Math. Ann. 284, 251–265 (1989)

Mathematische Annalen

Amenable group actions on infinite graphs

Wolfgang Woess

Dipartimento di Matematica, Università di Milano, Via C. Saldini, 50, I-20133 Milano, Italy

Dedicated to Professor Gert Sabidussi on the occasion of his sixtieth birthday

1. Introduction

Throughout this paper, $G=(X, E)$ will be a locally finite, infinite connected graph with vertex set X and edge set E; the edges are unoriented, loops are permitted, but no multiple edges. The graph structure induces an integer-valued metric on $X: d(x, y)$ is the minimal length (number of edges) of a path connecting x and y. An *automorphism* of G is an isometry of X onto itself with respect to this metric. The full group of automorphisms of G is denoted by $\mathrm{Aut}(G)$. With the topology of pointwise convergence, it is a locally compact Hausdorff group. A neighbourhood base of the identity is given by the family of all subgroups which stabilize pointwise some finite set of vertices. These are compact open subgroups, so that $\mathrm{Aut}(G)$ is totally disconnected. For a more detailed description, see e.g. Trofimov [Tr].

Tits [T 2] has shown that a solvable group of automorphisms of an infinite *tree* must fix a vertex, an edge, an end or a pair of ends of the tree. Nebbia [Ne] has made it clear that this is in fact a consequence of *amenability*: a closed group of automorphisms of the tree has one of the four properties given above if and only if it is amenable.

The purpose of this paper is to show how Nebbia's result can be extended to an arbitrary graph G. We study the relation between amenability and the action of a group Γ of automorphisms on the space of *ends* of G.

The relevant preliminaries are presented in Sect. 2. If Γ acts amenably on G then it fixes a finite set of vertices, an end or a pair of ends (Sect. 2, Theorem 1). The converse is not true in full generality (Sect. 4, Example 2). However, if Γ fixes a finite set, an end of *finite diameter* or a pair of ends, then it acts amenably (Sect. 2, Theorem 2). The proofs of the two theorems (Sect. 3) are based on a detailed graph-theoretical study of automorphism groups fixing an end (Sect. 3, Propositions 1–3) and an approach to amenability as it was originally introduced by J. von Neumann.

It is then a rather easy task to deduce that amenability of a closed subgroup of Aut(G) is related to the same conditions as in Theorems 1 and 2 (Sect. 4, Corollaries 1 and 2). In particular, Nebbia's result can be obtained as a special case. Furthermore, in Sect. 4 some examples are given which illustrate each of the possible cases.

Although the methods and spirit of this paper are mostly those of elementary (though not trivial) combinatorial graph theory, the results will be of interest in questions of analytic, probabilistic and potential theoretic type. Some of these aspects will be studied in two forthcoming papers with P. M. Soardi.

The author acknowledges stimulating discussions with W. Imrich and N. Seifter (Leoben) and P. M. Soardi (Milano).

2. Amenable group actions and the ends of a graph

There is a great variety of equivalent approaches to amenability, to a large extent in an analytical setting, see Pier [Pi]. However, in the present context it seems to be more natural to go back to the very roots of amenability, as introduced by von Neumann [vN] (see the beautiful book by Wagon [Wa]).

In the sequel, Γ will be an arbitrary group of automorphisms of G, not necessarily closed in Aut(G).

Definition 1. We say that *the action of Γ on G is amenable*, if there is a nonnegative set function μ (called *invariant measure*), defined for every subset of the vertex set X, such that

- $\mu(X)=1$,
- μ is finitely additive, and
- $\mu(\gamma M)=\mu(M)$ for every $\gamma\in\Gamma$ and every $M\subset X$.

Next, we describe the space of ends of G. An *infinite path* is a sequence $\pi=[x_0,x_1,x_2,\dots]$ of successively adjacent vertices without repetitions. Two infinite paths π,π' are said to be *equivalent* if, for every finite subset U of X, there is some finite path lying entirely outside U which connects some vertex of π with some vertex of π'. An *end* is an equivalence class under this relation. The set of all ends of G is denoted by Ω. If U is a finite subset of X, then $G\setminus U$ decomposes into finitely many connected components. If B is the set of vertices in one of these components, then we add to B all ends which have a representative infinite path lying entirely within the component. Thus, we obtain a set $C\subset\bar{G}=X\cup\Omega$. If $z\in(X\setminus U)\cup\Omega$, then z lies in exactly one set C of this type: this is the component of z after removing U, denoted by $C(U,z)$. If z' is another element of $(X\setminus U)\cup\Omega$ and $C(U,z)\neq C(U,z')$, then we say that *U separates z and z'*. Varying U (finite) and z, the family of all sets $C(U,z)$ becomes the basis of a topology. Thus $\bar{G}$ becomes a compact, totally disconnected Hausdorff space in which X is open and dense and Ω is compact.

The space of ends was originally introduced by Freudenthal [Fr]; in the present terminology it is due to Halin [H 1] and Jung [J 1]. Along a different line, see e.g. Stallings [St] for ends of groups (Cayley graphs).

If $\omega \in \Omega$, then one can always find a sequence $\{U_n\}$ of finite subsets of X such that

$$C(U_n, \omega) \supset U_{n+1} \cup C(U_{n+1}, \omega), \qquad n = 1, 2, \ldots. \tag{2.1}$$

Such a sequence is called *contracting towards* ω; $\{C(U_n, \omega)\}$ constitutes a neighbourhood base at ω.

Definition 2. The *size* and *diameter* of ω are defined respectively by

$$m_1(\omega) = \inf \liminf_{n \to \infty} |U_n|, \qquad \operatorname{diam}(\omega) = \inf \liminf_{n \to \infty} \operatorname{diam}(U_n),$$

where the infima extend over all sequences $\{U_n\}$ contracting towards ω.

Here, $|U_n|$ stands for the cardinality of U_n, whereas $\operatorname{diam}(U_n)$ refers to the diameter in the discrete metric of G. It is rather straightforward, by Menger's theorem (see Ore [Or, p. 204]), that our definition of $m_1(\omega)$ coincides with that of Halin [H 2]. For more details concerning diameters of ends, cf. Woess [Wo]. We remark that diameters of ends are also of some interest in the study of harmonic functions, see Picardello and Woess [PW].

Every automorphism γ of G extends continuously to Ω. In a little known, but very instructive paper, Halin [H 3, Theorems 4 and 9] showed that either

(i) γ is *elliptic*: it fixes a finite subset of X, or

(ii) γ is *parabolic*: it fixes a unique end ω_0 with $m_1(\omega_0) = \infty$, and for every $x \in X$, $\lim_{n \to \infty} \gamma^n x = \omega_0$ in the topology of $\bar{G}$, or

(iii) γ is *hyperbolic*: it fixes each of a unique pair of ends ω_0, ω_1 with $m_1(\omega_0) = m_1(\omega_1) < \infty$, and $\lim_{n \to \infty} \gamma^n x = \omega_0$, $\lim_{n \to \infty} \gamma^{-n} x = \omega_1$ for every $x \in X$.

