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Numbers with fixed sum of digits in linear recurrent number systems

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Abstract We study the set of integers with a given sum of digits with respect to a linear recurrent digit system. An asymptotic formula for the number of integers $\leq N$ with given sum of digits is determined, and the distribution in residue classes is investigated, thus generalizing results due to Mauduit and Sárközy. It turns out that numbers with fixed sum of digits are uniformly distributed in residue classes under some very general conditions. Namely, the underlying linear recurring sequence must have the property that there is no prime factor P of the modulus such that all but finitely many members of the sequence leave the same residue modulo P. The key step in the proof is an estimate for exponential sums using known theorems from Diophantine approximation.

Keywords Linear recurrent digit system · Sum of digits · Residue distribution

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1 Introduction and notation

Linear recurrent digit systems are a generalization of the usual radix representations; they have been studied, for example, in [3, 12, 14, 15, 21]. We start with a definition of these systems:

Let $G = (G_n)$ $(n = 0, 1, \dots)$ be a linear recurring sequence of order $d \geq 1$, i.e.

$$G_{n+d} = a_1 G_{n+d-1} + a_2 G_{n+d-2} + \cdots + a_d G_n \tag{1}$$

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with integral coefficients and integral initial values. We assume that the coefficients $a_1 \geq a_2 \geq \cdots \geq a_d > 0$ are non-increasing ($a_1 > 1$ if $d = 1$) and that $G_0 = 1$ and

$$G_n > a_1(G_0 + \cdots + G_{n-1}),\ n = 1, \ldots, d - 1.$$

For an arbitrary positive integer N, we define $L = L(N)$ by $G_L \leq N < G_{L+1}$ (and set $L(0) = 0$). Furthermore, set $N_L = N$,

$$\epsilon_j = \left\lfloor \frac{N_j}{G_j} \right\rfloor,\ N_{j-1} = N_j - G_j\epsilon_j\ (1 \leq j \leq L),$$

and finally $\epsilon_0 = N_0$, yielding a unique representation of N of the form

$$N = \sum_{j=0}^{L(N)} \epsilon_j G_j, \tag{2}$$

the G-ary representation of N with digits ϵ_j. If $d = 1$ and $a_1 = g$, we obtain the well-known base-g representation of N.

Now, the sum of digits is naturally defined as

$$s_G(N) = \sum_{j=0}^{L(N)} \epsilon_j.$$

The best-known instance of such a digit system is probably the Zeckendorf expansion [22], belonging to the Fibonacci sequence $G_0 = 1$, $G_1 = 2$, $G_{n+2} = G_{n+1} + G_n$.

In [21], Pethő and Tichy generalized a well-known result of Delange [5] on the mean value of the sum of digits to linear recurring sequences. For usual base-g expansions, numbers with fixed sum of digits were studied by Mauduit and Sárközy in [19]. Their first main result states that the number of integers with $\leq \nu$ digits and sum of digits $k \leq \frac{g-1}{2}\nu$ (for reasons of symmetry, this case is obviously sufficient) is, uniformly for $k \to \infty$,

$$r^{-k}(1 + r + \cdots r^{g-1})^{\nu}\pi^{1/2}(D\nu)^{-1/2}(1 + O(D\nu)^{-1/2}), \tag{3}$$

where the implied constant depends only on the base g; r is defined as the unique positive zero of

$$Q(x) = -k(1 + x + \cdots + x^{g-1}) + \nu x(1 + 2x + \cdots + (g - 1)x^{g-2}),$$

and $D = 2\pi^2(B - A^2)$, where

$$A = \left(\sum_{j=1}^{g-1} jr^j\right)\left(\sum_{j=0}^{g-1} r^j\right)^{-1} = \frac{k}{\nu} \text{ and } B = \left(\sum_{j=1}^{g-1} j^2r^j\right)\left(\sum_{j=0}^{g-1} r^j\right)^{-1}.$$

Secondly, they showed that the integers with fixed sum of digits are uniformly distributed in residue classes if the modulus is not too large and relatively prime to $(g-1)g$—this theorem was further generalized in a very recent paper of Mauduit, Pomerance and Sárközy [17], relaxing the condition that the modulus is relatively prime to $(g-1)g$. Furthermore, they were able to prove an Erdős-Kac-type theorem for integers with fixed sum of digits.

Similar results for other kinds of digitally restricted sets are due to Erdős, Mauduit and Sárközy ([8, 9], integers with missing digits), Fouvry and Mauduit resp. Mauduit and Sárközy ([10, 11, 18], integers with congruence conditions for the sum of digits).

In this paper, we are going to prove a generalization of formula (3) to linear recurrent digit systems and study the distribution in residue classes. It turns out that we have uniform distribution if there is no prime divisor P of the modulus such that (G_n) is constant modulo P for all but finitely many values of n.

We will make use of the following notational conventions: we write $e(\alpha) = \exp(2\pi i\alpha)$, we use $c_1(G), c_2(G), \ldots$ for constants which depend only on the basis G of our digital system, and we write $f(N) = O_G(g(N))$, if there is a constant $C(G)$ depending only on G such that, for sufficiently large N, $f(N) \leq C(G)g(N)$ holds.

2 Asymptotic enumeration

We start with a characterization of admissible digital expansions given by Pethő and Tichy in [21]:

Lemma 1. *The* $(t+1)$*-tuple* $(\epsilon_0, \ldots, \epsilon_t) \in \mathbb{N}_0^{t+1}$ *is the sequence of* G*-ary digits of an integer if and only if*

$$\sum_{j=0}^{n} \epsilon_j G_j < G_{n+1} \tag{4}$$

for all $0 \leq n < d-1$ *and*

$$(\epsilon_n, \ldots, \epsilon_{n-d+1}) < (a_1, \ldots, a_d) \tag{5}$$

lexicographically (*i.e. there is an* i *such that* $\epsilon_{n+1-j} = a_j$ *for* $j < i$ *and* $\epsilon_{n+1-i} < a_i$) *for all* $d-1 \leq n \leq t$.

This lemma enables us to establish a generating function for the integers with fixed sum of digits:

Proposition 2. *Let* $F(k, \nu)$ *be the set of integers with* $\leq \nu$ *base-*G *digits and sum of digits* k*. Then we have*

$$|F(k, \nu)| = [x^\nu y^k]\,\frac{p(x, y)}{q(x, y)},$$

where $p(x, y)$ and $q(x, y)$ are polynomials and $q(x, y)$ is given by

$$q(x, y) = 1 - \sum_{i=1}^{d} \left(\sum_{j=0}^{a_i - 1} y^j \right) \left(\prod_{l=1}^{i-1} y^{a_l} \right) x^i. \tag{6}$$

Proof: By the preceding lemma, we have to consider sequences satisfying the two conditions (4) and (5). We call such sequences *good*. Let a good sequence $(\epsilon_0, \ldots, \epsilon_t)$ be given. By (5), there is an i such that $\epsilon_{t+1-j} = a_j$ for $j < i$ and $\epsilon_{t+1-i} < a_i$. The remaining digits $(\epsilon_0, \ldots, \epsilon_{t-i})$ obviously form a good sequence. Conversely, a sequence $(b, a_{i-1}, \ldots, a_1)$ with $b < a_i$ may be appended to any good sequence of length $\geq d$ to form another good sequence. Thus, if

$$g(t) = \sum_{\epsilon} y^{s(\epsilon)},$$

where the sum is over all good sequences $\epsilon = (\epsilon_0, \ldots, \epsilon_t)$ and $s(\epsilon) = \epsilon_0 + \cdots + \epsilon_t$, we have

$$g(t) = \sum_{i=1}^{d} \left(\sum_{j=0}^{a_i - 1} y^j \right) \left(\prod_{l=1}^{i-1} y^{a_l} \right) g(t-i)$$

if t is large enough. This shows that the generating function for our problem is given by a rational function of the form $\frac{p(x,y)}{q(x,y)}$, with $q(x, y)$ as in (6). $\square$

Lemma 3. *Let $q(x, y)$ be given by (6), and define $\lambda = \lambda(y)$ for positive y as the unique positive solution to $q(\lambda, y) = 0$. Furthermore, define*

$$\mu(y) = -\frac{y\lambda'(y)}{\lambda(y)} = \frac{y q_y(\lambda(y), y)}{\lambda(y) q_x(\lambda(y), y)}. \tag{7}$$

Then $\mu(y)$ is a continuous, strictly increasing function with $\lim_{y\to 0} \mu(y) = 0$ and $\lim_{y\to\infty} \mu(y) = A = \max_i \frac{a_1 + \cdots + a_i - 1}{i}$. Furthermore, there exists a constant $c_1(G) > 0$ depending on G such that $\mu'(y) \geq c_1(G)$ for all $y \in [0, 1]$.

