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# The Number of Spanning Trees in Self-Similar Graphs

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**Abstract.** The number of spanning trees of a graph, also known as the complexity, is computed for graphs constructed by a replacement procedure yielding a self-similar structure. It is shown that under certain symmetry conditions exact formulas for the complexity can be given. These formulas indicate interesting connections to the theory of electrical networks. Examples include the well-known Sierpiński graphs and their higher-dimensional analogues. Several auxiliary results are provided on the way — for instance, a property of the number of rooted spanning forests is proven for graphs with a high degree of symmetry.

*Keywords*: spanning trees, self-similar graphs

## 1. Introduction

The number of spanning trees of a finite graph or multigraph $X$, also known as the *complexity* $\tau(X)$, is certainly one of the most important graph-theoretical parameters, and also one of the oldest. Its applications range from the theory of networks, where the number of spanning trees is used as a measure for network reliability [8] to theoretical chemistry, in connection with the enumeration of certain chemical isomers [4]. Kirchhoff's celebrated matrix tree theorem [12] relates the properties of an electrical network to the number of spanning trees in the underlying graph. There is a large variety of proofs for the matrix tree theorem, see for instance [3, 6, 10], and several extensions and generalizations have been provided in the past. One of them, due to Moon [16], which gives a general formula for *spanning forests*, will be of vital importance within this paper.

In view of the large number of interpretations and applications, it is not surprising that many papers deal with exact formulas for the number of spanning trees in

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certain graph classes. Cayley's well-known enumeration of labelled trees [5], which is equivalent to the enumeration of spanning trees in a complete graph $K_n$, can be seen as the starting point for this path of investigation: Cayley's theorem states that $\tau(K_n) = n^{n-2}$, and this formula has been generalized in many ways. Further examples of closed formulas include those for wheels, fans, ladders, prisms, and other special families. A collection of formulas can be found in Berge's book [2].

Recently, Chang, Chen, and Yang [7] studied the complexity of Sierpiński graphs and their higher-dimensional analoga; they determined explicit formulas for small dimensions and conjectured a general formula (see [7, Conjectures 6.1 and 6.2]). This is motivated by applications in statistical physics, where the complexity is of use in the study of lattices (cf. [20]). The quantity

$$h = \lim_{n\to\infty} \frac{\log(\tau(X_n))}{|VX_n|},$$

where $X_n$ is an increasing sequence of graphs (such as finite sections of a lattice) approaching an infinite graph, is a useful descriptor in this context. In [14] this quantity is termed *tree entropy* and its relation to the simple random walk is studied.

In the present paper, we will prove the conjecture of Chang, Chen, and Yang, and extend the result to a considerably larger class of graphs of *self-similar nature* obtained from a recursive construction that was essentially described in [22], where enumeration problems are treated from a more general point of view.

Even though these fractal-like graphs are quite popular in the study of electrical networks and random walks (see the lecture notes of Barlow [1], Kigami's book [11], and the references therein), it seems that the enumeration of spanning trees (especially exact enumeration) has been somewhat neglected up to now in spite of the obvious connections. The main result indicates the relation between the tree counting problem and electrical networks, Laplacians and random walks, as the number of spanning trees involves the so-called *resistance scaling factor* (equivalently, the *spectral dimension*). Using the method of "spectral decimation" (see [19]), it is possible to determine the spectrum rather explicitly by iterating rational functions for certain examples. However, "spectral decimation" is a restricted tool: On the one hand it is not always applicable — Example 2.3.2 is one instance where it does not work — on the other hand some additional work is needed for each example considered.

Finally, let us mention that there are several notions of self-similarity for graphs. The one used in [9] is a broad generalization of the notion used here. A different notion of self-similarity was developed in [17].

## 2. Self-Similar Graphs

### 2.1. Construction

In the following we describe a replacement procedure for graphs, which is a simplification of the construction considered and explained in [22]. Briefly, given a graph $X$, a new graph $Y$ is constructed by amalgamating several copies of $X$. This replacement procedure is the basis for an inductive construction of self-similar graphs.

Let $G$ be an edgeless graph with $\theta \geq 2$ distinguished vertices given by a map $\eta\colon \Theta \to VG$ ($\Theta = \{1,\ldots,\theta\}$). Let $s \geq 2$ substitutions be defined by injective maps $\sigma_i\colon \Theta \to VG$ for $i \in S = \{1,\ldots,s\}$. For any multigraph $X$ and any injective map $\varphi\colon \Theta \to VX$ a new multigraph $Y$ together with an injective map $\psi\colon \Theta \to VY$ is constructed as follows.

For each $i \in S$, let $Z_i$ be an isomorphic copy of the multigraph $X$, so that the vertex sets $VZ_1,\ldots,VZ_s$, and $VG$ are mutually disjoint. The isomorphism between $X$ and $Z_i$ is denoted by $\zeta_i\colon VX \to VZ_i$. Let $Z$ be the disjoint union of $G$ and $Z_1,\ldots,Z_s$, and define the relation $\sim$ on $VZ$ as the reflexive, symmetric, and transitive hull of

$$\bigcup_{i=0}^{s} \big\{(\sigma_i(j), \zeta_i(\varphi(j)))\colon\, j \in \Theta\big\} \subseteq VZ \times VZ.$$

Then the multigraph $Y$ is defined by its vertex set $VY = VZ/{\sim}$ and edge multiset

$$EY = \big\{\{[v],[w]\}\colon\, \{v,w\} \in EZ\big\},$$

where $[v]$ denotes the equivalence class of a vertex $v$. The map $\psi\colon \Theta \to VY$ is defined by $\psi(i) = [\eta(i)] \in VY$.

If the pair $(Y,\psi)$ is constructed as above from $(X,\varphi)$, then we write $(Y,\psi) = \mathsf{Copy}(X,\varphi)$. Since we fix $G$, $\eta$, and $\{\sigma_i\colon\, i \in S\}$, the dependence on these items is suppressed. Note that $Y$ is the amalgamation of $s$ isomorphic copies of $X$: For $i \in S$, define $\bar{Z}_i$ by

$$V\bar{Z}_i = \big\{[v]\colon\, v \in VZ_i\big\} \quad \text{and} \quad E\bar{Z}_i = \big\{\{[v],[w]\}\colon\, \{v,w\} \in EZ_i\big\}.$$

Then $\bar{Z}_i$ is isomorphic to $X$ and the isomorphism is given by

$$\bar{\zeta}_i\colon VX \to V\bar{Z}_i, \quad v \mapsto [\zeta_i(v)].$$

The subgraph $\bar{Z}_i$ is called the *i-th part* of $Y$. On the $i$-th part of $Y$ distinguished vertices are given by $\Theta \to V\bar{Z}_i$, $j \mapsto \bar{\zeta}_i(\varphi(j)) = [\sigma_i(j)]$.

## 2.2. Assumptions

We say that a graph $X$ is *strongly symmetric* with respect to a vertex subset $D$, if those automorphisms of $X$ stabilizing $D$ act on $D$ like the alternating or symmetric group.

We always assume that the initial data $G$, $\eta$, and $\sigma_i$ satisfy the following conditions:

(C) *Connectedness*: If $(Y,\psi) = \mathsf{Copy}(X,\varphi)$ and $X$ is connected, then $Y$ is connected, too.

(S) *Strong symmetry*: If $(Y,\psi) = \mathsf{Copy}(X,\varphi)$, where $X$ is strongly symmetric with respect to $\varphi(\Theta)$, then $Y$ is strongly symmetric with respect to $\psi(\Theta)$.

Set $(Y,\psi) = \mathsf{Copy}(X,\varphi)$, where $X$ is a connected graph. Define the constant $\kappa$ by

$$\kappa = s(\theta - 1) + 1 - |VG|.$$

Then $\kappa \geq 0$ has a geometric interpretation: If $H$ is a subgraph of $Y$, so that the restriction of $H$ on each part of $Y$ is a spanning tree, then the cyclomatic number of $H$ is $\kappa$. It is easy to derive the following recurrences:

$$|VY| = s(|VX| - \theta) + |VG| = s(|VX| - 1) - \kappa + 1 \quad \text{and} \quad |EY| = s|EX|.$$

If $c(X)$ and $c(Y)$ are the cyclomatic numbers of $X$ and $Y$, respectively, then $c(Y) = sc(X) + \kappa$.

2.3. Examples

2.3.1. Sierpiński Graphs

Fix some $d \in \mathbb{N}_0$ and let $s = \theta = d+1$. Define the edgeless graph $G$ by

$$VG = \left\{ \mathbf{x} \in \mathbb{N}_0^{d+1} : \sum_i x_i = 2 \right\}$$

and the map $\eta\colon \Theta \to VG$ by $\eta(i) = 2\mathbf{e}_i$, where $\mathbf{e}_i$ is the $i$-th canonical basis vector of $\mathbb{R}^{d+1}$. In addition, set $\sigma_i(j) = \mathbf{e}_i + \mathbf{e}_j \in VG$ for $i \in S$ and $j \in \Theta$. Note that $\Theta = S = \{1, \ldots, d+1\}$. It is easy to see that

$$|VG| = \frac{1}{2}(d+2)(d+1) \quad \text{and} \quad \kappa = d(d+1) + 1 - \frac{1}{2}(d+2)(d+1) = \frac{1}{2}d(d-1).$$

The usual finite $d$-dimensional Sierpiński graphs are then constructed as follows: Let $X_0 = K_{d+1}$, $\varphi_0\colon \Theta \to VX_0$ be injective, and inductively define $(X_n, \varphi_n)$ by $(X_n, \varphi_n) = \mathsf{Copy}\,(X_{n-1}, \varphi_{n-1})$ for $n \in \mathbb{N}$. See Figure 1 for the case $d = 2$.

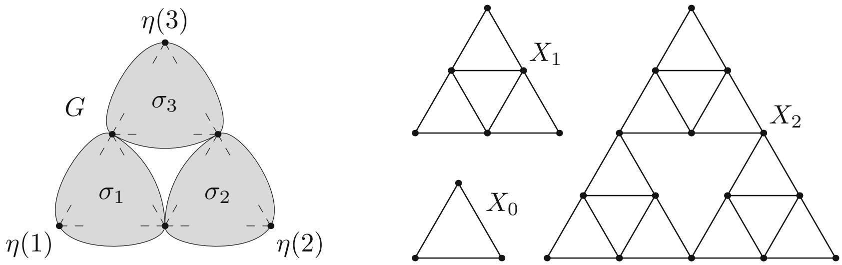


Figure 1: Initial data and finite 2-dimensional Sierpiński graphs.

