ON A FAMILY OF q-BINOMIAL DISTRIBUTIONS

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Abstract. We introduce a family of q-analogues of the binomial distribution, which generalises the Stieltjes-Wigert-, Rogers-Szegö-, and Kemp-distribution. Basic properties of this familiy are provided and several convergence results involving the classical binomial, Poisson, discrete normal distribution, and a family of q-analogues of the Poisson distribution are established. These results generalise convergence properties of Kemp's-distribution, and some of them are q-analogues of classical convergence properties.

1. Introduction

In [7] Kemp studied many q-analogues of the classical binomial distribution, in particular she investigated Kemp's distribuion, the Rogers-Szegö and the Stieljes-Wigert distribution, which all are of the form

$$\mathbb{P}(X = x) = C_\alpha \cdot \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta^x \qquad x = 0 \dots n, 0 < \theta,$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q,q)_n}{(q,q)_k (q,q)_{n-k}} \qquad \text{and} \qquad (z,q)_n = \prod_{i=0}^{n-1} (1 - zq^i)$$

are the q-binomial coefficient and the q-shifted factorial, and where C_α is a normalising constant. In this paper we are interested in the convergence properties of this family of q-binomial distributions. We will see that the behaviour in the case $\alpha = 0$ is very different from the case $\alpha > 0$. For Kemp's distribution (i.e. $\alpha = \frac{1}{2}$) the limit distributions are the Heine distribution and the discrete normal distribution. This was done by Gerhold and Zeiner [5]. We will show that these results can be generalized to the case $\alpha > 0$.

This paper is organised as follows: In Section 2 we give the definitions of the q-binomial distributions mentioned above and sum up their basic convergence properties. Afterwards we introduce the family $\mathcal{B}$ of q-binomial distributions we are interested in and a family of q-Poisson distributions. Afterwards we study basic properties of the family $\mathcal{B}$ in Section 3. In Sections 4-5 we investigate sequences of random variables X_n with $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$. In particular we show that there are analogues to the convergence of the classical binomial distribution to the Poisson distribution and the normal distribution, and that the limits $q \to 1$ and $n \to \infty$ can be exchanged. Section 4 deals with convergent parameter sequences, in particular with the case of constant parameter and constant mean, and contains a detailed analysis of the behaviour of the RS-distribution in the limit $\theta_n \to 1$. In Section 5 we examine the case of an increasing parameter sequence θ_n. We show that, if

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$\alpha > 0$ and θ_n grows not too fast, the normalised X_n converge to a discrete normal distribution.

2. Preliminaries

Throughout this paper we use the notation of [4]. Kemp's distribution $KB(n, \theta, q)$ was introduced in [7] and is defined as

$$\mathbb{P}(X_{KB} = x) = \begin{bmatrix} n \\ x \end{bmatrix}_q \frac{\theta^x q^{x(x-1)/2}}{(-\theta, q)_n}, \qquad 0 \le x \le n,\ 0 < \theta.$$

For properties and applications of this distribution see [5, 6, 9, 11]. In the limit $q \to 1$ Kemp's distribution converges to a binomial distribution:

$$KB(n, \theta, q) \to B\left(n, \frac{\theta}{1+\theta}\right).$$

If n goes to ∞, Kemp's distribution tends to the Heine distribution $H(\theta)$, which probabilities are given by

$$\mathbb{P}(X_H = x) = \frac{q^{x(x-1)/2}\theta^x}{(q,q)_x} e_q(-\theta), \qquad x \ge 0,$$

where

$$e_q(z) = \frac{1}{(z,q)_\infty}, \qquad z \in \mathbb{C} \setminus \{q^{-i} :\ i = 0, 1, 2, \dots\},$$

is a q-analogue of the exponential function, since $e_q((1-q)z) \to e^z$. The Heine distribution converges to the Poisson-distribution in the sense that $H((1-q)\theta) \to P(\theta)$ for $q \to 1$ and can therefore be seen as a q-analogue of the Poisson distribution.

Kemp [9] also introduced two other q-analogues of the binomial distribution, namely the Rogers-Szegö- (RS) and the Stieltjes-Wigert-distribution (SW), which probabilities are very similar to those of Kemp's-distribution:

$$\mathbb{P}(X_{RS} = x) = C_{RS} \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x, \qquad 0 \le x \le n,\ 0 < \theta,$$

$$\mathbb{P}(X_{SW} = x) = C_{SW} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{x(x-1)}\theta^x, \qquad 0 \le x \le n,\ 0 < \theta,$$

where C_{RS} and C_{SW} are normalising constants. For $q \to 1$ these distributions tend to a binomial distribuion with parameter $\frac{\theta}{1+\theta}$. In the limit $n \to \infty$ the RS-distribution converges for $\theta < 1$ to an Euler distribution with parameter θ, which is given by

$$\mathbb{P}(X_E = x) = \frac{\theta^x}{(q,q)_x} E_q(-\theta), \qquad x \ge 0,$$

where

$$E_q(z) = (-z, q)_\infty, \qquad z \in \mathbb{C},$$

is an other q-analogue of the exponential function, since $E_q((1-q)z) \to e^z$. Moreover, we have $e_q(z)E_q(-z) = 1$. The Euler distribution is a q-analogue of the Poisson distribution since $E((1-q)\theta) \to P(\theta)$.

Because of the similarities of these distributions we introduce a family $\mathcal{B}$ of q-analogues of binomial distributions which covers the distributions mentioned above as special cases: We say a random variable X is $\mathcal{B}(\alpha, \theta, n, q)$-distributed iff

$$\mathbb{P}(X = x) = \frac{\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2}\theta^x}{\sum_{y=0}^n \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2}\theta^y}, \qquad x = 0, \dots, n,\ 0 < \theta,\ 0 \le \alpha.$$

For $\alpha = 0$ this is the RS-distribution, $\alpha = \frac{1}{2}$ gives a $KB(n, \theta q^{1/2}, q)$-distribution and $\alpha = 1$ a $SW(n, \theta q, q)$-distribution.

Moreover, we define a family $\mathcal{P}$ of q-analogues of the Poisson distribution by

$$\mathbb{P}(X=x)=\frac{q^{\alpha x^2}\theta^x}{(q,q)_x}\frac{1}{E_q^{2\alpha}(\theta)},\qquad 0\le x,$$

where $0<\theta<1$ if $\alpha=0$, and $0<\theta$ if $\alpha>0$, and E_q^α is a q-analogue of the exponential function (which was introduced by [3] and studied by [1] and also appears in [2]) defined by

$$E_q^\alpha(z)=\sum_{x\ge 0}\frac{q^{\frac{\alpha}{2}x^2}}{(q,q)_x}z^x, \tag{1}$$

since $E_q^\alpha((1-q)z)\to e^z$. We then write $X\sim\mathcal{P}(\alpha,\theta,q)$. For $\alpha=0$ we obtain the Euler distribuion, and $\alpha=\frac{1}{2}$ gives a $H(\theta q^{1/2})$-distribution. The sum in (1) has a different behaviour for $\alpha=0$ and $\alpha>0$: In the case $\alpha=0$ it is convergent only for $0\le|z|<1$, but for $\alpha>0$ it converges for all $z\in\mathbb{C}$. This is why we restricted the parameter θ in the definition of our q-Poisson familiy. Consequently there is a big difference in the behaviour of the RS-distribution and the other members of this q-binomial-family. So we will often distinguish between $\alpha=0$ and $\alpha>0$ in the convergence results.

3. Properties of the Family $\mathcal{B}$

As noted above we study basic properties of our family $\mathcal{B}$. We show that it is in fact a q-analogue of the binomial distribution and logconcave. These properties hold for the family $\mathcal{P}$ too. Then we give a characterisation of a $\mathcal{B}(\alpha,\theta,n,q)$-distribution and a random walk model for $\mathcal{B}$ and then we turn to the study of the behaviour of the mean of a $\mathcal{B}(\alpha,\theta,n,q)$-distribution in dependence on n, θ and α. In the present section we always allow $\alpha\ge 0$.

The following two theorems show that our families $\mathcal{B}$ and $\mathcal{P}$ tend to the classical binomial and Poisson distribution. This generalises the results for the Kemp-, SW-, RS-, Heine, and Euler distributions.

Theorem 3.1. *For $q\to 1$ we have*

$$\mathcal{B}(\alpha,\theta,n,q)\to B\left(n,\frac{\theta}{1+\theta}\right).$$

Proof. By definition,

$$\begin{aligned}\mathbb{P}(X=x)&=\frac{{n\brack x}_q q^{\alpha x^2}\theta^x}{\sum_{y=0}^n {n\brack y}_q q^{\alpha y^2}\theta^y}\\ &\to\frac{\binom{n}{x}\theta^x}{\sum_{y=0}^n\binom{n}{y}\theta^y}=\frac{\binom{n}{x}\theta^x}{(1+\theta)^n}\\ &=\binom{n}{x}\left(\frac{\theta}{1+\theta}\right)^x\left(\frac{1}{1+\theta}\right)^{n-x}.\end{aligned}$$

□

Theorem 3.2. *In the limit $q\to 1$ we have $\mathcal{P}(\alpha,(1-q)\theta,q)\to P(\theta)$.*

Proof. By definition,

$$\mathbb{P}(X=x)=\frac{q^{\alpha x^2}(1-q)^x\theta^x}{(q,q)_x}\frac{1}{E_q^{2\alpha}((1-q)\theta)}\to\frac{\theta^x}{x!}\exp(-\theta).$$

□

Kemp showed in [9] that the RS-, SW-, and Kemp-distribution are logconcave, i.e.,

$$\Delta(x):=\frac{\mathbb{P}(X=x+1)}{\mathbb{P}(X=x)}-\frac{\mathbb{P}(X=x+2)}{\mathbb{P}(X=x+1)}>0$$

for $x = 0, \dots, n-2$. We can generalise this as follows:

Theorem 3.3. *$\mathcal{B}(\alpha, \theta, n, q)$ is logconcave.*

Proof. We have

$$\begin{aligned}
\Delta(x) &= \frac{q^{\alpha(x+1)^2}\theta^{x+1}(q,q)_x(q,q)_{n-x}}{(q,q)_{x+1}(q,q)_{n-x-1}q^{\alpha x^2}\theta^x} - \frac{q^{\alpha(x+2)^2}\theta^{x+2}(q,q)_{x+1}(q,q)_{n-x-1}}{(q,q)_{x+2}(q,q)_{n-x-2}q^{\alpha(x+1)^2}\theta^{x+1}} \\
&= \theta\left(\frac{q^{2\alpha x+\alpha}\left(1-q^{n-x}\right)}{1-q^{x+1}} - \frac{\left(1-q^{n-x-1}\right)q^{2\alpha x+3\alpha}}{1-q^{x+2}}\right) \\
&= \theta q^{2\alpha x+\alpha}\left(\frac{1-q^{x+2}-q^{n-x}+q^{n+2}-\left(1-q^{n-x-1}-q^{x+1}+q^n\right)q^{2\alpha}}{\left(1-q^{x+1}\right)\left(1-q^{x+2}\right)}\right).
\end{aligned}$$

For $\alpha = 0$ we have $\Delta(x) > 0$ by [9], and the numerator is increasing in α since

$$1-q^{n-x-1}-q^{x+1}+q^n = q^{n-x-1}\left(q^{x+1}-1\right)-\left(q^{x+1}-1\right) > 0$$

for $x < n-1$. □

In the same way we obtain the same property of the family $\mathcal{P}$.