Proof: Obviously, $q(x, y)$ is strictly decreasing in x and y, and $q(0, y) = 1$, whereas $q(x, y) \to -\infty$ as $x \to \infty$. Therefore, $\lambda(y)$ is well-defined, and so is $\mu(y)$. Clearly, $\lambda(y)$ and $\mu(y)$ are continuous. As $q(x, 0) = 1 - x$, we know that $\lambda(0) = 1$. Furthermore, $q_x(x, 0) = -1$, which means that $\mu(0) = 0$.

Since $\lambda(y)$ is an algebraic function with no branch points on $[0, \infty)$ (note that the derivative $q_x(\lambda(y), y)$ is strictly negative on this interval), $\lambda(y)$ has a holomorphic continuation and is thus infinitely often differentiable. Since $\lambda(y) \neq 0$ for all y, this also holds for $\mu(y)$.

$r(x, y) = 1 - q(x, y)$ is a polynomial in x, y with positive coefficients and constant coefficient 0. We write $r(x, y) = \sum_{k,l} r_{kl} x^k y^l$. Implicit differentiation yields

$$\mu(y) = \frac{y q_y(\lambda(y), y)}{\lambda(y) q_x(\lambda(y), y)} = \frac{y r_y(\lambda(y), y)}{\lambda(y) r_x(\lambda(y), y)}$$

and

$$\mu'(y) = \frac{1}{x^3 y r_x(x, y)^3} (y^2 r_y(x, y)^2 (x r_x(x, y) + x^2 r_{xx}(x, y)) + x^2 r_x(x, y)^2 (y r_y(x, y) \\ + y^2 r_{yy}(x, y)^2) - 2x^2 y^2 r_x(x, y) r_y(x, y) r_{xy}(x, y))|_{x=\lambda(y)}.$$

The denominator is positive for $y > 0$. The numerator can be written as

$$\left(\sum_{k,l} l r_{kl} x^k y^l\right)^2 \left(\sum_{k,l} k^2 r_{kl} x^k y^l\right) + \left(\sum_{k,l} k r_{kl} x^k y^l\right)^2 \left(\sum_{k,l} l^2 r_{kl} x^k y^l\right) \\ -2\left(\sum_{k,l} k r_{kl} x^k y^l\right)\left(\sum_{k,l} l r_{kl} x^k y^l\right)\left(\sum_{k,l} k l r_{kl} x^k y^l\right).$$

We set $u_{kl} = \sqrt{r_{kl} x^k y^l}$, $v_{kl} = k\sqrt{r_{kl} x^k y^l}$ and $w_{kl} = l\sqrt{r_{kl} x^k y^l}$. Then this equals

$$\left(\sum_{k,l} u_{kl} w_{kl}\right)^2 \left(\sum_{k,l} v_{kl}^2\right) + \left(\sum_{k,l} u_{kl} v_{kl}\right)^2 \left(\sum_{k,l} w_{kl}^2\right) - 2\left(\sum_{k,l} u_{kl} v_{kl}\right) \\ \left(\sum_{k,l} u_{kl} w_{kl}\right)\left(\sum_{k,l} v_{kl} w_{kl}\right) = \langle \mathbf{u}, \mathbf{w}\rangle^2 \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle^2 \langle \mathbf{w}, \mathbf{w}\rangle - 2\langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle \langle \mathbf{v}, \mathbf{w}\rangle,$$

where $\langle ., .\rangle$ denotes the scalar product. Combining the inequality between the arithmetic and geometric mean and the Cauchy-Schwarz inequality yields

$$\begin{aligned} \langle \mathbf{u}, \mathbf{w}\rangle^2 \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle^2 \langle \mathbf{w}, \mathbf{w}\rangle &- 2\langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle \langle \mathbf{v}, \mathbf{w}\rangle \\ &\geq 2\sqrt{\langle \mathbf{u}, \mathbf{w}\rangle^2 \langle \mathbf{v}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{v}\rangle^2 \langle \mathbf{w}, \mathbf{w}\rangle} - 2\langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle \langle \mathbf{v}, \mathbf{w}\rangle \\ &\geq 2\sqrt{\langle \mathbf{u}, \mathbf{w}\rangle^2 \langle \mathbf{u}, \mathbf{v}\rangle^2 \langle \mathbf{v}, \mathbf{w}\rangle^2} - 2\langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle \langle \mathbf{v}, \mathbf{w}\rangle \\ &= 0 \end{aligned}$$

with equality if and only if $\mathbf{v}$, $\mathbf{w}$ are linearly dependent. In our case, this can only be if $r_{kl} \neq 0$ happens only for one value of $\frac{k}{l}$. By our conditions on the a_i, this is impossible. Therefore, $\mu'(y) > 0$ for all $y \in (0, \infty)$, which implies that $\mu(y)$ is strictly increasing. Direct calculation shows that $\mu'(0) = 1$. So $\mu'(y)$ is continuous and positive on the compact interval $[0, 1]$ and has thus a minimum $c_1(G) > 0$.

Finally, we note that $r(x, y)$ behaves like

$$\sum_i y^{\sum_{l=1}^{i} a_l - 1} x^i$$

for $y \to \infty$. Now it is easy to see that

$$q_y(\lambda(y), y) \sim \sum_i \frac{iA}{y} \text{ and } q_x(\lambda(y), y) \sim \sum_i \frac{i}{\lambda(y)},$$

where the sum is over all i (there might be more than one) for which $\frac{a_1+\cdots+a_i-1}{i} = A$. It follows immediately that $\lim_{y\to\infty} \mu(y) = A$. $\square$

Remark 1. It is easily proved that $A = a_1 - \frac{1}{M} \geq \frac{1}{2}$, where M is the largest index such that $a_1 = a_M$.

Lemma 4. *Let $\lambda_1(y)$ be the solution of smallest modulus of $q(x, y) = 0$ for arbitrary complex y, and let $\lambda_2(y)$ be one of the solutions of second-smallest modulus. Then there exist constants $\phi(G), c_2(G), c_3(G), \kappa_1(G)$ depending only on the sequence G such that $c_2(G) < 1$, $\kappa_1(G) > 0$ and*

$$\left|\frac{\lambda_1(y)}{\lambda_2(y)}\right| \leq \min\left(c_2(G), c_3(G)|y|^{\kappa_1(G)}\right) \tag{8}$$

for all $y \in B = \{z \in \mathbb{C} : |z| \leq 1, |\arg z| \leq \phi(G)\}$ and λ_1 coincides with the branch λ on B.

Proof: $\lambda_1(y)$ coincides with $\lambda(y)$ on the compact interval $[0, 1]$, since we already know that $\lambda(y)$ is the unique solution of minimal modulus on this interval. Note that all branches of the equation $q(x, y) = 0$ except λ tend to ∞ with some negative power of y as $y \to 0$. Therefore, there exists some $\delta > 0$ such that $\lambda_1(y) = \lambda(y)$ and

$$\left|\frac{\lambda_1(y)}{\lambda_2(y)}\right| \leq c_4(G)|y|^{\kappa_1(G)} \tag{9}$$

for all y with $|y| \leq \delta$, where $c_4(G), \kappa_1(G)$ are constants depending on G.

The absolute distance to the second-smallest solution is a continuous function on $(0, 1]$, and it tends to ∞ as $y \to 0$, so it has a minimum on $[0, 1]$.

Furthermore, if we choose ϵ_1 small enough to avoid all the (finitely many) branch points of the equation $q(x, y) = 0$ – there are none on $[\delta/2, 1]$ –, all branches are holomorphic on $[\delta/2, 1] \times [-\epsilon_1, \epsilon_1]$, so they satisfy a Lipschitz condition. This means that we can find $\epsilon_2 > 0$ such that λ is the unique branch of smallest modulus on $[\delta/2, 1] \times [-\epsilon_2, \epsilon_2]$.

Choose $\phi(G)$ small enough such that B is contained in

$$\{y \in \mathbb{C} : |y| \leq \delta\} \cup [\delta/2, 1] \times [-\epsilon_2, \epsilon_2].$$

B is a compact set, and the function $f(y) = |\frac{\lambda_1(y)}{\lambda_2(y)}|$ is continuous on this set, if we take $f(0) = 0$. Thus it has a maximum, which must be < 1. Take this as the constant $c_2(G)$. Then, (8) holds for some constant $c_3(G)$. □

Corollary 5.

$$f(x, y) = \frac{p(x, y)}{q(x, y)} - \frac{p(\lambda_1(y), y)}{q_x(\lambda_1(y), y)(x - \lambda_1(y))}$$

is a holomorphic function on $\{x \in \mathbb{C} : |x| < |\lambda_2(y)|\}$ *for all* $y \in B$, *and there exist constants* $c_5(G)$, $\kappa_2(G)$ *depending only on* G *such that*

$$|f(x, y)| \le c_5(G) y^{-\kappa_2(G)} \tag{10}$$

holds on $\{x \in \mathbb{C} : |x| \le \sqrt{|\lambda_1(y)||\lambda_2(y)|}\}$. *As a consequence,*

$$[x^\nu] \frac{p(x, y)}{q(x, y)} = -\frac{p(\lambda_1(y), y)}{q_x(\lambda_1(y), y)} \lambda_1(y)^{-\nu-1} \big(1 + O_G\big(\eta_G^{-\nu}\big)\big), \tag{11}$$

where $\eta_G > 1$ *depends only on* G.