2.3.2. Austria Graphs

The "Austria" graphs are studied in [13] (their shape resembles a map of Austria). Let $\theta = 2$, $s = 4$, and $VG = \{1, 2, 3, 4\}$. Define $\eta$ and $\sigma_1, \ldots, \sigma_4$ as follows

| $i$ | $\eta(i)$ | $\sigma_1(i)$ | $\sigma_2(i)$ | $\sigma_3(i)$ | $\sigma_4(i)$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 4 | 4 |
| 2 | 4 | 2 | 3 | 2 | 3 |

Obviously, we have $\kappa = 1$. The finite Austria graphs are inductively constructed by $X_0 = K_2$ and $(X_n, \varphi_n) = \mathsf{Copy}(X_{n-1}, \varphi_{n-1})$ for $n \in \mathbb{N}$, see Figure 2 for an illustration of the initial data and some finite Austria graphs. The orientation of each of the

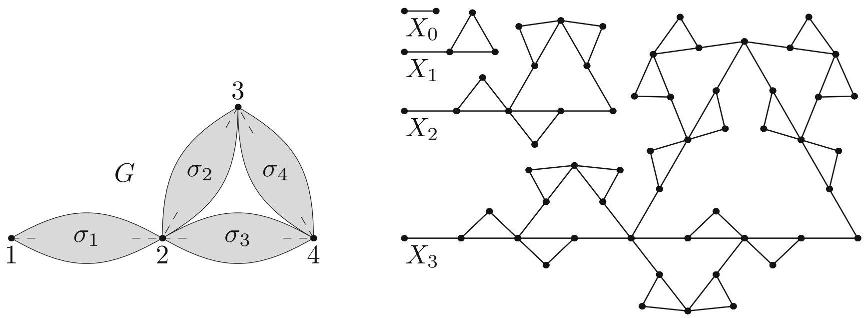


Figure 2: Initial data and finite Austria graphs.

four substitutions (defined by $\sigma_1, \ldots, \sigma_4$) can be flipped. For example, $\sigma_1$ could also be defined by $\sigma_1(1) = 2$ and $\sigma_1(2) = 1$. Note that two distinct choices yield different graph sequences $X_0, X_1, \ldots$ (the specific configuration can be identified in $X_2$). Here the substitutions $\sigma_1, \ldots, \sigma_4$ are chosen in such a way that the vertex degrees in $X_0, X_1, \ldots$ are uniformly bounded.

## 3. Electrical Networks

Let $F$ be a finite non-empty set and $X$ be a multigraph with vertex set $VX = F$. In addition, let $c\colon EX \to (0, \infty)$ be conductances on the edges of $X$. Then the pair $(F, c)$ is called an *electrical network*. The *Laplace operator* (or *Laplacian*) $\Delta\colon \mathbb{R}^F \to \mathbb{R}^F$ of a network $(F, c)$ is defined by

$$\Delta(f)(x) = \sum_{\substack{e \in EX \\ e = \{x, y\}}} \big(f(x) - f(y)\big) c(e).$$

For $B \subseteq F$, denote by $H_B^F\colon \mathbb{R}^B \to \mathbb{R}^F$ the linear operator, so that $H_B^F g\colon F \to \mathbb{R}$ is the harmonic extension of $g\colon B \to \mathbb{R}$, i.e., $\Delta H_B^F g|_{F \setminus B} = 0$ and $H_B^F g|_B = g$, and write $\Pi_B\colon f \mapsto f|_B$ for the canonical restriction onto $B$.

Two networks $(F, c_F)$ and $(G, c_G)$ with $\varnothing \neq B \subseteq F \cap G$ are called *electrically equivalent* with respect to $B$, if they cannot be distinguished by applying voltages to $B$ and measuring the resulting currents on $B$. In terms of the associated Laplace operators $\Delta_F$ and $\Delta_G$ electrical equivalence means $\Pi_B \Delta_F H_B^F = \Pi_B \Delta_G H_B^G$.

Let $D \subseteq VX = F$ be a non-empty vertex subset, so that $X$ is strongly symmetric with respect to $D$, and let $c\colon EX \to (0, \infty)$ be the unit conductances on $X$: $c(e) = 1$ for all edges $e \in EX$. It is easy to show that there exists a number $\rho$ such that the network $(F, c)$ is electrically equivalent to $(D, \rho^{-1} c_D)$ with respect to $D$, where $c_D$ are the unit conductances on the complete graph with vertex set $D$. The number $\rho$ is called *resistance scaling factor* of $X$ with respect to $D$.

### 3.1. Resistance Scaling in Self-Similar Graphs

Besides the obvious parameters $s$, $\theta$, and $\kappa$ there are two further intrinsic parameters of the initial data: The resistance scaling factor $\lambda$ and the tree scaling factor $\mu$. The first one will be introduced in the following. The existence and meaning of the second one form the content of Theorem 4.1. We remark that the notion of resistance (or conductance) scaling is used frequently in the context of self-similar graphs, see for instance [1, 11, 15].

Let $X$ be a connected multigraph and $\varphi\colon \Theta \to VX$ be an injective map, so that $X$ is strongly symmetric with respect to $\varphi(\Theta)$. Additionally, set $(Y, \psi) = \mathsf{Copy}(X, \varphi)$. Denote by $\rho(X)$ and $\rho(Y)$ the resistance scaling factor of $X$ with respect to $\varphi(\Theta)$ and of $Y$ with respect to $\psi(\Theta)$, respectively.

**Lemma 3.1.** *With the above notation the quotient $\rho(Y)/\rho(X)$ is independent of the specific choice of the multigraph $X$ and will be denoted by $\lambda$, called the* resistance scaling factor *of the initial data.*

*Proof.* We define $\lambda$ by $\lambda = \rho(Y)$, where $(Y, \psi) = \mathsf{Copy}(X, \varphi)$ and $X$ is the complete graph with $\theta$ vertices. For general multigraphs $X$ we have to prove that $\rho(Y) = \lambda\rho(X)$: Since $\rho(X)$ is the resistance scaling factor of $X$, each copy of $X$ in $Y$ can be replaced by a complete graph with constant conductances $\rho(X)^{-1}$. Therefore $\rho(Y) = \lambda\rho(X)$ by definition of $\lambda$. ∎

### 3.2. Examples

#### 3.2.1. Sierpiński Graphs

It is not difficult to derive the resistance scaling factor $\rho(X_1)$ of $X_1$ with respect to its distinguished vertices: $\rho(X_1) = \frac{d+3}{d+1}$. As $\rho(X_0) = 1$ with respect to $\varphi_0(\Theta)$, the resistance scaling factor $\lambda$ of the initial data in this case is given by $\lambda = \frac{d+3}{d+1}$.

#### 3.2.2. Austria Graphs

The resistance scaling factor $\rho(X_1)$ of $X_1$ with respect to its distinguished vertices is given by $\rho(X_1) = \frac{5}{3}$. Therefore, $\lambda = \frac{5}{3}$.

## 4. Main Results

Let $X$ be a connected graph which is strongly symmetric with respect to $\varphi(\Theta)$. Set $(Y, \psi) = \mathsf{Copy}(X, \varphi)$ and denote by $\rho(X)$ and $\rho(Y)$ the resistance scaling factors of $X$ and $Y$ with respect to $\varphi(\Theta)$ and $\psi(\Theta)$, respectively.

**Theorem 4.1.** *There exists a number $\mu$, the* tree scaling factor*, depending on the initial data only, such that*

$$(\rho(Y), \tau(Y)) = \mathbf{T}(\rho(X), \tau(X)),$$

*where $\mathbf{T}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $\mathbf{T}(a, b) = (\lambda a, \mu a^\kappa b^s)$.*

Let $X_0$ be a connected graph which is strongly symmetric with respect to $\varphi_0(\Theta)$. We define a sequence of pairs $(X_n, \varphi_n)$ inductively by

$$(X_{n+1}, \varphi_{n+1}) = \mathsf{Copy}(X_n, \varphi_n).$$

Note that the cardinalities of $VX_n$ and $EX_n$ are given by

$$|VX_n| = s^n(|VX_0| - 1) - \kappa\frac{s^n - 1}{s-1} + 1 \quad \text{and} \quad |EX_n| = s^n|EX_0|.$$

The next theorem follows by induction from the previous one.

**Theorem 4.2.** *The complexity $\tau(X_n)$ of $X_n$ is given by*

$$\tau(X_n) = \lambda^{\kappa\frac{s^n-1-n(s-1)}{(s-1)^2}} \left(\mu\rho(X_0)^\kappa\right)^{\frac{s^n-1}{s-1}} \tau(X_0)^{s^n}.$$

Note that this can be rewritten in terms of the spectral dimension which is related to the resistance scaling factor by the formula

$$d_s = \frac{2\log(s)}{\log(s\lambda)}.$$

The spectral dimension is a useful descriptor in the spectral theory of self-similar structures as well as in the study of random walks and Brownian motion.

Furthermore, there are $s$ possibilities to embed $X_n$ in $X_{n+1}$. Hence for each infinite sequence $\boldsymbol{\iota} = (\iota_1, \iota_2, \ldots) \in S^{\mathbb{N}}$, there exists an infinite limit graph $X_\infty(\boldsymbol{\iota})$, so that the embeddings

$$X_0 \xhookrightarrow{\iota_1} X_1 \xhookrightarrow{\iota_2} X_2 \cdots \xhookrightarrow{\iota_n} X_n \cdots \hookrightarrow X_\infty(\boldsymbol{\iota})$$

hold. In this sense the multigraph sequence $X_0, X_1, \ldots$ approaches the infinite multigraph $X_\infty(\boldsymbol{\iota})$. The *tree entropy* $h$ of $X_\infty(\boldsymbol{\iota})$ (see [14]) is defined to be

$$h = \lim_{n\to\infty} \frac{\log(\tau(X_n))}{|VX_n|}.$$

**Corollary 4.3.** *The tree entropy $h$ of the infinite limit graph $X_\infty(\boldsymbol{\iota})$ for some $\boldsymbol{\iota} \in S^{\mathbb{N}}$ is then given by*

$$h = \frac{\frac{\kappa}{s-1}\log(\lambda) + \log(\mu) + \kappa\log(\rho(X_0)) + (s-1)\log(\tau(X_0))}{(s-1)(|VX_0| - 1) - \kappa}.$$

The proof of Theorem 4.1 proceeds in the following major steps:

(1) First of all a decomposition of certain spanning forests of $Y$ into spanning forests of $X$ of the same type is developed. This yields a multi-dimensional polynomial recursion for the numbers of these spanning forests. The symmetry condition is essential to decrease the number of variables, see Section 5.

(2) In the next step the numbers of rooted spanning forests, where all roots are contained in the set of distinguished vertices, are deduced from the numbers of spanning forests mentioned before. The crucial point here are the correct coefficients for these relations, see Section 6.

