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A Note on Bounded Automorphisms of Infinite Graphs

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Abstract. Let X be a connected locally finite graph with vertex-transitive automorphism group. If X has polynomial growth then the set of all bounded automorphisms of finite order is a locally finite, periodic normal subgroup of $AUT(X)$ and the action of $AUT(X)$ on $V(X)$ is imprimitive if X is not finite. If X has infinitely many ends, the group of bounded automorphisms itself is locally finite and periodic.

1. Introduction

If a connected locally finite graph X with polynomial growth admits a fixed-point free vertex-transitive group G of automorphisms, then X is a Cayley graph of G. In this case the growth rate of G is the growth rate of X. By Gromov's characterization of groups with polynomial growth [3] this implies, in particular, that G is a finite extension of a nilpotent group if X has polynomial growth.

If the action of G is vertex-transitive but not fixed-point free, then X need not be a Cayley graph of G. However, Trofimov's characterization of vertex-transitive connected locally finite graphs with polynomial growth [12] shows that the automorphism groups of these graphs are still very close to nilpotent groups. We exhibit further properties of graphs of this type: the set of all bounded automorphisms of finite order constitutes a locally finite periodic subgroup of $AUT(X)$ (§3, Theorem 3) and the action of $AUT(X)$ on $V(X)$ is imprimitive (§3, Theorem 4).

Connected, locally finite, vertex-transitive graphs with polynomial growth have one or two ends. On the other hand, the group of bounded automorphisms of a locally finite, connected, vertex-transitive graph with infinitely many ends has bounded orbits (§4, Theorem 5). In particular, by Lemma 1 (§2) these automorphisms form a locally finite periodic group.

2. Preliminaries

In the sequel, all graphs under consideration are vertex-transitive, locally finite, and connected. An element g of the automorphism group $AUT(X)$ of such a graph X is called *bounded* if there is a constant M, depending upon g, such that

$$d(x, gx) \leqq M \qquad \text{for every } x \in V(X). \tag{1}$$

Bounded automorphisms are called *translations* by Gromov [4]. They have been studied, for example, by Trofimov [11] and Jung and Watkins [7]. The bounded automorphisms of X constitute a normal subgroup of $AUT(X)$, denoted by $B(X)$. We are interested in properties of the set $B_0(X)$ of bounded automorphisms of *finite order* for certain classes of vertex-transitive graphs.

We say a vertex-transitive graph X has *polynomial growth* if there are constants $C, k > 0$ such that $\beta_n(X) \leqq Cn^k$, where

$$\beta_n(X) = |\{y \in V(X) | d(x,y) \leqq n\}|, \qquad x \in V(X). \tag{2}$$

Note that $\beta_n(X)$ is independent of x.

If a subgroup G of $AUT(X)$ acts transitively on $V(X)$, then an *imprimitivity system* of G on X is a partition σ of $V(X)$ into subsets called *blocks*, such that every element of G is a permutation of the blocks of σ. The corresponding quotient graph of X and the homomorphic image of G are denoted by X^σ and G^σ, respectively. If b_1, b_2 are two blocks, then $[b_1, b_2]$ is an edge of X^σ if some vertex in b_1 is adjacent in X to some vertex in b_2. The partitions of $V(X)$ into singletons and into $\{V(X)\}$ itself give rise to the so-called *trivial* imprimitivity systems. We say the action of G is *imprimitive* if there is a nontrivial imprimitivity system of G.

We shall need the following important result:

Theorem 1 (Trofimov [12]). *The following assertions are equivalent for an infinite vertex-transitive connected locally finite graph X:*

(i) *X has polynomial growth.*
(ii) *There exists an imprimitivity system σ of $AUT(X)$ on $V(X)$ with finite blocks such that $AUT(X^\sigma)$ is a finitely generated nilpotent-by-finite group and the stabilizer in $AUT(X^\sigma)$ of a vertex of the graph X^σ is finite.*

If X is a graph and n is a cardinal, then the graph nX is defined on the cartesian product of $V(X)$ by a set $\mathcal{N}$ of cardinality n, and

$$E(nX) = \{[(x,\mu),(y,\nu)] | [x,y] \in E(X), \mu, \nu \in \mathcal{N}\}. \tag{3}$$

Trofimov's theorem combined with the following result shows how close vertex-transitive graphs with polynomial growth are to Cayley graphs of nilpotent-by-finite groups.