Proof: Note that $\frac{p(\lambda_1(y), y)}{q_x(\lambda_1(y), y)(x-\lambda_1(y))}$ is the principal part of $\frac{p(x,y)}{q(x,y)}$ at $x = \lambda_1(y)$, so $f(x, y)$ is indeed holomorphic, since $\frac{p(x,y)}{q(x,y)}$ has a single pole at $\lambda_1(y)$ and no other singularity for $|x| < |\lambda_2(y)|$.

Now, we write

$$q(x, y) = r(y)(x - \lambda_1(y))(x - \lambda_2(y)) \dots (x - \lambda_d(y))$$

for $y \in B \setminus \{0\}$ and note that

$$q_x(\lambda_1(y), y) = r(y)(\lambda_1(y) - \lambda_2(y)) \dots (\lambda_1(y) - \lambda_d(y)),$$

yielding

$$\Bigg[f(x, y) = \frac{1}{r(y)(x - \lambda_1(y))} \Bigg(\frac{p(x, y)}{(x - \lambda_2(y)) \dots (x - \lambda_d(y))} - \frac{p(\lambda_1(y), y)}{(\lambda_1(y) - \lambda_2(y)) \dots (\lambda_1(y) - \lambda_d(y))} \Bigg) \Bigg].$$

y is bounded on B, and $|x| < |\lambda_2(y)|$ can be bounded by a power of y. Furthermore, the factors $(x - \lambda_i(y))$ are bounded below by $|\lambda_2(y)||1 - \sqrt{|\frac{\lambda_1(y)}{\lambda_2(y)}|}|$ for $x \le \sqrt{|\lambda_1(y)||\lambda_2(y)|}$, and the factors $(\lambda_1(y) - \lambda_i(y))$ by $|\lambda_2(y)||1 - |\frac{\lambda_1(y)}{\lambda_2(y)}||$.

Altogether, we see that (10) holds for some constant $c_5(G)$ if $y \in B \setminus \{0\}$ and $|x| \le \sqrt{|\lambda_1(y)||\lambda_2(y)|}$. For $y = 0$, however, the claim is essentially trivial.

Now, we have

$$[x^\nu]\frac{p(x,y)}{q(x,y)} = [x^\nu]\frac{p(\lambda_1(y),y)}{q_x(\lambda_1(y),y)(x-\lambda_1(y))} + [x^\nu]f(x,y)$$

and

$$[x^\nu]f(x,y) = \oint_{\mathcal{C}} x^{-\nu-1}f(x,y)\,dx \le 2\pi c_5(G) y^{-\kappa_2(G)}\sqrt{|\lambda_1(y)||\lambda_2(y)|}^{\,-\nu},$$

where $\mathcal{C}$ is the circle of radius $\sqrt{|\lambda_1(y)||\lambda_2(y)|}$ around 0. Finally,

$$[x^\nu]\frac{p(\lambda_1(y),y)}{q_x(\lambda_1(y),y)(x-\lambda_1(y))} = -\frac{p(\lambda_1(y),y)}{q_x(\lambda_1(y),y)}\lambda_1(y)^{-\nu-1}$$

The claim now follows from the preceding lemma. □

Next, we need a lemma from [19]:

Lemma 6 (Mauduit/Sárközy [19]). *For $g > 1$, $0 < r \le 1$ and all $\alpha \in \mathbb{R}$ we have*

$$\left|\frac{1 + re(\alpha) + r^2e(2\alpha) + \cdots + r^{g-1}e((g-1)\alpha)}{1 + r + r^2 + \cdots + r^{g-1}}\right| \le 1 - \frac{2r}{g}\|\alpha\|^2. \tag{12}$$

Lemma 7. *There exist constants $c_6(G)$, $c_7(G)$ depending only on G such that*

$$\left|[x^{-\nu}]\frac{p(x,re(\alpha))}{q(x,re(\alpha))}\right| \le c_6(G)\exp(-c_7(G)r\nu\|\alpha\|^2)[x^\nu]\frac{p(x,r)}{q(x,r)} \tag{13}$$

for all $0 < r \le 1$ and all $\alpha \in \mathbb{R}$.

Proof: Note that $z_\nu(y) := [x^\nu]\frac{p(x,y)}{q(x,y)}$ is a polynomial with positive coefficients in y. So, obviously, $z_\nu(re(\alpha)) \le z_\nu(r)$ for all ν. Furthermore, $z_\nu(y)$ satisfies a recurrence relation of the form

$$z_\nu(y) = \sum_{i=1}^{d}\left(\sum_{j=0}^{a_i-1} y^j\right)\left(\prod_{l=1}^{i-1} y^{a_l}\right) z_{\nu-i}(y).$$

It follows that

$$|z_\nu(y)| \le \sum_{i=1}^{d}\left|\sum_{j=0}^{a_i-1} y^j \prod_{l=1}^{i-1} y^{a_l}\right| |z_{\nu-i}(y)|.$$

First, we assume that $a_1 > 1$. Then, by the previous lemma,

$$\left|\sum_{j=0}^{a_1-1}(re(\alpha))^j\right| \le \left(1-\frac{2r}{a_1}\|\alpha\|^2\right)\sum_{j=0}^{a_1-1}r^j.$$

Trivially,

$$\left|\sum_{j=0}^{a_i-1}(re(\alpha))^j\left(\prod_{l=1}^{i-1}(re(\alpha))^{a_l}\right)\right| \le \sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}$$

for all $i > 1$. Now, if we define $Z_\nu(r,\alpha)$ by $Z_\nu(r,\alpha) = z_\nu(r)$ for $\nu < d$ and

$$Z_\nu(r,\alpha) = \left(1-\frac{2r}{a_1}\|\alpha\|^2\right)\sum_{j=0}^{a_1-1}r^j Z_{\nu-1}(r,\alpha) + \sum_{i=2}^{d}\sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}Z_{\nu-i}(r,\alpha),$$

we know that $Z_\nu(r,\alpha) \ge |z_\nu(re(\alpha))|$ for all ν. Since

$$\left(1-\frac{2r}{a_1}\|\alpha\|^2\right)\sum_{j=0}^{a_1-1}r^j \ge \left(1-\frac{r}{4}\right)(1+r) = 1+\frac{r(3-r)}{4} \ge 1,$$

$Z_\nu(r,\alpha)$ is an increasing sequence. Furthermore,

$$\sum_{j=0}^{a_1-1}r^j \ge \sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}$$

for all $i \ge 2$, since $r \le 1$ and the a_i are decreasing. It follows that

$$\begin{aligned}
Z_\nu(r,\alpha) &\le \left(1-\frac{2r}{a_1 d}\|\alpha\|^2\right)\sum_{i=1}^{d}\sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}Z_{\nu-i}(r,\alpha)\\
&\le \exp\left(-\frac{2r}{a_1 d}\|\alpha\|^2\right)\sum_{i=1}^{d}\sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}Z_{\nu-i}(r,\alpha)\\
&\le \sum_{i=1}^{d}\exp\left(-\frac{2ri}{a_1 d^2}\|\alpha\|^2\right)\sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}Z_{\nu-i}(r,\alpha)
\end{aligned}$$

and thus

$$Z_\nu(r,\alpha) \le c_6(G)\exp(-c_7(G)r\nu\|\alpha\|^2)z_\nu(r)$$

for constants $c_6(G)$, $c_7(G) = \frac{2}{a_1 d^2}$ by simple induction on ν. This proves the claim in the case of $a_1 > 1$. If $a_1 = a_2 = \cdots = a_d = 1$, iterate the recurrence equation for z_ν

once to obtain

$$z_\nu(y) = \sum_{i=2}^{d} y^{i-2}(1+y)z_{\nu-i}(y) + y^{d-1}z_{\nu-d-1}$$

and apply the same method to this equation (note that we have at least one term of the form $(1+y)$, as $d \geq 2$ in this case). $\square$

Now, we are ready to prove our first main theorem following the same line of proof as Mauduit and Sárközy:

Theorem 8. *Let $F(k,\nu)$ be defined as in Proposition 2 and take A as in Lemma 3. Then, uniformly for $l = \min(k, A\nu - k) \to \infty$, we have*

$$|F(k,\nu)| = \frac{p(\lambda(r), r)}{-\lambda(r)q_x(\lambda(r), r)} \pi^{1/2}(D\nu)^{-1/2} r^{-k} \lambda(r)^{-\nu}(1 + O_G((D\nu)^{-1/2})), \quad (14)$$

where r is defined by $\mu(r) = \frac{k}{\nu}$ and $D = 2\pi^2 r\mu'(r)$.