(3) Using a certain factorization (Corollary 7.2), we transfer the polynomial recursion for the numbers of spanning forests to a polynomial recursion for the numbers of rooted spanning forests. This is done using a general result concerning multi-dimensional polynomials (Proposition 7.3), see Section 7. It seems to be difficult to obtain this second recursion without the first step.

(4) Due to the symmetry condition it turns out that there are relations between the numbers of these rooted spanning forests (Theorem 8.1). This allows us to reduce the number of variables to two, see Section 8.

For illustration we discuss each step in the following sections for the sequence of finite Sierpiński graphs in more detail.

## 5. Spanning Forests

In order to formulate the construction scheme for spanning forests we need some notations for number and set partitions and spanning forests.

### 5.1. Number Partitions and Set Partitions

For $n \in \mathbb{N}$, denote by $\mathcal{P}(n)$ the set of number partitions of $n$, and write $\nu_k(p)$ for the number of occurrences of $k \in \mathbb{N}$ in a partition $p \in \mathcal{P}(n)$, so that

$$n = \sum_{k \in \mathbb{N}} k \cdot \nu_k(p)$$

and $\nu_k(p) = 0$ for $k > n$. In addition, define $|p|$ by

$$|p| = \sum_{k \in \mathbb{N}} \nu_k(p)$$

and set $\mathcal{P}_r(n) = \{p \in \mathcal{P}(n) \colon |p| = r\}$ for $r \in \mathbb{N}$. If a number partition has $k_1, \dots, k_r$ as its distinct parts, we write

$$p = k_1^{\nu_{k_1}(p)} \cdots k_r^{\nu_{k_r}(p)}$$

as a shorthand. Usually, the summands $k_1, \dots, k_r$ are sorted in descending order. For example, $3^1 2^3 1^2$ means the number partition $3+2+2+2+1+1$.

Let $M$ be a finite set. A *set partition* $B$ of $M$ is a family of non-empty and disjoint subsets of $M$, so that their union is equal to $M$. The elements of $M$ are called *blocks*. The block sizes of $B$ define a number partition $p$ of $|M|$, and the set partition $B$ is said to be of *type* $p$ in this case. For convenience, the type $p$ of $B$ is denoted $\ell(B) = p$. Let $\mathcal{B}(M)$ be the set of all set partitions of $M$ and denote by $\mathcal{B}_p(M) \subseteq \mathcal{B}(M)$ those partitions of type $p$. Of course,

$$\mathcal{B}(M) = \biguplus_{p \in \mathcal{P}(|M|)} \mathcal{B}_p(M).$$

If $K$ is a subset of $M$, then the *restriction* $B|_K \in \mathcal{B}(K)$ of $B \in \mathcal{B}(M)$ is given by

$$B|_K = \big\{ b \cap K \colon b \in B, b \cap K \neq \varnothing \big\}.$$

Finally, set $\varphi(B) = \{\varphi(b) \colon b \in B\}$ for any $B \in \mathcal{B}(M)$ and any map $\varphi \colon M \to R$, where $R$ is any set.

Let $I$ be an index set. For $i \in I$, let $M_i \subseteq M$ be a non-empty subset of $M$, so that the union of all $M_i$ is equal to $M$. Let $B_i$ be a set partition of $M_i$ and denote by $\mathcal{B} = \{B_i \colon i \in I\}$ the family of these partitions. Then define a multigraph $X_{\mathcal{B}}$ as follows:

$$VX_{\mathcal{B}} = \big\{(B, b) \colon B \in \mathcal{B}, b \in B\big\},$$

and two distinct vertices $(B_1, b_1)$ and $(B_2, b_2)$ are joined by $|b_1 \cap b_2|$ edges in $EX_{\mathcal{B}}$. By definition $X_{\mathcal{B}}$ does not contain any loops. We call $\mathcal{B}$

- *cycle-free*, if the multigraph $X_{\mathcal{B}}$ is cycle-free (and hence simple);
- *connected*, if $X_{\mathcal{B}}$ is so.

Notice that two blocks from distinct partitions of a cycle-free family have at most one point in common. The connected components of $X_{\mathcal{B}}$ naturally define a set partition on $M$, which is called the *transitive union* $\mathsf{Union}(\mathcal{B}) \in \mathcal{B}(M)$ of $\mathcal{B}$: Each block of $\mathsf{Union}(\mathcal{B})$ is given as the union of all blocks of the family $\mathcal{B}$, which are contained in one connected component of $X_{\mathcal{B}}$. In other words, $\mathsf{Union}(\mathcal{B})$ is the finest partition of $M$, such that each block of the family $\mathcal{B}$ is contained in one block of $\mathsf{Union}(\mathcal{B})$.

## 5.2. Spanning Forests

Let $X$ be a multigraph with $\theta$ distinguished vertices $D \subseteq VX$. Every spanning forest $F$ of $X$ induces a set partition $B$ on $D$: The distinguished vertices in one connected component of $F$ form a block of $B$. Let $\mathcal{S}_X$ be the set of spanning forests of $X$, which only have components containing at least one distinguished vertex each. For $B \in \mathcal{B}(D)$, write $\mathcal{S}_X(B)$ for the set of those forests in $\mathcal{S}_X$ whose induced set partition is $B$. For $p \in \mathcal{P}(\theta)$, $\mathcal{S}_X(p)$ denotes the set of spanning forests in $\mathcal{S}_X$ defining a set partition of type $p$. Then,

$$\mathcal{S}_X(p) = \biguplus_{B \in \mathcal{B}_p(D)} \mathcal{S}_X(B) \quad \text{and} \quad \mathcal{S}_X = \biguplus_{p \in \mathcal{P}(\theta)} \mathcal{S}_X(p).$$

## 5.3. Decomposition of Spanning Forests

Fix some initial data $G$, $\eta$, and $\sigma_i$. Consider an element $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_s)$ in the Cartesian product $\prod_{i \in S} \mathcal{B}(\Theta)$ and denote by $\sigma(\boldsymbol{\omega})$ the family

$$\sigma(\boldsymbol{\omega}) = \big\{\sigma_i(\omega_i) \colon i \in S\big\}.$$

Then $\mathsf{Union}(\sigma(\boldsymbol{\omega}))$ is a set partition of $VG$. Moreover, define the following counting functions:

$$\chi_p(\boldsymbol{\omega}) = \big|\big\{i \in S \colon \omega_i \in \mathcal{B}_p(\Theta)\big\}\big| \quad \text{and} \quad \chi(\boldsymbol{\omega}) = \sum_{i \in S} |\omega_i| = \sum_{p \in \mathcal{P}(\theta)} |p|\chi_p(\boldsymbol{\omega}),$$

for $p \in \mathcal{P}(\theta)$ and $\boldsymbol{\omega} \in \prod_{i \in S} \mathcal{B}(\Theta)$.

Now let $X$ be a connected multigraph, which is strongly symmetric with respect to $\varphi(\Theta)$, and set $(Y,\psi)=\mathsf{Copy}(X,\varphi)$. For the sake of notation, define $\psi_i\colon \Theta\to VY$ by $\psi_i(j)=\bar{\zeta}_i(\varphi(j))$ $(i\in S)$. Let $F$ be a spanning forest in $\mathcal{S}_Y(\psi(B))$ for some $B\in\mathcal{B}(\Theta)$ and denote by $F_i$ the restriction of $F$ on the $i$-th part of $Y$ $(i\in S)$. Then, for each $i\in S$, there exists exactly one spanning forest $L_i$ of $X$ such that

$$F_i=\bar{\zeta}_i(L_i). \tag{5.1}$$

Define the *trace* $\boldsymbol{\omega}=\mathsf{Tr}(F)$ of $F$, so that the forest $L_i$ is contained in the set $\mathcal{S}_X(\varphi(\omega_i))$. From this we can draw some conclusions:

- For each $b\in\mathsf{Union}(\sigma(\boldsymbol{\omega}))$, the intersection $b\cap\eta(\Theta)$ is not empty.
- The restriction $\mathsf{Union}(\sigma(\boldsymbol{\omega}))|_{\eta(\Theta)}$ equals $\eta(B)$.
- The family $\sigma(\boldsymbol{\omega})$ is cycle-free.

The above discussion motivates the definition of $\Omega(B)$ for $B\in\mathcal{B}(\Theta)$: $\Omega(B)$ is the set of all $\boldsymbol{\omega}\in\prod_{i\in S}\mathcal{B}(\Theta)$ such that $b\cap\eta(\Theta)\neq\varnothing$ for $b\in\mathsf{Union}(\sigma(\boldsymbol{\omega}))$, $\mathsf{Union}(\sigma(\boldsymbol{\omega}))|_{\eta(\Theta)}=\eta(B)$, and $\sigma(\boldsymbol{\omega})$ is a cycle-free family. Then we have $\mathsf{Tr}(\mathcal{S}_Y(\psi(B)))=\Omega(B)$, and there is a bijective correspondence between

$$\mathcal{S}_Y(\psi(B)) \quad \text{and} \quad \biguplus_{\boldsymbol{\omega}\in\Omega(B)}\prod_{i\in S}\mathcal{S}_X(\varphi(\omega_i)), \tag{5.2}$$

for $B\in\mathcal{B}(\Theta)$, which is determined by (5.1).

With the symmetry condition in mind let us define the set $\Omega(p)$ for a number partition $p\in\mathcal{P}(\theta)$ by

$$\Omega(p)=\biguplus_{B\in\mathcal{B}_p(\Theta)}\Omega(B).$$

It is remarkable that, for any tuple $\boldsymbol{\omega}\in\Omega(p)$, the number of blocks $\chi(\boldsymbol{\omega})$ in $\boldsymbol{\omega}$ satisfies an identity, which only involves $|p|$:

**Lemma 5.1.** *Suppose that the connectedness condition is satisfied. Then, for $p\in\mathcal{P}(\theta)$, we have*

$$\chi(\boldsymbol{\omega})=\kappa+s+|p|-1,$$

*for all $\boldsymbol{\omega}\in\Omega(p)$.*

*Proof.* Suppose that $X=K_\theta$ and $(Y,\psi)=\mathsf{Copy}(X,\varphi)$. We prove that $\chi(\mathsf{Tr}(F))=\kappa+s+|p|-1$ holds for all spanning forests $F\in\mathcal{S}_Y(p)$, which implies the statement, since each $\boldsymbol{\omega}\in\Omega(p)$ has a representation as a spanning forest in $\mathcal{S}_Y(p)$.

