conditions hold:

(i) If $\mathbf{N} \in I[\mathbf{N}_k, \mathbf{N}_{k+1})$, then there is a point μ on the linear connection between μ_k and μ_{k+1} with $d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}), \mu) < C\varepsilon_k$, where C is a constant depending only on the dimension d.

(ii) There is a finite sequence $\mathbf{y}_{k+1} = (\mathbf{y}_{\mathbf{i}}^{k+1})_{\mathbf{1} \leq \mathbf{i} \leq (K',\ldots,K')}$ such that for the periodic continuation of $\mathbf{x}_{k+1}$ with $\mathbf{y}_{k+1}$, which we denote by $\mathbf{x}$, we have $d(\lambda_{\mathbf{N}}(\mathbf{x}), \mu_{k+1}) < \varepsilon_{k+1}$ for all $\mathbf{N} \geq \mathbf{N}_{k+1}$.

Then the limit sequence $\lim_{k\to\infty} \mathbf{x}_{k+1}$, generated by the above induction, has the desired properties. $\square$

Lemma 5.4. *Let M be a nonempty closed and connected subset of $\mathcal{M}(X)$. Then there is a net $(\mu_{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}^d}$ in M, whose set of cluster points equals M and with the additional properties $\lim_{\mathbf{k}\to\infty} d_{\mathbf{k}} = 0$, where $d_{\mathbf{k}}$ is the maximum of the distances of $\mu_{\mathbf{k}}$ to its successors, i.e., $d_{\mathbf{k}} = \max_{\mathbf{k}' \in I_{\mathbf{k}}} d(\mu_{\mathbf{k}}, \mu_{\mathbf{k}'})$, where $I_{\mathbf{k}} = \{(K_1, \ldots, K_d) : \ K_i \in \{k_i, k_i + 1\}, i = 1, \ldots, d\}$ if $\mathbf{k} = (k_1, \ldots k_d)$, and $\mu_{\mathbf{k}'} = \mu_{(k,k,\ldots,k)}$, where $\mathbf{k}'$ runs over all indices which coincide with $(k, \ldots, k)$ in at least one coordinate.*

Example 5.5. In two dimensions such a net has the following form:

$$\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_4 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \mu_3 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_2 & \mu_2 & \mu_2 & \cdots \\ \mu_1 & \mu_1 & \mu_1 & \mu_1 & \mu_1 & \cdots \end{array}$$

P r o o f. By [18, Lemma 3.3], there exists a sequence $(\mu_k)_{k\in\mathbb{N}}$ in M whose set of accumulation points equals M and with $\lim_{k\to\infty} d(\mu_k, \mu_{k+1}) = 0$. The net determined by $\mu_{(k,\ldots,k)} = \mu_k$ has the desired properties. □

Lemma 5.6. *Let μ_k, $\mu_{k+1} \in \mathcal{M}(X)$ and $\varepsilon_k > \varepsilon_{k+1} > 0$ be given. Assume that $\mathbf{x}_k = (x^k_{\mathbf{i}})_{\mathbf{i}\in I[\mathbf{1},\mathbf{N}_k]}$ with $\mathbf{N}_k = (N_k, \ldots, N_k)$ is an angle-sequence with the following properties:*

(1) $d(\lambda_{\mathbf{N}_k}(\mathbf{x}_k), \mu_k) < \varepsilon_k$

(2) *There exists a finite sequence $\mathbf{y}_k = (y^k_{\mathbf{i}})_{\mathbf{1}\le\mathbf{i}\le(K,\ldots,K)}$ such that for the periodic continuation $\mathbf{x}^c_k$ of $\mathbf{x}_k$ with period $\mathbf{y}_k$ the following property holds: $d(\lambda_{\mathbf{n}}(\mathbf{x}^c_k), \mu_k) < \varepsilon_k$ for all $\mathbf{n} \ge \mathbf{N}_k$.*

Then there is an angle-sequence $\mathbf{x}' = (x'_{\mathbf{i}})_{\mathbf{i}\in I(\mathbf{N}_k,\mathbf{N}_{k+1}]}$ such that for the angle-sequence $\mathbf{x}_{k+1} = (x^{k+1}_{\mathbf{i}})_{\mathbf{i}\in I[\mathbf{1},\mathbf{N}_{k+1}]}$ defined by

$$x^{k+1}_{\mathbf{i}} = \begin{cases} x^k_{\mathbf{i}} & \text{if } \mathbf{i} \in I[1, \mathbf{N}_k], \\ x'_{\mathbf{i}} & \text{if } \mathbf{i} \in I(\mathbf{N}_k, \mathbf{N}_{k+1}] \end{cases}$$

the following conditions hold:

(i) *If $\mathbf{n} \in I[\mathbf{N}_k, \mathbf{N}_{k+1})$ then there is a point μ on the linear connection between μ_k and μ_{k+1} with $d(\lambda_{\mathbf{n}}(\mathbf{x}_{k+1}), \mu) < C\varepsilon_k$, where the constant C depends only on the dimension d.*

(ii) *There is a finite sequence $\mathbf{y}_{k+1} = (y^{k+1}_{\mathbf{i}})_{\mathbf{1}\le\mathbf{i}\le(K',\ldots,K')}$ such that for the sequence $\mathbf{x}$, which denotes the periodic continuation of $\mathbf{x}_{k+1}$ with period $\mathbf{y}_{k+1}$, we have $d(\lambda_{\mathbf{n}}(\mathbf{x}), \mu_{k+1}) < \varepsilon_{k+1}$ for all $\mathbf{n} \ge \mathbf{N}_{k+1}$.*

P r o o f. By Section 4 there is a sequence $\mathbf{y}$ with limit distribution μ_{k+1}. Take the initial part $\mathbf{y}_{k+1} = (y^{k+1}_{\mathbf{i}})_{\mathbf{1}\le\mathbf{i}\le(K',\ldots,K')}$ in such a way that the induced measure $\lambda = \lambda_{(K',\ldots,K')}(\mathbf{y}_{k+1})$ satisfies

$$d(\lambda, \mu_{k+1}) < \varepsilon_{k+1} < \varepsilon_k$$

and $K|K'$. Consider the angle-sequence $\mathbf{x}_{k+1} = (x^{k+1}_{\mathbf{i}})_{\mathbf{i}\in I[\mathbf{1},\mathbf{N}_{k+1}]}$ constructed in the following way:

$$x^{k+1}_{\mathbf{i}} = \begin{cases} x^k_{\mathbf{i}} & \text{if } \mathbf{i} \in I[\mathbf{1}, \mathbf{N}_k], \\ y^k_{\mathbf{i}-N_k \bmod K} & \text{if } \mathbf{i} \in I(\mathbf{N}_k, \mathbf{N}_k + m_k K], \\ y_{\mathbf{i}-(N_k+m_kK) \bmod K'} & \text{if } \mathbf{i} \in I(\mathbf{N}_k + m_k K, \mathbf{N}_{k+1}], \end{cases}$$

where $\mathbf{N}_{k+1} = (N_{k+1}, \ldots, N_{k+1})$ and $N_{k+1} = N_k + m_k K + m_{k+1} K'$ with suitable chosen m_k and m_{k+1}; this is done below. Let $\mathbf{x}$ denote the sequence obtained from $\mathbf{x}_{k+1}$ by periodic continuation with period $\mathbf{y}_{k+1}$. We first prove the second statement of the lemma. Given m_k, we can choose m_{k+1} large enough that

$$d(\lambda_{\mathbf{n}}(\mathbf{x}), \mu_{k+1}) < \varepsilon_{k+1} \quad \text{for all} \quad \mathbf{n} \geq \mathbf{N}_{k+1}$$

since for $\mathbf{n} = (n_1, \dots, n_d)$ with $n_i = N_k + m_k K + m_{k+1} K' + s_i K' + c_i$ with $0 \leq c_i < K'$ and $s_i \geq 0$ we have

$$\begin{aligned} d(\lambda_{\mathbf{N}}(\mathbf{x}), \mu_{k+1}) &= d\left(\frac{1}{|\mathbf{n}|}\left(K'^d \prod_{i=1}^{d}(m_{k+1}+s_i)\lambda + \sum\right), \mu_{k+1}\right) \\ &\leq \frac{1}{|\mathbf{n}|} K'^d \prod_{i=1}^{d}(m_{k+1}+s_i) d(\lambda, \mu_{k+1}) \\ &\quad + \frac{|\mathbf{n}| - K'^d \prod_{i=1}^{d}(m_{k+1}+s_i)}{|\mathbf{n}|} \\ &= 1 - (1 - d(\lambda, \mu_{k+1}))\frac{1}{|\mathbf{n}|} K'^d \prod_{i=1}^{d}(m_{k+1}+s_i), \end{aligned}$$

where $\sum$ is a sum over $|\mathbf{n}| - K'^d \prod_{i=1}^{d}(m_{k+1}+s_i)$ indicator functions. For m_{k+1} so large that

$$\frac{1}{|\mathbf{n}|} K'^d \prod_{i=1}^{d}(m_{k+1}+s_i) > \frac{1-\varepsilon_{k+1}}{1-d(\lambda, \mu_{k+1})},$$

the statement is true.

Now we turn to the first assertion. Firstly we give a detailed proof of the two-dimensional case, afterwards we prove the general case. Indeed, the general case uses the same idea as the two-dimensional case, but it is not necessary (and in higher dimension also very awful to write things down) to be so accurate as we are in the two-dimensional case, but this accuracy will be very helpful to understand what's going on.

So we have to show that for all $\mathbf{n} \in I[\mathbf{N}_k, \mathbf{N}_{k+1})$ we have

$$d(\lambda_{\mathbf{n}}(\mathbf{x}_{k+1}), \mu) < C\varepsilon_k$$

for some μ on the linear connection between μ_k and μ_{k+1}. For

$$\mathbf{n} \in I[\mathbf{N}, \mathbf{N}_k + m_k K]$$

this is true by assumption. Now consider a point

$$\mathbf{N} = (N_1, N_2) = (N_k + m_k K + sK' + d, N_k + m_k K + tK' + e)$$

with

$$0 \leq s, t \leq m_{k+1} \quad \text{and} \quad 0 \leq d, e < K'.$$

We can write $\lambda_{\mathbf{N}}$ as

$$\begin{aligned}
\mathbf{N}\lambda_{\mathbf{N}} = N_{k+1}^2 \frac{st}{m_{k+1}^2} \lambda_{\mathbf{N}_{k+1}} \\
&- (N_k + m_k K)\left(N_k + m_k K + m_{k+1}K'\right) \frac{st}{m_{k+1}^2} \lambda_{(N_k+m_kK, N_{k+1})} \\
&- (N_k + m_k K)\left(N_k + m_k K + m_{k+1}K'\right) \frac{st}{m_{k+1}^2} \lambda_{(N_{k+1}, N_k+m_kK)} \\
&+ (N_k + m_k K)\left(N_k + m_k K + tK' + e\right) \lambda_{(N_k+m_kK, N_2)} \\
&+ (N_k + m_k K)\left(N_k + m_k K + sK' + d\right) \lambda_{(N_1, N_k+m_kK)} \\
&+ \left(\frac{st}{m_{k+1}^2} - 1\right) (N_k + m_k K)^2 \lambda_{(N_k+m_kK, N_k+m_kK)} \\
&+ \sum \\
=: a_1\lambda_1 + \sum_{i=2}^{6} a_i \lambda_i + \sum \quad ,
\end{aligned}$$

where $\sum$ is a sum over $de + esK' + dtK'$ indicator functions. The first term is needed to count the indicator functions induced by the complete $\mathbf{y}_{k+1}$-blocks. In $\lambda_{\mathbf{N}_{k+1}}$ we have m_{k+1}^2 such blocks and we need st blocks, so we multiply with $\frac{st}{m_{k+1}^2}$. But with this measure we count too many indicator functions, namely those in the areas

$$A = I((1, N_2), (N_k + m_k K, N_{k+1})] \quad \text{and} \quad B = I((N_1, 1), (N_{k+1}, N_k + m_k K)]$$

(see Figure 3). This error is corrected by subtracting the terms with the measures $\lambda_{(N_k+m_kK, N_{k+1})}$ and $\lambda_{(N_{k+1}, N_k+m_kK)}$. But now we have eliminated all contributions from the areas

$$I((1, N_k + m_k K), (N_k + m_k K, N_2)] \quad \text{and} \quad I((N_k + m_k K, 1), (N_1, N_k + m_k K)]$$

too, so we add the terms with $\lambda_{(N_k+m_kK, N_2)}$ and $\lambda_{(N_1, N_k+m_kK)}$. Last we correct the contribution of $I[\mathbf{1}, (N_k + m_k K, N_k + m_k K)]$. The measure $\sum$ contains all the indicator functions from the incomplete $\mathbf{y}_{k+1}$-blocks.

Figure 4 illustrates this procedure (except the error $\sum$): We have the thick-border area and want to construct the grey area, using only rectangles starting in the origin. So we substract the vertical and horizontal dotted areas first, then we have to add their intersection, the thick-bordered square again. Afterwards we add the diagonally lined areas and correct the error we made by substracting their intersections, again thick-bordered square.