For *trees*, compare with Tits [T1]. In fact, Halin [H3] uses "Type 1" for elliptic and "Type 2" for parabolic and hyperbolic automorphisms; the present terminology seems to be more suggestive in accordance with classcial geometric concepts. In cases (ii) and (iii) we say that ω_0 is the *direction* of γ, denoted by $\operatorname{dir}(\gamma) = \omega_0$, and ω_0 is called a parabolic or hyperbolic end, respectively. It is easy to see from Theorem 9 and its proof in [H 3] that besides finite size, a hyperbolic end also has finite diameter. If $\Gamma \subset \operatorname{Aut}(G)$ has no hyperbolic element, then we say that it is *nonhyperbolic*; if Γ has elliptic elements only then we call it an *elliptic group*. All these notions depend, of course, on the graph G and the compactification under consideration (which in our case is always the end compactification).

Now consider the following four statements.

(a) Γ fixes a finite subset of X.

(b) Γ fixes an end of G.

(b′) Γ fixes an end of G with finite diameter.

(c) Γ fixes a set of two ends of G which are the directions of a hyperbolic automorphism in Γ and its inverse, respectively.

Theorem 1. *If the action of* Γ *on* G *is amenable, then one of* (a), (b), (c) *holds.*

Theorem 2. *If one of* (a), (b′), (c) *holds, then the action of* Γ *on* G *is amenable.*

If G has only one end, then of course the theorems do not tell much. There are good reasons for the gap between (b) and (b′) in Theorems 1 and 2. Indeed, Γ may fix a connected subgraph of G whose only end has infinite diameter, see the examples in Sect. 4. Thus, the gap arises from the problem of a graph-theoretical characterization of amenable group actions on graphs with one end of infinite diameter: this is beyond the scope of the present paper, which is devoted to studying the interplay between amenability and the space of ends.

3. Fixed ends and Banach limits

The main purpose of this section is to prove Theorems 1 and 2. For this purpose, we first have to study in detail the action of groups which fix an end of G. We shall always have in mind the case when $|\Omega| \geqq 2$. The following three propositions may be of some purely graph-theoretical interest by themselves.

Speaking of a connected subset or of a component of $\bar{G}$, we do not refer to connectedness in the topology of $\bar{G}$ but mean that its intersection with X induces a connected subgraph of G.

Definition 3. An automorphism σ of G is called a *shift*, if there is a finite subset U of X, such that $\bar{G} \backslash U$ splits into components C_i,

$$\bar{G} \backslash U = C_0 \cup C_1 \cup \ldots \cup C_k, \qquad k \geqq 1, \tag{3.1}$$

and

$$\sigma(U \cup C_i) \subset C_i \quad \text{for some } i. \tag{3.2}$$

Observe that every shift is hyperbolic and that every hyperbolic automorphism has a power which is a shift (this is implicit in [H 3, Theorem 9]). Our first proposition generalizes [T 1, Proposition 3.4].

Proposition 1. *If Γ is nonhyperbolic and does not fix a finite subset of X, then*

- *for every finite $U \subset X$, there is a unique component $C_0(U)$ of $\bar{G} \backslash U$, such that $\Gamma x \cap C_0(U)$ is infinite for some (all) $x \in X$, and*
- *Γ fixes a unique end of G.*

Proof. By assumption, for every $x \in X$, the orbit Γx is infinite. Let $U \subset X$ be any finite set such that (3.1) holds. (If $\bar{G} \backslash U$ is connected then there is nothing to prove.) We may assume that U is connected: otherwise, U can be replaced by a larger, connected set V, and if the proposition is true for V then it is also true for U. Now $\Gamma U = \bigcup \{\Gamma u \mid u \in U\}$ is infinite. There must be a component C_i such that $\Gamma U \cap C_i$ is infinite. Without loss of generality, we assume that this holds for $C_0 = C_0(U)$, and set

$$R = R(U) = \bar{G} \backslash (U \cup C_0). \tag{3.3}$$

Then $\Gamma U \cap \alpha C_0$ is infinite for every $\alpha \in \Gamma$, and $\{\gamma U \mid \gamma \in \Gamma, \gamma U \subset \alpha C_0\}$ must be infinite. Now let $\alpha, \beta \in \Gamma$ and suppose that $\alpha U \cap \beta U = \emptyset$. By connectedness, $\beta U \subset \alpha C_i$ and $\alpha U \subset \beta C_{i'}$ for some i, i'. We obtain

$$\alpha C_i \supset \beta\Big(U \cup \bigcup_{j \neq i'} C_j\Big)$$

as the latter set is connected and does not intersect αU. If $i \neq i'$, then $\alpha C_i \supset \beta(U \cup C_i)$ and $\alpha^{-1}\beta$ is a shift, a contradiction. Hence $i = i'$. If $i \neq 0$, then take $\gamma \in \Gamma$ such that $\gamma U \subset \alpha C_0$. Then, by the same reasoning as above, $\alpha U \subset \gamma C_0$ and

$\alpha C_0 \supset \gamma\left(U \cup \bigcup_{j \neq 0} C_j\right)$. In particular, $\beta C_i \supset \alpha(U \cup C_0) \supset \gamma(U \cup C_i)$, so that $\beta^{-1}\gamma$ is a shift, a contradiction. Hence $i=0$. We have obtained the following for connected $U \subset X$ satisfying (3.1):

$$\text{If} \quad \alpha, \beta \in \Gamma \quad \text{and} \quad \alpha U \cap \beta U = \emptyset \quad \text{then} \quad \alpha U \subset \beta C_0 \quad \text{and} \quad \beta U \subset \alpha C_0. \tag{3.4}$$

This implies that $\Gamma U \cap \alpha R$ is finite for every $\alpha \in \Gamma$, so that C_0 is indeed the unique component of $\bar{G} \setminus U$ which has infinite intersection with Γx, $x \in X$.

We now choose some reference vertex x and define

$$U_n = \{y \in X \mid d(y, x) \leqq n\}.$$

From some n_0 onwards, U_n will have property (3.1), with the components and their number depending on n. Every component $C_j(U_{n+1})$ is contained in some $C_i(U_n)$. We have seen that for $\alpha \in \Gamma$, $\Gamma x \cap \alpha C_i(U_n)$ is infinite if and only if $i=0$. Hence, $C_0(U_{n+1}) \subset C_0(U_n)$ for every n. If we set $V_n = C_0(U_{n-1}) \cap U_n$, $n \geqq 1$, then $C_0(U_n)$ is a component of $\bar{G} \setminus V_n$, so that $C_0(U_n) = C_0(V_n)$. We have

$$C_0(V_n) \supset V_{n+1} \cup C_0(V_{n+1}), \qquad n \geqq 1, \tag{3.5}$$

and $\{V_n \mid n \geqq 1\}$ is contracting towards some end ω_0 of G with $C_0(V_n) = C(V_n, \omega_0)$. If $\gamma \in \Gamma$, then $\{\gamma V_n \mid n \geqq 1\}$ is contracting towards $\gamma\omega_0$. Note that $\gamma C_0(U_n) = C_0(\gamma U_n)$. Suppose that $\gamma\omega_0 \neq \omega_0$. Then there must be m, n such that $C_0(U_m) \cap C_0(\gamma U_n) = \emptyset$. Thus, $C_0(\gamma U_n) \subset U_m \cup R(U_m)$. Now, Γx has finite intersection with $U_m \cup R(U_m)$, while $\Gamma x \cap C_0(\gamma U_n)$ is infinite, a contradiction. Thus, $\Gamma\omega_0 = \omega_0$.