Let $\boldsymbol{\omega}\in\Omega(p)$ and let $F$ be a spanning forest with $\boldsymbol{\omega}=\mathsf{Tr}(F)$. If $\omega_i\in\mathcal{B}_q(\Theta)$, then $F$ has exactly $\theta-|q|$ edges in the $i$-th part of $Y$ (since $F$ induces a spanning forest with $|q|$ components). Similarly, $F$ has a total of exactly $|VY|-|p|$ edges. Therefore, we have two expressions for the number of edges of $F$:

$$|VY|-|p|=\sum_{q\in\mathcal{P}(\Theta)}(\theta-|q|)\chi_q(\boldsymbol{\omega})=\theta\sum_{q\in\mathcal{P}(\Theta)}\chi_q(\boldsymbol{\omega})-\chi(\boldsymbol{\omega})=\theta s-\chi(\boldsymbol{\omega}).$$

Additionally, we know that $|VY|=s(\theta-1)-\kappa+1$. Now, solving the equation for $\chi(\boldsymbol{\omega})$ yields the lemma. ∎

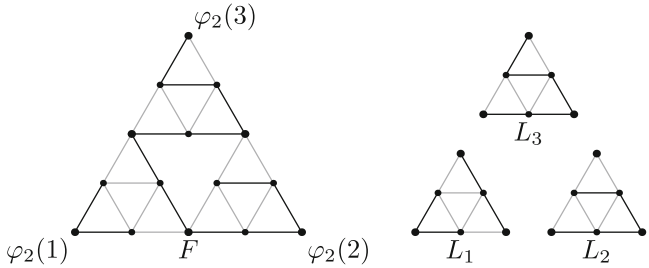


Figure 3: Decomposition of spanning forests.

Finally, we illustrate the correspondence given in (5.2) for the case of finite 2-dimensional Sierpiński graphs. Figure 3 depicts the decomposition of a spanning forest $F$ of $X_2$ into a triple $(L_1, L_2, L_3)$, so that the relation (5.1) holds. It is readily seen that $F \in \mathcal{S}_{X_2}(\varphi_2(\{1, 23\}))$ and

$$L_1 \in \mathcal{S}_{X_1}(\varphi_1(\{1, 23\})), \quad L_2 \in \mathcal{S}_{X_1}(\varphi_1(\{12, 3\})), \quad L_3 \in \mathcal{S}_{X_1}(\varphi_1(\{123\})).$$

Thus the trace of $F$ is given by

$$\mathsf{Tr}(F) = (\{1, 23\}, \{12, 3\}, \{123\}) \in \Omega(\{1, 23\}).$$

Here and in the following we sometimes write $\{1, 23\}$ as a shorthand for the partition $\{\{1\}, \{2, 3\}\}$ and analogously for other partitions, if no ambiguity can occur.

By virtue of symmetry

$$|\mathcal{S}_X(B_1)| = |\mathcal{S}_X(B_2)|,$$

for all $B_1, B_2 \in \mathcal{B}(\varphi(\Theta))$ of the same type. Thus define $\tau_p(X)$ by

$$\tau_p(X) = \frac{|\mathcal{S}_X(p)|}{b(p)},$$

for $p \in \mathcal{P}(\theta)$, where $b(p) = |\mathcal{B}_p(\Theta)|$ is the number of set partitions of $\Theta$ of type $p$. Furthermore, write $\boldsymbol{\tau}(X)$ for the vector $(\tau_p(X))_{p\in\mathcal{P}(\theta)}$. Note that $\tau_p(X) = |\mathcal{S}_X(B)|$ for any $B \in \mathcal{B}_p(\varphi(\Theta))$ and $\tau_p(X)$ is equal to the complexity $\tau(X)$ of $X$ if $p$ is the trivial partition with one summand given by $p = \theta$. No confusion should occur between the complexity $\tau(X)$ and the vector $\boldsymbol{\tau}(X)$.

If $(Y, \psi) = \mathsf{Copy}(X, \varphi)$, then Equation (5.2) implies

$$\begin{aligned} b(p)\,\tau_p(Y) &= \sum_{B\in\mathcal{B}_p(\Theta)} |\mathcal{S}_Y(\psi(B))| = \sum_{\boldsymbol{\omega}\in\Omega(p)} \prod_{i\in S} |\mathcal{S}_X(\varphi(\omega_i))| \\ &= \sum_{\boldsymbol{\omega}\in\Omega(p)} \prod_{q\in\mathcal{P}(\theta)} \tau_q(X)^{\chi_q(\boldsymbol{\omega})}, \end{aligned}$$

for all $p \in \mathcal{P}(\theta)$. For a subset $\Omega \subseteq \prod_{i\in S} \mathcal{B}(\Theta)$, define the generating function $\mathsf{GF}(\Omega\,|\,\mathbf{x})$ by

$$\mathsf{GF}(\Omega\,|\,\mathbf{x}) = \sum_{\boldsymbol{\omega}\in\Omega} \prod_{i\in S} x_{\ell(\omega_i)} = \sum_{\boldsymbol{\omega}\in\Omega} \prod_{q\in\mathcal{P}(\theta)} x_q^{\chi_q(\boldsymbol{\omega})},$$

where $\ell(\omega_i)$ is the type of the set partition $\omega_i$. Now the following proposition is immediate.

**Proposition 5.2.** *The vectors* $\boldsymbol{\tau}(X)$ *and* $\boldsymbol{\tau}(Y)$ *satisfy the following identity*:

$$\boldsymbol{\tau}(Y) = \mathbf{Q}(\boldsymbol{\tau}(X)),$$

*where the s-homogeneous polynomial function* $\mathbf{Q}\colon \mathbb{R}^{\mathcal{P}(\theta)} \to \mathbb{R}^{\mathcal{P}(\theta)}$ *is given by its coordinates*

$$Q_p(\mathbf{x}) = \frac{1}{b(p)} \mathsf{GF}(\Omega(p) \,|\, \mathbf{x}),$$

*for* $p \in \mathcal{P}(\theta)$. *Additionally, strong symmetry implies that*

$$Q_p(\mathbf{x}) = \mathsf{GF}(\Omega(B) \,|\, \mathbf{x}),$$

*for* $p \in \mathcal{P}(\theta)$ *and any* $B \in \mathcal{B}_p(\Theta)$.

Lemma 5.1 implies a constraint for the monomials of $\mathbf{Q}$: Let $p \in \mathcal{P}(\theta)$ and let

$$\prod_{q \in \mathcal{P}(\theta)} x_q^{n_q}$$

be a monomial with nonzero coefficient in $Q_p$, then there is some $\boldsymbol{\omega} \in \Omega(p)$ with $\chi_q(\boldsymbol{\omega}) = n_q$ for all $q \in \mathcal{P}(\theta)$ and the relation

$$\sum_{q \in \mathcal{P}(\theta)} |q| n_q = \chi(\boldsymbol{\omega}) = \kappa + s + |p| - 1$$

holds.

As an example, we study these results in the case of finite 2-dimensional Sierpiński graphs in more detail: Recall that $\theta = s = 3$ and note that $\mathcal{P}(3) = \{3^1, 2^1 1^1, 1^3\}$. Furthermore, $\mathcal{B}_p(\{1,2,3\})$ for $p \in \mathcal{P}(3)$ is given by

$$\mathcal{B}_{3^1}(\{1,2,3\}) = \{\{123\}\}, \qquad \mathcal{B}_{1^3}(\{1,2,3\}) = \{\{1,2,3\}\},$$
$$\mathcal{B}_{2^1 1^1}(\{1,2,3\}) = \{\{12,3\}, \{13,2\}, \{23,1\}\}.$$

The left part of Figure 4 shows the initial data with complete labelling, whereas the right part yields a table of all arrangements for the construction of spanning forests. (The shaded area indicates connected pieces.) For example, up to symmetry, there is one way to construct a spanning tree $F$ of $X_{n+1}$ from a triple $(L_1, L_2, L_3)$ of certain spanning forests of $X_n$, so that the relation (5.1) holds. This arrangement is illustrated in the first row and first column of the table. Therefore, $\Omega(3^1) = \Omega(\{123\})$ consists of the six tuples

$$(\{123\}, \{123\}, \{13,2\}), (\{123\}, \{123\}, \{23,1\}), (\{123\}, \{12,3\}, \{123\}),$$
$$(\{123\}, \{23,1\}, \{123\}), (\{12,3\}, \{123\}, \{123\}), (\{13,2\}, \{123\}, \{123\}).$$

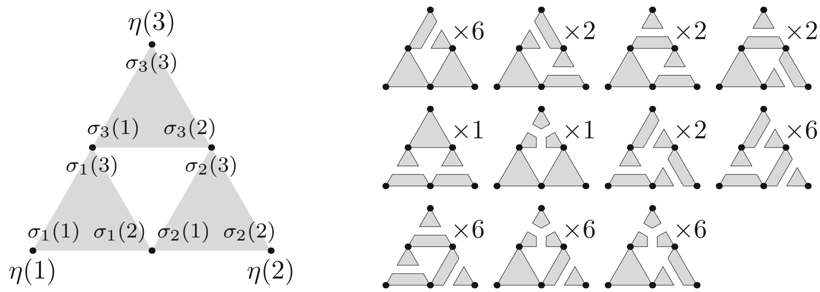


Figure 4: Initial data with complete labelling and (up to symmetry) all arrangements for the construction of spanning forests.

The next five arrangements of the table belong to $\Omega\left(2^1 1^1\right)$ and the last five to $\Omega(1^3) = \Omega(\{1, 2, 3\})$. Altogether, we get

$$\mathbf{Q}\begin{pmatrix} x_{3^1} \\ x_{2^1 1^1} \\ x_{1^3} \end{pmatrix} = \begin{pmatrix} 6x_{3^1}^2 x_{2^1 1^1} \\ 7x_{3^1} x_{2^1 1^1}^2 + x_{3^1}^2 x_{1^3} \\ 14x_{2^1 1^1}^3 + 12x_{3^1} x_{2^1 1^1} x_{1^3} \end{pmatrix}.$$

It is easy to see that the initial values are $\boldsymbol{\tau}(X_0) = (3, 1, 1)$. Therefore,

$$\boldsymbol{\tau}(X_1) = \mathbf{Q}(\boldsymbol{\tau}(X_0)) = (54, 30, 50),$$

$\boldsymbol{\tau}(X_2) = (524880, 486000, 1350000)$, and so forth.

A similar enumeration yields the polynomial $\mathbf{Q}$ in the 3-dimensional case, see Table 1. The initial values are $\boldsymbol{\tau}(X_0) = (16, 3, 1, 1, 1)$ and thus

$$\boldsymbol{\tau}(X_1) = (131072, 42996, 6156, 18432, 27648), \ldots$$

Note that there are five partitions of $\theta = 4$:

$$\mathcal{P}(4) = \{4^1, 3^1 1^1, 2^2, 2^1 1^2, 1^4\}$$

and $3^1 1^1$ and $2^2$ both have two terms. In Section 7, it is shown how such terms can be combined to reduce the number of variables. The following section provides the necessary auxiliary results for this step.