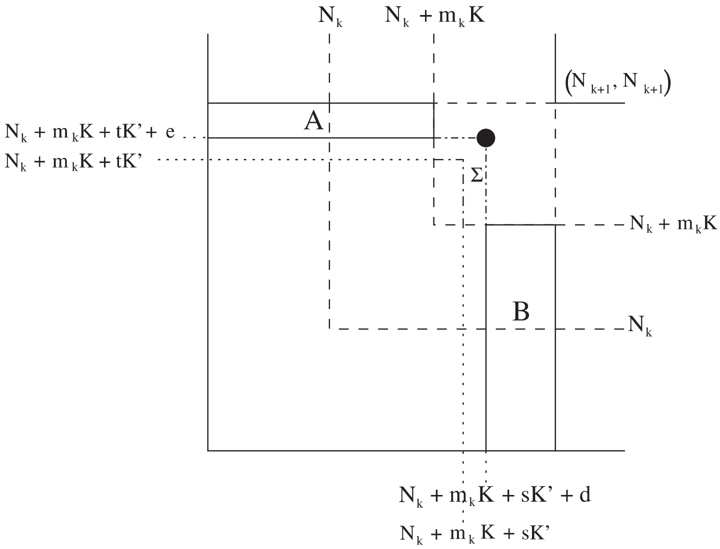

FIGURE 3.

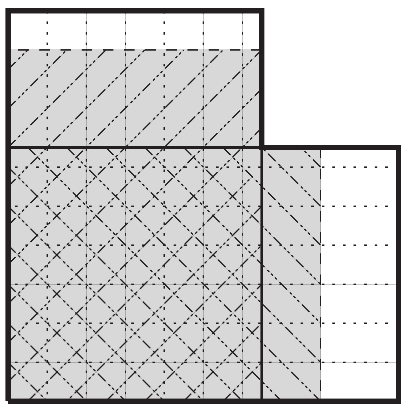

FIGURE 4.

Now define $\mu := \frac{1}{|\mathbf{N}|}\left(a_1\mu_{k+1} + \mu_k\left(\sum_{i=2}^{6} a_i + de + dtK' + esK'\right)\right)$. Then μ is on linear connection between μ_k and μ_{k+1} and we have

$$\begin{aligned} d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}),\mu) &\le \frac{a_1 d}{N_1N_2}(\lambda_1(\mathbf{x}_{k+1}),\mu_{k+1}) + \frac{1}{N_1N_2}\sum_{i=2}^{6}|a_i|d(\lambda_i(\mathbf{x}_{k+1}),\mu_k) \\ &+ 2\frac{de+dtK'+esK'}{N_1N_2} \\ &< 6\varepsilon_k + 2\frac{2K'^2(s+t+1)}{N_1N_2} \end{aligned}$$

Now we reduce the fraction by $\max(s,t)$, hence

$$d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}),\mu) < 6\varepsilon_k + 4K'^2\frac{3}{K'(N_k+m_kK)} < 7\varepsilon_k$$

if m_k is chosen large enough.

In a similar way we can decompose $\lambda_{\mathbf{N}}$ with $\mathbf{N}=(N_1,N_2)=(N_k+m_kK+sK'+d, N_k+m_kK+tK'+e)$, $s<m_{k+1}$, $t>m_{k+1}$ and $0\le d,e<K'$ (the case $t<m_{k+1}$, $s>m_{k+1}$ is symmetric) into

$$\begin{aligned} N_1N_2\lambda_{\mathbf{N}} &= N_{k+1}\left(N_k+m_kK+(t+1)K'\right)\frac{st}{m_{k+1}(t+1)}\lambda_{\left(N_{k+1},N_k+m_kK+(t+1)K'\right)} \\ &- \frac{st}{m_{k+1}(t+1)}(N_k+m_kK)\left(N_k+m_kK+(t+1)K'\right)\lambda_{(N_k+m_kK,N_k+m_kK+(t+1)K')} \\ &- \frac{st}{m_{k+1}(t+1)}N_{k+1}\left(N_k+m_kK\right)\lambda_{\left(N_{k+1},N_k+m_kK\right)} \\ &+ \left(N_k+m_kK\right)N_2\lambda_{(N_k+m_kK,N_2)} \\ &+ N_1\left(N_k+m_kK\right)\lambda_{(N_1,N_k+m_kK)} \\ &+ \left(\frac{st}{m_{k+1}(t+1)}-1\right)\left(N_k+m_kK\right)^2\lambda_{(N_k+m_kK,N_k+m_kK)} \\ &+ \sum\quad, \end{aligned}$$

where $\sum$ is a sum over $de+esK'+dtK'$ indicator functions and obtain that for suitable chosen μ on the linear connection between μ_k and μ_{k+1}

$$d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}),\mu) < 7\varepsilon_k.$$

Now we turn to the general case. Therefor consider a point $\mathbf{N}=(N_1,\dots,N_d)$ with $N_i=N_k+m_kK+s_iK'+d_i$ and $s_i<m_{k+1}$ and $0\le d_i<K'$ first. Then we can write

$$|\mathbf{N}|\lambda_{\mathbf{N}} = |\mathbf{N}_{k+1}|\frac{\prod_{i=1}^d s_i}{m_{k+1}^d}\lambda_{\mathbf{N}_{k+1}} + \sum_{i=1}^{T}a_i\lambda_{\mathbf{n}_i} + \sum\quad,$$

where $\mathbf{n}_i$ is of the form $\mathbf{n}_i = (n_1, \dots n_d)$ with all $n_i \in \{N_{k+1}, N_k + m_k K\}$ or all $n_i \in \{N_i, N_k + m_k K\}$ but not all $n_i = N_i$. The coefficients

$$a_i = v_i |\mathbf{n}_i| c_i \quad \text{with} \quad v_i \in \{1, -1\} \quad \text{and} \quad c_i \in \left\{1, \frac{\prod_{i=1}^d s_i}{m_{k+1}^d}\right\}.$$

The above formula is true, since after taking $\lambda_{\mathbf{N}_{k+1}}$, we have to substract the error we made. Therefore we substract the $\lambda_{\mathbf{n}_i}$ with exactly one $n_i = N_k + m_k K$ and for all the other $j \neq i$ with $n_j = N_{k+1}$; there are $p_1 = d$ such measures. Each two of them have an intersection, so we have the correct this, which leads to $p_2 = \binom{d}{2}$ summands (each such index has exactly two entries $N_k + m_k K$). Each of them have again an intersection (now there are $p_3 = \binom{p_2}{2}$ of them) and so on ($p_{i+1} = \binom{p_i}{2}$). After d steps this procedure must end. Afterwards we start adding the terms with those $\mathbf{n}_i$ with exactly one entry equals $N_k + m_k K$ and the other entries equal N_i. There are p_1 of them. Then we correct the intersections again and so on. Last we add the term due to the non-complete $\mathbf{y}_k$ blocks, this is denoted by $\sum$ and is a sum over

$$S := 1 + \sum_{j=1}^{d} \sum_{\substack{A \subseteq \{1,\dots,d\} \\ |A|=j}} K'^{d-j} \prod_{p \in A} d_p \prod_{q \in \{1,\dots,d\} \setminus A} s_q$$

indicator functions.