Finally, if $\omega \in \Omega$, $\omega \neq \omega_0$, then $\omega \notin C_0(U_n)$ for some $n \geqq n_0$ by (3.5). Hence $\omega \in R(U_n)$. Now choose $\gamma \in \Gamma$ such that $\gamma U_n \cap U_n = \emptyset$. By (3.4), $U_n \subset \gamma C_0(U_n)$ and $\gamma U_n \subset C_0(U_n)$, so that $\gamma R(U_n) \subset C_0(U_n)$. This yields $\gamma\omega \neq \omega$. □

We remark that along the lines of Proposition 1 and its proof one can deduce the following, slightly stronger version of a result of Jung [J 2]: suppose that G has more than one end and that Γ has only finitely many orbits on X. If U has property (3.1) with at least two infinite components and C_i is one of them, then there is a shift $\sigma \in \Gamma$ with $\sigma(U \cup C_i) \subset C_i$. Indeed, it suffices to assume that Γ acts *densely* on G, i.e., every infinite component of $\bar{G}$ has infinite intersection with Γx, $x \in X$.

We now refine Proposition 1 in the case when the end fixed has finite diameter.

Proposition 2. *Suppose that Γ is nonhyperbolic and does not fix a finite subset of X. Let ω_0 be the end of G fixed by Γ. If* $\operatorname{diam}(\omega_0) < \infty$ *then there exists a sequence $\{V_n\}$ of finite connected subsets of X contracting towards ω_0, such that*

- *the V_n have bounded diameters, and*
- *for every $\gamma \in \Gamma$ there is an index $n(\gamma) \geqq 1$ such that $\gamma V_n = V_n$ for every $n \geqq n(\gamma)$.*

Proof. We take up the notation of Proposition 1 and its proof. By the definition of $\operatorname{diam}(\omega_0)$, we can find a sequence $\{U_n\}$ of finite connected sets in X, such that

$$\operatorname{diam}(U_n) \leqq D < \infty \quad \text{for every} \quad n \geqq 0,$$

and $\{U_n\}$ is contracting towards ω_0. We must have

$$C(U_n, \omega_0) = C_0(U_n) \supset U_{n+1} \cup C_0(U_{n+1}). \tag{3.6}$$

By deleting some of the U_n, we may also assume that

$$d(U_n, U_{n+1}) \geqq 2D+1 \quad \text{for every} \quad n \geqq 0. \tag{3.7}$$

If $\gamma \in \Gamma$, then (3.6) and (3.7) hold with $\{\gamma U_n\}$ in the place of $\{U_n\}$, and $\{\gamma U_n\}$ is contracting towards ω_0. In particular, there is a minimal index $k(\gamma) \geqq 1$ such that

$$U_0 \cap (\gamma U_n \cup C_0(\gamma U_n)) = \emptyset \quad \text{for every} \quad n \geqq k(\gamma).$$

Suppose that $\gamma U_n \cap U_n = \emptyset$ for $n \geqq k(\gamma)$. Then by (3.4), $\gamma U_n \subset C_0(U_n)$ and $U_n \subset C_0(\gamma U_n)$. On the other hand, $U_0 \subset C_i(U_n)$ for some $i \neq 0$, and $C_i(U_n) \cap \gamma U_n = \emptyset$, so that $U_n \cup C_i(U_n) \subset C_j(\gamma U_n)$ for some j. It must be $j=0$, and $U_0 \subset C_0(\gamma U_n)$, a contradiction. We have obtained:

$$\text{if} \quad n \geqq k(\gamma) \quad \text{then} \quad \gamma U_n \cap U_n \neq \emptyset. \tag{3.8}$$

Claim 1. If $\alpha, \beta \in \Gamma$, then $k(\alpha^{-1}\beta) \leqq \max\{k(\alpha)+1, k(\beta)\}$.

Proof of Claim 1. Let $n \geqq \max\{k(\alpha)+1, k(\beta)\}$. From (3.8) we infer that $U_{n-1} \cap \alpha U_{n-1}$, $U_n \cap \alpha U_n$, and $U_n \cap \beta U_n$ are nonvoid. Thus, $U_{n-1} \cup \alpha U_{n-1}$ and $U_n \cup \alpha U_n \cup \beta U_n$ are connected sets which do not intersect by (3.7). As $U_n \subset C_0(U_{n-1})$, we must have $\beta U_n \subset C_0(U_{n-1})$. Similarly, $\beta U_n \subset C_0(\alpha U_{n-1})$. On the other hand, $U_{n-1} \cup \alpha U_{n-1} \subset C_i(\beta U_n)$ for some i. By the above, we have $U_0 \subset C_i(\beta U_n)$ and $\alpha U_0 \subset C_i(\beta U_n)$. The choice of $n \geqq k(\beta)$ implies $i \neq 0$, and we obtain $\alpha U_0 \cap (\beta U_n \cup C_0(\beta U_n)) = \emptyset$.

We now define for $n \geqq 1$

$$V_n = \bigcup \{\beta U_n | \beta \in \Gamma,\ k(\beta) \leqq n\}.$$

From (3.8) and (3.7) we see that the V_n are connected, pairwise disjoint sets containing U_n, and $\operatorname{diam}(V_n) \leqq 3D$, $n = 1, 2, \ldots$. If $\gamma \in \Gamma$, then by Claim 1

$$\gamma V_n = V_n \quad \text{for every} \quad n \geqq n(\gamma) = k(\gamma^{-1}) + 1. \quad \square$$

In particular, if Γ is as in Proposition 2, then it is an elliptic group of automorphisms. Observe that ω_0 need not be a hyperbolic end, even if it has finite diameter and size. Next, we prove a hyperbolic version of Proposition 2. Although the spirit is similar, the details are quite different.

Let $\gamma \in \operatorname{Aut}(G)$ and $A \subset X$. We say that the action of γ is *bounded on A*, if

$$\sup\{d(x, \gamma x) | x \in A\} < \infty. \tag{3.9}$$

If Γ has a hyperbolic element and fixes an end ω_0, then we shall show that there exists a two-sided infinite "chain" $A \subset X$ which has only two accumulation points in $\bar{G}$, one of which is ω_0. For every $\gamma \in \Gamma$, from some point onward in the direction of ω_0, γ behaves on A like a translation, i.e., its action is bounded.