## 6. Rooted Spanning Forests

### 6.1. Set Partitions Forming a Cycle-Free, Connected Family

Let $M$ be a set and let $o$, $p$ be number partitions of $|M|$. Fix a set partition $O$ of $M$ with $\ell(O) = o$. Denote by $\mathcal{A}(O)$ the family of set partitions $P$ of $M$ with the property that $\{O, P\}$ forms a cycle-free, connected family, and set $\mathcal{A}(O, p) = \mathcal{A}(O) \cap \mathcal{B}_p(M)$. Then the number $\alpha(O, p) = |\mathcal{A}(O, p)|$ only depends on the type of $O$ and not on $O$

Table 1: The polynomial $\mathbf{Q}$ for finite 3-dimensional Sierpiński graphs.

$$
\begin{aligned}
Q_{4^1}(\mathbf{x}) &= 56x_{4^1}x_{3^1 1^1}^3 + 168x_{4^1}x_{3^1 1^1}^2 x_{2^2} + 168x_{4^1}x_{3^1 1^1}x_{2^2}^2 + 56x_{4^1}x_{2^2}^3 \\
&\quad + 72x_{4^1}^2 x_{3^1 1^1}x_{2^1 1^2} + 72x_{4^1}^2 x_{2^2}x_{2^1 1^2} \\
Q_{3^1 1^1}(\mathbf{x}) &= 20x_{3^1 1^1}^4 + 96x_{3^1 1^1}^2 x_{2^2}^2 + 108x_{4^1}x_{3^1 1^1}^2 x_{2^1 1^2} + 192x_{4^1}x_{3^1 1^1}x_{2^2}x_{2^1 1^2} \\
&\quad + 72x_{3^1 1^1}^3 x_{2^2} + 56x_{3^1 1^1}x_{2^2}^3 + 84x_{4^1}x_{2^2}^2 x_{2^1 1^2} + 24x_{4^1}^2 x_{2^1 1^2}^2 \\
&\quad + 6x_{4^1}^2 x_{3^1 1^1}x_{1^4} + 6x_{4^1}^2 x_{2^2}x_{1^4} \\
Q_{2^2}(\mathbf{x}) &= 2x_{3^1 1^1}^4 + 16x_{3^1 1^1}^3 x_{2^2} + 36x_{3^1 1^1}^2 x_{2^2}^2 + 32x_{3^1 1^1}x_{2^2}^3 + 12x_{4^1}x_{3^1 1^1}^2 x_{2^1 1^2} \\
&\quad + 22x_{2^2}^4 + 48x_{4^1}x_{3^1 1^1}x_{2^2}x_{2^1 1^2} + 36x_{4^1}x_{2^2}^2 x_{2^1 1^2} + 2x_{4^1}^2 x_{2^1 1^2}^2 \\
Q_{2^1 1^1}(\mathbf{x}) &= 88x_{3^1 1^1}^3 x_{2^1 1^2} + 264x_{3^1 1^1}^2 x_{2^2}x_{2^1 1^2} + 264x_{3^1 1^1}x_{2^2}^2 x_{2^1 1^2} + 88x_{2^2}^3 x_{2^1 1^2} \\
&\quad + 120x_{4^1}x_{3^1 1^1}x_{2^1 1^2}^2 + 120x_{4^1}x_{2^2}x_{2^1 1^2}^2 + 14x_{4^1}x_{3^1 1^1}^2 x_{1^4} \\
&\quad + 28x_{4^1}x_{3^1 1^1}x_{2^2}x_{1^4} + 14x_{4^1}x_{2^2}^2 x_{1^4} + 6x_{4^1}^2 x_{2^1 1^2}x_{1^4} \\
Q_{1^4}(\mathbf{x}) &= 720x_{3^1 1^1}^2 x_{2^1 1^2}^2 + 1440x_{3^1 1^1}x_{2^2}x_{2^1 1^2}^2 + 720x_{2^2}^2 x_{2^1 1^2}^2 + 208x_{4^1}x_{2^1 1^2}^3 \\
&\quad + 56x_{3^1 1^1}^3 x_{1^4} + 168x_{3^1 1^1}^2 x_{2^2}x_{1^4} + 168x_{3^1 1^1}x_{2^2}^2 x_{1^4} + 56x_{2^2}^3 x_{1^4} \\
&\quad + 144x_{4^1}x_{3^1 1^1}x_{2^1 1^2}x_{1^4} + 144x_{4^1}x_{2^2}x_{2^1 1^2}x_{1^4}
\end{aligned}
$$

itself. Thus we may define $\alpha(o,p) = \alpha(O,p)$ for any $O \in \mathcal{B}_o(M)$. If $O$ and $P$ are set partitions of $M$, so that $\mathcal{B} = \{O,P\}$ is cycle-free and connected, then $|O|+|P| = |M|+1$, since the associated graph $X_{\mathcal{B}}$ is a tree, $|VX_{\mathcal{B}}| = |O|+|P|$, and $|EX_{\mathcal{B}}| = |M|$. This implies

$$\mathcal{A}(O) = \biguplus_{p \in \mathcal{P}_k(|M|)} \mathcal{A}(O,p), \tag{6.1}$$

where $k$ is given by $k = |M|+1-|O|$.

**Theorem 6.1.** *Let $M$ be a finite set and $o,p \in \mathcal{P}(|M|)$ with $|o|+|p| = |M|+1$. Then the formula*

$$\alpha(o,p) = (|o|-1)!\,(|p|-1)! \prod_{k \in \mathbb{N}} \frac{k^{\nu_k(o)}}{\nu_k(p)!\big((k-1)!\big)^{\nu_k(p)}} \tag{6.2}$$

*holds.*

*Proof.* For the moment write $\mathcal{P}_r$ for the set of all number partitions with exactly $r$ terms and set $\nu(p) = n$ if $p \in \mathcal{P}(n)$. Fix some set partition $O \in \mathcal{B}_o(M)$ and some integer $r \geq 1$ with $\nu_r(p) > 0$. Let $P$ be a set partition of type $p$, such that $\{O,P\}$ is a cycle-free, connected family. Then each element of a block $b \in P$ of size $r$ (there are $\nu_r(p)$ possibilities to choose such a block) is contained in exactly one of $r$ pairwise different blocks $c_1,\ldots,c_r$ of $O$. Denote by $q \in \mathcal{P}_r$ the type of $\{c_1,\ldots,c_r\}$. There are

$$\prod_{k \in \mathbb{N}} \binom{\nu_k(o)}{\nu_k(q)} k^{\nu_k(q)}$$

choices for $\{c_1,\ldots,c_r\}$ and the $r$ elements of the block $b$, if the type of the "neighboring" blocks $\{c_1,\ldots,c_r\}$ is given by $q$. Now consider $P' = P \setminus \{b\}$ and

$$O' = (O \setminus \{c_1,\ldots,c_r\}) \cup \{(c_1 \cup \cdots \cup c_r) \setminus b\}.$$

Both $P'$ and $O'$ are set partitions of $M \setminus b$, and the family $\{O', P'\}$ is cycle-free and connected. The type $p'$ of $P'$ is obtained from $p$ by removing one part of size $r$: $\nu_r(p') = \nu_r(p) - 1$. On the other hand, the type $o'$ of $O'$ is given by

$$\nu_k(o') = \begin{cases} \nu_k(o) - \nu_k(q) + 1, & \text{if } k = \nu(q) - r, \\ \nu_k(o) - \nu_k(q), & \text{otherwise,} \end{cases}$$

since the partition $O'$ emerges from $O$ by removing $\nu_k(q)$ blocks of size $k$ and adding one block of size

$$\sum_{k \in \mathbb{N}} (k-1)\nu_k(q) = \nu(q) - r. \tag{6.3}$$

Obviously, $p'$ and $o'$ are both number partitions of $|M| - r$, and $|p'| = |p| - 1$, $|o'| = |o| - r + 1$. We point out the dependency of $o'$ on $q$.

The considerations above yield a recursive formula for $\alpha(o, p)$:

$$\alpha(o, p) = \frac{1}{\nu_r(p)} \sum_{q \in \mathcal{P}_r} \alpha(o', p') \prod_{k \in \mathbb{N}} \binom{\nu_k(o)}{\nu_k(q)} k^{\nu_k(q)}. \tag{6.4}$$

Now, we may proceed by induction. Equation (6.2) is trivial if $|o| = 1$ or $|p| = 1$. The hypothesis for $\alpha(o', p')$ implies

$$\alpha(o', p') = (|o| - r)!\,(|p| - 2)!\,(r-1)!\,\nu_r(p)(\nu(q) - r) \prod_{k \in \mathbb{N}} \frac{k^{\nu_k(o) - \nu_k(q)}}{\nu_k(p)!\big((k-1)!\big)^{\nu_k(p)}}.$$

Inserting this into (6.4) shows that it suffices to prove

$$(|p| - 1)\binom{|o| - 1}{r - 1} = \sum_{q \in \mathcal{P}_r} (\nu(q) - r) \prod_{k \in \mathbb{N}} \binom{\nu_k(o)}{\nu_k(q)}.$$

The term on the right-hand side, which we denote by $A$, can be written as

$$A = [y^r]\frac{\partial}{\partial x} \prod_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}_0} \binom{\nu_k(o)}{j} x^{(k-1)j} y^j \bigg|_{x=1}$$

bearing the identity (6.3) in mind. However, some elementary transformations yield

$$\begin{aligned} A &= [y^r]\frac{\partial}{\partial x} \prod_{k \in \mathbb{N}} \left(1 + x^{k-1}y\right)^{\nu_k(o)} \bigg|_{x=1} \\ &= [y^r] \sum_{j \in \mathbb{N}} \frac{\nu_j(o)(j-1)x^{j-2}y}{1 + x^{j-1}y} \prod_{k \in \mathbb{N}} \left(1 + x^{k-1}y\right)^{\nu_k(o)} \bigg|_{x=1} \\ &= [y^r] \sum_{j \in \mathbb{N}} \nu_j(o)(j-1)y(1+y)^{|o|-1} \\ &= (|M| - |o|)\binom{|o| - 1}{r - 1}, \end{aligned}$$

which proves the theorem, since $|o| + |p| = |M| + 1$. ∎

Define number partitions $p_k \in \mathcal{P}(|M|)$ for $k \in \{1, \ldots, |M|\}$ as follows: For $k = 1$, set $p_1 = 1^{|M|}$, and for $k \geq 2$ set $p_k = k^1\, 1^{|M|-k}$. Thus,

$$p_k = k + \underbrace{1 + \cdots + 1}_{|M|-k \text{ times}}$$

and $|p_k| = |M| + 1 - k$. Let $p \in \mathcal{P}(|M|)$ with $|p| = k$. Then we set $\alpha_p = \alpha(p_k, p)$. Last, but not least, we remark that the quotient

$$\frac{\alpha(o, p)}{\alpha_p} = \frac{1}{|M| + 1 - |o|} \prod_{k \in \mathbb{N}} k^{\nu_k(o)}$$

is independent of $p$ for all $o \in \mathcal{P}(|M|)$ with $|o| + |p| = |M| + 1$ and will be denoted by $\beta_o$. This implies $\alpha(o, p) = \beta_o \alpha_p$.