Theorem 3.4. *$\mathcal{P}(\alpha, \theta, q)$ is logconcave.*

For the Heine- and the Euler-distribution this property was proven by Kemp [8].

In [10] Kemp characterised some q-analogues of the binomial distribution as the conditional distribution of $U|(U+V=m)$ where U and V are independent random variables. We can characterise our family $\mathcal{B}$ in an analogous way and generalise some of Kemp's results.

Theorem 3.5. *A $\mathcal{B}(\alpha, \theta/\lambda, m, q)$-distribution is the distribution of $U|(U+V=m)$, where U and V are independent, iff U has a $\mathcal{P}(\alpha, \beta, \theta)$-distribution and V has an Euler-distribution with parameter λ.*

Proof. The proof runs along the same lines as the proofs in [10]: If U and V have the postulated distributions, then

$$\begin{aligned}
\mathbb{P}(U = u|U+V=n) &= C\frac{\theta^u q^{\alpha u^2}}{(q,q)_u}\frac{\lambda^{m-u}}{(q,q)_{m-u}} \\
&= C\frac{\lambda^m}{(q,q)_u(q,q)_{m-u}}\left(\frac{\theta}{\lambda}\right)^u q^{\alpha u^2}.
\end{aligned}$$

To prove the other implication, we need the following theorem ([12]):

Let X and Y be independent discrete random variables and

$$c(x, x+y) = \mathbb{P}(X = x|X+Y = x+y).$$

If

$$\frac{c(x+y,x+y)c(0,y)}{c(x,x+y)c(y,y)} = \frac{h(x+y)}{h(x)h(y)},$$

where h is a nonnegative function, then

$$f(x) = f(0)h(x)e^{ax}, \qquad g(y) = g(0)k(y)e^{ay},$$

where a is an arbitrary parameter and

$$0 < f(x) = \mathbb{P}(X = x), \quad 0 < g(y) = \mathbb{P}(Y = y), \quad k(y) = \frac{h(y)c(0,y)}{c(y,y)}.$$

Here we have

$$\frac{c(u+v,u+v)c(0,v)}{c(u,u+v)c(u,v)} = \frac{\left(\frac{\theta}{\lambda}\right)^{u+v}q^{\alpha(u+v)^2}}{\frac{(q,q)_{u+v}}{(q,q)_v(q,q)_u}\left(\frac{\theta}{\lambda}\right)^u q^{\alpha u^2}\left(\frac{\theta}{\lambda}\right)^v q^{\alpha v^2}} = \frac{h(u+v)}{h(u)h(v)},$$

where

$$h(u) = \frac{q^{\alpha u^2}}{(q,q)_u}.$$

Thus $k(v) = (\theta/\lambda)^v/(q,q)_v$ and

$$\mathbb{P}(U = u) = C_1 \frac{q^{\alpha u^2} e^{au}}{(q,q)_u},$$
$$\mathbb{P}(V = v) = C_2 \left(\frac{\theta e^a}{\lambda}\right)^v \frac{1}{(q,q)_v},$$

yielding a $\mathcal{P}(\alpha, e^a, q)$-distribution and an Euler distribution. □

We now give a random-walk-model for the family $\mathcal{B}$ (the models for the Kemp-, RS-, and SW-distribution given in [9] are special cases of this model). Let a_x and b_x denote the probabilities to move up and down and choose

$$a_x = c\gamma q^{2\alpha x}\left(1 - q^{n-x}\right) \qquad \text{and} \qquad b_x = c(1 - q^x)$$

for $x = 0, \ldots, n$. Then $\mathcal{B}(\alpha, \gamma q^{-\alpha}, n, q)$ is a stationary distribution. To see this, note that for a stationary distribution we must have

$$\mathbb{P}(X = x) = \mathbb{P}(X = x)(1 - a_x - b_x) + \mathbb{P}(X = x+1)b_{x+1} + \mathbb{P}(X = x-1)a_{x-1}.$$

So we have to show that

$$\Delta(x) := -\mathbb{P}(X = x)(a_x + b_x) + \mathbb{P}(X = x+1)b_{x+1} + \mathbb{P}(X = x-1)a_{x-1} = 0$$

if $X \sim \mathcal{B}(\alpha, \gamma q^{-\alpha}, n, q)$. For $1 \le x \le n-1$ we have

$$\begin{aligned}\Delta(x) = C \Bigg(& -\begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \gamma^x q^{-\alpha x} \left(c(1 - q^x) + c\gamma q^{2\alpha x}(1 - q^{n-x})\right) \\ & + \begin{bmatrix} n \\ x-1 \end{bmatrix}_q q^{\alpha(x-1)^2} \gamma^{x-1} q^{-\alpha(x-1)} c\gamma q^{2\alpha(x-1)} (1 - q^{n-x+1}) \\ & + \begin{bmatrix} n \\ x+1 \end{bmatrix}_q q^{\alpha(x+1)^2} \gamma^{x+1} q^{-\alpha(x+1)} c(1 - q^{x+1}) \Bigg).\end{aligned}$$

Using the relation

$$\begin{bmatrix} n \\ x-1 \end{bmatrix}_q \left(1 - q^{n-x+1}\right) = \begin{bmatrix} n \\ x \end{bmatrix}_q (1 - q^x)$$

we obtain that the terms with γ^x and γ^{x+1} vanish. Similarly $\Delta(0)$ and $\Delta(n)$ can be treated.

Now we study the means; for this purpose let us denote by $\mu_n(\alpha, \theta, q)$ the mean of a random variable $X \sim \mathcal{B}(\alpha, \theta, n, q)$. The following lemmas are devoted to the behaviour of $\mu_n(\alpha, \theta, q)$ in dependence on n, α and θ. The first result shows that the means are increasing in n.

Lemma 3.6. *For all $\alpha \ge 0$ $\mu_n(\alpha, \theta, q)$ is increasing in n.*

Proof. For $0 \le x < y \le n$ we have

$$q^{-x} < q^{-y}.$$

By elementary calculations, this can be written as

$$\frac{1}{1 - q^{n+1-x}} < \frac{1}{1 - q^{n+1-y}}.$$

This is equivalent to

$$\begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q (y-x) < \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q (y-x)$$

and

$$\begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q y + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q x < \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q x + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q y.$$

Multiplication with $\theta^{x+y} q^{\alpha(x^2+y^2)}$ yields

$$\begin{aligned} &\begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q y\theta^{x+y} q^{\alpha(x^2+y^2)} + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q x\theta^{x+y} q^{\alpha(x^2+y^2)} \\ &\qquad < \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q x\theta^{x+y} q^{\alpha(x^2+y^2)} + \begin{bmatrix} n+1 \\ y \end{bmatrix}_q \begin{bmatrix} n \\ x \end{bmatrix}_q y\theta^{x+y} q^{\alpha(x^2+y^2)}. \end{aligned}$$

Now we sum over all pairs (x, y) with $x < y$:

$$\sum_{\substack{x,y=0 \\ x\neq y}}^{n} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \begin{bmatrix} n \\ y \end{bmatrix}_q y\theta^y q^{\alpha y^2} < \sum_{\substack{x,y=0 \\ x\neq y}}^{n} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} x \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}.$$

By adding the terms for $x = y$ and an extra-sum we get

$$\begin{aligned} &\theta^{n+1} q^{\alpha(n+1)^2} \sum_{y=0}^{n} y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} + \sum_{x=0}^{n}\sum_{y=0}^{n} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \begin{bmatrix} n \\ y \end{bmatrix}_q y\theta^y q^{\alpha y^2} \\ &\qquad < (n+1)\theta^{n+1} q^{\alpha(n+1)^2} \sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} + \sum_{x=0}^{n}\sum_{y=0}^{n} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} x \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}. \end{aligned}$$

This can be written as

$$\sum_{x=0}^{n+1} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^{n} y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} < \sum_{x=0}^{n+1} x \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2},$$

and so we have

$$\frac{\sum_{y=0}^{n} y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}}{\sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2}} < \frac{\sum_{x=0}^{n+1} x \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}{\sum_{x=0}^{n+1} \begin{bmatrix} n+1 \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}. \qquad \square$$

The means are increasing in the parameter θ too:

Lemma 3.7. *The means $\mu_n(\alpha, \theta, q)$ are increasing in θ for all $\alpha \geq 0$.*

Proof. We show that $\frac{\partial}{\partial\theta}\mu_n(\alpha, \theta, q) > 0$. Differentiating gives

$$\begin{aligned} &\frac{\partial}{\partial\theta}\left(\frac{\sum_{x=0}^{n} x \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}{\sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}}\right) = \\ &= \frac{\sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^{n} y^2 \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^{y-1} q^{\alpha y^2} - \sum_{x=0}^{n} x \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^{n} y \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^{y-1} q^{\alpha y^2}}{\left(\sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2}\right)^2}. \end{aligned}$$

Thus it suffices to show that

$$\left(\sum_{x=1}^{n} x \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta^{x-1}\right)^2 < \sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^{x-1} q^{\alpha x^2} \sum_{y=0}^{n} y^2 \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^{y-1} q^{\alpha y^2}.$$

The left-hand side can be written as

$$\sum_{x=1}^{n} x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q^2 q^{2\alpha x^2} \theta^{2(x-1)} + \sum_{\substack{x,y=0\\x\neq y}}^{n} xy \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y-2} =: A_1 + B_1$$

and the right-hand side as

$$\sum_{x=1}^{n} x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q^2 q^{2\alpha x^2} \theta^{2(x-1)} + \sum_{\substack{x,y=0\\x\neq y}}^{n} x^2 \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y-2} =: A_2 + B_2.$$

Since $A_1 = A_2$, it suffices to show that $B_1 < B_2$. For this purpose we consider the pairs (x, y) and (y, x) with $x < y$: In B_1 we have the term

$$2xy \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y-2} \tag{2}$$

and in B_2

$$\begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y-2} (x^2 + y^2). \tag{3}$$

Since $2xy < x^2 + y^2$ for $x \neq y$, we have (2) < (3) and so $B_1 < B_2$. □

For α the situation is a little bit different:

Lemma 3.8. *$\mu_n(\alpha, \theta, q)$ is decreasing in α if $\alpha \in (0, 1]$ and increasing in α if $\alpha \geq 1$.*

Proof. Assume $\alpha > 1$ (in the same way we can treat the case $0 < \alpha < 1$). We show that $\frac{\partial}{\partial \alpha} \mu_n(\alpha, \theta, q) > 0$. This is equivalent to

$$\sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^{n} y^3 \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} \log \alpha > \sum_{x=0}^{n} x \begin{bmatrix} n \\ x \end{bmatrix}_q \theta^x q^{\alpha x^2} \sum_{y=0}^{n} y^2 \begin{bmatrix} n \\ y \end{bmatrix}_q \theta^y q^{\alpha y^2} \log \alpha.$$

So it is sufficient to show that

$$\sum_{\substack{x,y=0\\x\neq y}}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y} y^3 \log \alpha > \sum_{\substack{x,y=0\\x\neq y}}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha(x^2+y^2)} \theta^{x+y} x y^2 \log \alpha.$$

Considering the pairs (x, y) and (y, x), it is sufficient that $x^3 + y^3 > xy^2 + yx^2$. This is true because this can be written as $(y^2 - x^2)(y - x) = (y + x)(y - x)^2 > 0$. □

Finally, a straightforward calculation shows that our family $\mathcal{B}$ is closed under reversing, i.e., $n - X$ has the same form as X.