Hence $T \leq 2 \sum_{i=1}^d p_i < F(d)$, where $F(d)$ is a constant only depending on the dimension d. Taking

$$\mu = \frac{1}{\mathbf{N}} |\mathbf{N}_{k+1}| \frac{\prod_{i=1}^d s_i}{m_{k+1}^d} \mu_{k+1} + \frac{1}{\mathbf{N}} \left(\sum_{i=1}^{T} a_i + S \right) \mu_k$$

we find that

$$d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}), \mu) \leq F(d) \varepsilon_k + \frac{2S}{|\mathbf{N}|}.$$

By reducing the fraction on the right-hand side by the product of the $(d-1)$ greatest s_i and estimating $d_i \leq K'$ we see

$$d(\lambda_{\mathbf{N}}(\mathbf{x}_{k+1}), \mu) \leq F(d) \varepsilon_k + 2 \frac{2^d K'^d}{K'^{d-1}(N_k + m_k K)}.$$

So we have to choose m_k in such a way that the fraction becomes small. A similar construction holds for the other points $\mathbf{N} \in I[\mathbf{N}_k + m_k K, \mathbf{N}_{k+1})$. □

After this characterization of $M(\mathbf{x})$ we will study the distribution of certain subnets of a given net and generalize results due to Goldstern, Winkler and Schmeling [6]. We study subnets as studied in Losert and Tichy [12]: Choose d sequences $\mathbf{a}_1, \dots, \mathbf{a}_d \in \{0,1\}^{\mathbb{N}}$ and define $\mathbf{a} = (a_{\mathbf{n}})_{n \in \mathbb{N}^d}$ by

$$a_{(n_1,\dots,n_d)} = \prod_{i=1}^{d} \mathbf{a}_{i,n_i}.$$

Then the subnet $\mathbf{ax}$ of $\mathbf{x}$ is the net obtained by taking those elements $x_{\mathbf{n}}$ for which $a_{\mathbf{n}} = 1$ and using the given relation $\leq$.

The next theorem is a consequence of Theorem 5.3 and generalizes [6, Theorem 1.2]:

Theorem 5.7. *Let* $\mathbf{x} \in X^{\mathbb{N}^d}$ *and* $M \subseteq \mathcal{M}(X)$. *Then there exists a subsequence* $\mathbf{ax}$ *with* $M(\mathbf{ax}) = M$ *iff* M *is closed and connected with* $\emptyset \neq M \subseteq \mathcal{M}(A(\mathbf{x}))$, *where* $A(\mathbf{x})$ *is the set of cluster points of the net* $\mathbf{x}$.

P r o o f. This proof runs along the same lines as the one in [6]: First assume $M = M(\mathbf{ax})$. Using Lemma 5.2 we get that M is nonempty, closed and connected. It remains to show that $M \subseteq \mathcal{M}(A(\mathbf{x}))$. For this purpose it suffices to show that every $x \in X \setminus A(\mathbf{x})$ has a neighborhood U with $\lim_{\mathbf{N}\to\infty} \mu_{\mathbf{N},\mathbf{ax}}(U) = 0$. Therefore take a neighborhood U with compact closure $\overline{U}$ and with $\overline{U} \cap A(\mathbf{x}) = \emptyset$. If $x_{\mathbf{n}} \in \overline{U}$ for an infinite increasing sequence of indices $\mathbf{n}_1 < \mathbf{n}_2 < \cdots$, $\overline{U}$ would contain a cluster point of $\mathbf{x}$, which is a contradiction. Hence $x_{\mathbf{n}} \notin U$ for all $\mathbf{n} \geq \mathbf{N}_0$. Thus

$$\lim_{\mathbf{N}\to\infty} \mu_{\mathbf{N},\mathbf{ax}}(U) \leq \lim_{\mathbf{N}\to\infty} \frac{|\mathbf{N}_0|}{|\mathbf{N}|} = 0.$$

The other direction is completely analogous to [6]. □

Similarly to [6, Theorem 1.3] we get that a typical subsequence of a given sequence is maldistributed in $A(\mathbf{x})$:

Theorem 5.8. $M(\mathbf{ax}) = \mathcal{M}(A(\mathbf{x}))$ *holds for all* $\mathbf{a} \in R$ *from a residual set* $R \subseteq [0,1)^d$.

6. $\mathbf{n}\alpha$-nets over $\mathbb{N}^d$

In this section we specialize on $\mathbf{n}\alpha$-nets over $\mathbb{N}^d$, i.e., we consider $X = [0,1)$ and μ the Lebesgue-measure. Besides the two notions of uniform distribution mod 1 according to Section 4 we consider the $(s_1,\ldots,s_d)$-u.d. (see Kirschenhofer and Tichy [10]). After some elementary properties and examples of these three concepts we turn to the generalization of results given in Goldstern, Schmeling and Winkler [7] and Ajtai, Havas and Komlós [2].

In Section 4 we introduced two special notions of uniform distribution. In the context of this section we call a net $\mathbf{x}$ uniformly distributed mod 1 iff for any a and b with $0 \leq a < b \leq 1$,

$$\lim_{N_1,\ldots,N_d\to\infty} \frac{A([a,b);\mathbf{N})}{|\mathbf{N}|} = b - a,$$

where $A([a,b);\mathbf{N})$ is the number of $x_{\mathbf{k}}$, $\mathbf{1} \leq \mathbf{k} \leq \mathbf{N}$ with $a \leq \{x_{\mathbf{k}}\} < b$.

This definition is a direct extension of uniform distribution in the case $d = 2$ given by Kuipers and Niederreiter [11] and a special case of the concept studied in Losert and Tichy [12]. Following [11] one gets immediately the theorems given below.