## 6.2. Rooted Spanning Forests and Spanning Forests

A *rooted spanning forest* $(F, R)$ of a multigraph $X$ is a spanning forest $F$ of $X$ together with a collection $R \subseteq VX$ of roots, such that $F$ has exactly $|R|$ components and each component contains exactly one element of $R$. We denote by $\mathcal{R}_X(R)$ the set of all rooted spanning forests of $X$ with roots $R \subseteq VX$.

Suppose that $X$ is connected and strongly symmetric with respect to a set of distinguished vertices $\varphi(\Theta)$. Then the number $|\mathcal{R}_X(W)|$ of rooted spanning forests with roots $W \subseteq \varphi(\Theta)$ depends only on the size of $W$. Hence define

$$r_k(X) = |\mathcal{R}_X(W)|,$$

for some $W \subseteq \varphi(\Theta)$ with $k \in \Theta$ elements. Thus $\mathbf{r}(X) = (r_1(X), \ldots, r_\theta(X))$ is a vector in $\mathbb{R}^\theta$. We remark that $r_1(X)$ is precisely the complexity $\tau(X)$ of $X$ and $r_\theta(X) = \tau_p(X)$, where the number partition $p$ is given by $p = 1^\theta$.

Let $W \subseteq \varphi(\Theta)$ be a $k$-set for $k \in \Theta$ and $p \in \mathcal{P}_k(\theta)$, then by Theorem 6.1 there are $\alpha_p$ set partitions $B \in \mathcal{B}_p(\varphi(\Theta))$, so that $|b \cap W| = 1$ for all $b \in B$. If $B$ is such a set partition and $F$ is a spanning forest in $\mathcal{S}_X(B)$, then $(F, W)$ is a rooted spanning forest in $\mathcal{R}_X(W)$.

This motivates the following definitions: For a set $K \subseteq \Theta$ with $k \in \Theta$ elements define the set partition $P_K$ of $\Theta$ by

$$P_K = \{K\} \uplus \big\{\{j\} \colon j \in \Theta \setminus K\big\}.$$

Notice that the type of $P_K$ is given by the number partition $p_k$. Then, for a spanning forest $F \in \mathcal{S}_X(\varphi(B))$ with $B \in \mathcal{A}(P_K)$, the tuple $(F, \varphi(K))$ is a rooted spanning forest in $\mathcal{R}_X(\varphi(K))$. Hence,

$$\mathcal{R}_X(\varphi(K)) = \biguplus_{B \in \mathcal{A}(P_K)} \big\{(F, \varphi(K)) \colon F \in \mathcal{S}_X(\varphi(B))\big\}$$

and

$$r_k(X) = \sum_{p \in \mathcal{P}_k(\theta)} \sum_{B \in \mathcal{A}(P_K, p)} |\mathcal{S}_X(\varphi(B))| = \sum_{p \in \mathcal{P}_k(\theta)} \alpha_p \tau_p(X),$$

using the decomposition (6.1) and $|\mathcal{A}(P_K, p)| = \alpha_p$. Thus define the map $\Sigma\colon \mathbb{R}^{\mathcal{P}(\theta)} \to \mathbb{R}^{\theta}$ by its coordinates

$$\Sigma_k(\mathbf{x}) = \sum_{p \in \mathcal{P}_k(\theta)} \alpha_p x_p.$$

**Corollary 6.2.** *Suppose $X$ is a connected multigraph and $\varphi\colon \Theta \to VX$ is an injective map. If $X$ is strongly symmetric with respect to $\varphi(X)$, then $\mathbf{r}(X) = \Sigma(\boldsymbol{\tau}(X))$.*

## 7. A Recursion for Rooted Spanning Forests

Let the initial data $G$, $\eta$, $\sigma_i$ be given and recall that we always require the connectedness and symmetry condition.

For a $k$-set $K \subseteq \Theta$, define $O(K)$ to be the set of all $\boldsymbol{\omega}$, such that the family $\sigma(\boldsymbol{\omega}) \cup \{\eta(P_K)\}$ is connected and cycle-free. Then the following partition is immediate:

$$O(K) = \biguplus_{B \in \mathcal{A}(P_K)} \Omega(B).$$

This implies

$$\begin{aligned}\Sigma_k(\mathbf{Q}(\mathbf{x})) &= \sum_{p \in \mathcal{P}_k(\theta)} \alpha_p Q_p(\mathbf{x}) = \sum_{p \in \mathcal{P}_k(\theta)} \sum_{B \in \mathcal{A}(P_K, p)} Q_p(\mathbf{x}) \\ &= \sum_{B \in \mathcal{A}(P_K)} Q_{\ell(B)}(\mathbf{x}) = \sum_{B \in \mathcal{A}(P_K)} \mathsf{GF}(\Omega(B) \,|\, \mathbf{x}) = \mathsf{GF}(O(K) \,|\, \mathbf{x}),\end{aligned}$$

using Equation (6.1), Proposition 5.2 and $\alpha_p = |\mathcal{A}(P_K, p)|$.

**Lemma 7.1.** *Let $j \in S$ and fix partitions $B_i \in \mathcal{B}(\Theta)$ for $i \in S \setminus \{j\}$, so that*

$$(B_1, \ldots, B_{j-1}, B_j, B_{j+1}, \ldots, B_s) \in O(K)$$

*for some $B_j \in \mathcal{B}(\Theta)$. Now consider the cycle-free family*

$$\mathcal{B} = \big\{\sigma_i(B_i)\colon\, i \in S \setminus \{j\}\big\} \cup \{\eta(P_K)\}$$

*and set $O = \sigma_j^{-1}\big(\mathsf{Union}(\mathcal{B})|_{\sigma_j(\Theta)}\big)$. Then the s-tuple*

$$\boldsymbol{\omega} = (B_1, \ldots, B_{j-1}, \omega_j, B_{j+1}, \ldots, B_s)$$

*is contained in $O(K)$ if and only if $\omega_j \in \mathcal{A}(O)$.*

*Proof.* By definition $\boldsymbol{\omega} \in O(K)$ if and only if $\sigma(\boldsymbol{\omega}) \cup \{\eta(P_K)\}$ is connected and cycle-free. Note that $\mathcal{B}$ is cycle-free and $\sigma_j(O)$ reflects the connected components of $\mathcal{B}$ on $\sigma_j(\Theta)$. Therefore,

$$\sigma(\boldsymbol{\omega}) \cup \{\eta(P_K)\} = \mathcal{B} \cup \{\sigma_j(\omega_j)\}$$

is connected and cycle-free if and only if $\{O, \omega_j\}$ is, which is equivalent to $\omega_j \in \mathcal{A}(O)$. ∎

**Corollary 7.2.** *Let $\mathbf{B} = (B_1, \dots, B_s) \in O(K)$ be an $s$-tuple of set partitions. For $j \in S$ define $O(K, \mathbf{B}, j)$ to be the set of all $\boldsymbol{\omega} \in O(K)$, such that $\omega_i = B_i$ for $i \in S \setminus \{j\}$. Then, for each $j \in S$, there exists a constant $c_{\mathbf{B},j}$, so that*

$$\mathsf{GF}(O(K, \mathbf{B}, j) \,|\, \mathbf{x}) = c_{\mathbf{B},j} \Sigma_m(\mathbf{x}) \prod_{i \in S \setminus \{j\}} x_{\ell(B_i)},$$

*where $m = |B_j|$.*

*Proof.* Consider the family $\mathcal{B}$ and the set $O = \sigma_j^{-1}\big(\mathsf{Union}(\mathcal{B})|_{\sigma_j(\Theta)}\big)$ defined in Lemma 7.1. Notice that $m + |O| = \theta + 1$. Lemma 7.1 states that

$$O(K, \mathbf{B}, j) = \{B_1\} \times \cdots \times \{B_{j-1}\} \times \mathcal{A}(O) \times \{B_{j+1}\} \times \cdots \times \{B_s\}.$$

Using Equation (6.1) and $|\mathcal{A}(O, p)| = \alpha(\ell(O), p) = \beta_{\ell(O)} \alpha_p$ for $p \in \mathcal{P}_m(\theta)$ we obtain

$$\sum_{\omega \in \mathcal{A}(O)} x_{\ell(\omega)} = \sum_{p \in \mathcal{P}_m(\theta)} \sum_{\omega \in \mathcal{A}(O,p)} x_p = \sum_{p \in \mathcal{P}_m(\theta)} \beta_{\ell(O)} \alpha_p x_p = \beta_{\ell(O)} \Sigma_m(\mathbf{x}),$$

which implies the statement for $c_{\mathbf{B},j} = \beta_{\ell(O)}$. ∎

This implies that $\mathbf{Q}$ has a remarkable structure: The terms corresponding to a single part can be rewritten in terms of $\Sigma_m(\mathbf{x})$, $m \in \Theta$. The following proposition shows how such polynomials can be simplified in general.

**Proposition 7.3.** *Let $I$ be an index set and $\{x_\iota : \iota \in I\}$ be a set of variables. Define the polynomial $P$ by*

$$P = \sum_{\mathbf{u} \in U} a(\mathbf{u}) \prod_{j \in S} x_{u_j},$$

*where $U \subseteq I^s$ is a finite set of $s$-tuples, and $a(\mathbf{u})$ is a real number for $\mathbf{u} \in U$. If $\mathbf{v} \in U$ and $j \in S$ are given, we denote by $U(\mathbf{v}, j)$ the set of all $\mathbf{u} \in U$ with $u_i = v_i$ for $i \in S \setminus \{j\}$. Now suppose that there are finite-dimensional subspaces $\mathcal{L}_1, \dots, \mathcal{L}_s$ of the vector space $\sum_{\iota \in I} \mathbb{R} x_\iota$ such that*

$$\sum_{\mathbf{u} \in U(\mathbf{v}, j)} a(\mathbf{u}) \prod_{i \in S} x_{u_i} = L_{\mathbf{v},j} \prod_{i \in S \setminus \{j\}} x_{v_i}$$

*holds with $L_{\mathbf{v},j} \in \mathcal{L}_j$ for all $\mathbf{v} \in U$ and all $j \in S$. Then $P$ can be written in the form*

$$P = \sum_{m=1}^{M} \prod_{i \in S} L'_{m,i}$$

*for some $M \in \mathbb{N}$ and some $L'_{m,i} \in \mathcal{L}_i$.*

*Proof.* We use simultaneous induction on $s$ and $d = \dim \mathcal{L}_s$. The claim is trivial for $s = 1$ as well as for $d = 0$ ($P$ is identically 0 in the latter case). Choose a basis $\Lambda_1, \dots, \Lambda_d$ of $\mathcal{L}_s$ in reduced echelon form. Hence $\Lambda_1$ contains a variable $x_c$ for some $c \in I$ that is not contained in $\Lambda_2, \dots, \Lambda_d$. We may suppose that the coefficient of $x_c$ in $\Lambda_1$ is 1.