Theorem 3.9. *If $X \sim \mathcal{B}(\alpha, \theta, n, q)$ then $n - X \sim \mathcal{B}(\alpha, \theta^{-1} q^{-2\alpha n}, n, q)$.*

4. Convergent Parameter

In this section we consider sequences $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$ where the parameter sequence θ_n tends to a finite limit as $n \to \infty$. This will lead to the family $\mathcal{P}$ as limit law. In particular we prove that the convergence of the classical binomial distribution with constant mean has a q-analogue. But in the case $\alpha = 0$ and $\theta_n \to 1$ these results fail. In this case we obtain - depending on the limit of θ_n - a uniform distribution or exponential-like distributions. In the following we need the two auxiliary results below.

Lemma 4.1. *For $\alpha > 0$ we have for all $z \in \mathbb{C}$*

$$\sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} z^x \to E_q^{2\alpha}(z), \qquad \text{as } n \to \infty.$$

For $\alpha = 0$ this holds for $|z| < 1$.

Proof. We estimate the difference

$$\left| \sum_{x=0}^{\infty} \frac{q^{\alpha x^2}}{(q,q)_x} z^x - \sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} z^x \right| \le \sum_{x=n+1}^{\infty} \frac{q^{\alpha x^2}}{(q,q)_x} |z|^x + \sum_{x=1}^{n} q^{\alpha x^2} |z|^x \left| \begin{bmatrix} n \\ x \end{bmatrix}_q - \frac{1}{(q,q)_x} \right|.$$

Estimating in the first sum the q-shifted factorial by the q-exponential function yields

$$\le e_q(q) \sum_{x=n+1}^{\infty} (q^{\alpha n}|z|)^x + \sum_{x=1}^{n} \frac{q^{\alpha x^2}}{(q,q)_x} |z|^x \left(1 - \prod_{i=1}^{x} (1 - q^{n-i+1}) \right);$$

the same estimate we use for the second sum, split it and compute the first sum to obtain

$$\le e_q(q) \left(\frac{(q^{\alpha n}|z|)^{n+1}}{1 - q^{\alpha n}|z|} + \sum_{x=1}^{\lfloor \frac{n}{2} \rfloor} q^{\alpha x^2} |z|^x \left(1 - \prod_{i=1}^{x} (1 - q^{n-i+1}) \right) \right.$$
$$\left. + \sum_{x=\lfloor \frac{n}{2} \rfloor}^{n} q^{\alpha x^2} |z|^x \left(1 - \prod_{i=1}^{x} (1 - q^{n-i+1}) \right) \right).$$

The first term is obviously $\mathrm{o}(1)$. Estimating the products gives

$$\le e_q(q) \left(\mathrm{o}(1) + \left(1 - \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (1 - q^{n-i}) \right) \sum_{x=1}^{\infty} q^{\alpha x^2} |z|^x + \sum_{x=\lfloor \frac{n}{2} \rfloor}^{\infty} q^{\alpha x^2} |z|^x \right)$$

and further

$$\le e_q(q) \left(\mathrm{o}(1) + \left(1 - \left(1 - q^{\lfloor \frac{n}{2} \rfloor} \right)^{\lfloor \frac{n}{2} \rfloor} \right) \sum_{x=1}^{\infty} q^{\alpha x^2} |z|^x + \sum_{x=\lfloor \frac{n}{2} \rfloor}^{\infty} \left(q^{\alpha \lfloor \frac{n}{2} \rfloor} |z| \right)^x \right);$$

the latter sum is $\mathrm{o}(1)$ as before, thus

$$= \mathrm{o}(1) + \mathrm{O}(n^2 q^n) \sum_{x=1}^{\infty} q^{\alpha x^2} |z|^x = \mathrm{o}(1).$$ □

Lemma 4.2. *Assume $\alpha > 0$ and let (θ_n) be a sequence of real numbers with limit $\theta \ge 0$. Then*

$$\lim_{n \to \infty} \sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta_n^x = E_q^{2\alpha}(\theta).$$

If $\theta < 1$, this holds for $\alpha = 0$ as well.

Proof. For small $\epsilon > 0$ and n large enough we have

$$\sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} (\theta - \varepsilon)^x \le \sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} \theta_n^x \le \sum_{x=0}^{n} \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2} (\theta + \varepsilon)^x,$$

hence, with use of Lemma 4.1,

$$\begin{aligned} E_q^{2\alpha}(\theta-\varepsilon) &= \lim_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}(\theta-\varepsilon)^x \le \liminf_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}\theta_n^x \\ &\le \limsup_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}\theta_n^x \le \lim_{n\to\infty}\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}(\theta+\varepsilon)^x \\ &= E_q^{2\alpha}(\theta+\varepsilon). \end{aligned}$$

By continuity of $E_q^{2\alpha}$, the lemma follows. □

The first result is a generalisation of the fact that Kemp's distribution converges to the Heine distribution (see [11]).

Proposition 4.3. *If $X_n \sim \mathcal{B}(\alpha,\theta_n,n,q)$, $\alpha>0$, then for $n\to\infty$*

$$X_n \to \mathcal{P}(\alpha,\theta,q),$$

if $\theta_n\to\theta$. This still remains true in the case $\alpha=0$ and $\theta<1$.

Proof. This follows immediately from the fact that

$$\begin{bmatrix}n\\x\end{bmatrix}_q \to \frac{1}{(q,q)_x}$$

for $n\to\infty$ and from Lemma 4.2. □

In the case $\alpha=0$ and $\theta>1$ the situation is slightly different:

Proposition 4.4. *If $X_n\sim\mathcal{B}(0,\theta_n,n,q)$, then for $n\to\infty$, if $\theta_n\to\theta>1$,*

$$n-X_n\to\mathcal{P}\left(0,\tfrac{1}{\theta},q\right),$$

which is an Euler distribution.

Proof. Define $Y_n=n-X_n$. Then

$$\mathbb{P}(Y_n=x)=\frac{\left[\begin{smallmatrix}n\\x\end{smallmatrix}\right]_q\theta_n^{n-x}}{\sum_{y=0}^{n}\left[\begin{smallmatrix}n\\y\end{smallmatrix}\right]_q\theta_n^{n-y}}=\frac{\left[\begin{smallmatrix}n\\x\end{smallmatrix}\right]_q\theta_n^{-x}}{\sum_{y=0}^{n}\left[\begin{smallmatrix}n\\y\end{smallmatrix}\right]_q\theta_n^{-y}}\to\frac{\theta^{-x}}{(q,q)_x}\frac{1}{\sum_{y=0}^{n}\frac{1}{(q,q)_y}\theta^{-y}}$$

by Lemma 4.2. □

In particular we are interested in sequences X_n such that the limits $q\to1$ and $n\to\infty$ can be exchanged. The propositions above immediately yield

Corollary 4.5. *For each $\alpha>0$ let $X_n\sim\mathcal{B}(\alpha,\theta_n(q),n,q)$ with $\theta_n(q)\to\theta(q)$. Additionally assume that $\theta_n(q)\to\lambda/n$ and $\theta(q)/(1-q)\to\lambda$ as $q\to1$. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{B}(\alpha,\theta_n(q),n,q) & \xrightarrow{n\to\infty} & \mathcal{P}(\alpha,(1-q)\theta(q),q) \\ {\scriptstyle q\to1}\big\downarrow & & \big\downarrow{\scriptstyle q\to1} \\ B\left(n,\frac{\lambda}{n}\right) & \xrightarrow[n\to\infty]{} & P(\lambda) \end{array}$$

One very natural way to choose the parameter sequence is to set $\theta_n(q)=\frac{\lambda}{[n-\lambda]_q}$, $\lambda>0$.

The convergence $\mathcal{B}(\alpha,\theta_n(q),n,q)\to\mathcal{P}(\alpha,(1-q)\theta(q),q)$ still remains true for $\alpha=0$ if we require $(1-q)\theta(q)<1$. Moreover, the commutative diagram remains correct for given $\lambda>0$, if we restrict q to values greater than or equal to $\max(0,1-\frac{1}{\lambda})$.

The next result is a q-analogue of the convergence of the classical binomial distribution with constant mean to the Poisson distribution.

Theorem 4.6. *Fix $\mu > 0$ and $\alpha > 0$. Consider a sequence of random variables $X_n \sim \mathcal{B}(\alpha, \theta_n, n, q)$ with parameter sequence $\theta_n = \theta_n(q, \mu)$ chosen such that the means μ_n of X_n are equal to μ. Then we have*

(i) *The sequence X_n converges to the limit law $\mathcal{P}(\alpha, \theta, q)$, where θ is the limit of the sequence θ_n.*
(ii) *As $q \to 1$, X_n tends to a binomial distribution with parameters n and μ/n.*
(iii) *In the limit $q \to 1$, $\mathcal{P}(\alpha, \theta(q, \mu), q)$ converges to a Poisson distribution with parameter μ.*

Thus the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{B}(\alpha, \theta_n(q, \mu), n, q) & \xrightarrow{n\to\infty} & \mathcal{P}(\alpha, \theta(q, \mu), q) \\ {\scriptstyle q\to 1}\big\downarrow & & \big\downarrow{\scriptstyle q\to 1} \\ B\left(n, \frac{\mu}{n}\right) & \xrightarrow[n\to\infty]{} & P(\mu) \end{array}$$

Proof. First we check, that for given μ, q and large n there is a unique $\theta_n(q)$, such that $\mu_n(\theta_n(q), q) = \mu$. The function $\mu_n(\theta, q)$ is continuous and increasing in θ (see Lemma 3.7). Moreover $\lim_{\theta\to 0} \mu_n(\theta, q) = 0$. From Corollary 5.1 we see that for sufficiently large n and suitable θ_n, $\mu_n(\theta_n, q) \geq \frac{n}{2}$. Consequently there exists a unique solution $\theta_n(q)$ of $\mu_n(\theta, q) = \mu$. By [14, Lemma 3.3], $\theta_n(q)$ converges to a limit $\theta(q)$, where $\theta(q)$ is the unique solution of $\mu_\infty(\theta, q) = \mu$. Hence $\mathcal{B}(\alpha, \theta_n(q), n, q) \to \mathcal{P}(\alpha, \theta(q), q)$ by Lemma 4.2.