Proof: From Proposition 2, we know that

$$|F(k,\nu)| = [x^\nu y^k]\,\frac{p(x,y)}{q(x,y)}.$$

First, let $\frac{k}{\nu} \leq \mu(1)$. Choose $0 < r \leq 1$ in such a way that $\mu(r) = \frac{k}{\nu}$ – this is possible by Lemma 3. Now, we have

$$|F(k,\nu)| = r^{-k}\int_{-1/2}^{1/2} [x^\nu]\frac{p(x, re(\alpha))}{q(x, re(\alpha))} e(-k\alpha)\,d\alpha.$$

We split the integral in two parts: define

$$J_1 = \int_{-\delta}^{\delta} [x^\nu]\frac{p(x, re(\alpha))}{q(x, re(\alpha))} e(-k\alpha)\,d\alpha$$

and

$$J_2 = \int_{\delta < |\alpha| \leq 1/2} [x^\nu]\frac{p(x, re(\alpha))}{q(x, re(\alpha))} e(-k\alpha)\,d\alpha,$$

where $\delta = k^{-1/2}\log k$. We will deal with J_1 first. If k is large enough, we have $\delta < \phi(G)$, so we may apply Corollary 5. This means that

$$J_1 = \left(\int_{-\delta}^{\delta} \frac{p(\lambda_1(re(\alpha)), re(\alpha))}{-q_x(\lambda_1(re(\alpha)), re(\alpha))} \lambda_1(re(\alpha))^{-\nu-1} e(-k\alpha)\,d\alpha\right)\left(1 + O_G\left(\eta_G^{-\nu}\right)\right).$$

We expand $\frac{p(\lambda_1(y),y)}{q_x(\lambda_1(y),y)}$ in a Taylor series around $y = r$; $p(x, y)$ and $-q_x(x, y)$ are polynomials with positive coefficients, and we have $-q_x(1, 0) = 1$ and $p(1, 0) = 1$ (note that $\frac{p(x,0)}{q(x,0)}$ is the counting series for integers with sum of digits 0). This means that $p(\lambda(y), y)$ and $-q_x(\lambda(y), y)$ can be bounded above and below for $y \le 1$ (the bounds depending only on G), and their derivatives are also bounded. Therefore, we have

$$\frac{p(\lambda_1(re(\alpha)), re(\alpha))}{-q_x(\lambda_1(re(\alpha)), re(\alpha))} = \frac{p(\lambda(r)), r)}{-q_x(\lambda(r), r)}(1 + b(r)\alpha + O_G(\alpha^2)).$$

Likewise, we have

$$\lambda_1(re(\alpha)) = \lambda(r) + 2\pi i\alpha r\lambda'(r) - 2\pi^2 r(\lambda'(r) + r\lambda''(r))\alpha^2 + O_G(r\alpha^3).$$

Inserting yields

$$\begin{aligned} J_1 = \lambda(r)^{-\nu-1}\bigl(1 + O_G\bigl(\eta_G^{-\nu}\bigr)\bigr) \int_{-\delta}^{\delta} \frac{p(\lambda(r)), r)}{-q_x(\lambda(r), r)} \bigl(1 + b(r)\alpha + O_G(\alpha^2)\bigr) \\ \exp\Biggl(- \frac{2\pi i\alpha\nu r\lambda'(r)}{\lambda(r)} + \frac{2\pi^2 r\alpha^2\nu(\lambda(r)\lambda'(r) + r\lambda(r)\lambda''(r) - r\lambda'(r)^2)}{\lambda(r)^2} \\ + O_G(r\alpha^3\nu) - 2\pi i k\alpha\Biggr) d\alpha. \end{aligned}$$

r was chosen in such a way that $\mu(r) = -\frac{r\lambda'(r)}{\lambda(r)} = \frac{k}{\nu}$. Thus, the coefficients of α in the exponent cancel out. Furthermore, note that

$$\frac{2\pi^2 r\nu(\lambda(r)\lambda'(r) + r\lambda(r)\lambda''(r) - r\lambda'(r)^2)}{\lambda(r)^2} = -2\pi^2 r\nu\mu'(r) \le -2\pi^2 r\nu c_1(G) < 0$$

by Lemma 3. We write $D = 2\pi^2 r\mu'(r)$ and use the standard estimates

$$\begin{aligned} &\int_{-\delta}^{\delta} (b(r)\alpha + O_G(\alpha)^2) \exp(-D\nu\alpha^2 + O_G(r\alpha^3\nu))\, d\alpha \\ &\quad = \int_{-\delta}^{\delta} (b(r)\alpha + O_G(\alpha^2) + O_G(r\alpha^4\nu)) \exp(-D\nu\alpha^2)\, d\alpha \\ &\quad = O_G\left(\int_0^{\delta} \alpha^2 \exp(-D\nu\alpha^2)\, d\alpha\right) + O_G\left(r\nu \int_0^{\delta} \alpha^4 \exp(-D\nu\alpha^2)\, d\alpha\right), \end{aligned}$$

$$\begin{aligned}
&\int_{-\delta}^{\delta} \exp(-D\nu\alpha^2 + O_G(r\nu\alpha^3))\, d\alpha \\
&\quad = \int_{-\delta}^{\delta} \exp(-D\nu\alpha^2)\, d\alpha + O_G\left(r\nu \int_0^{\delta} \alpha^3 \exp(-D\nu\alpha^2)\right) \\
&\quad = \frac{\sqrt{\pi}}{\sqrt{D\nu}} - 2\int_{\delta}^{\infty} \exp(-D\nu\alpha^2)\, d\alpha + O_G\left(r\nu \int_0^{\delta} \alpha^3 \exp(-D\nu\alpha^2)\right),
\end{aligned}$$

$$\begin{aligned}
\int_0^{\delta} \alpha^p \exp(-D\nu\alpha^2)\, d\alpha &= (D\nu)^{-(p+1)/2} \int_0^{\sqrt{D\nu}\delta} x^p \exp(-x^2)\, dx \\
&\le (D\nu)^{-(p+1)/2} \int_0^{\infty} x^p \exp(-x^2)\, dx \\
&= O\big((D\nu)^{-(p+1)/2}\big)
\end{aligned}$$

and

$$\begin{aligned}
\int_{\delta}^{\infty} \exp(-D\nu\alpha^2)\, d\alpha &= \frac{1}{2\sqrt{D\nu}} \int_{D\nu\delta^2}^{\infty} x^{-1/2} \exp(-x)\, dx \\
&\le \frac{1}{2D\nu\delta} \exp(-D\nu\delta^2).
\end{aligned}$$

Since $\mu'(y)$ is bounded on [0, 1] by Lemma 3, there are constants $c_8(G)$ and $c_9(G)$ such that

$$c_8(G)\frac{k}{\nu} \le r \le c_9(G)\frac{k}{\nu}.$$

Therefore, these estimates imply that

$$J_1 = \frac{p(\lambda(r), r)}{-\lambda(r) q_x(\lambda(r), r)} (2\pi r\nu\mu'(r))^{-1/2} \lambda(r)^{-\nu} (1 + O_G((D\nu)^{-1/2})).$$

Finally, we estimate J_2: by Lemma 7,

$$\begin{aligned}
|J_2| &= \left| \int_{\delta \le |\alpha| \le 1/2} [x^\nu] \frac{p(x, re(\alpha))}{q(x, re(\alpha))} e(-k\alpha)\, d\alpha \right| \\
&\le 2c_6(G)[x^\nu] \frac{p(x, r)}{q(x, r)} \int_{\delta}^{1/2} \exp(-c_7(G) r\nu \|\alpha\|^2)\, d\alpha \\
&= O_G(\lambda(r)^{-\nu} \exp(-c_7(G) r\nu\delta^2)).
\end{aligned}$$

Altogether, we have established formula (14) for $\frac{k}{\nu} \leq \mu(1)$. We indicate how to extend it to the case $\frac{k}{\nu} \geq \mu(1)$: if A is taken as in Lemma 3 and $l = A\nu - k$, we have

$$|F(k,\nu)| = [x^\nu y^l] \frac{p(xy^A, y^{-1})}{q(xy^A, y^{-1})}.$$

The proof now goes along the same lines, with $\mu(y)$ replaced by $A - \mu(y^{-1})$ and the roles of y and y^{-1} interchanged. □

Corollary 9. *There is a constant $c_{10}(G)$ depending only on G such that the number of integers $\leq N$ with sum of digits k is bounded below by*

$$c_{10}(G) \cdot \frac{p(\lambda(r), r)}{-\lambda(r) q_x(\lambda(r), r)} r^{-k} \lambda(r)^{-\nu} k^{-1/2} \tag{15}$$

uniformly for $k \leq \mu(1)\nu$, $k \to \infty$, where $\nu + 1$ is the number of digits of N.