Theorem 2 (Sabidussi [9, Theorem 4]). *Let X be a connected vertex-transitive graph, G a transitive subgroup of $AUT(X)$ and n the cardinality of the stabilizer in G of a vertex of X. Then nX is a Cayley graph of G.*

Recall that a group is called *locally finite* if every finitely generated subgroup is finite. Our final prerequisite is the following simple lemma.

Lemma 1. *Let G be a group of permutations of a set V. If the lengths of the orbits of the elements of V under the action of G are bounded, then G is locally finite and periodic.*

Proof. Let $\Sigma = \{S_i | i \in I\}$ be the set of orbits of G in V. Set $n_i = |S_i|$. By assumption, $n_i \leqq M < \infty$ for all $i \in I$. In particular, the orders of the elements of G are bounded by $M!$, and G is periodic. For each $i \in I$, label the elements of S_i by the integers 1, 2, ..., n_i. Then we may consider the restriction $G|S_i$ as a group of permutations of the set $\{1, 2, \ldots, M\}$ which fixes $\{n_i + 1, \ldots, M\}$ pointwise. In view of this identification, for each $i \in I$, there are at most $M!$ different ways in which elements of G can act on S_i.

Now consider a finitely generated subgroup $H = \langle g_1, \ldots, g_k \rangle$ of G. We partition Σ into classes Σ_l, such that for every l and every j $(1 \leqq j \leqq k)$, $g_j|S_i$ and $g_j|S_m$ represent the same permutation with respect to the above labelling whenever S_i, $S_m \in \Sigma_l$. There are $N \leqq M!^k$ different classes, and we see that the action of H on V is determined by a single representative S_l of each of the classes. As $\bigcup_{l=1}^{N} S_l$ is finite, H must be finite. In fact, the order of H is bounded by $M!^N$. □

3. Graphs with Polynomial Growth

We start with some observations on bounded group actions. If X is a Cayley graph of a group G, then any group element $g \in G$ gives rise to a bounded automorphism if and only if the conjugacy class of g in G is finite. In fact, if e denotes the unit element of G, then condition (1) is equivalent with $d(e, x^{-1}gx) \leqq M$ for all $x \in V(X) = G$, which holds if and only if $\{x^{-1}gx | x \in G\}$ is finite.

Hence, the boundedness of an element g of G is a property of the group G itself. Since it is independent of whatever Cayley graph may represent G, we say in this case that g is a *bounded element* of G.

Lemma 2 (B.H. Neumann [8, Theorem 5.1]). *Let G be a finitely generated group of bounded elements. Then its commutator group G' is finite.*

Now we can prove the following result.

Theorem 3. *Let X be a vertex-transitive connected locally finite graph with polynomial growth, and let $B(X)$ be the group of all bounded automorphisms of X. Then the set $B_0(X)$ of elements of finite order in $B(X)$ forms a normal subgroup of $AUT(X)$. It is locally finite, periodic and has finite orbits on $V(X)$.*

Proof. By Theorem 1, there exists an imprimitivity system σ of X with respect to $AUT(X)$, consisting of finite blocks, such that $AUT(X^\sigma)$ is a finitely generated nilpotent-by-finite group acting vertex-transitively on X^σ, and the vertex-stabilizers of X^σ have cardinality $n < \infty$. By Theorem 2, nX^σ is a Cayley graph of $AUT(X^\sigma)$.

If $\tilde{g} \in AUT(X^\sigma)$ and $\tilde{g}x = y$ for some $x, y \in V(X^\sigma)$, and if $\mu \in \mathcal{N}$, where $|\mathcal{N}| = n$, then on nX^σ, $\tilde{g}(x, \mu) = (y, \nu)$ for some $\nu \in \mathcal{N}$. In particular,

$$d_{X^\sigma}(\tilde{g}x, x) \leqq d_{nX^\sigma}(\tilde{g}(x, \mu), (x, \mu)) \leqq d_{X^\sigma}(\tilde{g}x, x) + 2,$$

and $\tilde{g}$ is bounded on nX^σ if and only it is bounded on X^σ. Thus, the group $B(X^\sigma)$ of bounded automorphisms is also a group of bounded elements. As a subgroup of a finitely generated nilpotent-by-finite group, it is also finitely generated (see e.g. Wolf [13]).

If $\tilde{g}_1, \tilde{g}_2 \in B(X^\sigma)$ have finite order and $G = \langle \tilde{g}_1, \tilde{g}_2 \rangle$, then G/G' (being abelian) must be finite. By Lemma 2, G' is finite. Hence, G is finite, and $\tilde{g}_1\tilde{g}_2$ has finite order. Thus, $B_0(X^\sigma)$ is a group. Again, it must be finitely generated, and another application of Lemma 2 yields that $B_0(X^\sigma)$ is finite.