Again by [14, Lemma 3.3] we get that $\theta_n(q) \to \frac{\mu}{n-\mu}$ for $q \to 1$ and so $\frac{\theta_n(q)}{1+\theta_n(q)} \to \frac{\mu}{n}$. Consequently $\mathcal{B}(\alpha, \theta_n(q), n, q) \to B\left(n, \frac{\mu}{n}\right)$.

It remains to check that $\theta(q)/(1-q)$ converges to μ for $q \to 1$ (then $\mathcal{P}(\alpha, \theta(q), q) \to P(\mu)$). The value $\theta(q)/(1-q)$ is the unique solution of $\mu_\infty((1-q)\theta, q) = \mu$. Moreover, $\mu_\infty((1-q)\theta, q)$ converges pointwise to θ for $q \to 1$, so we can apply [14, Lemma 3.3]. □

In the case $\alpha = 0$ an analogous result holds for X_n or $n - X_n$ depending on the values of the parameters, i.e., if $\theta(q, \mu) < 1$ then the theorem holds for the sequence X_n, and if $\theta(q, \mu) > 1$ then this is true for $n - X_n$.

Now we turn our attention to the case $\alpha = 0$. To finish the analysis of the RS-distribution we consider $\theta_n \to 1$. It is worthwhile to point out that the limit distributions only depend on the growth rate of the parameter sequences and are independent of q. This is why we will distinguish three cases in dependence on the speed of the convergence of the parameters θ_n to the limit 1. First we will provide a result of fast growing θ_n. In order to do so we start with an auxiliary result.

Lemma 4.7. *If $f(n) \leq n$, $\theta_n \leq 1$ and $f(n) \to \infty$ and $\theta_n^{f(n)} \to 1$ for $n \to \infty$, then for $k \in \mathbb{N}$*

$$\sum_{0\leq i\leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i \sim e_q(q) \frac{f(n)^{k+1}}{k+1}, \qquad n \to \infty.$$

Proof. Write

$$\sum_{0\leq i\leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i = \sum_{i=0}^{\lfloor\sqrt{f(n)}\rfloor} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i + \sum_{i=\lfloor\sqrt{f(n)}\rfloor+1}^{f(n)-\lfloor\sqrt{f(n)}\rfloor-1} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i$$
$$+ \sum_{n-\lfloor\sqrt{n}\rfloor\leq i\leq f(n)} \begin{bmatrix} n \\ i \end{bmatrix}_q i^k \theta_n^i.$$

The first and the third term on the right-hand side can be estimated by

$$(\sqrt{f(n)}+1)f(n)^k \frac{(q,q)_n}{(q,q)_{\lfloor n/2\rfloor}(q,q)_{n-\lfloor n/2\rfloor}}$$

and are therefore negligible. The middle term can be bounded by

$$\begin{aligned}\frac{(q,q)_n}{(q,q)_{\lfloor\sqrt{f(n)}\rfloor+1}(q,q)_{n-\lfloor\sqrt{f(n)}\rfloor-1}}\theta^{f(n)} \sum_{\lfloor\sqrt{f(n)}\rfloor+1}^{f(n)-\lfloor\sqrt{f(n)}\rfloor-1} i^k &\le \sum_{\lfloor\sqrt{f(n)}\rfloor+1}^{f(n)-\lfloor\sqrt{f(n)}\rfloor-1} \begin{bmatrix} n\\ i\end{bmatrix}_q i^k\\ &\le \frac{(q,q)_n}{(q,q)_{\lfloor n/2\rfloor}(q,q)_{n-\lfloor n/2\rfloor}} \sum_{\lfloor\sqrt{f(n)}\rfloor+1}^{f(n)-\lfloor\sqrt{f(n)}\rfloor-1} i^k\end{aligned}$$

and has the asserted asymptotic. □

This lemma implies that under the assumption $\theta_n^n \to 1$ the limit law is uniform on the interval $[-\sqrt{3},\sqrt{3}]$.

Theorem 4.8. *If $X_n \sim RS(n,\theta_n,q)$ with $\theta_n \le 1$ and $\theta_n^n \to 1$, then $(X_n-\mu_n)/\sigma_n$ converges for $n\to\infty$ to the uniform distribution on the interval $[-\sqrt{3},\sqrt{3}]$.*

Proof. We start with an asymptotic of the means and the variances. By Lemma 4.7 we have

$$\mu_n = \frac{\sum_{i=0}^n \left[{n \atop i}\right]_q i\theta_n^i}{\sum_{i=0}^n \left[{n \atop i}\right]_q \theta_n^i} \sim \frac{e_q(q)\frac{n^2}{2}}{e_q(q)n} = \frac{n}{2}$$

and

$$\sigma_n^2 = \frac{\sum_{i=0}^n \left[{n \atop i}\right]_q i^2\theta_n^i}{\sum_{i=0}^n \left[{n \atop i}\right]_q \theta_n^i} - \mu_n^2 \sim \frac{n^2}{3}-\frac{n^2}{4} = \frac{n^2}{12}.$$

From these two fact one can easily see that the support of the limiting distribution is

$$\lim_{n\to\infty}[-\mu_n/\sigma_n,(n-\mu_n)/\sigma_n] = [-\sqrt{3},\sqrt{3}].$$

Now we compute

$$\begin{aligned}\mathbb{P}(X\le x) &= \lim_{n\to\infty} \sum_{-\frac{\mu_n}{\sigma_n}\le\frac{k-\mu_n}{\sigma_n}\le x} \frac{\left[{n \atop k}\right]_q \theta_n^k}{\sum_{i=0}^n \left[{n \atop i}\right]_q \theta_n^i} = \lim_{n\to\infty}\frac{1}{\sum_{i=0}^n \left[{n \atop i}\right]_q \theta_n^i}\sum_{0\le k\le\sigma_n x+\mu_n}\begin{bmatrix} n\\ k\end{bmatrix}_q \theta_n^k\\ &= \lim_{n\to\infty}\frac{1}{e_q(q)n}e_q(q)(\sigma_n x+\mu_n) = \lim_{n\to\infty}\frac{1}{n}\left(\frac{n}{2\sqrt{3}}x+\frac{n}{2}\right) = \frac{1}{2\sqrt{3}}x+\frac{1}{2},\end{aligned}$$

which is the distribution function of the uniform distribution on $[-\sqrt{3},\sqrt{3}]$. □

Using the fact that a $RS(n,\theta,q)$-distribution corresponds to a $(n-RS(n,1/\theta,q))$-distribution or following the above proofs we immediately get the following corollary:

Corollary 4.9. *If $X_n \sim RS(n,\theta_n,q)$ with $\theta_n \ge 1$ and $\theta_n^n \to 1$, then $(\mu_n - X_n)/\sigma_n$ and $(X_n-\mu_n)/\sigma_n$ converge for $n\to\infty$ to the uniform distribution on the interval $[-\sqrt{3},\sqrt{3}]$.*

Now we turn to the case that $\theta_n^n \to c$ with $0<c<1$. For this purpose we start with the following lemma, which supplements [14, Lemmas 4.4 and 4.5] and is crucial for the analysis of the variances.

Lemma 4.10. *For $\theta_n \le 1$ and $\theta_n \to 1$, $\theta_n^n \to c$ with $0 < c < 1$ and $f(n)/n \sim \beta > 0$ we have*

$$\sum_{i=0}^{n} i^2 \theta_n^i \sim -2\frac{1-c+c\log c-\frac{1}{2}c\log^2 c}{\log^3 c}n^3$$

as $n \to \infty$.

$$\sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q i^2 \theta_n^i \sim -2e_q(q)\frac{1-c+c\log c-\frac{1}{2}c\log^2 c}{\log^3 c}n^3$$

as $n \to \infty$.

Proof. Using

$$\sum_{i=0}^{n} i^2 t^i = \frac{t(-1-t+t^n+2nt^n(1-t)+n^2t^n(1-t)^2+t^{n+1})}{(t-1)^3}$$

we obtain for the first sum

$$\sum_{i=0}^{n} i^2 \theta_n^i \sim (-2+2c-2c\log c+c\log^2 c)\frac{n^3}{\log^3 c}.$$

The second sum follows immediately as in [14, Lemma 4.4]. $\square$

Now we are able to establish the convergence result in this case.

Theorem 4.11. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \le 1$, $\theta_n \to 1$ and $\theta_n^n \to c$ with $0 < c < 1$, then $(X_n - \mu_n)/\sigma_n$ converges to a limit distribution X with*

$$\mathbb{P}(X \le x) = \frac{c^{\alpha(c,x)}-1}{c-1},$$

where

$$\alpha(c,x) = \frac{\sqrt{(c-1)^2-c\log^2 c}}{(c-1)\log c}x + \frac{1-c+c\log c}{(c-1)\log c}$$

and $x \in [-\gamma_1, \gamma_2]$ with

$$\gamma_1 = \frac{1-c+c\log c}{\sqrt{(c-1)^2-c\log^2 c}} \qquad \text{and} \qquad \gamma_2 = \frac{c-1-\log c}{\sqrt{(c-1)^2-c\log^2 c}}.$$

Proof. Using [14, Lemmas 4.4 and 4.5] and Lemma 4.10 we get for the means μ_n

$$\mu_n = \frac{\sum_{i=0}^{n} i\theta^i \begin{bmatrix} n \\ i \end{bmatrix}_q}{\sum_{i=0}^{n} \theta^i \begin{bmatrix} n \\ i \end{bmatrix}_q} \sim \frac{(1-c+c\log c)n^2}{\log^2 c}\frac{\log c}{(c-1)n} = \frac{1-c+c\log c}{(c-1)\log c}n$$

and for the variances σ_n^2

$$\begin{aligned}\sigma_n^2 &= \frac{\sum_{i=0}^{n} i^2\theta^i \begin{bmatrix} n \\ i \end{bmatrix}_q}{\sum_{i=0}^{n} \theta^i \begin{bmatrix} n \\ i \end{bmatrix}_q} - \mu_n^2 \\ &\sim \frac{-2(1-c+c\log c-\frac{1}{2}c\log^3 c)n^3}{\log^3 c}\frac{\log c}{(c-1)n} - \frac{(1-c+c\log c)^2}{(c-1)^2\log^2 c}n^2 \\ &= \frac{c^2+1-2c-c\log^2 c}{(c-1)^2\log^2 c}n^2.\end{aligned}$$