Now, consider those tuples $\mathbf{v} \in U$, for which the coefficient of $\Lambda_1$ in $L_{\mathbf{v},s}$ with respect to the basis $\Lambda_1, \dots, \Lambda_d$ is nonzero. By the choice of $\Lambda_1$, such tuples $\mathbf{v}$ are characterized by the property that $L_{\mathbf{v},s}$ has a nonzero coefficient with respect to $x_c$, which is equivalent to $\bar{\mathbf{v}} = (v_1, \dots, v_{s-1}, c) \in U$ and $a(\bar{\mathbf{v}}) \neq 0$. As a consequence, the nonzero coefficient of $\Lambda_1$ in $L_{\mathbf{v},s}$ is given by $a(\bar{\mathbf{v}})$. This motivates the following definition: Let $W \subseteq I^{s-1}$ be the set of all tuples $\mathbf{w} = (w_1, \dots, w_{s-1})$, so that $\bar{\mathbf{w}} = (w_1, \dots, w_{s-1}, c) \in U$ and $a(\bar{\mathbf{w}}) \neq 0$, and set

$$P^* = \sum_{\mathbf{w} \in W} a(\bar{\mathbf{w}}) \prod_{i=1}^{s-1} x_{w_i} = \sum_{\substack{\mathbf{u} \in U \\ u_s = c}} a(\mathbf{u}) \prod_{i=1}^{s-1} x_{u_i}.$$

Then, we have $P = (P - P^* \cdot \Lambda_1) + P^* \cdot \Lambda_1$. The second representation of $P^*$ shows that $P^*$ satisfies the condition of the proposition (with $s$ replaced by $s-1$). Therefore, by the induction hypothesis, $P^*$ can be written in the claimed form. Furthermore, $P - P^* \cdot \Lambda_1$ also satisfies the condition of the proposition, but instead of $\mathcal{L}_s$, we can take $\mathcal{L}_s^*$, the space spanned by $\Lambda_2, \dots, \Lambda_d$. Since $\dim \mathcal{L}_s^* = \dim \mathcal{L}_s - 1$, we may employ the induction hypothesis again, which shows that $P - P^* \cdot \Lambda_1$ can also be written in the desired form. Altogether, we obtain a representation for $P = (P - P^* \cdot \Lambda_1) + P^* \cdot \Lambda_1$ of the form

$$P = \sum_{m=1}^{M} \prod_{i \in S} L'_{m,i},$$

which finishes the proof. ∎

Now we combine all ingredients to obtain the key result of this section.

**Theorem 7.4.** *There exists an s-homogeneous polynomial* $\mathbf{R}\colon \mathbb{R}^\theta \to \mathbb{R}^\theta$ *satisfying* $\Sigma \circ \mathbf{Q} = \mathbf{R} \circ \Sigma$, *i.e.,*

$$\sum_{p \in \mathcal{P}_k(\theta)} \alpha_p Q_p(\mathbf{x}) = R_k \left( \sum_{p \in \mathcal{P}_1(\theta)} \alpha_p x_p, \dots, \sum_{p \in \mathcal{P}_\theta(\theta)} \alpha_p x_p \right),$$

*for* $k \in \Theta$.

*Proof.* For each $k \in \Theta$ apply Proposition 7.3 to the polynomial $\Sigma_k \circ \mathbf{Q}$: For $i \in S$ let $\mathcal{L}_i$ be spanned by the linear combinations $\Sigma_1, \dots, \Sigma_\theta$. Then Corollary 7.2 yields exactly the required condition for Proposition 7.3. Hence, for each $k \in \Theta$, there exists an $s$-homogeneous polynomial $R_k\colon \mathbb{R}^\theta \to \mathbb{R}$, so that $\Sigma_k \circ \mathbf{Q} = R_k \circ \Sigma$ holds. ∎

**Corollary 7.5.** *Let* $k \in \Theta$ *and* $z_1^{n_1} \cdots z_\theta^{n_\theta}$ *be a monomial which occurs in the polynomial* $R_k(\mathbf{z})$, *then*

$$\sum_{i \in \Theta} i n_i = \kappa + s + k - 1.$$

*Proof.* The monomial $z_1^{n_1} \cdots z_\theta^{n_\theta}$ in $R_k(\mathbf{z})$ corresponds to the term

$$\left(\Sigma_1(\mathbf{x})\right)^{n_1} \cdots \left(\Sigma_\theta(\mathbf{x})\right)^{n_\theta}$$

in $\Sigma_k(\mathbf{Q}(\mathbf{x}))$. Hence for some number partitions $p_1 \in \mathcal{P}_1(\theta), \dots, p_\theta \in \mathcal{P}_\theta(\theta)$ and $p \in \mathcal{P}_k(\theta)$, the monomial $x_{p_1}^{n_1} \cdots x_{p_\theta}^{n_\theta}$ occurs in $Q_p(\mathbf{x})$. However, all monomials in $Q_p(\mathbf{x})$ are of the form

$$\prod_{q \in \mathcal{P}(\theta)} x_q^{\chi_q(\boldsymbol{\omega})},$$

for some $\boldsymbol{\omega} \in \Omega(p)$. Now the assertion follows from Lemma 5.1. ∎

**Corollary 7.6.** *Let* $(Y, \psi) = \mathsf{Copy}(X, \varphi)$, *then the following relation between* $\mathbf{r}(X)$ *and* $\mathbf{r}(Y)$ *holds*:

$$\mathbf{r}(Y) = \mathbf{R}(\mathbf{r}(X)).$$

*Proof.* Proposition 5.2 states that $\boldsymbol{\tau}(Y) = \mathbf{Q}(\boldsymbol{\tau}(X))$. Using Corollary 6.2 and Theorem 7.4 we get

$$\mathbf{r}(Y) = \Sigma(\boldsymbol{\tau}(Y)) = \Sigma(\mathbf{Q}(\boldsymbol{\tau}(X))) = \mathbf{R}(\Sigma(\boldsymbol{\tau}(X))) = \mathbf{R}(\mathbf{r}(X)),$$

which proves the statement. ∎

Note that Theorem 7.4 is trivial for $\theta \leq 3$, since $|\mathcal{P}_k(\theta)| = 1$ for $k \in \Theta$ in this case. For the sake of completeness, we give the map $\Sigma$ for $\theta = 2$ and $\theta = 3$:

$$\Sigma(x_{2^1}, x_{1^2}) = \left(x_{2^1}, x_{1^2}\right) \quad \text{and} \quad \Sigma\left(x_{3^1}, x_{2^1 1^1}, x_{1^3}\right) = \left(x_{3^1}, 2x_{2^1 1^1}, x_{1^3}\right).$$

However, if $\theta \geq 4$, the situation is more complicated. As an example, we consider 3-dimensional Sierpiński graphs $X_0, X_1, \dots$. Here $\theta = s = 4$, and the polynomial $\mathbf{Q}$ is given in Table 1. A simple computation yields

$$\Sigma\left(x_{4^1}, x_{3^1 1^1}, x_{2^2}, x_{2^1 1^2}, x_{1^4}\right) = \left(x_{4^1}, 2x_{3^1 1^1} + 2x_{2^2}, 3x_{2^1 1^2}, x_{1^4}\right),$$

and therefore we obtain the polynomial

$$\mathbf{R}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 7z_1 z_2^3 + 12 z_1^2 z_2 z_3 \\ \frac{11}{4} z_2^4 + 20 z_1 z_2^2 z_3 + \frac{52}{9} z_1^2 z_3^2 + 6 z_1^2 z_2 z_4 \\ 11 z_2^3 z_3 + 20 z_1 z_2 z_3^2 + \frac{21}{2} z_1 z_2^2 z_4 + 6 z_1^2 z_3 z_4 \\ 20 z_2^2 z_3^2 + \frac{208}{27} z_1 z_3^3 + 7 z_2^3 z_4 + 24 z_1 z_2 z_3 z_4 \end{pmatrix}$$

satisfying $\Sigma \circ \mathbf{Q} = \mathbf{R} \circ \Sigma$. This is a considerable simplification compared to the polynomial $\mathbf{Q}$ given in Table 1. In the following section, it is shown how this can be reduced even further.

## 8. The Final Step

### 8.1. A Theorem on Rooted Spanning Forests

An extension of Kirchhoff's famous matrix tree theorem states that the number of rooted spanning forests $(F, R)$ of a finite multigraph $X$ with given roots $\varnothing \neq R \subseteq VX$ is

$$|\mathcal{R}_X(R)| = \det\left(\Pi_H \Delta \Pi_H^*\right),$$

where $H = VX \setminus R$, see [16]. (If $H = \varnothing$ the above determinant is defined to be 1.) Note that $\Pi_H \Delta \Pi_H^*$ is the Dirichlet-Laplace operator with respect to the boundary $R$.

If $X$ is strongly symmetric with respect to $D \subseteq VX$, then

$$|\mathcal{R}_X(R_1)| = |\mathcal{R}_X(R_2)|, \tag{8.1}$$

for any two $k$-sets $R_1, R_2 \subseteq D$.