Proposition 3. *Suppose that Γ contains a hyperbolic automorphism and fixes an end ω_0 of G. Then there exist a shift $\sigma \in \Gamma$ and a finite connected subset V of X such that the following holds.*

- $\omega_0 = \operatorname{dir}(\sigma)$,
- *$\bar{G} \backslash V$ has at least two infinite components,*
- *if $\gamma \in \Gamma$ then there are $k(\gamma)$, $m(\gamma) \in \mathbb{Z} \cup \{-\infty\}$ such that*

$$\gamma\left(\bigcup\{\sigma^n V | n \in \mathbb{Z},\ n \geqq k(\gamma)\}\right) \subset \bigcup\{\sigma^n V | n \in \mathbb{Z},\ n \geqq m(\gamma)\},$$

and

- *the action of γ on $\bigcup\{\sigma^n V | n \geqq 0\}$ is bounded.*

Proof. As Γ has a hyperbolic element, G has more than one end. Thus, we can find a shift and a corresponding finite connected set U such that (3.1) holds with at least two infinite components. We fix U once and for all and assume that $C(U, \omega_0) = C_0$.

We claim that we can choose a shift $\sigma \in \Gamma$ such that

$$C_0 \supset \sigma(U \cup C_0),$$

and

$$\text{if} \quad C_i \supset \gamma(U \cup C_i) \quad \text{for some} \quad \gamma \in \Gamma, \quad \text{then} \quad d(u, \gamma U) \geqq d(U, \sigma U). \tag{3.10}$$

Indeed, we may choose a shift $\tau \in \Gamma$ such that $C_i \supset \tau(U \cup C_i)$ for some i, and $d(U, \tau U)$ is minimal with respect to this property. If $i = 0$ then (3.10) is satisfied for $\sigma = \tau$. Assume that $i \neq 0$. Then $C_0 \cup U \subset \tau C_j$ for some j. As $\tau\omega_0 = \omega_0 \in C_0$, we must have $j = 0$. But then (3.10) holds with $\sigma = \tau^{-1}$.

From now on, we fix σ as given by (3.10). Then $\omega_0 = \text{dir}(\sigma)$, $\{\sigma^n U | n \geqq 0\}$ is contracting towards ω_0, and

$$C(\sigma^n U, \omega_0) = \sigma^n C_0 \supset \sigma^{n+1}(U \cup C_0) \quad \text{for every} \quad n \in \mathbb{Z}. \tag{3.11}$$

By (3.10), $\sigma^{-1}(U \cup C_i) \subset C_i$ for some $i \neq 0$; without loss of generality we assume $i = 1$. Let $\omega_1 = \text{dir}(\sigma^{-1})$. Then $\{\sigma^{-n} U | n \geqq 0\}$ is contracting towards ω_1, and

$$C(\sigma^n U, \omega_1) = \sigma^n C_1 \supset \sigma^{n-1}(U \cup C_1) \quad \text{for every} \quad n \in \mathbb{Z}. \tag{3.12}$$

Now let $\gamma \in \Gamma$. Then $\gamma\omega_0 = \omega_0$. In particular, we obtain from (3.11) and (3.12) that

$$\begin{gathered} \bigcap_{n \in \mathbb{Z}} \gamma\sigma^n C_0 = \{\omega_0\}, \qquad \bigcap_{n \in \mathbb{Z}} \gamma\sigma^n C_1 = \{\gamma\omega_1\}, \\ \bigcup_{n \in \mathbb{Z}} \gamma\sigma^n C_1 = \bar{G} \setminus \{\omega_0\} \quad \text{and} \quad \bigcup_{n \in \mathbb{Z}} \gamma\sigma^n C_0 = \bar{G} \setminus \{\gamma\omega_1\}. \end{gathered} \tag{3.13}$$

The sets $\sigma^n U$, $n \in \mathbb{Z}$ constitute an "infinite zip". We shall see that every $\gamma \in \Gamma$ leaves closed at least one half of the "zip" in the direction of ω_0. We now define the index $m(\gamma)$ where it opens.

$$m(\gamma) = \inf\{n \in \mathbb{Z} | \gamma\omega_1 \in \sigma^n C_1\} + 1.$$

Then by (3.13), $m(\gamma) \in \mathbb{Z} \cup \{-\infty\}$, $m(\gamma) = -\infty$ if and only if $\gamma\omega_1 = \omega_1$, and in this case also $m(\gamma^{-1}) = -\infty$. We now define for $k, m \in \mathbb{Z}$, $n \in \mathbb{N}$

$$\begin{gathered} \Gamma_m = \{\beta \in \Gamma | \beta U \cap \sigma^m U \neq \emptyset, \ \omega_1 \notin \beta C_0\}, \\ V = \Gamma_0 U, \quad A_k = \bigcup_{l \geqq k} \sigma^l V \quad \text{and} \quad A_{k,n} = \bigcup_{l=0}^{n-1} \sigma^{k+l} V. \end{gathered} \tag{3.14}$$

First of all, observe that $\sigma^m \in \Gamma_m$ and that

$$\sigma^{-m}\Gamma_m = \Gamma_0 \quad \text{and} \quad \sigma^m V = \Gamma_m U \quad \text{for every} \quad m \in \mathbb{Z}. \tag{3.15}$$

Indeed, if $\beta U \cap \sigma^m U \neq \emptyset$ and $\omega_1 \notin \beta C_0$, then $\sigma^{-m}\beta U \cap U \neq \emptyset$ and $\omega_1 = \sigma^{-m}\omega_1 \notin \sigma^{-m}\beta C_0$ and conversely.

Claim 2. If $\gamma \in \Gamma$ then there exists $k(\gamma) \in \mathbb{Z} \cup \{-\infty\}$, such that $\gamma A_{k(\gamma)} \subset A_{m(\gamma)}$.