As an immediate consequence we get that the support of the limit distribution

$$[-\gamma_1, \gamma_2] = \lim_{n\to\infty}[-\mu_n/\sigma_n, (n-\mu_n)/\sigma_n]$$

is as stated in the theorem. Now we compute the distribution function of X:

$$\mathbb{P}(X \le x) = \lim_{n\to\infty} \sum_{\frac{k-\mu_n}{\sigma_n} \le x} \frac{\left[{n \atop k}\right]_q \theta_n^k}{\sum_{i=0}^n \left[{n \atop i}\right]_q \theta_n^i} = \lim_{n\to\infty} \frac{1}{e_q(q)\frac{c-1}{\log c} n} \sum_{k \le \sigma_n x + \mu_n} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^k.$$

Since $\sigma_n x + \mu_n \sim n\alpha(c,x)$ we have further

$$\begin{aligned} \mathbb{P}(X \le x) &= \lim_{n\to\infty} \frac{1}{e_q(q)\frac{c-1}{\log c} n} \sum_{k \le n\alpha(c,x)} \begin{bmatrix} n \\ k \end{bmatrix}_q \theta_n^q \\ &= \lim_{n\to\infty} \frac{\frac{c^{\alpha(c,x)}-1}{\log c} n e_q(q)}{e_q(q)\frac{c-1}{\log c} n} = \frac{c^{\alpha(c,x)}-1}{c-1}, \end{aligned}$$

what completes the proof of this theorem. □

Again we get the following immediate consequence:

Corollary 4.12. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \ge 1$, $\theta_n \to 1$ and $\theta_n \to \tilde{c}$ with $1 < \tilde{c} < \infty$, then $(\mu_n - X_n)/\sigma_n$ and $(X_n - \mu_n)/\sigma_n$ converge to a limit X, whose distribution is given in Theorem 4.11 with $c = 1/\tilde{c}$ resp. $\tilde{c}$.*

Finally we study the case that $\theta_n^{f(n)} \to c$ with $0 < c < 1$ and $f(n) = o(n)$. The analysis of this case is very similar to that of the previous case. So we start again with a lemma which is useful to find the asymptotic behaviour of the means and variances.

Lemma 4.13. *Let $f(n) \to \infty$ for $n \to \infty$, $\frac{f(n)}{n} \to 0$, $\theta_n^{f(n)} \to c$ with $0 < c < 1$ and. Then*

$$\sum_{i=0}^n i^2 \theta_n^i \sim \frac{f(n)^3}{\log^3 c}$$

and

$$\sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q i^2 \theta_n^i \sim e_q(q) \frac{f(n)^3}{\log^3 c}.$$

Proof. Similar to [14, Lemma 4.8]. □

The following theorem shows that in this case the limiting distribution is an exponential distribution.

Theorem 4.14. *If $X_n \sim RS(n, \theta_n, q)$ with $\theta_n \le 1$, $\theta_n \to 1$, $\theta_n^{f(n)} \to c$ with $f(n) = o(n)$ and $0 < c < 1$, then $(X_n - \mu_n)/\sigma_n$ converges to a normalised exponential distribution with parameter $\lambda = 1$, i.e.,*

$$\mathbb{P}(X \le x) = 1 - e^{-x-1}, \quad x \ge -1.$$

Proof. From [14, Lemmas 4.4 and 4.8] and Lemma 4.13 we get

$$\mu_n \sim \frac{-f(n)}{\log c} \qquad \text{and} \qquad \sigma_n^2 \sim \frac{2f(n)^2}{\log^2 c} - \frac{f(n)^2}{\log^2 c} = \frac{f(n)^2}{\log^2 c}.$$

Computing the distribution function of the limit distribution yields

$$\begin{aligned}
\mathbb{P}(X \le x) &= \lim_{n\to\infty} \sum_{k\le \sigma_n x+\mu_n} \frac{\begin{bmatrix} n \\ k\end{bmatrix}_q \theta_n^k}{\sum_{i=0}^n \begin{bmatrix} n \\ i\end{bmatrix}_q \theta_n^i} \\
&= \lim_{n\to\infty} \frac{1}{\frac{-e_q(q)}{\log c} n} \sum_{k\le \frac{-f(n)}{\log c}x + \frac{-f(n)}{\log c}} \begin{bmatrix} n \\ k\end{bmatrix}_q \theta_n^k \\
&= \lim_{n\to\infty} \frac{1-c^{\frac{1+x}{-\log c}}}{\log c} f(n) e_q(q) \frac{\log c}{e_q(q) f(n)} \\
&= 1 - c^{\frac{1+x}{-\log c}} = 1 - e^{-x-1}. \qquad \square
\end{aligned}$$

Corollary 4.15. *If $X_n \sim RS(n,\theta_n,q)$ with $\theta_n \ge 1$, $\theta_n \to 1$, $\theta_n^{f(n)} \to c$ with $f(n) = o(n)$ and $1 < c < \infty$, then $(\mu_n - X_n)/\sigma_n$ converges to a normalised exponential distribution with parameter $\lambda = 1$.*

5. Unbounded Parameter

Now we turn our attention to sequences of random variables X_n with $X_n \sim \mathcal{B}(\alpha,\theta_n,n,q)$, where the parameter sequence $\theta_n = \theta_n(q)$ tends to infinity. We start with fast growing parameters θ_n, i.e., $\theta_n = q^{-2\alpha n - g(n)}$ with $g(n)$ convergent or $g(n) \to \infty$. Due to the reversing property 3.9 we conclude immediately from Lemma 4.2:

Corollary 5.1. *Let $X_n \sim \mathcal{B}(\alpha,\theta_n,n,q)$ with $\theta_n = q^{-2\alpha n - g(n)}$.*
- (i) *If $g(n) \to \gamma$ then for $\alpha > 0$ we have $n - X_n \to \mathcal{P}(\alpha, q^{-\gamma}, q)$.*
- (ii) *If $g(n) \to \infty$ then for all $\alpha \ge 0$ we have $n - X_n \to \delta_0$.*

Now we consider parameter sequences $\theta_n(q) = q^{-f(n)}$ with $f(n) \to \infty$ and $2\alpha n - f(n) \to \infty$ for $n \to \infty$ and $\alpha > 0$. These assumptions will be on force throughout the section. We will prove in Theorem 5.7 that a suitable chosen subsequence of the normalised sequence of random variables X_n converges to a discrete normal distribution. Theorem 5.2 and Lemmas 5.3 and 5.4 are devoted to the asymptotic behaviour of the sequence (μ_n) of means. Afterwards we study the sequence (σ_n) of variances in Lemmas 5.5 and 5.6 and then we establish the convergence result.

To simplify notation, we define

$$\Sigma_1(z) := \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha} \rfloor} z \begin{bmatrix} n \\ \lfloor \frac{f(n)}{2\alpha} \rfloor - x \end{bmatrix}_q q^{\alpha(a+x)^2}, \qquad \Sigma_1 := \Sigma_1(1),$$

$$\Sigma_2(z) := \sum_{x=0}^{n-\lfloor \frac{f(n)}{2\alpha} \rfloor - 1} z \begin{bmatrix} n \\ x + \lfloor \frac{f(n)}{2\alpha} \rfloor + 1 \end{bmatrix}_q q^{\alpha(a-(x+1))^2}, \qquad \Sigma_2 := \Sigma_2(1),$$

$$\Sigma_1^\infty(z) := \sum_{x=0}^{\infty} z q^{\alpha(a+x)^2}, \qquad \Sigma_1^\infty := \Sigma_1^\infty(1),$$

$$\Sigma_2^\infty(z) := \sum_{x=0}^{\infty} z q^{\alpha(a-(x+1))^2}, \qquad \Sigma_2^\infty := \Sigma_2^\infty(1),$$

where $a = \left\{ \frac{f(n)}{2\alpha} \right\}$.

Now we turn to the study of the sequence of means.

Lemma 5.2. *For $n \to \infty$ we have*

$$\mu_n = \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor + c(a,\alpha,q) + \mathrm{o}(1),$$

where

$$c(a,\alpha,q)=\frac{\sum_{x=1}^{\infty} x\left(q^{\alpha(a-x)^2}-q^{\alpha(a+x)^2}\right)}{\sum_{x=0}^{\infty}\left(q^{\alpha(a+x)^2}+q^{\alpha(a-(x+1))^2}\right)}.$$

Proof. We have to study the behaviour of

$$\frac{\sum_{x=0}^{n} x\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}q^{-f(n)x}}{\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2}q^{-f(n)x}}.$$

For this purpose we expand the fraction by $q^{\frac{f(n)^2}{4\alpha}}$ and analyse the denominator D and the numerator N separately.

$$D=\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\alpha x^2-f(n)x+\frac{f(n)^2}{4\alpha}}=\sum_{x=0}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\frac{(-2\alpha x+f(n))^2}{4\alpha}};$$

splitting the sum into two parts gives

$$=\sum_{x=0}^{\lfloor\frac{f(n)}{2\alpha}\rfloor}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\frac{(-2\alpha x+f(n))^2}{4\alpha}}+\sum_{x=\lfloor\frac{f(n)}{2\alpha}\rfloor+1}^{n}\begin{bmatrix}n\\x\end{bmatrix}_q q^{\frac{(-2\alpha x+f(n))^2}{4\alpha}}.$$

By reversing the order of summation in the first sum and shifting the summation index in the second sum we obtain

$$\begin{aligned}=&\sum_{x=0}^{\lfloor\frac{f(n)}{2\alpha}\rfloor}\begin{bmatrix}n\\ \lfloor\frac{f(n)}{2\alpha}\rfloor-x\end{bmatrix}_q q^{\frac{\left(-2\alpha\lfloor\frac{f(n)}{2\alpha}\rfloor+f(n)+2\alpha x\right)^2}{4\alpha}}\\ &+\sum_{x=0}^{n-\lfloor\frac{f(n)}{2\alpha}\rfloor-1}\begin{bmatrix}n\\ x+\lfloor\frac{f(n)}{2\alpha}\rfloor+1\end{bmatrix}_q q^{\frac{\left(-2\alpha\lfloor\frac{f(n)}{2\alpha}\rfloor-2\alpha-2\alpha x+f(n)\right)^2}{4\alpha}};\end{aligned}$$

simplifying the exponents of q leads to

$$\begin{aligned}=&\sum_{x=0}^{\lfloor\frac{f(n)}{2\alpha}\rfloor}\begin{bmatrix}n\\ \lfloor\frac{f(n)}{2\alpha}\rfloor-x\end{bmatrix}_q q^{\alpha(a+x)^2}\\ &+\sum_{x=0}^{n-\lfloor\frac{f(n)}{2\alpha}\rfloor-1}\begin{bmatrix}n\\ x+\lfloor\frac{f(n)}{2\alpha}\rfloor+1\end{bmatrix}_q q^{\alpha(a-(x+1))^2}.\end{aligned}$$