Theorem 6.1. *The sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is u.d.* mod 1 *if and only if for every Riemann-integrable function f on* $[0, 1]$

$$\lim_{N_1,\dots,N_d\to\infty} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} f(\{x_{\mathbf{k}}\}) = \int_0^1 f(x)\mathrm{d}x,$$

where $\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} = \sum_{\mathbf{k}:\mathbf{1}\le\mathbf{k}\le\mathbf{N}}$.

Theorem 6.2. *The sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is u.d.* mod 1 *if and only if*

$$\lim_{N_1,\dots,N_d\to\infty} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} e^{2\pi i h x_{\mathbf{k}}} = 0$$

for all integers $h \neq 0$.

Moreover, a net $\mathbf{x} = (x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is said to be strongly uniformly distributed (s.u.d.) mod 1 iff for any a and b with $0 \le a < b \le 1$,

$$\lim_{|\mathbf{N}|\to\infty} \frac{A([a,b);\mathbf{N})}{|\mathbf{N}|} = b - a.$$

Here $\lim_{|\mathbf{N}|\to\infty} f(\mathbf{N}) = f$ means that $\forall\varepsilon > 0\ \exists N \in \mathbb{N} : \forall\mathbf{N}$ with $|\mathbf{N}| \ge N$: $|f(\mathbf{N}) - f| < \varepsilon$.

The following theorems hold:

Theorem 6.3. *The sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is s.u.d.* mod 1 *iff for every Riemann--integrable function f on* $[0, 1]$

$$\lim_{|\mathbf{N}|\to\infty} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} f(\{x_{\mathbf{k}}\}) = \int_0^1 f(x)\mathrm{d}x.$$

Theorem 6.4. *The sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is s.u.d.* mod 1 *if and only if*

$$\lim_{|\mathbf{N}|\to\infty} \frac{1}{|\mathbf{N}|}\sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N}} e^{2\pi i h x_{\mathbf{k}}} = 0$$

for all integers $h \neq 0$.

As in the one-dimensional case, a sequence with multidimensional sequences is strongly uniformly distributed modulo 1 if and only if the multidimensional discrepancy introduced by Aistleitner [1] tends to 0.

Clearly, strong uniform distribution implies uniform distribution. The converse is not true: Consider the double sequence $\mathbf{x}$ defined by $x_{j,k} = j\theta$ with θ irrational. Then this sequence is u.d. mod 1 (this follows easily from Theorem 6.2), but not s.u.d., since this sequence is constant for fixed k. Thus $\mathbf{x}$ is not s.u.d. mod 1 by the following theorem:

Theorem 6.5. *Let $x_{\mathbf{k}}$ be s.u.d.* mod 1. *Then all "one-dimensional sequences", i.e., sequences $(x_{(k_1,\ldots,k_j,\ldots,k_d)})_{k_j\in\mathbb{N}}$ with fixed k_s, $s \neq j$, are u.d.* mod 1.

Proof. By the criterion of Weyl, we have to show that

$$\lim_{k_j\to\infty} \frac{1}{k_j} \sum_{n=1}^{k_j} e^{2\pi i h x_{(k_1,\ldots,k_{j-1},n,k_{j+1},\ldots,k_d)}} = 0 \tag{6}$$

for all $k_s \in \mathbb{N}$, $s \neq j$ and $h \in \mathbb{Z} \setminus \{0\}$. We use induction. From Theorem 6.4 we get readily that (6) holds for $k_s = 1$, $s \neq j$ for all integers $h \neq 0$ and all j. Assume that (6) holds for all

$$\mathbf{k}'_j := (k'_1, \ldots, k'_{j-1}, k'_{j+1}, \ldots, k'_d) < (k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_d) := \mathbf{k}_j.$$

Again by Theorem 6.4 we have

$$\varepsilon > \left| \frac{1}{|\mathbf{k}|} \sum_{\mathbf{k}'_j \le \mathbf{k}_j} \sum_{n=1}^{k_j} e^{2\pi i h x_{(k'_1,\ldots,k'_{j-1},n,k'_{j+1},\ldots,k'_d)}} \right| \tag{7}$$

for k_j big enough. Hence

$$\varepsilon > \frac{1}{|\mathbf{k}|} \left| \left| \sum_{n=1}^{k_j} e^{2\pi i h x\left(k_1,\ldots,k_{j-1},n,k_{j+1},\ldots,k_d\right)} \right| - \left| \sum_{n=1}^{k_j} \sum_{\mathbf{1} \le \mathbf{k}'_j < \mathbf{k}_j} e^{2\pi i h x\left(k'_1,\ldots,k'_{j-1},n,k'_{j+1},\ldots,k'_d\right)} \right| \right|.$$

The second term on the right hand side tends to 0 by (7). Thus

$$\frac{1}{|\mathbf{k}|} \left| \sum_{n=1}^{k_j} e^{2\pi i h x_{(k_1,\ldots,k_{j-1},n,k_{j+1},\ldots,k_d)}} \right| \to 0$$

for $k_j \to \infty$. Therefore (6) holds. □

We give an example of a sequence which is s.u.d. mod 1. This sequence can be seen as a generalization of the one-dimensional sequence $(n\theta)_{n\in\mathbb{N}}$. This sequence is u.d. mod 1 for all irrational θ. Choose now $n_{\mathbf{k}} = \sum_{i=1}^{d} k_i - (d-1)$. In two dimensions this sequence is

$$
\begin{array}{|cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\
4 & 5 & 6 & 7 & 8 & \dots \\
3 & 4 & 5 & 6 & 7 & \dots \\
2 & 3 & 4 & 5 & 6 & \dots \\
1 & 2 & 3 & 4 & 5 & \dots \\
\hline
\end{array}
$$

We will prove now that this sequence is s.u.d. mod 1 for all irrational α. The proof is similar to the proof of the one-dimensional case (see [11]).We have to show that

$$\lim_{|\mathbf{n}|\to\infty} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} e^{2\pi i h\alpha \sum_{s=1}^{d} k_s} = 0$$

for all integers $h \neq 0$ and irrational α. Here we assume $n_1 \geq n_2 \geq \cdots \geq n_d$. Therefore $n_1 \to \infty$. With $S(\mathbf{n}) = \sum_{i=1}^{s} n_i$ we have

$$
\begin{aligned}
\frac{1}{|\mathbf{n}|} \left| \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}} e^{2\pi i h\alpha \sum_{s=1}^{d} k_s} \right| &= \frac{1}{|\mathbf{n}|} \left| \sum_{\mathbf{k}=\mathbf{1}}^{(n_2,\dots n_d)} \sum_{j=S(\mathbf{k})}^{S(\mathbf{n})-S(\mathbf{k})} e^{2\pi i h\alpha j} \right| \\
&= \frac{1}{|\mathbf{n}|} \left| \sum_{\mathbf{k}=\mathbf{1}}^{(n_2,\dots n_d)} \frac{e^{2\pi i h\alpha (S(\mathbf{n})-S(\mathbf{k})+1)} - e^{2\pi i h\alpha S(\mathbf{k})}}{1-e^{2\pi i h\alpha}} \right| \\
&\leq \frac{1}{|\mathbf{n}|} \frac{2\prod_{s=2}^{d} n_s}{|1-e^{2\pi i h\alpha}|} \\
&= \frac{1}{n_1} \frac{1}{|1-e^{2\pi i h\alpha}|} \to 0.
\end{aligned}
$$