Proof of Claim 2. If $\gamma\omega_1=\omega_1$ then we set $k(\gamma)=-\infty$. If $\gamma\omega_1\neq\omega_1$, then observe that $\lim\limits_{n\to\infty}\sigma^n x=\omega_0$ for every $x\in V$, while $\gamma^{-1}\sigma^{m(\gamma)-1}C_0$ is a neighbourhood of ω_0. As V is finite, we can choose $k(\gamma)\in\mathbb{Z}$ (minimal) such that

$$\sigma^k V\subset\gamma^{-1}\sigma^{m(\gamma)-1}C_0 \quad \text{for every} \quad k\geqq k(\gamma).$$

Now fix $k\geqq k(\gamma)$, $k\in\mathbb{Z}$, and let $\beta\in\Gamma_k$. By (3.13), each of the numbers

$$n'=\max\{n\in\mathbb{Z}\mid\gamma^{-1}\sigma^n C_0\supset\beta U\},$$

$$n''=\min\{n\in\mathbb{Z}\mid\beta C_0\supset\gamma^{-1}\sigma^n U\}$$

is well defined and finite. We show that $m(\gamma)\leqq n'+1\leqq n''-1$. –

If $\gamma\omega_1=\omega_1$, then $n'>-\infty=m(\gamma)$. If $\gamma\omega_1\neq\omega_1$, then $\sigma^{-k}\beta U\subset V$, and

$$\beta U\subset\gamma^{-1}\sigma^{m(\gamma)-1}C_0$$

by (3.15) and the definition of $k(\gamma)$. Thus $n'\geqq m(\gamma)-1$. Furthermore, $\beta C_i\supset\gamma^{-1}\sigma^{n'}C_1$ for some i. As $n'\geqq m(\gamma)-1$, $\omega_1\in\gamma^{-1}\sigma^{n'}C_1$ by the definition of $m(\gamma)$. By (3.14), we must have $i\neq 0$. Now, $\beta(U\cup C_0)$ is connected and does not intersect $\gamma^{-1}\sigma^{n'}U$. We infer that

$$\gamma^{-1}\sigma^{n'}C_0\supset\beta(U\cup C_0)\supset\gamma^{-1}\sigma^{n''}U. \tag{3.16}$$

Hence, we must have $n''\geqq n'+1$.

Suppose that $n''=n'+1$. Then we see from (3.16) that βU separates $\gamma^{-1}\sigma^{n'}U$ from $\gamma^{-1}\sigma^{n'+1}U$, and

$$d(U,\sigma U)\geqq d(U,\sigma^{-n'}\gamma\beta U),$$

in contradiction with (3.10): $n''\geqq n'+2$.

Now set $l=l(\beta,\gamma)=n'+1$. We neither have $\gamma^{-1}\sigma^l C_0\supset\beta U$ nor $\beta C_0\supset\gamma^{-1}\sigma^l U$.

Suppose that $\gamma^{-1}\sigma^l U\cap\beta U=\emptyset$. Then $\gamma^{-1}\sigma^l U\subset\beta C_i$ for some i, and $i\neq 0$ by the above. But then we must have $\beta(U\cup C_0)\subset\gamma^{-1}\sigma^l C_j$ for some j. Again, $j\neq 0$ by the above. However, $\omega_0\in\beta(U\cup C_0)$, while $\omega_0\notin\gamma^{-1}\sigma^l C_j$, a contradiction.

Thus, $\gamma\beta U\cap\sigma^l U\neq\emptyset$. Furthermore, by (3.16), $\gamma\beta C_0\subset\sigma^{n'}C_0$, which does not contain ω_1. In other words, $\gamma\beta\in\Gamma_l$. This holds for every $\beta\in\Gamma_k$, $k\geqq k(\gamma)$, and Claim 2 is proved.

Claim 3. The action of $\gamma\in\Gamma$ on A_0 is bounded.

Proof of Claim 3. It is enough to show boundedness of γ on some A_k, $k\geqq\max\{k(\gamma),0\}$: the set $A_k\backslash A_0$ is finite. Note that γ cannot be parabolic.

Case 1. γ is elliptic, it fixes some finite $W\subset X$. Each of the sets

$$W_n=\{x\in X\mid d(x,W)\leqq n\}$$

is also fixed by γ, and one of them must separate ω_0 from ω_1 and be connected. We may assume that this is already true for W itself. By construction of the A_k, we must have $A_{r(W)}\subset C(W,\omega_0)$ for some minimal $r(W)\in\mathbb{Z}$. If $k=\max\{r(W),k(\gamma),0\}$ then $\gamma A_k\subset\gamma C(W,\omega_0)=C(W,\omega_0)$ and $\gamma A_k\subset A_{m(\gamma)}$. Note that $F=\gamma A_k\backslash A_{r(W)}$ is finite (γA_k has only one accumulation point in Ω). Hence, γ must fix all but finitely many of the "cross sections"

$$\{x\in A_k | d(x,W)=n\}, \qquad n\geqq 1$$

of A_k, compare also with the proof of Proposition 2. It is a rather obvious exercise to show that there is a finite upper bound on the diameters of these sets, so that the action of γ on A_k is bounded.

Case 2. γ is hyperbolic. Then either $\mathrm{dir}(\gamma)=\omega_0$ or $\mathrm{dir}(\gamma^{-1})=\omega_0$. If the latter holds, suppose that we have already shown that the action of γ^{-1} on A_0 is bounded. Then also γ is bounded on A_0. Hence, we may assume without loss of generality that $\mathrm{dir}(\gamma)=\omega_0$. Now let $k=\max\{k(\gamma),0\}$, fix $x\in\sigma^k V=\Gamma_k U$ [see (3.15)] and let $D=d(x,\gamma x)$. By Halin's theorems $\lim\limits_{n\to\infty}\gamma^n x=\omega_0$, see Sect. 2. In particular, $D>0$ and $\gamma^n x\in A_{k(\gamma)}\cap A_{m(\gamma)}$ for all but finitely many n.

Let $y\in\sigma^m V$, where $m\geqq k$. Consider the set $A_{m,D}$ as defined in (3.14). Then $\mathrm{diam}(A_{m,D})\leqq K$, where $K=d(V,\sigma^D V)+2\,\mathrm{diam}(V)$. We must have $\gamma^n x\in\sigma^l V$ for some $l\geqq k+D$ and for some n. If we choose this n to be minimal, then it must be $u\in A_{m,D}$ for at least one $u\in\{\gamma^{n-1}x,\gamma^n x\}$ (note that $\sigma^{l-1}V$ and $\sigma^l V$ may intersect). We obtain

$$d(\gamma y,y)\leqq d(\gamma y,\gamma u)+d(\gamma u,u)+d(u,y)\leqq 2K+D\,,$$

and Claim 3 is proved.

Together with (3.10) and (3.15), Claims 2 and 3 prove the proposition. □

We now have collected enough structural details in order to prove the two theorems on amenable group actions.

Proof of Theorem 1. Let $\Gamma\subset\mathrm{Aut}(G)$ be a group which acts amenably on G. If Γ is nonhyperbolic, then by Proposition 1 one of statements (a) or (b) holds. We are left with the case when Γ contains a hyperbolic element. But then there is a shift $\sigma\in\Gamma$.

Claim 4. Let $\omega_0=\mathrm{dir}(\sigma)$ and $\omega_1=\mathrm{dir}(\sigma^{-1})$. Then Γ fixes ω_0,ω_1 or $\{\omega_0,\omega_1\}$.

Proof of Claim 4. Let μ be an invariant measure as in Definition 1. The orbit Γx is infinite for every $x\in X$. Hence, $\mu(U)=0$ for every finite $U\subset X$. If $C\subset\bar{G}$, then we write $\mu(C)$ for $\mu(C\cap X)$ to facilitate notation.