This tends to

$$e_q(q)\left(\sum_{x=0}^{\infty}q^{\alpha(a+x)^2}+\sum_{x=0}^{\infty}q^{\alpha(a-(x+1))^2}\right)=:\gamma \tag{4}$$

since we can bound the first sum as follows:

$$\begin{aligned}e_q(q)\sum_{x=0}^{\infty}q^{\alpha(a+x)^2}&\geq\sum_{x=0}^{\lfloor\frac{f(n)}{2\alpha}\rfloor}\begin{bmatrix}n\\ \lfloor\frac{f(n)}{2\alpha}\rfloor-x\end{bmatrix}_q q^{\alpha(a+x)^2}\\ &\geq\sum_{x=0}^{\frac{1}{2}\lfloor\frac{f(n)}{2\alpha}\rfloor}\begin{bmatrix}n\\ \lfloor\frac{f(n)}{2\alpha}\rfloor-x\end{bmatrix}_q q^{\alpha(a+x)^2},\end{aligned}$$

estimating the q-binomial coefficient yields

$$\geq \frac{\left(1-q^{n-\lfloor\frac{f(n)}{2\alpha}\rfloor+1}\right)^{\lfloor\frac{f(n)}{2\alpha}\rfloor+1}}{(q,q)_{\frac{1}{2}\lfloor\frac{f(n)}{2\alpha}\rfloor}} \sum_{x=0}^{\frac{1}{2}\lfloor\frac{f(n)}{2\alpha}\rfloor} q^{\alpha(a+x)^2}$$
$$\to e_q(q) \sum_{x=0}^{\infty} q^{\alpha(a+x)^2}.$$

Here we used that

$$\left(1-q^{n-\lfloor\frac{f(n)}{2\alpha}\rfloor+1}\right)^{\lfloor\frac{f(n)}{2\alpha}\rfloor+1} = 1+\mathcal{O}\left(\left(\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1\right) n q^{n-\lfloor\frac{f(n)}{2\alpha}\rfloor+1}\right).$$

Similar arguments hold for the second sum. Now we turn to the numerator N.

$$N = \sum_{x=0}^{n} x \begin{bmatrix} n \\ x \end{bmatrix}_q q^{\alpha x^2 - f(n)x + \frac{f(n)^2}{4\alpha}},$$

we split the sum again, reverse the order of summation resp. shift the summation index and get

$$= \sum_{x=0}^{\lfloor\frac{f(n)}{2\alpha}\rfloor} \left(\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor - x\right) \begin{bmatrix} n \\ \lfloor\frac{f(n)}{2\alpha}\rfloor - x \end{bmatrix}_q q^{\alpha(a+x)^2} +$$
$$+ \sum_{x=0}^{n-\lfloor\frac{f(n)}{2\alpha}\rfloor-1} \left(x+\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+1\right) \begin{bmatrix} n \\ x+\lfloor\frac{f(n)}{2\alpha}\rfloor+1 \end{bmatrix}_q q^{\alpha(a-(x+1))^2}.$$

Using the same arguments as above yields

$$= \left\lfloor\frac{f(n)}{2\alpha}\right\rfloor \gamma - e_q(q)\left(\Sigma_1^\infty(x) - \Sigma_2^\infty(x)\right) + e_q(q)\Sigma_2^\infty + \mathrm{o}(1). \tag{5}$$

Combining (4) and (5) we obtain

$$\mu_n = \left\lfloor\frac{f(n)}{2\alpha}\right\rfloor + \frac{\sum_{x=0}^\infty q^{\alpha(a-(x+1))^2} - \sum_{x=0}^\infty x\left(q^{\alpha(a+x)^2} - q^{\alpha(a-(x+1))^2}\right)}{\sum_{x=0}^\infty \left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2}\right)} + o(1).$$

Simplifying the fraction yields the theorem. □

Now we provide an estimate for the O(1)-term in the preceding theorem.

Lemma 5.3. *Let $c(a,\alpha,q)$ be defined as in Theorem 5.2. Then*

(i) $0 \leq c(a,\alpha,q) < 1$,
(ii) $c(a,\alpha,q) = 0 \Leftrightarrow a = 0$,
(iii) $c(a,\alpha,q) + c(1-a,\alpha,q) = 1$.

Proof. Since for all $x \geq 0$

$$q^{\alpha(-a+x)^2} \geq q^{\alpha(a+x)^2}, \tag{6}$$

$0 \leq c(a,\alpha,q)$. Moreover, $c(a,\alpha,q) = 0$ iff in (6) equality holds for all $x \geq 1$. But this is the case iff $(x-a)^2 = (x+a)^2$ for all x. So $c(a,\alpha,q) = 0$ iff $a = 0$. For (i) it remains to show that

$$\sum_{x=1}^{\infty} x\left(q^{\alpha(-a+x)^2} - q^{\alpha(a+x)^2}\right) < \sum_{x=1}^{\infty}\left(q^{\alpha(-a+x)^2} + q^{\alpha(a+x)^2}\right) + q^{\alpha a^2}.$$

We can rewrite this as

$$\sum_{x=1}^{\infty}(x-1)q^{\alpha(x-a)^2}-\sum_{x=0}^{\infty}(x+1)q^{\alpha(x+a)^2}<0.$$

The left-hand side is increasing in a, and for $a=1$ we have

$$\sum_{x=1}^{\infty}(x-1)q^{\alpha(x-1)^2}-\sum_{x=0}^{\infty}(x+1)q^{\alpha(x+1)^2}=0.$$

Since $0\le a<1$, (ii) follows.

To see (iii), note that the denominators of $c(a,\alpha,q)$ and $c(1-a,\alpha,q)$ are invariant under the substitution $a\mapsto 1-a$. Then add the numerators:

$$\sum_{x=1}^{\infty}x\left(q^{\alpha(1-a-x)^2}-q^{\alpha(1-a+x)^2}\right)+\sum_{x=1}^{\infty}x\left(q^{\alpha(a-x)^2}-q^{\alpha(a+x)^2}\right),$$

by splitting the first sum and shifting the summation index we obtain

$$\begin{aligned}
&=\sum_{x=0}^{\infty}(x+1)q^{\alpha(a+x)^2}-\sum_{x=2}^{\infty}(x-1)q^{\alpha(a-x)^2}+\sum_{x=1}^{\infty}x\left(q^{\alpha(a-x)^2}-q^{\alpha(a+x)^2}\right)\\
&=q^{\alpha a^2}+\sum_{x=1}^{\infty}q^{\alpha(a+x)^2}+q^{\alpha(a-1)^2}+\sum_{x=2}^{\infty}q^{\alpha(a-x)^2}\\
&=\sum_{x=0}^{\infty}\left(q^{\alpha(a+x)^2}+q^{\alpha(a-(x+1))^2}\right),
\end{aligned}$$

which is exactly the denominator of $c(a,\alpha,q)$. □

Lemma 5.4. *Let $c(a,\alpha,q)$ be defined as in Theorem 5.2.*

(i) *If $a>0$, then* $\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+c(a,\alpha,q)\notin\mathbb{Z}$.

(ii) *If $a=0$, then*

$$\mu_n\begin{cases}\ge\frac{f(n)}{2\alpha} & \text{if } n\ge\frac{f(n)}{\alpha}\\ <\frac{f(n)}{2\alpha} & \text{if } n<\frac{f(n)}{\alpha}\end{cases}.$$

Proof. Lemma 5.3 implies (i). To see (ii) we use

$$\mu_n=\frac{f(n)}{2\alpha}+\frac{\sum_{x=1}^{n-\frac{f(n)}{2\alpha}}x\begin{bmatrix}n\\x+\frac{f(n)}{2\alpha}\end{bmatrix}_q q^{\alpha x^2}-\sum_{x=1}^{\frac{f(n)}{2\alpha}}x\begin{bmatrix}n\\\frac{f(n)}{2\alpha}-x\end{bmatrix}_q q^{\alpha x^2}}{\sum_{x=0}^{\frac{f(n)}{2\alpha}}\begin{bmatrix}n\\\frac{f(n)}{2\alpha}-x\end{bmatrix}_q q^{\alpha x^2}+\sum_{x=1}^{n-\frac{f(n)}{2\alpha}}\begin{bmatrix}n\\x+\frac{f(n)}{2\alpha}\end{bmatrix}_q q^{\alpha x^2}}.$$

Now consider the case $n\ge\frac{f(n)}{\alpha}$: We have to prove that

$$\sum_{x=1}^{n-\frac{f(n)}{2\alpha}}x\begin{bmatrix}n\\x+\frac{f(n)}{2\alpha}\end{bmatrix}_q q^{\alpha x^2}\ge\sum_{x=1}^{\frac{f(n)}{2\alpha}}x\begin{bmatrix}n\\\frac{f(n)}{2\alpha}-x\end{bmatrix}_q q^{\alpha x^2}.$$

We will see that for all $1\le x\le\frac{f(n)}{2\alpha}$, the term on the right-hand side is less than or equal to the corresponding term on the left-hand side (there are enough terms on the left-hand side by our assumption), i.e.,

$$\begin{bmatrix}n\\x+\frac{f(n)}{2\alpha}\end{bmatrix}_q\ge\begin{bmatrix}n\\\frac{f(n)}{2\alpha}-x\end{bmatrix}_q:$$

Our assumption implies

$$n-x-\frac{f(n)}{2\alpha}+1+i\ge\frac{f(n)}{2\alpha}-x+1+i\qquad\text{for } 0\le i\le 2x-1.$$

Taking the product over all i on both sides yields

$$\left(1-q^{n-x-\frac{f(n)}{2\alpha}+1}\right)\cdots\left(1-q^{n-\frac{f(n)}{2\alpha}+x}\right)\geq\left(1-q^{\frac{f(n)}{2\alpha}-x+1}\right)\cdots\left(1-q^{x+\frac{f(n)}{2\alpha}}\right)$$

and therefore

$$\frac{1}{(q,q)_{x+\frac{f(n)}{2\alpha}}(q,q)_{n-x-\frac{f(n)}{2\alpha}}}\geq\frac{1}{(q,q)_{\frac{f(n)}{2\alpha}-x}(q,q)_{n-\frac{f(n)}{2\alpha}+x}},$$

and this leads to the assertion in this case immediately.