Theorem 8 is a consequence of general theorems of Bender and Richmond [1, 2] (see also Drmota [6]) in the case when r is bounded above and below by positive constants. Equivalenty, $\frac{k}{\nu} \in [a, b]$ for constants $0 < a < b < A$. It is easy to see that the sum of digits asymptotically follows a normal distribution with mean $\mu(1)\nu$ and variance $\mu'(1)\nu$: note first that $r^{-k}\lambda(r)^{-\nu} = (r^{\mu(r)}\lambda(r))^{-\nu}$. The maximal value of $-\log(r^{\mu(r)}\lambda(r))$ is achieved when the derivative is 0, i.e.

$$\frac{\mu(r)}{r} + \mu'(r)\log(r) + \frac{\lambda'(r)}{\lambda(r)} = \mu'(r)\log(r) = 0,$$

which happens if $r = 1$. The following corollary of Theorem 8 gives precise information:

Corollary 10. *When k is near the mean value, i.e. $\Delta = \mu(1)\nu - k = o(\nu)$, we have*

$$\begin{aligned} |F(k,\nu)| = \frac{p(\lambda(1), 1)}{-\lambda(1) q_x(\lambda(1), 1)} \lambda(1)^{-\nu} \cdot (2\pi\nu\mu'(1))^{-1/2} \\ \exp\left(-\frac{\Delta^2}{2\nu\mu'(1)}\right)\left(1 + O_G\left(\frac{\Delta}{\nu} + \nu^{-1/2}\right)\right). \end{aligned} \tag{16}$$

Proof: We set $\eta = 1 - r$ and use the Taylor expansion of μ around 1 to find that

$$\eta = \frac{\Delta}{\nu\mu'(1)} + O_G\left(\frac{\Delta^2}{\nu^2}\right).$$

Then,

$$r^{-k} = \exp(-k\log(1-\eta)) = \exp\left(k\eta + \frac{1}{2}k\eta^2 + O(k\eta^3)\right)$$

and

$$\lambda(r)^{-\nu} = \lambda(1)^{-\nu} \exp\left(\nu\left(\frac{\lambda'(1)}{\lambda(1)}\eta + \frac{\lambda'(1)^2 - \lambda(1)\lambda''(1)}{2\lambda(1)^2}\eta^2 + O_G(\eta^3)\right)\right).$$

Furthermore,

$$\begin{aligned}&\frac{p(\lambda(r), r)}{-\lambda(r)q_x(\lambda(r), r)}(2\pi r\nu\mu'(r))^{-1/2}\\ &\quad = \frac{p(\lambda(1), 1)}{-\lambda(1)q_x(\lambda(1), 1)}(2\pi\nu\mu'(1))^{-1/2}\left(1 + O_G\left(\frac{\Delta}{\nu}\right)\right).\end{aligned}$$

We insert $k = \mu(1)\nu - \Delta$ and use the formula

$$\mu'(y) = \frac{y\lambda'(y)^2 - y\lambda(y)\lambda''(y) - \lambda(y)\lambda'(y)}{\lambda(y)^2}$$

to obtain the stated result. □

Remark 2. Note that $\frac{p(\lambda(1),1)}{-\lambda(1)q_x(\lambda(1),1)}\lambda(1)^{-\nu}$ is (asymptotically) the number of all integers with an expansion of $\leq \nu$ digits.

Corollary 11. *If k is small, i.e. $k = o(\nu)$, we have*

$$|F(k, \nu)| = (2\pi k)^{-1/2} \exp\left(-k\log\frac{k}{\nu} + k + \frac{1 - \lambda''(0)}{2}\cdot\frac{k^2}{\nu} + O_G\left(\frac{k^3}{\nu^2} + \frac{1}{\sqrt{k}}\right)\right). \tag{17}$$

Remark. It is easy to check that

$$\lambda''(0) = \begin{cases} 4 & d = 2,\ a_1 = a_2 = 1,\\ 2 & d = 1,\ a_1 = 2 \text{ or } d > 2,\ a_1 = a_2 = \cdots = a_d = 1,\\ 0 & \text{otherwise.}\end{cases}$$

Proof: We see that

$$r = \frac{k}{\nu} - \frac{\mu''(0)}{2}\left(\frac{k}{\nu}\right)^2 + O_G\left(\left(\frac{k}{\nu}\right)^3\right),$$

since $\mu'(0) = 1$. This gives us

$$r\mu'(r) = \frac{k}{\nu} + \frac{\mu''(0)}{2}\left(\frac{k}{\nu}\right)^2 + O_G\left(\left(\frac{k}{\nu}\right)^3\right)$$

and

$$\lambda(r) = 1 - r + \frac{\lambda''(0)}{2} r^2 + O_G(r^3) = 1 - \frac{k}{\nu} + \frac{\lambda''(0) + \mu''(0)}{2} \left(\frac{k}{\nu}\right)^2 + O_G\left(\left(\frac{k}{\nu}\right)^3\right).$$

Therefore,

$$\frac{p(\lambda(r), r)}{-\lambda(r) q_x(\lambda(r), r)} = \frac{p(\lambda(0), 0)}{-\lambda(0) q_x(\lambda(0), 0)} \left(1 + O_G\left(\frac{k}{\nu}\right)\right) = 1 + O_G\left(\frac{k}{\nu}\right),$$

$$2\pi r \mu'(r) \nu = 2\pi k \left(1 + O_G\left(\frac{k}{\nu}\right)\right),$$

$$-k \log r = -k \log \frac{k}{\nu} + \frac{\mu''(0)}{2} \cdot \frac{k^2}{\nu} + O_G\left(\frac{k^3}{\nu^2}\right),$$

and

$$-\nu \log \lambda(r) = k - \frac{\lambda''(0) + \mu''(0) - 1}{2} \cdot \frac{k^2}{\nu} + O_G\left(\frac{k^3}{\nu^2}\right).$$

Inserting in (14) yields the stated result. □

Example 1. It is not difficult to check that our result agrees with (3) in the case $d = 1$, $a_1 = g$. We will consider the classical Zeckendorf expansion ($d = 2$, $a_1 = a_2 = 1$, $G_0 = 1$, $G_1 = 2$) as another example. In this case, we have

$$p(x, y) = 1 + xy, \; q(x, y) = 1 - x - yx^2,$$

yielding

$$\lambda(y) = \frac{1}{2y}(\sqrt{1 + 4y} - 1), \; \mu(y) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + 4y}}\right).$$

If we set $\frac{k}{n} = \gamma$, we obtain

$$|F(k, \nu)| \sim \sqrt{\frac{(1-\gamma)^3}{2\pi\gamma(1-2\gamma)^3 \nu}} \cdot \left(\frac{(1-\gamma)^{1-\gamma}}{\gamma^\gamma (1-2\gamma)^{1-2\gamma}}\right)^\nu. \tag{18}$$

The mean value is given by $\mu\nu = \mu(1)\nu = \frac{1}{2}(1 - \frac{1}{\sqrt{5}})\nu$, the variance by $\sigma^2 \nu = \mu'(1)\nu = 5^{-3/2}\nu$.

3 Distribution in residue classes

The aim of this section is to prove that $F(k, \nu)$ is well-distributed in residue classes modulo m provided that m is not too large and there is no prime divisor P of m such that G_n is constant modulo P for all but finitely many values of n.

Theorem 12. *Let $V(k, N)$ be the set of integers $\leq N$ with G-ary sum of digits k. There exist positive constants $k_0(G)$, $c_{11}(G)$, $c_{12}(G)$, $c_{13}(G)$ (depending on G only) such that for all $l = \max(k, A\nu - k) \geq k_0(G)$ (ν denoting the number of G-ary digits of N), $2 \leq m < \exp(c_{13}(G)l^{1/2})$, $h \in \mathbb{Z}$, for which there is no prime divisor P of m such that (G_n) is constant modulo P for all but finitely many values of n, we have*

$$\begin{aligned}\Big| |\{n \in V(k, N) : n \equiv h \mod m\}| - \frac{1}{m}|V(k, N)| \Big| \\ < \frac{c_{11}(G)}{m}|V(k, N)| \exp\left(- c_{12}(G) \frac{k}{\log m}\right). \qquad (19)\end{aligned}$$

Remark. The condition on the prime factors of m is a necessary one. If (G_n) was constant modulo P for all but finitely many values of n, the restriction on the sum of digits would imply a condition on the residues modulo P. Note that $(g^n)_{n\geq 0}$ is constant modulo P for all but finitely many values of n if and only if $P|g(g-1)$.