We give another example: In the one-dimensional case the sequence $(\{k!e\})_{k\in\mathbb{N}}$ has 0 as the only limit point (see [11]). Consider now the sequence $n_{\mathbf{k}} = (S(\mathbf{k}) - d + 1)! =: \mathbf{k}!$. Then

$$\mathbf{k}!e = A + \frac{e^{\alpha}}{S(\mathbf{k}) - d + 1}, \qquad 0 < \alpha < 1,\ A \in \mathbb{N}.$$

Thus $\{\mathbf{k}!e\} = e^{\alpha}/(S(\mathbf{k}) - d + 1) \to 0$ in the first sense. Therefore it is not u.d. mod 1 (and hence not s.u.d. mod 1).

The third concept is the $(s_1, \dots, s_d)$-uniform distribution introduced by Kirschenhofer and Tichy [10]. According to the definitions above we gave an equivalent definition to that one stated in [10].

Definition 6.6. A sequence $(x_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ is $(s_1,\dots,s_d)$-u.d. iff for all $a_{i_1\dots i_d}$ and $b_{i_1\dots i_d}$ with $0\le a_{i_1\dots i_d}<b_{i_1\dots i_d}\le 1$ and $1\le i_j\le s_j$ for $1\le j\le d$

$$\lim_{N_1,\dots,N_d\to\infty}\prod_{i=1}^{d}\binom{N_i}{s_i}^{-1}A\left([a_{11\dots1},b_{11\dots1}),\dots,[a_{s_1\dots s_d},b_{s_1\dots s_d});N_1,\dots,N_d;s_1,\dots s_d\right)$$
$$=\prod_{i=1}^{d}\prod_{j_i=1}^{s_i}b_{j_1\dots j_d}-a_{j_1\dots j_d},$$

where $A\left([a_{11\dots1},b_{11\dots1}),\dots,[a_{s_1\dots s_d},b_{s_1\dots s_d});N_1,\dots,N_d;s_1,\dots s_d\right)$ is the number of $(s_1\cdots s_d)$-tuples $(x_{i_{11},\dots i_{d1}},\dots x_{i_{1s_1},\dots i_{ds_d}})$ with $1\le i_{j\,1}<\dots<i_{j\,s_d}\le N_j$ for all $1\le j\le d$ in $[a_{11\dots1},b_{11\dots1})\times\dots\times[a_{s_1\dots s_d},b_{s_1\dots s_d})$.

As in [11] we have that the set S of $(s_1,\dots,s_d)$-u.d. sequences is everywhere dense in $X^{\omega\times\dots\times\omega}$. By [10], $(s_1,\dots,s_d)$-uniform distribution implies uniform distribution. Thus from Section 4 we conclude: If X contains more than one element, then the set S of $(s_1,\dots,s_d)-\mu$-u.d. sequences is a set of first category in $X^{\omega\times\dots\times\omega}$.

After these examples and elementary properties of uniform distribution of sequences with multidimensional indices, we turn to the generalization of [7, Theorem 2.4] for these cases. For the sake of completeness we mention that in [16] Šalát proved that for a sequence $(n_k)_{k\in\mathbb{N}}$ with $n_k=\prod_{j=1}^{k}q_j$, where $(q_j)_{j\in\mathbb{N}}$ is a sequence of integers greater than 1, then the set $\mathcal{U}:=\{\alpha\in\mathbb{R}:(n_k\alpha)$ is u.d. mod 1$\}$ is meager. By modifying the proof slightly, we get

Theorem 6.7. *Let $(q_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ be a sequence of integers greater than 1. Put*

$$a_{\mathbf{n}}=\prod_{\mathbf{k}=\mathbf{1}}^{\mathbf{n}}q_{\mathbf{k}},\qquad \mathbf{n}\in\mathbb{N}^d.$$

Then the set

$$\mathcal{U}:=\{\alpha\in\mathbb{R}:\ (n_{\mathbf{k}}\alpha)\ \textit{is u.d.}\bmod 1\}$$

is meager. Consequently the sets

$$\mathcal{U}':=\{\alpha\in\mathbb{R}:\ (n_{\mathbf{k}}\alpha)\ \textit{is s.u.d.}\bmod 1\}$$

and

$$\mathcal{U}'':=\{\alpha\in\mathbb{R}:\ (n_{\mathbf{k}}\alpha)\ \textit{is}(s_1,\dots,s_d)\textit{-u.d.}\}$$

are meager.

Now we turn to the stronger result. We will generalize [7, Theorem 2.4]. For this purpose we will follow [7] again. Recall the definitions of Section 4. Moreover, let λ denote the Lebesgue measure on $\mathbb{R}/\mathbb{Z}$.

We start with an elementary property.

Theorem 6.8. *Given a sequence* $\mathbf{x} = (x_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^d}$ *we have* $\emptyset \neq M(\mathbf{x}) \subseteq M_s(\mathbf{x})$.

Now we can establish the main result of this section.

Theorem 6.9. *Let* $\mathbf{n} = (n_{\mathbf{k}})_{\mathbf{k}\in\mathbf{N}^d}$ *be a sequence of nonnegative integers and assume that there exists a constant* Q *such that*

$$\#\left\{\mathbf{k} : 2^r \leq n_{\mathbf{k}} < 2^{r+1}\right\} \leq Q, \qquad \forall r = 0,1,2,\dots$$

Then the set

$$\mathcal{U} := \{\alpha \in \mathbb{R}/\mathbb{Z} :\ \mathbf{n}\alpha \textit{ is uniformly distributed w.r.t. } \lambda\}$$

is meager. Moreover, there is a number $P > 0$ *such that for all intervals* I *the set*

$$\left\{\alpha : \overline{\mu}_{\mathbf{n}\alpha}(I) > \frac{P}{-\log \lambda(I)}\right\}$$

is residual (here $\overline{\mu}_{\mathbf{n}\alpha}$ *is defined analogously to Definition 2.4).*

Consequently, the sets

$$\mathcal{U}' := \{\alpha \in \mathbb{R}/\mathbb{Z} :\ \mathbf{n}\alpha \textit{ is s.u.d.} \bmod 1\}$$

and

$$\mathcal{U}'' := \{\alpha \in \mathbb{R}/\mathbb{Z} :\ (n_{\mathbf{k}}\alpha) \textit{ is } (s_1,\dots,s_d)\textit{-u.d.}\}$$

are meager.