**Theorem 8.1.** *Let $X$ be a connected, finite multigraph and let $D \subseteq VX$ be a vertex subset with $\theta$ vertices. Suppose that $X$ is strongly symmetric with respect to $D$. Then*

$$|\mathcal{R}_X(R)| = k\rho^{k-1}\theta^{1-k}\tau(X)$$

*for all $k$-sets $R \subseteq D$, where $\rho$ is the resistance scaling factor of $X$ with respect to $D$.*

*Proof.* Let $B, C$ be non-empty subsets of $D$ with $B \uplus \{u\} = C$. We prove that

$$|\mathcal{R}_X(C)| = \frac{\rho|C|}{\theta|B|}\,|\mathcal{R}_X(B)|$$

holds, which implies the statement by an easy induction. As before, let $\Delta$ be the Laplace operator associated with the unit conductances on $X$. For convenience, set $\Delta_A = \Pi_{VX\setminus A}\Delta\Pi_{VX\setminus A}^*$ for any non-empty set $A \subseteq D$. Then

$$\frac{|\mathcal{R}_X(B \cup \{x\})|}{|\mathcal{R}_X(B)|} = \frac{\det \Delta_{B\cup\{x\}}}{\det \Delta_B} = \left\langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{x\}} \right\rangle,$$

for all $x \in D \setminus B$. Thus (8.1) yields

$$\left\langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{x\}} \right\rangle = \left\langle 1_{\{y\}}, \Delta_B^{-1} 1_{\{y\}} \right\rangle,$$

for all $x, y \in D \setminus B$. Furthermore, if $v, w, x, y \in D \setminus B$ with $v \neq w$ and $x \neq y$, then there is an automorphism $\gamma$ of $X$, which stabilizes the set $B$ and satisfies $\{\gamma v, \gamma w\} = \{x, y\}$. This implies

$$\left\langle 1_{\{v\}}, \Delta_B^{-1} 1_{\{w\}} \right\rangle = \left\langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{y\}} \right\rangle,$$

since $\Delta_B$ is symmetric. Hence all diagonal entries, as well as all non-diagonal entries of the matrix representing $\Delta_B^{-1}$ corresponding to indices from $D \setminus B$ are equal: There are numbers $a$ and $b$, so that

$$\left\langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{x\}} \right\rangle = a \quad \text{and} \quad \left\langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{y\}} \right\rangle = b,$$

for all distinct $x, y \in D \setminus B$. Set

$$h = H_D^{VX} 1_{\{u\}}, \qquad g = \Delta h = \Delta H_D^{VX} 1_{\{u\}}.$$

Note that $\Pi_{VX\setminus D}\, g = 0$. Using the symmetry condition once again, $g(u) = (\theta - 1)\rho^{-1}$ and $g(x) = -\rho^{-1}$ for $x \in D \setminus \{u\}$. The definition of $h$ implies $\Pi_B h = 0$, and therefore,

$$\Delta_B \left(\Pi_{VX\setminus B} h\right) = \Pi_{VX\setminus B}\, g \quad \text{and} \quad \Pi_{VX\setminus B} h = \Delta_B^{-1}\left(\Pi_{VX\setminus B}\, g\right).$$

For $x \in D \setminus B$, a short computation yields

$$h(x) = \left(\Delta_B^{-1}\left(\Pi_{VX\setminus B}\, g\right)\right)(x) = \sum_{y \in D\setminus B} \left\langle 1_{\{x\}}, \Delta_B^{-1} 1_{\{y\}} \right\rangle g(y),$$

since $\Pi_{VX\setminus D}\, g = 0$. If $x = u$ and $x \neq u$, respectively, we obtain a simple linear system of equations from the last identity:

$$1 = (\theta - 1)\rho^{-1} a - (\theta - |C|)\rho^{-1} b,$$

$$0 = -\rho^{-1} a + |C|\rho^{-1} b,$$

with the solution

$$a = \frac{\rho|C|}{\theta|B|} \qquad \text{and} \qquad b = \frac{\rho}{\theta|B|}$$

using $|C| = |B| + 1$, which finishes the proof. ∎

Now we are able to prove our main result by combining Corollary 7.6 with the formula of Theorem 8.1 that was just established.

### 8.2. A Simplified Recursion

**Lemma 8.2.** *Consider the polynomial* $\mathbf{R}\colon \mathbb{R}^\theta \to \mathbb{R}^\theta$ *given in Theorem 7.4. Let* $R_1$ *be given by*

$$R_1(\mathbf{z}) = \sum_{\mathbf{n}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}}$$

*using multi-index notation and define* $\mu$ *by*

$$\mu = \theta^{-\kappa} \sum_{\mathbf{n}} a_{\mathbf{n}} \prod_{k\in\Theta} k^{n_k}.$$

*Then the complexity of* $Y$ *is given by* $\tau(Y) = \mu\rho(X)^\kappa \tau(X)^s$.

*Proof.* Note that $\tau(X) = r_1(X)$ and $\tau(Y) = r_1(Y)$ by definition. Theorem 8.1 yields the relation $r_k(X) = k\rho(X)^{k-1}\theta^{1-k}\tau(X)$ for $k \in \Theta$. Inserting this into the recursion $\tau(Y) = R_1(\mathbf{r}(X))$ (see Corollary 7.6) implies

$$\tau(Y) = \sum_{\mathbf{n}} a_{\mathbf{n}} \prod_{k\in\Theta} \left(k\rho(X)^{k-1}\theta^{1-k}\tau(X)\right)^{n_k}.$$

By the $s$-homogeneity of $\mathbf{R}$ and Corollary 7.5, the identities

$$\sum_{k\in\Theta} n_k = s \qquad \text{and} \qquad \sum_{k\in\Theta} (k-1) n_k = \kappa$$

hold. Therefore, we obtain

$$\tau(Y) = \rho(X)^\kappa \tau(X)^s\, \theta^{-\kappa} \sum_{\mathbf{n}} a_{\mathbf{n}} \prod_{k\in\Theta} k^{n_k} = \mu\rho(X)^\kappa\tau(X)^s,$$

finishing the proof. ∎

Now Lemmas 3.1 and 8.2 imply Theorem 4.1.

## 8.3. Examples

In the following we continue studying the examples from Section 2.3: Using Theorem 4.2 closed formulas for the complexity are derived. For these examples the parameters $\theta$, $s$, and $\kappa$ were mentioned before. In addition, the resistance scaling factor $\lambda$ was given in Section 3.2. It remains to compute the tree scaling factor $\mu$, which is done by means of Lemma 8.2.

### 8.3.1. Sierpiński Graphs

As the first example, we derive a formula for the complexity of $d$-dimensional Sierpiński graphs, see Section 2.3.1 for their definition. Recall that

$$s=\theta=d+1, \quad |VG|=\tfrac{1}{2}(d+2)(d+1), \quad \kappa=\tfrac{1}{2}d(d-1), \quad \lambda=\tfrac{d+3}{d+1}.$$

In order to apply Theorem 4.2, we have to determine the tree scaling factor $\mu$ first. To this end, apply the substitution procedure to $X=K_{1,d+1}$, the star. This method can be seen as an analogue to the Wye-Delta-transform for electrical networks. Then, the resulting graph $Y$ is bipartite and its vertices can be divided into the following types:

- the centers of the parts $\bar{Z}_i \simeq X$,
- the corner vertices, each of which is attached to exactly one of the centers, and
- linking vertices between the centers: Each of these vertices has exactly two neighbors (which are center vertices), and for each pair of center vertices, there is exactly one vertex linking them.

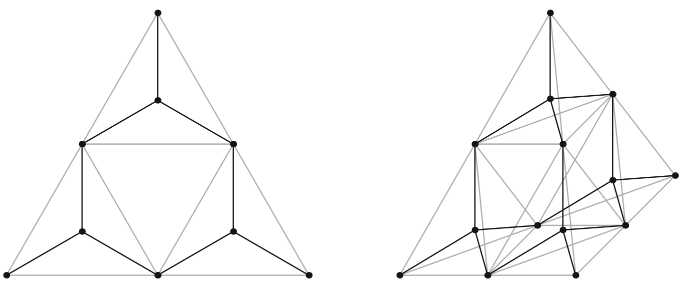

Figure 5: The graph $Y$ for $d=2$ and $d=3$.

We regard $Y$ as a complete graph with $d+1$ vertices whose edges are subdivided, with an additional pendant vertex attached to each of the $d+1$ vertices (see Figure 5).

Obviously, $\tau(X)=1$, since $X$ is a tree. Now, the main task is to calculate $\tau(Y)$: A spanning tree of $Y$ has to contain each of the $d+1$ edges incident to the pendant vertices. Furthermore, we can choose any of the $(d+1)^{d-1}$ spanning trees of the complete graphs $K_{d+1}$ (each of the $d$ edges is represented by two edges in view of the subdivisions), and add one of the two possible edges for each of the remaining $\binom{d+1}{2} - d = \frac{d(d-1)}{2}$ linking vertices. Therefore, we have

$$\tau(Y) = (d+1)^{d-1} \cdot 2^{d(d-1)/2}.$$

Simple computations yield $\rho(X) = \rho(K_{1,d+1}) = d+1$. Using Lemma 8.2 and the formula for $\kappa$ we obtain

$$\mu = \frac{\tau(Y)}{\tau(X)^s \rho(X)^\kappa} = \left(2^d (d+1)^{2-d}\right)^{\frac{d-1}{2}}.$$

Now, Theorem 4.2 can be applied: It is well known that $\tau(X_0) = \tau(K_{d+1}) = (d+1)^{d-1}$, which gives

$$\tau(X_n) = \left(2^{d((d+1)^n-1)} (d+1)^{(d+1)^{n+1}+d(n+1)-1} (d+3)^{(d+1)^n-dn-1}\right)^{\frac{d-1}{2d}}.$$

Note that this is a generalization of the formula for spanning trees of 2-dimensional Sierpiński graphs obtained by the authors in [21]. This formula was computed for low dimensions and conjectured to be true for high dimensions in [7].

8.3.2. Austria Graphs

The Austria graphs of Section 2.3.2 provide an example for the fact that no symmetry at all is needed in the case $\theta = 2$. Furthermore, two distinct orientations of the substitutions $\sigma_1, \dots, \sigma_4$ yield different graph sequences, but this does not alter the complexity by our considerations. It is not difficult to determine the polynomial **Q**:

$$\mathbf{Q}\begin{pmatrix} x_{21} \\ x_{12} \end{pmatrix} = \begin{pmatrix} 3x_{21}^3 x_{12} \\ 5x_{21}^2 x_{12}^2 \end{pmatrix}.$$

This leads to the closed formula

$$\tau(X_n) = 3^{\frac{1}{9}(2\cdot 4^n+3n-2)} \cdot 5^{\frac{1}{9}(4^n-3n-1)},$$

which also follows from Theorem 4.2 using the parameters $\theta = 2$, $s = 4$, $\kappa = 1$, $\lambda = \frac{5}{3}$, $\mu = 3$, and $\rho(X_0) = \tau(X_0) = 1$.

## 9. Conclusions

Our main result, Theorem 4.2, indicates connections between the complexity on finite self-similar graphs and the study of Laplace operators. The polynomials **Q** and **R** both carry information on the resistance scaling factor. Hence it is likely that these polynomials are closely related to the renormalization map, which is usually used in the definition of the resistance scaling factor (see [15]). The Dirichlet- or Neumann-spectrum of Laplace operators on self-similar graphs are well understood and described by the dynamics of a multi-dimensional polynomial, see [18]. Likewise, the complexity is governed by the polynomial **Q**. It is plausible that these two dynamical systems are linked.

Finally, we remark that in [22] it was conjectured that the number of connected subgraphs of $X_n$ asymptotically involves the resistance scaling factor. Our main result proves this conjecture for the number of spanning trees.

We conjecture that the results of this paper can be generalized to sequences of self-similar graphs with an even lesser amount of symmetry (maybe 2-homogeneity on the set of distinguished vertices is already sufficient). In even further generality similar formulas for the complexity might hold asymptotically.

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