The case $n<\frac{f(n)}{4\alpha}$ can be treated similarly. □

In [5] Gerhold and Zeiner studied the behaviour of the means of Kemp's q-binomial distribution in the limit $q\to 0$ and $c(a,\alpha,q)$ in the limit $q\to 1$. We will do the same analysis here. First we will show that for $q\to 0$

$$\mu_n\to\left\lfloor\frac{f(n)}{2\alpha}\right\rfloor+\begin{cases}0 & \text{if } 0\leq a<\frac{1}{2}\\ \frac{1}{2} & \text{if } a=\frac{1}{2}\\ 1 & \text{if } \frac{1}{2}<a<1\end{cases}.$$

For this purpose we estimate $c(a,\alpha,q)$:

$$\begin{aligned}
c(a,\alpha,q)&=\frac{q^{\alpha(1-a)^2}+2q^{\alpha(2-a)^2}+\sum_{x=3}^{\infty}xq^{\alpha(a-x)^2}-\sum_{x=1}^{\infty}xq^{\alpha(a+x)^2}}{q^{\alpha a^2}+q^{\alpha(a-1)^2}+\sum_{x=1}^{\infty}q^{\alpha(a+x)^2}+\sum_{x=2}q^{\alpha(a-x)^2}}\\
&\leq\frac{q^{\alpha(1-a)^2}+2q^{\alpha(2-a)^2}+\sum_{x=3}^{\infty}xq^{\alpha(1-x)^2}-\sum_{x=1}^{\infty}xq^{\alpha(1+x)^2}}{q^{\alpha a^2}+q^{\alpha(a-1)^2}+\sum_{x=1}^{\infty}q^{\alpha(1+x)^2}+\sum_{x=2}^{\infty}q^{\alpha x^2}}\\
&=\frac{q^{\alpha(1-a)^2}+2q^{\alpha(2-a)^2}+2\sum_{x=2}^{\infty}q^{\alpha x^2}}{q^{\alpha a^2}+q^{\alpha(a-1)^2}+2\sum_{x=2}^{\infty}q^{\alpha x^2}}\\
&=\frac{1+2q^{\alpha(3-2a)}+2q^{-\alpha(1-a)^2}\sum_{x=2}^{\infty}q^{\alpha x^2}}{q^{\alpha(-1+2a)}+1+2q^{-\alpha(1-a)^2}\sum_{x=2}^{\infty}q^{\alpha x^2}}.
\end{aligned}$$

For $a\in[0,\frac{1}{2})$ we have $2a-1<0$ and therefore the denominator tends to infinity while the numerator goes to 1. Consequently $c(a,\alpha,q)\to 0$. Lemma 5.3 (iii) implies that $c(a,\alpha,q)\to 1$ if $a\in(\frac{1}{2},1)$ and $c(\frac{1}{2},\alpha,q)=\frac{1}{2}$. Moreover, from the estimates in the proof of Theorem 5.2 we get easily that the o(1)-term vanishes in the limit $q\to 0$.

In the limit $q\to 1$ we have $c(a,\alpha,q)\to a$. To see this, apply the Euler-Maclaurin formula to

$$f^+(x)=q^{Ax^2+Bx}\qquad\text{and}\qquad g^+(x)=xq^{Ax^2+Bx}$$

first, which yields

$$\sum_{\ell\geq 0}f^+(\ell)=I_f^++\frac{f^+(0)}{2}+R_f^+$$

with

$$I_f^+=\int_0^{\infty}f^+(x)\mathrm{d}x\qquad\text{and}\qquad R_f^+=\int_0^{\infty}\left(x-\lfloor x\rfloor-\frac{1}{2}\right)f^{+'}(x)\mathrm{d}x.$$

Computing I_f^+ gives

$$I_f^+=\frac{\sqrt{\pi}q^{-\frac{B^2}{4A}}\left(1+\operatorname{erf}\left(\frac{B\log q}{2\sqrt{-A\log q}}\right)\right)}{2\sqrt{-A\log q}},$$

where $\operatorname{erf}(z)$ denotes the error-function. Similarly, we get for $g^+(x)$

$$\sum_{\ell\geq 0} g^+(\ell) = I_g^+ + \frac{g^+(0)}{2} + R_g^+$$

with

$$I_g^+ = \int_0^\infty g^+(x)\mathrm{d}x \qquad \text{and} \qquad R_g^+ = \int_0^\infty \left(x - \lfloor x\rfloor - \frac{1}{2}\right) g^{+'}(x)\mathrm{d}x.$$

Computing I_g^+ gives

$$I_g^+ = \frac{-B\sqrt{\pi}q^{-\frac{B^2}{4A}}\left(1+\operatorname{erf}\left(\frac{B\log q}{2\sqrt{-A\log q}}\right)\right)}{4A\sqrt{-A\log q}} - \frac{1}{2}\frac{1}{A\log q}.$$

In an analogous way we treat the functions

$$f^-(x) = q^{Ax^2-Bx} \qquad \text{and} \qquad g^-(x) = xq^{Ax^2-Bx}.$$

Note that $f^-(0) = f^+(0) = \frac{1}{2}$ and $g^-(0) = g^+(0) = 0$. Putting things together we obtain with $A = \alpha$ and $B = 2\alpha a$

$$\begin{aligned}
c(a,\alpha,q) &= \frac{I_g^- + R_g^- - I_g^+ - R_g^+}{I_f^+ + I_f^- + R_f^+ + R_f^- + q^{-a^2}} \\
&= \frac{\frac{B\sqrt{\pi}q^{-\frac{B^2}{4A}}}{2A\sqrt{-A\log q}} + R_g^- - R_g^+}{\frac{\sqrt{\pi}q^{-\frac{B^2}{4A}}}{\sqrt{-A\log q}} + R_f^- + R_f^+ + q^{-a^2}} \\
&= \frac{a + q^{\frac{B^2}{4A}}\frac{\sqrt{-A\log q}}{\sqrt{\pi}}\left(R_g^- - R_g^+\right)}{1 + q^{\frac{B^2}{4A}}\frac{\sqrt{-A\log q}}{\sqrt{\pi}}\left(R_f^- + R_f^+ + q^{-a^2}\right)}.
\end{aligned}$$

Thus it remains to show that $\sqrt{-\log q}(R_g^- - R_g^+)$ and $\sqrt{-\log q}(R_f^- + R_f^+)$ both tend to 0. We have

$$R_f^+ = \int_0^\infty \left(x - \lfloor x\rfloor - \frac{1}{2}\right) q^{Ax^2+Bx}\log q(2Ax+B)\mathrm{d}x.$$

The integral

$$J_1 := B\int_0^\infty \left(x - \lfloor x\rfloor - \frac{1}{2}\right) q^{Ax^2+Bx}\mathrm{d}x$$

is bounded uniformly for all $q \in [0,1)$, since q^{Ax^2+Bx} is decreasing in x:

$$\begin{aligned}
-\frac{1}{4} &= -\frac{1}{4} + \sum_{i=0}^\infty 0 \\
&\leq \int_0^{\frac{1}{2}} \left(x - \lfloor x\rfloor - \frac{1}{2}\right) q^{Ax^2+Bx}\mathrm{d}x + \sum_{i=0}^\infty \int_{\frac{1}{2}+i}^{\frac{1}{2}+i+1} \left(x - \lfloor x\rfloor - \frac{1}{2}\right) q^{Ax^2+Bx}\mathrm{d}x \\
&= J_1 \\
&= \int_0^1 \left(x - \lfloor x\rfloor - \frac{1}{2}\right) q^{Ax^2+Bx}\mathrm{d}x + \sum_{i=1}^\infty \int_i^{i+1} \left(x - \lfloor x\rfloor - \frac{1}{2}\right) q^{Ax^2+Bx}\mathrm{d}x \\
&\leq 0 + \sum_{i=1}^\infty 0 = 0.
\end{aligned}$$

Thus $(-\log q)^{3/2}J_1 \to 0$. With the same idea we want to estimate

$$J_2 := 2A\int\limits_0^\infty \left(x-\lfloor x\rfloor-\frac{1}{2}\right)q^{Ax^2+Bx}x\mathrm{d}x.$$

Unfortunately $h(x) := q^{Ax^2+Bx}x$ must not be decreasing in x for $x \geq 0$. Differentiating gives

$$h'(x) = q^{Ax^2+Bx}\left(1+(2Ax^2+Bx)\log q\right).$$

Obvious $h'(0) > 0$ and $\lim_{x\to-\infty} h'(x) = \lim_{x\to\infty} h'(x) = -\infty$ since $\log q < 0$. Consequently there exists a single positive root r of $h'(x)$. For q near at 1 we have $r \leq 1/\sqrt{-A\log q}$ since

$$\begin{aligned} h'\left(\frac{1}{\sqrt{-A\log q}}\right) &= 1+\log q\left(-\frac{2A}{A\log q}+\frac{B}{\sqrt{-A\log q}}\right) \\ &= 1-2+\frac{B\sqrt{-\log q}}{\sqrt{A}} < 0. \end{aligned}$$

Thus $h(x)$ is decreasing for $x \geq r$. Split J_2 into integrals over $[0,\lceil r\rceil]$ and $[\lceil r\rceil,\infty)$. The second integral is bounded by same arguments as above. The first integral can be estimated trivially by

$$2A\left|\int\limits_0^{\lceil r\rceil}\left(x-\lfloor x\rfloor-\frac{1}{2}\right)q^{Ax^2+Bx}x\mathrm{d}x\right| \leq A\int\limits_0^{\lceil r\rceil} x \leq A\lceil r\rceil^2.$$

Therefore $\sqrt{-\log q}R_f^+ \to 0$ for $q \to 1$. Analogously we get $\sqrt{-\log q}R_f^- \to 0$. In order to show that the term with

$$R_g^+ = \int\limits_0^\infty \left(x-\lfloor x\rfloor-\frac{1}{2}\right)\left(q^{Ax^2+Bx}+xq^{Ax^2+Bx}\log q(2Ax+B)\right)\mathrm{d}x$$

vanishes, it remains to consider the integral

$$J_3 := \int\limits_0^\infty \left(x-\lfloor x\rfloor-\frac{1}{2}\right)q^{Ax^2+Bx}qx^2\mathrm{d}x.$$

Again we compute where $H(x) := x^2q^{Ax^2+Bx}$ is decreasing. We have

$$H'(x) = q^{Ax^2+Bx}\left(2x+(2Ax^3+Bx^2)\log q\right)$$

and therefore $\lim_{x\to-\infty} H'(x) = +\infty$, $\lim_{x\to\infty} H'(x) = -\infty$ and $H'(0) = 0$. Since $H''(0) > 0$, there exists a single positive root s of $H'(x)$. Moreover $s \leq 1/\sqrt{-A\log q}$ since

$$\begin{aligned} H'\left(\frac{1}{\sqrt{-A\log q}}\right) &= \frac{2}{\sqrt{-A\log q}}+\log q\left(\frac{2a}{(-A\log q)^{3/2}}-\frac{B}{A\log q}\right) \\ &= \frac{2}{\sqrt{-A\log q}}-\frac{2}{\sqrt{-A\log q}}-\frac{B}{A} \leq 0. \end{aligned}$$

Thus $H(x)$ is decreasing for $x \geq s$. Split the integral into integrals over $[0,\lfloor s\rfloor]$, $[\lfloor s\rfloor,\lceil s\rceil]$ and $[\lceil s\rceil,\infty)$. The third integral is bounded as above. The second integral is trivially bounded by $\frac{1}{2}\lceil s\rceil^2$. And the first integral - the increasing part - we estimate with the same ideas as for the decreasing part:

$$\begin{aligned}
0 &\le \sum_{i=0}^{\lfloor s\rfloor-1} \int_i^{i+1} \left(x-\lfloor x\rfloor-\frac{1}{2}\right) q^{Ax^2+Bx} q x^2 \mathrm{d}x \\
&= \int_0^{\lfloor s\rfloor} \left(x-\lfloor x\rfloor-\frac{1}{2}\right) q^{Ax^2+Bx} x^2 \mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \left(x-\lfloor x\rfloor-\frac{1}{2}\right) q^{Ax^2+Bx} x^2 \mathrm{d}x + \sum_{i=0}^{\lfloor s\rfloor-2} \int_{\frac{1}{2}+i}^{\frac{1}{2}+i+1} \left(x-\lfloor x\rfloor-\frac{1}{2}\right) q^{Ax^2+Bx} x^2 \mathrm{d}x \\
&\quad + \int_{\lfloor s\rfloor-\frac{1}{2}}^{\lfloor s\rfloor} \left(x-\lfloor x\rfloor-\frac{1}{2}\right) q^{Ax^2+Bx} x^2 \mathrm{d}x \\
&\le 0 + \sum_{i=0}^{\lfloor s\rfloor-2} 0 + \frac{1}{4}\lfloor s\rfloor^2
\end{aligned}$$

Therefore $\sqrt{-\log q}R_g^+ \to 0$ for $q \to 1$. In a similar way we find $\sqrt{-\log q}R_g^- \to 0$.