For our shift σ we may assume without loss of generality that in (3.1), U is connected, all components of $\bar{G}\setminus U$ are infinite and $i=0$ in (3.2). (Otherwise, we may enlarge U and replace σ by a power of σ.):

$$\sigma(U\cup C_0)\subset C_0\,,\quad U \text{ connected}\,.$$

Now, $\bar{G}\setminus C_0$ is connected in $\bar{G}\setminus\sigma U$ and must be contained in a component σC_i, $i\neq 0$. Without loss of generality, we assume $i=1$:

$$U\cup C_1\subset\sigma C_1\,.$$

Thus, $\{\sigma^n U | n\geqq 0\}$ is contracting towards ω_0 and $C(\sigma^n U,\omega_0)=\sigma^n C_0$, while $\{\sigma^{-n}U | n\geqq 0\}$ is contracting towards ω_1 and $C(\sigma^{-n}U,\omega_1)=\sigma^{-n}C_1$. We have

$$\sigma^{-1}C_0\supset\bigcup_{i\neq 1}C_i\,,\quad \sigma C_1\supset\bigcup_{i\neq 0}C_i\,,\quad\text{and}\quad \sum_{i=0}^{k}\mu(C_i)=1\,. \tag{3.17}$$

If $k \geqq 2$ in (3.1), then (3.17) gives rise to "paradox" decompositions, where one part of G contains a copy of itself plus some disjoint copies of other parts, compare with [Wa] and de la Harpe and Skandalis [HS]. We obtain from (3.17)

$$\mu(C_0) \geqq \mu(C_0) + \sum_{i=2}^{k} \mu(C_i),$$

so that $\mu(C_i) = 0$, $i = 2, \ldots, k$.

Now let $\gamma \in \Gamma$. Observe that $\{\gamma\sigma^n U \mid n \geqq 0\}$ is contracting towards $\gamma\omega_0$, and $C(\gamma\sigma^n U, \gamma\omega_0) = \gamma\sigma^n C_0$.

Case 1. $\mu(C_0) = 1$, $\mu(C_1) = 0$. Suppose that $\gamma\omega_0 \neq \omega_0$. Then there are $m, n \geqq 0$ such that $\sigma^m C_0 \cap \gamma\sigma^n C_0 = \emptyset$, and $\mu(X) \geqq \mu(\sigma^m C_0) + \mu(\gamma\sigma^n C_0) = 2$, a contradiction. Hence, Γ fixes ω_1.

Case 2. $\mu(C_0) = 0$, $\mu(C_1) = 1$. Then Γ fixes ω_1.

Case 3. $\mu(C_0) > 0$, $\mu(C_1) > 0$. Suppose that $\gamma\omega_0 \notin \{\omega_0, \omega_1\}$. Then there is an $m \geqq 0$, such that

$$\gamma\omega_0 \notin \sigma^m C_0 \cup \sigma^{-m} C_1 = C, \quad \mu(C) = 1. \tag{3.18}$$

Now, $\bar{G} \setminus C$ is open and contains $\gamma\omega_0$, so that

$$\gamma\sigma^n C_0 \cap C = \emptyset \quad \text{for some} \quad n \geqq 0. \tag{3.19}$$

Combining (3.18) and (3.19), we obtain

$$\mu(X) \geqq \mu(\gamma\sigma^n C_0) + \mu(C) > 1,$$

a contradiction. In the same way, we must have $\gamma\omega_1 \in \{\omega_0, \omega_1\}$, and Γ fixes $\{\omega_0, \omega_1\}$. This concludes the proofs of Claim 4 and Theorem 1. ☐

We remark that it should also be possible to prove Theorem 1, with similar effort, by combining the result of [Ne] with the methods of Dunwoody [Du]. However, the direct use of the invariant measure appears to be more natural.

In addition, it may be of some interest to see where the invariant measure is concentrated in the case when Γ acts amenably and is nonhyperbolic. For the following lemma, $C_0(U)$ and $R(U)$ are as defined as in Proposition 1 and its proof.

Lemma 1. *If Γ is nonhyperbolic and has infinite orbits, and if μ is an invariant measure on X with respect to the action of Γ, then $\mu(C_0(U)) = 1$ whenever $U \subset X$ is finite.*

Proof. As in the proof of Proposition 1, we may assume without loss of generality that U is connected and that (3.1) holds. As $\Gamma U \cap C_0(U)$ is infinite, we can find $\alpha_0 = \iota$ (the identity), $\alpha_1, \alpha_2, \ldots$ in Γ such that $\alpha_m U \cap \alpha_n U = \emptyset$ whenever $m \neq n$. From (3.4) we infer $\alpha_n R(U) \subset \alpha_m C_0(U)$, so that $\alpha_m R(U) \cap \alpha_n R(U) = \emptyset$ for $m \neq n$. In particular,

$$C_0(U) \supset \bigcup_{n \geqq 1} \alpha_n R(U), \tag{3.20}$$

a union of disjoint sets. Thus, for every N,

$$\mu(C_0(U)) \geqq \mu\left(\bigcup_{n=1}^{N} \alpha_n R(U)\right) = N \cdot \mu(R(U)),$$

and $\mu(R(U))=0$. Recall that $\mu(U)=0$ for every finite $U \subset X$. We obtain

$$\mu(C_0(U))=\mu(X)-\mu(U \cup R(U))=1\,. \quad \square$$

Note that, as (3.17), also (3.20) is a kind of "paradox" decomposition.

In order to prove Theorem 2, we shall actually construct an invariant measure with respect to the action of Γ. In the case of a fixed end, $\mu(M)$ will be the "density" of $M \subset X$ near that end. However, such a construction is usually somehow transcendental, relying on some variant of the Axiom of Choice. In our proof, the Axiom of Choice will be present behind the following well known fact. –

There is a functional B-lim, called *Banach limit*, defined on the space of all bounded real sequences, such that

- B-lim is linear,
- $\liminf_{n\to\infty} a_n \leqq B\text{-}\lim a_n \leqq \limsup_{n\to\infty} a_n$, and
- $B\text{-}\lim a_{n+1} = B\text{-}\lim a_n$,

whenever $\{a_n\}$ is a bounded real sequence. The proof that such a Banach limit exists is a typical exercise of applying the Hahn-Banach extension theorem. We remark that a first version of the latter theorem was in fact developed by Banach [Ba] in order to prove amenability of the group of isometries of the unit circle. In the sequel, we shall fix some Banach limit with the above properties.

Proof of Theorem 2. We subdivide the proof according to four possible cases.

Case 1. Γ fixes a finite subset U of X. Then $\mu(M)=|M\cap U|/|U|$ $(M\subset X)$ fulfills the requirements of Definition 1.