Proof: We follow the lines of [19] again. Again, we consider the case $k \leq \mu(1)\nu$ only. If

$$D(z, \gamma) = \sum_{n=1}^{N} z^{s_G(n)} e(n\gamma),$$

where $z \in \mathbb{C}$, $\gamma \in \mathbb{R}$, we have

$$\frac{1}{m}\sum_{p=1}^{m} e\left(- \frac{hp}{m}\right) D\left(z, \frac{p}{m}\right) = \sum_{\substack{1\leq n\leq N \\ n\equiv h \mod m}} z^{s_G(n)}.$$

Now we take r as in the proof of Theorem 8 and obtain

$$\begin{aligned}&|\{n \leq N : s_G(n) = k, n \equiv h \mod m\}| \\ &= r^{-k} \int_0^1 e(-k\beta) \sum_{\substack{1\leq n\leq N \\ n\equiv h \mod m}} (re(\beta))^{s_G(n)}\, d\beta \\ &= \frac{1}{m} r^{-k} \sum_{p=1}^{m} \int_0^1 e\left(- k\beta - \frac{hp}{m}\right) D\left(re(\beta), \frac{p}{m}\right) d\beta.\end{aligned}$$

Obviously, the summand corresponding to $p = m$ equals $\frac{1}{m}|V(k, N)|$. Thus we have to estimate

$$\frac{1}{m} r^{-k} \sum_{p=1}^{m-1} \int_0^1 \left| D\left(re(\beta), \frac{p}{m} \right) \right| d\beta.$$

We write N in base-G representation:

$$N = \sum_{j=0}^{L(N)} \epsilon_j G_j = \sum_{i=1}^{t} \epsilon_{\nu_i} G_{\nu_i},$$

where $\nu_1 > \nu_2 > \cdots > \nu_t$ and all ϵ_{ν_i} are positive (i.e., we neglect all digits 0 in the base-G representation). Then, the set $\{0, \ldots, N\}$ can be partitioned into sets A_l, where A_l is the set of integers representable as

$$\sum_{i=1}^{l-1} \epsilon_{\nu_i} G_{\nu_i} + a G_{\nu_l} + b,$$

where $0 \leq a \leq \epsilon_{\nu_l} - 1$ and b is an arbitrary integer with $\leq \nu_l$ G-ary digits. Let the set of all such integers be denoted by B_{ν_l}. Additionally, we set $A_{t+1} = \{N\}$. Then we have

$$\begin{aligned}
1 + D(re(\beta), \gamma) &= \sum_{n=0}^{N} (re(\beta))^{s_G(n)} e(n\gamma) \\
&= \sum_{l=1}^{t+1} \sum_{n \in A_l} (re(\beta))^{s_G(n)} e(n\gamma) \\
&= (re(\beta))^{s_G(N)} e(N\gamma) + \sum_{l=1}^{t} \sum_{a=0}^{\epsilon_{\nu_l}-1} \sum_{b \in B_{\nu_l}} (re(\beta))^{\epsilon_{\nu_1} + \cdots + \epsilon_{\nu_{l-1}} + a + s_G(b)} \\
&\qquad e\left(\left(\sum_{i=1}^{l-1} \epsilon_{\nu_i} G_{\nu_i} + a G_{\nu_l} + b \right) \gamma \right),
\end{aligned}$$

from which it follows that

$$\begin{aligned}
|D(re(\beta), \gamma)| &\leq 2 + \sum_{l=1}^{t} r^{\epsilon_{\nu_1} + \cdots + \epsilon_{\nu_{l-1}}} \left| \sum_{a=0}^{\epsilon_{\nu_l}-1} (re(\beta + G_{\nu_l}\gamma))^a \right| \left| \sum_{b \in B_{\nu_l}} (re(\beta))^{s_G(b)} e(b\gamma) \right| \\
&\leq 2 + \sum_{l=1}^{t} r^{l-1} \epsilon_{\nu_l} \left| \sum_{b \in B_{\nu_l}} (re(\beta))^{s_G(b)} e(b\gamma) \right|.
\end{aligned}$$

We write

$$u_\nu(\beta,\gamma) := \sum_{b\in B_\nu} (re(\beta))^{s_G(b)} e(b\gamma).$$

Then we see that u_ν satisfies a recursive relation:

Lemma 13. *For $\nu \geq 2d$, we have*

$$u_\nu(\beta,\gamma) = \sum_{i=1}^{d} \left(\sum_{j=0}^{a_i-1} (re(\beta + G_{\nu-i}\gamma))^j \right) \left(\prod_{l=1}^{i-1} (re(\beta + G_{\nu-l}\gamma))^{a_l} \right) u_{\nu-i}(\beta,\gamma). \quad (20)$$

Proof: This is proved in the same way as Proposition 2: note that appending a sequence of the form $(\epsilon, a_{i-1}, \dots, a_1)$ with $\epsilon < a_i$ to a good sequence of length $\nu - i$ gives a factor of

$$(re(\beta))^{a_1+\cdots+a_{i-1}+\epsilon} e((G_{\nu-1}a_1 + \cdots + G_{\nu-i+1}a_{i-1} + G_{\nu-i}\epsilon)\gamma).$$

$\square$

The recurrence can be used to prove an analogue of Lemma 7:

Lemma 14. *There exist constants $c_{14}(G)$, $c_{15}(G)$ depending only on G such that*

$$u_\nu(\beta,\gamma) \leq c_{14}(G) \exp\left(-c_{15}(G) r \sum_{n=0}^{\nu-1} \|\beta + G_n\gamma\|^2 \right) u_\nu(0,0) \quad (21)$$

for all $0 < r \leq 1$ and all $\beta, \gamma \in \mathbb{R}$.

Proof: This is done almost analogously to the proof of Lemma 7. For $a_1 > 1$ (the other case is similar), we have

$$|u_\nu(\beta,\gamma)| \leq \left(1 - \frac{2r}{a_1}\|\beta + G_{\nu-1}\gamma\|^2\right) \sum_{j=0}^{a_1-1} r^j |u_{\nu-1}(\beta,\gamma)|$$
$$+ \sum_{i=2}^{d} \sum_{j=0}^{a_i-1} r^j \prod_{l=1}^{i-1} r^{a_l} |u_{\nu-i}(\beta,\gamma)|$$

by the same argument as in Lemma 7. If we define $U_\nu(\beta,\gamma)$ by $U_\nu(\beta,\gamma)=u_\nu(0,0)$ for $\nu<2d$ and

$$U_\nu(\beta,\gamma)=\left(1-\frac{2r}{a_1}\|\beta+G_{\nu-1}\gamma\|^2\right)\sum_{j=0}^{a_1-1}r^jU_{\nu-1}(\beta,\gamma)$$
$$+\sum_{i=2}^{d}\sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}U_{\nu-i}(\beta,\gamma),$$

we know that $|u_\nu(\beta,\gamma)|\le U_\nu(\beta,\gamma)$ for all ν, and the argument of Lemma 7 shows that

$$U_\nu(\beta,\gamma)\le\left(1-\frac{2r}{a_1d}\|\beta+G_{\nu-1}\gamma\|^2\right)\sum_{i=1}^{d}\sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}U_{\nu-i}(\beta,\gamma).$$