After this analysis of the means, we turn our attention to the sequence of variances.

Lemma 5.5. *For $n \to \infty$ we have*

$$\sigma_n^2 = \phi(a,\alpha,q) - c(a,\alpha,q)^2 + \mathrm{o}(1),$$

where

$$\phi(a,\alpha,q) := \frac{e_q(q)}{\gamma} \sum_{x=1}^{\infty} x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2}\right).$$

Proof. By definition we have

$$\mathbb{E}\left(X_n^2\right) = \frac{\sum_{x=0}^n x^2 \begin{bmatrix} n \\ x\end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}{\sum_{x=0}^n \begin{bmatrix} n \\ x\end{bmatrix}_q q^{\alpha x^2} q^{-f(n)x}}.$$

Now we proceed as in the proof of Theorem 5.2 and study the numerator $\tilde{N}$ after expansion by $q^{\frac{f(n)^2}{4\alpha}}$.

$$\tilde{N} = \sum_{x=0}^{n} x^2 \begin{bmatrix} n \\ x\end{bmatrix}_q q^{\alpha x^2 - f(n)x + \frac{f(n)^2}{4\alpha}};$$

we split the sum and reverse the order of summation resp. shift the summation index and get

$$\begin{aligned}
&= \sum_{x=0}^{\lfloor \frac{f(n)}{2\alpha}\rfloor} \left(\left\lfloor \frac{f(n)}{2\alpha}\right\rfloor - x\right)^2 \begin{bmatrix} n \\ \lfloor \frac{f(n)}{2\alpha}\rfloor - x\end{bmatrix}_q q^{\alpha(a+x)^2} + \\
&\quad + \sum_{x=0}^{n-\lfloor \frac{f(n)}{2\alpha}\rfloor-1} \left(x + \left\lfloor \frac{f(n)}{2\alpha}\right\rfloor + 1\right)^2 \begin{bmatrix} n \\ x + \lfloor \frac{f(n)}{2\alpha}\rfloor + 1\end{bmatrix}_q q^{\alpha(a-(x+1))^2},
\end{aligned}$$

which can be written as

$$= \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \Sigma_1 - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1(x) + \Sigma_1(x^2) + \Sigma_2(x^2) + \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \Sigma_2 + \Sigma_2$$
$$+ 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2(x) + 2\Sigma_2(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2.$$

Using similar arguments as above yields

$$= \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \gamma + e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) \right.$$
$$\left. + \Sigma_1^\infty(x^2) + \Sigma_2^\infty(x^2) + \Sigma_2^\infty + 2\Sigma_2^\infty(x) \right) + \mathrm{o}(1).$$

Thus

$$\mathbb{E}\left(X_n^2\right) = \frac{1}{\gamma} \left(\left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 \gamma + e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) + 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) \right) \right)$$
$$+ \phi(a, \alpha, q).$$

Since

$$\mu_n^2 = \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor^2 - \frac{e_q(q) \left(2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_1^\infty(x) - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty(x) - 2 \left\lfloor \frac{f(n)}{2\alpha} \right\rfloor \Sigma_2^\infty \right)}{\gamma}$$
$$+ c(a, \alpha, q)^2 + \mathrm{o}(1),$$

we obtain

$$\sigma_n^2 = \phi(a, \alpha, q) - c(a, \alpha, q)^2 + \mathrm{o}(1). \qquad \square$$

Lemma 5.6.

$$\phi(a, \alpha, q) > c(a, \alpha, q)^2.$$

Proof. We have to show that

$$\frac{\sum_{x=1}^\infty x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^\infty q^{\alpha(a+x)^2} + \sum_{x=1}^\infty q^{\alpha(a-x)^2}} > \left(\frac{\sum_{x=1}^\infty x \left(q^{\alpha(a-x)^2} - q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^\infty q^{\alpha(a+x)^2} + \sum_{x=1}^\infty q^{\alpha(a-x)^2}} \right)^2.$$

A sufficient condition for this is

$$\frac{\sum_{x=1}^\infty x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^\infty q^{\alpha(a+x)^2} + \sum_{x=1}^\infty q^{\alpha(a-x)^2}} > \left(\frac{\sum_{x=1}^\infty x \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right)}{\sum_{x=0}^\infty q^{\alpha(a+x)^2} + \sum_{x=1}^\infty q^{\alpha(a-x)^2}} \right)^2.$$

We show that

$$\left(\sum_{x=1}^\infty x^2 \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right) \right) \left(\sum_{x=1}^\infty q^{\alpha(a+x)^2} + \sum_{x=1}^\infty q^{\alpha(a-x)^2} \right) \geq$$
$$\geq \left(\sum_{x=1}^\infty x \left(q^{\alpha(a-x)^2} + q^{\alpha(a+x)^2} \right) \right)^2.$$

Expanding gives

$$\begin{aligned}\sum_{x,y=1}^{\infty} x^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} \\ + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} \\ \geq \sum_{x,y=1}^{\infty} x q^{\alpha(x+a)^2} y q^{\alpha(y+a)^2} + \sum_{x,y=1}^{\infty} x q^{\alpha(x-a)^2} y q^{\alpha(y-a)^2} \\ + 2 \sum_{x,y=1}^{\infty} x q^{\alpha(x+a)^2} y q^{\alpha(y-a)^2}.\end{aligned}$$

Now we consider the pairs (x, y) and (y, x) again and obtain

$$\begin{aligned}x^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + y^2 q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + x^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} + y^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} \\ + x^2 q^{\alpha(x-a)^2} q^{\alpha(y+a)^2} + y^2 q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} + x^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} + y^2 q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} \\ \geq 2xy q^{\alpha(x+a)^2} q^{\alpha(y+a)^2} + 2xy q^{\alpha(x-a)^2} q^{\alpha(y-a)^2} + 2xy q^{\alpha(x+a)^2} q^{\alpha(y-a)^2} \\ + 2xy q^{\alpha(x-a)^2} q^{\alpha(y+a)^2}.\end{aligned}$$

This is true since $x^2 + y^2 \geq 2xy$. □

Now we are able to establish the next convergence result. For this purpose recall that $c(a, \alpha, q)$ and $\phi(a, \alpha, q)$ depend on the fractional part of $\frac{f(n)}{2\alpha}$. Since convergent variances and convergent fractional parts of means are required for convergence to a discrete distribution, we will choose a subsequence (n_k) of (n) such that $\{\frac{f(n)}{2\alpha}\}$ remains constant.

Theorem 5.7. *Let (n_k) be an increasing sequence of natural numbers and $X_{n_k} \sim \mathcal{B}(\alpha, q^{-f(n_k)}, n_k, q)$ such that $\{\frac{f(n)}{2\alpha}\} =$ a constant. Recall that we always assume $f(n) \to \infty$, $2\alpha n - f(n) \to \infty$ and $\alpha > 0$. Then $(X_{n_k} - \mu_{n_k})/\sigma_{n_k}$ converges for $k \to \infty$ to a normalised discrete normal distribution, i.e., they converge to a limit X with*

$$\mathbb{P}\left(X = \frac{1}{\sigma}\left(x - c(a, \alpha, q)\right)\right) = \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2}},$$

where $\sigma = \lim_{k\to\infty} \sigma_{n_k}$ and $c(a, \alpha, q)$ is defined as in Theorem 5.2..

Proof. For simplicity we write in the following n instead of n_k. First we note that Lemmas 5.5 and 5.6 imply that the sequence of variances (σ_n^2) converges since $\{\frac{f(n)}{2\alpha}\}$ is constant by assumption. We define

$$H(\mu_n) := \begin{cases} \lfloor \mu_n \rfloor & \text{if } a > 0 \\ \lfloor \mu_n \rfloor & \text{if } a = 0, n \geq \frac{f(n)}{4\alpha} \\ \lceil \mu_n \rceil & \text{if } a = 0, n < \frac{f(n)}{4\alpha} \end{cases}.$$

Since $H(\mu_n) = \frac{f(n)}{2\alpha} - a$, we have

$$\begin{aligned}
\mathbb{P}\left(X_n = H(\mu_n) + x\right) &= \begin{bmatrix} n \\ \frac{f(n)}{2\alpha} - a + x \end{bmatrix}_q \frac{q^{\left(\frac{f(n)}{2\alpha} - a + x\right)^2 - f(n)\left(\frac{f(n)}{2\alpha} - a + x\right)}}{\sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2 - f(n) y}} \\
&= \begin{bmatrix} n \\ \frac{f(n)}{2\alpha} - a + x \end{bmatrix}_q \frac{q^{-\frac{f(n)^2}{4\alpha} + \alpha(x-a)^2}}{\sum_{y=0}^{n} \begin{bmatrix} n \\ y \end{bmatrix}_q q^{\alpha y^2 - f(n) y}} \\
&\to e_q(q) \frac{q^{\alpha(x-a)^2}}{e_q(q) \sum_{x=0}^{\infty} \left(q^{\alpha(a+x)^2} + q^{\alpha(a-(x+1))^2}\right)} \\
&= \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(a+x)^2}} \\
&= \frac{q^{\alpha(x-a)^2}}{\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2}}.
\end{aligned}$$

By normalising we get the theorem. □

For $\alpha = \frac{1}{2}$ this theorem reduces to the convergence property of Kemp's binomial distribution established in [5]

Using Jacobi's Triple Product we can rewrite the infinite sum as

$$\sum_{x=-\infty}^{\infty} q^{\alpha(x-a)^2} = q^{\alpha a^2} \left(q^{2\alpha}, q\right)_\infty \left(-q^{\alpha - 2\alpha a}, q\right)_\infty \left(-q^{\alpha + 2\alpha a}, q\right)_\infty.$$

In the limit $q \to 1$ these discrete normal distributions converge to the standard normal distribution, see [13].

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