Case 2. Γ is nonhyperbolic and has infinite orbits on X. By Proposition 1, Γ fixes a unique end of G. By assumption, this end has finite diameter. Let $\{V_n\}$ be as described in Proposition 2, and define

$$A_{1,n}=\bigcup_{l=1}^{n} V_l, \qquad \mu(M)=B\text{-}\lim_{n\to\infty}\frac{|M\cap A_{1,n}|}{|A_{1,n}|} \qquad (M\subset X).$$

If $\gamma\in\Gamma$, then the symmetric difference $A_{1,n}\Delta\gamma^{-1}A_{1,n}$ is finite by Proposition 2. Hence, for every $M\subset X$,

$$\lim_{n\to\infty}\frac{|M\cap A_{1,n}|}{|A_{1,n}|}-\frac{|\gamma M\cap A_{1,n}|}{|A_{1,n}|}=0,$$

and $\mu(M)=\mu(\gamma M)$.

Case 3. Γ contains a hyperbolic automorphism and fixes an end ω_0 of G. Let σ, V be as in Proposition 3 and $A_{k,n}$ as in (3.14), and define for $k\in\mathbb{Z}$

$$\mu(M)=B\text{-}\lim_{n\to\infty}\frac{|M\cap A_{k,n}|}{|A_{k,n}|} \qquad (M\subset X). \tag{3.21}$$

First of all, μ does not depend on k: $A_{k,n}\Delta A_{k+1,n}\subset\sigma^k V\cup\sigma^{k+n}V$ and $|A_{k,n}|=|A_{k+1,n}|$, so that

$$\left|\frac{|M\cap A_{k,n}|}{|A_{k,n}|}-\frac{|M\cap A_{k+1,n}|}{|A_{k+1,n}|}\right|\leqq 2\frac{|V|}{|A_{k,n}|},$$

which tends to zero as $n \to \infty$. If $\gamma \in \Gamma$ and $k = k(\gamma)$ is as given by Proposition 3, then there is some $s = s(\gamma) \in \mathbb{N}$ such that

$$\gamma A_{k+s,n} \subset A_{k,n+2s} \quad \text{for every} \quad n \in \mathbb{N},$$

as γ is bounded on $\bigcup_{l \geqq k} \sigma^l V$. We obtain for $M \subset X$

$$\frac{|M \cap A_{k+s,n}|}{|A_{k+s,n}|} \leqq \frac{|M \cap \gamma^{-1} A_{k,n+2s}|}{|A_{k+s,n}|} \leqq \frac{|\gamma M \cap A_{k,n}|}{|A_{k,n}|} + \frac{|A_{n,n+2s}|}{|A_{k,n}|}.$$

As $n \to \infty$, the last term tends to zero, and by applying B-lim we get $\mu(M) \leqq \mu(\gamma M)$. By symmetry, we must have equality.

Case 4. Γ contains a hyperbolic automorphism σ and fixes the set $\{\omega_0 = \operatorname{dir}(\sigma), \omega_1 = \operatorname{dir}(\sigma^{-1})\}$. If Γ fixes ω_0 or ω_1 then we are in Case 3. So assume that $\alpha\omega_0 = \omega_1$ for some $\alpha \in \Gamma$. Then the stabilizer Γ_{ω_0} of ω_0 has index two and is normal in Γ. From Case 3 we have the existence of an invariant measure μ_0 on G with respect to Γ_{ω_0}. Now it is easy to see that

$$\mu(M) = \tfrac{1}{2}(\mu_0(M) + \mu_0(\alpha M)) \qquad (M \subset X)$$

fulfills the requirements of Definition 1 with respect to Γ. □

4. Amenable groups of automorphisms

We recall the notion of an amenable *group*. Let Γ be a locally compact Hausdorff group and H a closed subgroup. The homogeneous space Γ/H is called *amenable*, if there is an invariant measure with respect to the action of Γ, with total mass one, given as in Definition 1 on the family of all Borel sets of Γ/H. If in particular H is trivial, then Γ is called amenable. If Γ is amenable, then so is every continuous action of Γ on some locally compact space, see e.g. [Pi, pp. 362–363]. Thus, from Theorem 1 we have the following

Corollary 1. *If Γ is an amenable, closed group of automorphisms of G, then Γ fixes a finite subset of X, an end of G, or a pair of hyperbolic ends of G.*

Note that we need closedness in order to have Γ locally compact. The analogous corollary to Theorem 2 needs some preliminaries. For $x \in X$, denote by Γ_x the stabilizer of x in Γ (while Γx is the orbit of x). As Γ is closed, Γ_x is compact open, and the homogeneous space Γ/Γ_x can be identified with Γx by the mapping $\gamma\Gamma_x \mapsto \gamma x$. If Γ acts amenably on G, then its action on Γx is not necessarily amenable: it might happen that $\mu(\Gamma x) = 0$ for the invariant measure μ on G. However, the action is amenable if $\mu(\Gamma x) > 0$.

Corollary 2. *Suppose that $\Gamma \subset \operatorname{Aut}(G)$ is closed. If Γ fixes a finite subset of X, an end of G with finite diameter or a pair of hyperbolic ends of G, then Γ is amenable.*

Proof. 1. The nonhyperbolic case. a) If Γ fixes a finite set of vertices, then Γ is compact and hence amenable.

b) If Γ is nonhyperbolic but has infinite orbits, then we may consider the sets V_n, $n \geqq 1$, as given by Proposition 2. For $k \geqq 1$, we define $\Gamma^{(k)} = \{\gamma \in \Gamma \mid \gamma V_n = V_n$ for every $n \geqq k\}$ (compare with [Ne]). By closedness of Γ, the $\Gamma^{(k)}$ constitute an increasing sequence of compact subgroups whose union is Γ. Thus Γ is amenable by [Ey, p. 17].

(We remark that this argument could have also been used in the "elliptic" part of the proof of Theorem 2. However, we have preferred to construct explicitly the invariant measure.)

2. The hyperbolic case. a) Suppose that Γ fixes an end ω_0, and that σ and V are as in Proposition 3. Choose $x \in V$ and consider the set $M = \{\sigma^k x \mid k \geqq 0\}$. For $A_{0,n}$ [as defined in (3.14)] we have $|A_{0,n}| \leqq n|V|$ and $|M \cap A_{0,n}| \geqq n$. Hence, for the invariant measure μ defined in (3.21) we get

$$\mu(\Gamma x) \geqq \mu(M) \geqq \frac{1}{|V|},$$

and the homogeneous space Γ/Γ_x is amenable. As Γ_x is compact, also Γ must be amenable by [Ey, p. 16].

b) If Γ fixes $\{\omega_0, \omega_1\} \subset \Omega$, but none of the two, then we may apply the above argument to the subgroup Γ_{ω_0} which fixes ω_0. This group is amenable and has index two in Γ, so that Γ is also amenable. □

For graphs where all ends have finite diameter, statements (b) and (b′) of Sect. 2 are of course equivalent. In particular, in trees all ends have diameter zero, and the result of [Ne] is a special case of Corollaries 1 and 2. To see this, the only remaining task is to show that a group of automorphisms which fixes a finite set of vertices in a tree must fix a vertex or an edge. We leave this to the reader, compare with [H 3, Lemma 2].