Write $C_i:=\sum_{j=0}^{a_i-1}r^j\prod_{l=1}^{i-1}r^{a_l}$. For a sequence $\mathbf{x}=(x_n)_{n\ge0}$ with $1\ge x_n\ge1-\frac{r}{2}$, define $W_\nu(\mathbf{x})$ by $W_\nu(\mathbf{x})=u_\nu(0,0)$ for $\nu<2d$ and

$$W_\nu(\mathbf{x})=x_\nu\sum_{i=1}^{d}C_iW_{\nu-i}(\mathbf{x}).$$

Since $x_\nu C_1\ge(1-\frac{r}{2})(1+r)=1+\frac{r(1-r)}{2}\ge1$, we know that $W_\nu(\mathbf{x})$ is increasing, and we also know that the C_i are decreasing, so $C_iW_{\nu-i}(\mathbf{x})$ is always decreasing. Let $\mathbf{x}^{(n)}$ be the sequence $\mathbf{x}$ with 1 at the place of x_n. We claim that

$$W_\nu(\mathbf{x})\le\left(1-\frac{1-x_n}{d}\right)W_\nu\big(\mathbf{x}^{(n)}\big)$$

holds for $\nu\ge n$. This is trivial for $\nu=n$, since we have

$$W_n(\mathbf{x})=x_nW_n\big(\mathbf{x}^{(n)}\big)$$

and $\big(1-\frac{1-x_n}{d}\big)\ge x_n$. We proceed by induction: for $1\le j\le d-1$, we have

$$\begin{aligned}W_{n+j}(\mathbf{x})&=\sum_{i=1}^{j-1}C_iW_{n+j-i}(\mathbf{x})+C_jW_n(\mathbf{x})+\sum_{i=j+1}^{d}C_iW_{n+j-i}(\mathbf{x})\\&\le\left(1-\frac{1-x_n}{d}\right)\sum_{i=1}^{j-1}C_iW_{n+j-i}\big(\mathbf{x}^{(n)}\big)+x_nC_jW_n\big(\mathbf{x}^{(n)}\big)+\sum_{i=j+1}^{d}C_iW_{n+j-i}\big(\mathbf{x}^{(n)}\big)\\&\le\left(1-\frac{1-x_n}{d}\right)\sum_{i=1}^{j-1}C_iW_{n+j-i}\big(\mathbf{x}^{(n)}\big)+\frac{d-j+x_n}{d-j+1}\sum_{i=j}^{d}C_iW_{n+j-i}\big(\mathbf{x}^{(n)}\big)\end{aligned}$$

$$\leq \left(1 - \frac{1 - x_n}{d}\right) \sum_{i=1}^{d} C_i W_{n+j-i}\left(\mathbf{x}^{(n)}\right)$$
$$= \left(1 - \frac{1 - x_n}{d}\right) W_{n+j}\left(\mathbf{x}^{(n)}\right).$$

For $j \geq d$, the induction is even simpler. Another straightforward induction shows that

$$W_\nu(\mathbf{x}) \leq \prod_{j=2d}^{\nu} \left(1 - \frac{1 - x_j}{d}\right) W_\nu(\mathbf{1}),$$

where $\mathbf{1}$ is the sequence consisting only of 1's. In our special case, we take $x_n = 1 - \frac{2r}{a_1 d}\|\beta + G_{n-1}\gamma\|^2$ to show that

$$\begin{aligned} U_\nu(\beta, \gamma) &\leq \prod_{n=2d}^{\nu} \left(1 - \frac{2r}{a_1 d^2}\|\beta + G_{n-1}\gamma\|^2\right) u_\nu(0, 0) \\ &\leq \left(1 - \frac{1}{2a_1 d^2}\right)^{1-2d} \prod_{n=1}^{\nu} \left(1 - \frac{2r}{a_1 d^2}\|\beta + G_{n-1}\gamma\|^2\right) u_\nu(0, 0) \\ &\leq \left(1 - \frac{1}{2a_1 d^2}\right)^{1-2d} \exp\left(-\frac{2r}{a_1 d^2}\sum_{n=0}^{\nu-1}\|\beta + G_n\gamma\|^2\right) u_\nu(0, 0), \end{aligned}$$

which finally proves the claim. □

Lemma 15. *Let $m, \rho \in \mathbb{N}$ and $1 \leq p \leq m - 1$. If there is no prime divisor P of m such that the sequence G_n is constant modulo P for all but finitely many values of n, we have*

$$\sum_{n=0}^{\rho-1} \left\| \beta + G_n \frac{j}{m} \right\|^2 \geq c_{16}(G) \frac{\rho}{\log m} + O_G(1). \tag{22}$$

Proof: Without loss of generality, we may assume that $(p, m) = 1$ (cancellation of common factors only improves the bound, and the conditions keep true). First, we show that there exist constants $c_{17}(G)$ and $c_{18}(G)$ such that, among any set of $c_{17}(G) + c_{18}(G) \log m$ consequent integers, there is an integer n such that

$$\left\| (G_{n+1} - G_n) \frac{p}{m} \right\| \geq \frac{1}{2(a_1 + \cdots + a_d)}.$$

For this purpose, we define a sequence $(A_n)_{n\geq 0}$ by $A_n \equiv (G_{n+1} - G_n)p \mod m$ and $-\frac{m}{2} < A_n \leq \frac{m}{2}$. We want to show that there are constants $c_{17}(G)$ and $c_{18}(G)$ such that for all $I \geq 0$, there is an $n < c_{17}(G) + c_{18}(G) \log m$ with

$$\left\| \frac{A_{I+n}}{m} \right\| \geq \frac{1}{2(a_1 + \cdots + a_d)}.$$

First of all, we will take $c_{17}(G) \geq d$. Consider the values $A_I, A_{I+1}, \ldots, A_{I+d-1}$. If one of them has absolute value $\geq \frac{m}{2(a_1+\cdots+a_d)}$, we are done. Otherwise, define the sequence $(B_n)_{n\geq 0}$ by $B_n = A_{I+n}$ $(n = 0, \ldots, d-1)$ and

$$B_{n+d} = a_1 B_{n+d-1} + a_2 B_{n+d-2} + \cdots + a_d B_n.$$

Note that $B_n \equiv A_{I+n}$ for all values of n. Now we use a result of Brauer [4] that was also applied in [21]: The characteristic polynomial

$$x^d - a_1 x^{d-1} - \cdots - a_d$$

has a dominating root $\theta \in [a_1, a_1 + 1)$ that is a Pisot number, i.e., all conjugates $\theta_2, \ldots, \theta_d$ (if $d > 1$) have modulus < 1. Thus, we can express B_n by an explicit formula:

$$B_n = \beta\theta^n + \sum_{i=2}^{d} \beta_i n^{\delta(i)} \theta_i^n,$$

where the β_i are linear combinations of the initial values $B_0, B_1, \ldots, B_{d-1}$ (with algebraic coefficients depending only on the characteristic polynomial). Therefore, there exist constants $c_{19}(G)$ and $\kappa_3(G)$ such that

$$|B_n - \beta\theta^n| \leq c_{19}(G) m n^{\kappa_3(G)} |\theta_2|^n.$$

The coefficient β is also a linear combination of the initial values, i.e. it is of the form

$$x_0 B_0 + \cdots + x_{d-1} B_{d-1},$$

where the x_i are algebraic numbers depending on the characteristic polynomial. By a result of Schmidt (cf. [7, Theorem 2.1]), the inequality

$$0 < |x_0 B_0 + \cdots + x_{d-1} B_{d-1}| < M^{-d+1-\epsilon}$$

with $|B_n| \leq M$ has only finitely many solutions for every $\epsilon > 0$; therefore, there are constants $c_{20}(G) > 0$ and $\kappa_4(G)$ such that either

$$\beta = x_0 B_0 + \cdots + x_{d-1} B_{d-1} = 0$$

or

$$|\beta| = |x_0 B_0 + \cdots + x_{d-1} B_{d-1}| \geq c_{20}(G) M^{-\kappa_4(G)}$$

whenever $|B_0|, \dots, |B_{d-1}| \le M$. We know that β cannot be 0, since then we would have $\lim_{n\to\infty} B_n = 0$, i.e. $A_n \equiv 0 \mod m$ for all but finitely many values of n. This contradicts the assumptions on G: as $(p, m) = 1$, G_n would be constant modulo m for all but finitely many values of n. Therefore, since $|B_n| \le \frac{m}{2(a_1+\cdots+a_d)}$ for $0 \le n \le d-1$, $|\beta| \ge c_{21}(G)m^{-\kappa_4}$, where $c_{21} > 0$ depends only on G. It follows that

$$|B_n| \ge c_{21}(G)m^{-\kappa_4}\theta^n - c_{19}(G)mn^{\kappa_3(G)}|\theta_2|^n$$

for all n; there are constants $c_{22}(G)$ and $c_{23}(G)$ such that

$$c_{21}(G)m^{-\kappa_4(G)}\theta^n - c_{19}(G)mn^{\kappa_3(G)}|\theta_2|^n \ge \frac{m}{2(a_1+\cdots+a_d)}$$

for all $n \ge c_{22}(G)\log m + c_{23}(G)$. Thus, $|B_n| \ge \frac{m}{2(a_1+\cdots+a_d)}$ for some $n \le c_{22}(G)\log m + c_{23}(G)$; for the smallest index n for which this is true, we must also have $|B_n| \le \frac{m}{2}$, so

$$\left\| \frac{A_{I+n}}{m} \right\| = \left\| \frac{B_n}{m} \right\| \ge \frac{1}{2(a_1+\cdots+a_d)}.$$

This proves the claim, and the lemma is a simple consequence if we make use of the inequality

$$\left\| \beta + G_{k+1}\frac{p}{m} \right\|^2 + \left\| \beta + G_k\frac{p}{m} \right\|^2 \ge \frac{1}{2}\left\| G_{k+1}\frac{p}{m} - G_k\frac{p}{m} \right\|^2.$$