Before proving the theorem we note the following lemma:

Lemma 6.10. *Assume that* $(n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ *is a sequence of positive integers with the property that whenever you choose* $T+1$ *elements* $n_{\mathbf{k}_1} \leq \dots \leq n_{\mathbf{k}_{T+1}}$ *you know that* $n_{\mathbf{k}_{T+1}}/n_{\mathbf{k}_1} > U$. *Then in a cuboid with* $X \geq T+1$ *elements there are* $D := \lfloor \frac{X-1}{T} \rfloor + 1$ *elements* $n_{\mathbf{k'}_1}, \dots, n_{\mathbf{k'}_D}$ *such that* $n_{\mathbf{k'}_{i+1}}/n_{\mathbf{k'}_i} > U$ *for* $i = 1,\dots,D-1$.

Proof. Let $n_{\mathbf{k}_1} \leq n_{\mathbf{k}_2} \leq \dots \leq n_{\mathbf{k}_X}$ be a sorting of the X elements in the cuboid and choose the elements with indices $\mathbf{k}_1, \mathbf{k}_{T+1}, \dots, \mathbf{k}_{\lfloor (X-1)/T \rfloor T+1}$. $\square$

Proof of the theorem. Choose $P > 0$ so small, that

$$\left(\frac{1}{2^{d+1}PQ} - 1\right) - 1 > 1$$

and assume $\lambda(I) =: \varepsilon < \frac{1}{2}$. Then there exists an integer c in the interval

$$\left(1 - \log\varepsilon, -\frac{1}{2^{d+1}PQ}\log\varepsilon\right).$$

Thus $\frac{1}{2^{d+1}Qc} > \frac{2P}{-\log\varepsilon}$ and $2^c > 2/\varepsilon$. Again we assume that the theorem is false. Since the set $\{\alpha : \overline{\mu}_{\mathbf{n}\alpha}(I) > \frac{P}{-\log\varepsilon}\}$ is a Borel set and not residual, its complement is residual in I, for some open interval I:

$$I \Vdash \left\{\alpha :\ \overline{\mu}_{\mathbf{n}\alpha}(I) \leq \frac{P}{-\log\varepsilon}\right\}.$$

As in Section 2 the set $\{\alpha :\ \overline{\mu}_{\mathbf{n}\alpha}(I) \leq \frac{P}{-\log\varepsilon}\}$ is contained in the set

$$\left\{\alpha :\ \exists\mathbf{m}\ \forall\mathbf{N} \geq \mathbf{m} :\ \mu_{\mathbf{n}\alpha,\mathbf{N}}(I) \leq \frac{2P}{-\log\varepsilon}\right\}.$$

Denote the set $\{\mathbf{j} \leq \mathbf{N} :\ n_{\mathbf{j}}\alpha \in I\}$ by $Z_{\mathbf{N}}(\alpha)$. So $\mu_{\mathbf{n}\alpha,\mathbf{N}}(I) = \frac{\#Z_{\mathbf{N}}(\alpha)}{|\mathbf{N}|}$. Therefore

$$I \Vdash \bigcup_{\mathbf{m}} \bigcap_{\mathbf{N}\geq\mathbf{m}} \left\{\alpha :\ \frac{\#Z_{\mathbf{N}}(\alpha)}{|\mathbf{N}|} \leq \frac{2P}{-\log\varepsilon}\right\}.$$

So, by Fact 2.7, we can find an open interval $J \subseteq I$ and a $\mathbf{k}^*$ such that

$$J \Vdash \bigcap_{\mathbf{N}\geq\mathbf{k}^*} \left\{\alpha :\ \frac{\#Z_{\mathbf{N}}(\alpha)}{|\mathbf{N}|} \leq \frac{2P}{-\log\varepsilon}\right\},$$

or equivalently, for all $\mathbf{N} \geq \mathbf{k}^*$:

$$J \Vdash \left\{\alpha :\ \frac{\#Z_{\mathbf{N}}(\alpha)}{|\mathbf{N}|} \leq \frac{2P}{-\log\varepsilon}\right\}, \tag{8}$$

Let $\delta := \lambda(J)$. Without loss of generality we assume $\mathbf{k}^* = (k, k, \ldots, k)$ and $n_{\mathbf{k}} > \varepsilon/\delta$ for all $\mathbf{k} \geq (kc, \ldots, kc)$. Now consider the cuboid starting at $(kc, \ldots, kc)$ and ending at $(kc(2Q+1), \ldots, kc(2Q+1)) =: \mathbf{K}$. Then, by Lemma 6.10 with $U = 2/\varepsilon$, $T = 2Qc$ and $X = (2Qkc+1)^d$, there are at least

$$\left\lfloor \frac{(2Qkc+1)^d - 1}{2Qc} \right\rfloor + 1 \geq \frac{2^d Q^d k^d c^d}{2Qc} =: D$$

elements $n_{\mathbf{k}_1}, \ldots, n_{\mathbf{k}_D}$ with $n_{\mathbf{k}_{i+1}}/n_{\mathbf{k}_i} > 2/\varepsilon$ for $i = 1, \ldots, D-1$. Thus the corresponding functions are ε-mixing in δ by Lemma [7, Lemma 2.13]. So there is an open interval $K \subseteq J$ such that for all $\alpha \in K$

$$\#Z_{\mathbf{K}}(\alpha) = \#\{\mathbf{j} \leq \mathbf{K} :\ \mathbf{n}_i\boldsymbol{\alpha} \in I\} \geq D.$$

Thus for all $\boldsymbol{\alpha} \in K$

$$\frac{\#Z_{\mathbf{K}}(\alpha)}{|\mathbf{K}|} = \frac{\#Z_{\mathbf{K}}(\alpha)}{k^d c^d (2Q+1)^d} \geq \frac{D}{k^d c^d 4^d Q^d} = \frac{1}{2^{d+1}Qc}. \tag{9}$$

Since $\frac{1}{2^{d+1}Qc} > \frac{2P}{-\log\varepsilon}$ and $K \subseteq J$, (8) with $\mathbf{N} := \mathbf{K}$ implies

$$K \Vdash \left\{\alpha : \ \frac{\#Z_{\mathbf{K}}(\alpha)}{|\mathbf{K}|} \leq \frac{1}{2^{d+1}Qc}\right\}. \tag{10}$$

Now consider the set $\{\alpha : \ \frac{\#Z_{\mathbf{K}}(\alpha)}{|\mathbf{K}|} < \frac{1}{2^{d+1}Qc}\} \cap K$. By (9), this set is empty, but by (10) it is residual in K, which is a contradiction. $\square$

To obtain the extension of [7, Theorem 2.6], the theorem about the fast growing sequences, we call - in analogy to the classical case - a sequence with multi-dimensional indices maldistributed in $[0,1]$, if $M(\mathbf{x}) = \mathscr{P}$.