We now give examples to illustrate each of the possible situations.

Example 1 (Compare with [Ne]). Let $G = T$ be a homogeneous tree of degree at least three. Choose a reference vertex o and an end $\omega \in \Omega$. Then there is a unique infinite path ("geodesic") $\pi = [x_0, x_1, \ldots]$ which represents ω, such that $x_0 = o$.

A) The group of all automorphisms γ of T satisfying $\gamma x_n = x_n$ for all but finitely many n is closed, fixes ω and contains only elliptic elements.

B) The group Γ_ω of all automorphisms of T which fix ω is closed, contains hyperbolic elements and acts vertex-transitively. We have $\gamma \in \Gamma_\omega$ if and only if there are $d, k \in \mathbb{Z}$, $k \geqq 0$, such that $\gamma x_n = x_{n+d}$ for every $n \geqq k$.

Example 2. Let Γ_1, Γ_2 be two infinite, finitely generated groups whose Cayley graphs have one end only. (This does not depend on the choice of the generating set, see [Fr, St].) Let A_i be finite symmetric sets of generators of Γ_i, $i = 1, 2$. Consider the free product $\Gamma = \Gamma_1 * \Gamma_2$ and its Cayley graph G with respect to $A_1 \cup A_2$. Choose $i \in \{1, 2\}$. As a subgroup of Γ, Γ_i induces a subgraph of G which gives rise to an end ω_i of G having infinite diameter. As a group of automorphisms of G, Γ_i fixes ω_i. An element of Γ_i is elliptic with respect to G and its ends if and only if it has finite order; it is parabolic otherwise. The action of Γ_i on G is amenable if and only if Γ_i is an amenable group.

A) Choose Γ_1, Γ_2 isomorphic with $\mathbb{Z} \times \mathbb{Z}$: we obtain a free product of two square lattices, and Γ_1 acts amenably on G.

B) Choose Γ_1, Γ_2 isomorphic with $\mathbb{F} \times \mathbb{F}$, where $\mathbb{F}$ is the free group on two generators: the action of Γ_1 on G is nonamenable.

C) Choose Γ_1, Γ_2 isomorphic with one of the torsion groups described by Grigorchuk [Gr]: these groups have nonpolynomial, subexponential growth and are amenable, so that the action of Γ_1 on G is elliptic and amenable.

D) Finally, choose Γ_1, Γ_2 isomorphic with a free periodic group with $m \geqq 2$ generators and odd order $n \geqq 665$: Adyan [Ad] has proved that these groups are nonamenable, so that the action of Γ_1 on G is elliptic and nonamenable.

All graphs in the above examples admit a *vertex-transitive* group action. The relation between the action on Ω, amenability and unimodularity of vertex-transitive groups of automorphisms on one hand and the spectral radius of the simple random walk on G on the other are studied in detail by Soardi and Woess [SW1] and are applied to electric currents in infinite networks in [SW2].

References

[Ad] Adyan, S.I.: Random walks on free periodic groups. Math. USSR Izv. **21**, 425–434 (1983)

[Ba] Banach, S.: Sur le problème de la mesure. Fundam. Math. **4**, 7–33 (1923)

[Du] Dunwoody, M.J.: Cutting up graphs. Combinatorica **2**, 15–23 (1982)

[Ey] Eymard, P.: Moyennes invariantes et représentations unitaires (Lecture Notes Mathematics, Vol. 300). Berlin Heidelberg New York: Springer 1972

[Fr] Freudenthal, H.: Über die Enden diskreter Räume und Gruppen. Comment. Math. Helv. **17**, 1–38 (1944)

[Gr] Grigrochuk, R.I.: Degrees of growth of finitely generated groups, and the theory of invariant means. Math. USSR Izv. **25**, 259–300 (1985)

[H 1] Halin, R.: Über unendliche Wege in Graphen. Math. Ann. **157**, 125–137 (1964)

[H 2] Halin, R.: Die Maximalzahl zweiseitig unendlicher Wege in Graphen. Math. Nachr. **44**, 119–127 (1970)

[H 3] Halin, R.: Automorphisms and endomorphisms of infinite locally finite graphs. Abh. Math. Sem. Univ. Hamb. **39**, 251–283 (1973)

[HS] De la Harpe, P., Skandalis, G.: Un résultat de Tarski sur les actions moyennables de groupes et les partitions paradoxales. Enseign. Math., II Sér. **32**, 121–138 (1986)

[J 1] Jung, H.A.: Connectivity in infinite graphs. In: Studies in Pure Math., pp. 137–143. L. Mirsky (ed.). New York London: Academic Press 1971

[J 2] Jung, H.A.: A note on fragments of infinite graphs. Combinatorica **1**, 285–288 (1981)

[Or] Ore, O.: Theory of graphs. Providence: Amer. Math. Soc. 1962

[Ne] Nebbia, C.: On the amenability and the Kunze-Stein property for groups acting on a tree. Pac. J. Math. (to appear)

[PW] Picardello, M.A., Woess, W.: Harmonic functions and ends of graphs. Proc. Edin. Math. Soc. **31**, 457–461 (1988)

[Pi] Pier, J.P.: Amenable locally compact groups. New York: Wiley 1984

[SW1] Soardi, P.M., Woess, W.: Amenability, unimodularity, and the spectral radius of random walks on infinite graphs. Preprint, Univ. Milano 1988

[SW2] Soardi, P.M., Woess, W.: Uniqueness of currents in infinite resistive networks. Discrete Appl. Math. (to appear)

[St] Stallings, J.: Group theory and three-dimensional manifolds. New Haven: Yale Univ. Press 1971

[T 1] Tits, J.: Sur le groupe des automorphismes d'un arbre. In: Essays on topology and related topics, mémoires dédiés a G. de Rham, pp. 188–211. Berlin Heidelberg New York: Springer 1970

[T 2] Tits, J.: A "theorem of Lie-Kolchin" for trees. In: Contributions to algebra, a collection of papers dedicated to Ellis Kolchin, pp. 377–388. New York London: Academic Press 1977

[Tr] Trofimov, V.I.: Automorphism groups of graphs as topological objects. Math. Notes **38**, 717–720 (1985)

[vN] Von Neumann, J.: Zur allgemeinen Theorie des Maßes. Fund. Math. **13**, 73–116 (1929)

[Wa] Wagon, S.: The Banach-Tarski paradox. Encyclopedia of Math. **24**. Cambridge: Cambridge Univ. Press 1985

[Wo] Woess, W.: Graphs and groups with tree-like properties. J. Combin. Th., Ser. B (to appear)

Received July 21, 1988