□

We turn back to the proof of Theorem 12. By the preceding lemmas, there are constants $c_{24}(G)$ and $c_{25}(G)$ such that

$$u_\nu\left(\beta, \frac{p}{m}\right) \le c_{24}(G)\exp\left(-c_{25}(G)\frac{r\nu}{\log m}\right)u_\nu(0,0).$$

Therefore, since $u_{\nu_l}(0,0) = \sum_{b\in B_{\nu_l}} r^{s_G(b)}$, we have

$$\left| D\left(re(\beta), \frac{p}{m}\right)\right| \le c_{24}(G)\left(\sum_{l=1}^{t} r^{l-1}\epsilon_{\nu_l}\exp\left(-c_{25}(G)\frac{r\nu_l}{\log m}\right)\sum_{b\in B_{\nu_l}} r^{s_G(b)}\right) + O_G(1).$$

We divide the sum on the right side into two parts by defining the integer q for which $\nu_q \ge \nu/2 > \nu_{q+1}$ (set $\nu_{t+1} = 0$): the first part is defined by

$$\begin{aligned} S_1 &:= \sum_{l=1}^{q} r^{l-1}\epsilon_{\nu_l}\exp\left(-c_{25}(G)\frac{r\nu_l}{\log m}\right)\sum_{b\in B_{\nu_l}} r^{s_G(b)} \\ &\le c_{26}(G)\exp\left(-c_{25}(G)\frac{r\nu/2}{\log m}\right)\sum_{l=1}^{q} r^{l-1}\sum_{b\in B_{\nu_l}} r^{s_G(b)}, \end{aligned}$$

where $c_{26}(G)$ is the largest possible digit that can appear in a G-ary expansion. Next, we observe that

$$\sum_{b\in B_{\nu_l}} r^{s_G(b)} = [x^{\nu_l}]\frac{p(x,r)}{q(x,r)}.$$

By Corollary 5, this equals

$$\sum_{b\in B_{\nu_l}} r^{s_G(b)} = -\frac{p(\lambda(r),r)}{q_x(\lambda(r),r)}\lambda(r)^{-\nu_l-1}\bigl(1+O_G\bigl(\eta_G^{-\nu_l}\bigr)\bigr),$$

so that we obtain

$$\begin{aligned}
S_1 \leq{}& c_{26}(G)\exp\left(-c_{25}(G)\frac{r\nu/2}{\log m}\right)\sum_{l=1}^{q} r^{l-1}\\
&\cdot\left(-\frac{p(\lambda(r),r)}{q_x(\lambda(r),r)}\right)\lambda(r)^{-\nu_l-1}\left(1+O_G\left(\eta_G^{-\nu_l}\right)\right)\\
={}& c_{26}(G)\exp\left(-c_{25}(G)\frac{r\nu/2}{\log m}\right)\left(1+O_G\bigl(\eta_G^{-\nu/2}\bigr)\right)\\
&\cdot\left(-\frac{p(\lambda(r),r)}{\lambda(r)q_x(\lambda(r),r)}\right)\lambda(r)^{-\nu}\cdot\sum_{l=1}^{q} r^{l-1}\lambda(r)^{\nu-\nu_l}\\
\leq{}& c_{26}(G)\exp\left(-c_{25}(G)\frac{r\nu/2}{\log m}\right)\left(1+O_G\bigl(\eta_G^{-\nu/2}\bigr)\right)\\
&\cdot\left(-\frac{p(\lambda(r),r)}{\lambda(r)q_x(\lambda(r),r)}\right)\lambda(r)^{-\nu}\cdot\sum_{j=0}^{\infty}(r\lambda(r))^{j}.
\end{aligned}$$

If $a_1 \geq 2$, we have $q((1+r+\cdots+r^{a_1-1})^{-1},r) < 0$ and thus $\lambda(r) \leq (1+r+\cdots+r^{a_1-1})^{-1} \leq (1+r)^{-1}$, which in turn means that $r\lambda(r) \leq \frac{r}{1+r} \leq \frac{1}{2}$. If $a_1 = 1$, we also have $a_2 = 1$ and thus $q(\frac{\sqrt{1+4r}-1}{2r}, r) \leq 0$, so we obtain $r\lambda(r) \leq \frac{\sqrt{1+4r}-1}{2} \leq \frac{\sqrt{5}-1}{2}$. This means that the infinite sum converges and is bounded by $\frac{3+\sqrt{5}}{2}$. Together with Corollary 9, we obtain

$$S_1 = O_G\left(|V(k,N)|r^k k^{1/2}\exp\left(-c_{25}(G)\frac{r\nu/2}{\log m}\right)\right).$$

The other part of the sum,

$$S_2 := \sum_{l=q+1}^{t} r^{l-1}\epsilon_{\nu_l}\exp\left(-c_{25}(G)\frac{r\nu_l}{\log m}\right)\sum_{b\in B_{\nu_l}} r^{s_G(b)},$$

can be estimated as follows:

$$\begin{aligned}
S_2 &\le c_{26}(G) \sum_{l=q+1}^{t} r^{l-1} \sum_{b \in B_{\nu_l}} r^{s_G(b)} \\
&= c_{26}(G) \sum_{l=q+1}^{t} r^{l-1} \cdot \left(-\frac{p(\lambda(r), r)}{q_x(\lambda(r), r)} \right) \lambda(r)^{-\nu_l - 1} \left(1 + O_G\left(\eta_G^{-\nu_l}\right)\right) \\
&\le c_{26}(G) \cdot \left(-\frac{p(\lambda(r), r)}{\lambda(r) q_x(\lambda(r), r)} \right) \sum_{i=1}^{t-q} r^{i-1} \lambda(r)^{-\nu/2 + (i-1)} (1 + O_G(1)) \\
&\le c_{26}(G) \cdot \left(-\frac{p(\lambda(r), r)}{\lambda(r) q_x(\lambda(r), r)} \right) \lambda(r)^{-\nu/2} \sum_{j=0}^{\infty} (r\lambda(r))^j (1 + O_G(1))
\end{aligned}$$

and thus

$$S_2 = O_G\left(|V(k, N)| r^k k^{1/2} \lambda(r)^{\nu/2}\right).$$

It is known that $\mu'(y)$ is bounded above and below by positive constants depending only on G, which means that there are constants c_8, c_9 such that

$$c_8(G) \frac{k}{\nu} \le r \le c_9(G) \frac{k}{\nu}.$$

Furthermore, $\lambda'(y) = -\frac{\lambda(y)\mu(y)}{y}$ is strictly negative on $(0, 1]$ with $\lim_{y \to 0+} \lambda'(y) = -1$, so it is bounded above and below by negative constants. So there are constants $c_{27}(G)$ and $c_{28}(G)$ such that

$$c_{27}(G) \frac{k}{\nu} \le \lambda(0) - \lambda(r) = 1 - \lambda(r) \le c_{28}(G) \frac{k}{\nu},$$

and thus

$$\lambda(r)^{\nu/2} \le \left(1 - c_{27}(G) \frac{k}{\nu}\right)^{\nu/2} \le \exp\left(-\frac{c_{27}(G)}{2} k\right).$$

Altogether, we obtain

$$S_1 + S_2 = O_G\left(|V(k, N)| r^k k^{1/2} \left(\exp\left(-c_{29}(G) \frac{k}{\log m}\right) + \exp\left(-\frac{c_{27}(G)}{2} k\right)\right)\right),$$

which proves Theorem 12. □

Remark. As an example, we note that, since the Fibonacci numbers clearly satisfy the condition for any modulus, the set of integers with a fixed number of 1's in the Zeckendorf representation is well-distributed modulo any integer modulus. As in [19], Theorem 12 can also be used to prove the following:

Corollary 16. *If $z \in \mathbb{N}$, $z \geq 2$, then there are constants $N_0(G)$, $c_{30}(G)$, $c_{31}(G)$ (depending on G and z) such that for all $N \geq N_0(G)$ and all k with*

$$|\mu(1)\nu - k| < c_{30}(G)(\log N)^{3/4},$$

where $\nu + 1$ is the number of G-ary digits of N, the number of integers in $V(k, N)$ which are not divisible by the z-th power of a prime P in the set

$$\mathcal{P} := \{P : P \text{ prime}, P \text{ satisfies the condition of Theorem 12}\}$$

is given by

$$\left(\zeta(z) \prod_{P \in \mathcal{P}} \left(1 - \frac{1}{p^z}\right)\right)^{-1} |V(k, N)| \left(1 + O\left(\exp(-c_{31}(G)(\log N)^{1/2})\right)\right). \quad (23)$$

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