Theorem 6.11. *Let $\mathbf{n} = (n_{\mathbf{k}})_{\mathbf{k}\in\mathbb{N}^d}$ be a sequence of nonnegative integers and assume that there are $R, Q \in \mathbb{N}$, such that*

$$Q_r := \{\mathbf{k} : \ 2^r \leq n_{\mathbf{k}} < 2^{r+1}\} \leq Q \qquad \forall\ r = 0,1,2,\ldots,$$

and that $Q_r \leq 1$ for all $r \geq R$. Moreover, let $(r_j)_{j\in\mathbb{N}}$ be the sequence of those indices r_j with $Q_{r_j} > 0$. Define a sequence $(\tilde{r}_j)_{j\in\mathbb{N}}$ by $\tilde{r}_j = r_j - r_{j-1}$ $(j \geq 0)$ and $\tilde{r}_0 = 0$. Suppose $\tilde{r}_j \to \infty$. Then the set

$$\{\alpha \in \mathbb{R}/\mathbb{Z} : \ \mathbf{n}\alpha \textit{ is maldistributed}\}$$

is residual. Consequently, the set

$$\{\alpha \in \mathbb{R}/\mathbb{Z} : \ \mathbf{n}\alpha \textit{ is strongly maldistributed}\}$$

is residual.

P r o o f. We follow [7] and adapt the notation. With similar arguments it suffices to show that for each list $\vec{e}$ and each η the set

$$\{\alpha : \ \text{for all tails there is an index } \mathbf{N} \text{ such that } \mu_{\mathbf{n}\alpha,\mathbf{N}} \in M_{\vec{e},\eta}\} \tag{11}$$

is residual. Now assume that this fails. Therefore we can find a nonempty interval I, an index $\mathbf{N}_0$, a sequence $\vec{e} = (e_0, \ldots, e_{\ell-1})$ of natural numbers and an $\eta \in \mathbb{R}$ with

$$I \Vdash \{\alpha : \ \forall \mathbf{N} \geq \mathbf{N}_0 : \mu_{\mathbf{n}\alpha,\mathbf{N}} \notin M_{\vec{e},\eta}\}.$$

W.l.o.g. we assume $\mathbf{N}_0 = (n_0, \ldots, n_0) > (\frac{d}{\eta}, \ldots, \frac{d}{\eta})$, that $e := \sum e_i$ divides $|\mathbf{N}_0|$, $n_{\mathbf{N}_0} > \frac{1}{\lambda(I)}$ and that $\frac{n_{\mathbf{k}'}}{n_{\mathbf{k}}} > 2\ell$ if $n_{\mathbf{k}'} > n_{\mathbf{k}}$.

Choose a sequence of intervals $(I_{\mathbf{j}} : 1 \leq \mathbf{j} \leq \mathbf{N}_0^2)$ where $\mathbf{N}_0^2 = (n_0^2, \ldots, n_0^2)$ such that for all $0 \leq i \leq \ell - 1$ we have

$$\left|\left\{\mathbf{j} : 1 \leq \mathbf{j} \leq \mathbf{N}_0^2, I_j = \left[\frac{i}{\ell}, \frac{i+1}{\ell}\right)\right\}\right| = \frac{e_i}{e}|\mathbf{N}_0|^2.$$

So each interval I_j has length $\frac{1}{\ell}$. Let $f_{\mathbf{j}}(x) = n_{\mathbf{j}}x$ for $\mathbf{N}_0 \leq \mathbf{j} \leq \mathbf{N}_0^2$. Let (f_j) be a sorting of these functions such that $n_{j+1}/n_j > 2\ell$. Then, by [7, Lemma 2.13], the f_j are $\frac{1}{\ell}$-mixing in $\lambda(I)$, i.e., we can find an interval

$$J \subseteq I \cap \bigcap_j f_j^{-1}(I_j).$$

We will show that $\mu_{\mathbf{n}\alpha,\mathbf{N}_0^2} \in M_{\vec{e},\eta}$ for all $\alpha \in J$, which is a contradiction to (11). Indeed, if $\alpha \in J$, then for all j we have $f_j(\alpha) \in I_j$. Consequently (writing $O(1)$ for a quantity between -1 and 1) we obtain

$$\mu_{\mathbf{n}\alpha,\mathbf{N}_0^2}\left(\left[\frac{i}{\ell}, \frac{i+1}{\ell}\right)\right) = \frac{1}{|\mathbf{N}_0|^2}\left(\frac{e_i}{e}|\mathbf{N}_0^2| + O(1)\cdot d \cdot n_0^{2(d-1)+1}\right) = \frac{e_i}{e} + \frac{dO(1)}{n_0},$$

so $\mu_{\mathbf{n}\alpha,\mathbf{N}_0^2} \in M_{\vec{e},\eta}$, since $\frac{d}{n_0} < \eta$. $\square$

Replacing in [2] N by $\mathbf{N}$ and the one-dimensional limits by the multidimensional limits, we get immediately

Theorem 6.12. *Given any sequence*

$$(\varepsilon_{i_1,\dots,i_j,\dots,i_d}^{i_1,\dots,i_j+1,\dots,i_d})_{i_1,\dots,i_d\in\mathbb{N},j=1,\dots,d} \quad \textit{with} \quad \varepsilon_{i_1,\dots,i_j,\dots,i_d}^{i_1,\dots,i_j+1,\dots,i_d} \to 0$$

in the classical (strong) sense, there is a sequence $n_{\mathbf{k}}$ of positive integers with

$$\frac{n_{k_1,\dots,k_j+1,\dots,k_d}}{n_{k_1,\dots,k_j,\dots,k_d}} > 1 + \varepsilon_{k_1,\dots,k_j,\dots,k_d}^{k_1,\dots,k_j+1,\dots,k_d}$$

such that for any irrational α the sequence $\mathbf{n}\alpha$ is (strongly) uniformly distributed mod 1.

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