Now, we clearly have $B(X)^\sigma \subseteqq B(X^\sigma)$. If $g_1, g_2 \in B_0(X)$, then $(g_1 g_2)^\sigma = g_1^\sigma g_2^\sigma$ has finite order. By finiteness of the blocks of σ and of the vertex-stabilizer of X^σ, $g_1 g_2$ also has finite order: $B_0(X)$ is a normal subgroup of $AUT(X)$ and $B_0(X)^\sigma$ is finite. We infer that the orbits of $B_0(X)$ on X all have the same finite size, and an application of Lemma 1 completes the proof. □

Note that $B_0(X)$ may well be infinite: a simple example is given by $X = n\mathbb{Z}$, where $\mathbb{Z}$ stands for the two way-infinite path and $2 \leqq n < \infty$.

Theorem 4. *Let X be an infinite connected locally finite graph which is vertex-transitive. If X has polynomial growth, then the action of $AUT(X)$ on X is imprimitive.*

Proof. If the blocks of the imprimitivity system σ of Theorem 1 have more than one element, then the statement is true. Hence, we may assume that σ is trivial, $X = X^\sigma$, and $AUT(X)$ is finitely generated nilpotent by finite and has finite vertex-stabilizers. In particular, the same arguments as in the proof of Theorem 3 show that $B(X)$ is a group of bounded elements.

If we can find a normal subgroup $K \neq 1$ of $AUT(X)$ which does not act transitively, then the orbits of K constitute a nontrivial imprimitivity system for $AUT(X)$. As subgroups of a finitely generated nilpotent-by-finite group, $B(X)$ and $B_0(X)$ are finitely generated. By Theorem 3, $B_0(X)$ must be finite. If $B_0(X)$ is nontrivial, or if $B(X)$ is not transitive then we have found a nontrivial imprimitivity system, since these groups are normal in $AUT(X)$.

So assume that $B(X)$ is transitive and that $B_0(X)$ is trivial. By Lemma 2, $B(X)'$ is finite and must be contained in $B_0(X)$, so that it is trivial. In other words, $B(X)$ is torsion-free abelian. Without loss of generality, we write $B(X) = \mathbb{Z}^k$, $k > 0$. As its vertex stabilizers must be finite, they must therefore be trivial, and X is a Cayley graph of $B(X)$. On one hand, the growth degree of X is that of $AUT(X)$, on the other hand it is that of $B(X)$, that is, k. We infer that the index of $B(X)$ in $AUT(X)$ must be finite. Now, $G = (2\mathbb{Z})^k$ is a subgroup of finite index in $B(X)$, and also in $AUT(X)$, which does not act transitively on X. There are finitely many different conjugacy classes $g^{-1}Gg(g \in AUT(X))$ of G and their intersection H is a normal subgroup of finite index of $AUT(X)$, $H \cong \mathbb{Z}^k$. It does not act transitively, and so we obtain an imprimitivity system of $AUT(X)$. □

We remark that an alternate proof of Theorem 4 is possible if X has polynomial growth of degree at least two: Since a transitive permutation group G acting on a set V is primitive if and only if the stabilizer G_x of every vertex x in V is a maximal subgroup of G (see e.g. Schwerdtfeger [10], Chap. V, Theorem 3) it suffices to show that the stabilizer of a vertex of X is not a maximal proper subgroup of $AUT(X)$. To see this, suppose that σ of Theorem 1 is trivial (otherwise X is imprimitive). Then $AUT(X)$ is finitely generated by finite. If $x \in V(X)$ and $S = AUT(X)_x$ is its stabilizer, then S is finite. The growth degree k of $AUT(X)$ is that of X: $k \geqq 2$. Choose $g \in AUT(X) \setminus S$. Then the group G generated by S and g has at most one element of infinite order (the element g) and is finitely generated by finite. Such a group must be finite or grow linearly, so that $G \neq AUT(X)$, and S is not maximal.

Of course, the case of linear growth has to be treated separately then.

4. Graphs with Infinitely Many Ends

We recall the definition of an *end* of an infinite graph X (Hopf [6], Freudenthal [1] and Halin [5]): two one-sided infinite paths π, π' in X are called *equivalent* if, for every finite subset U of $V(X)$, there is a finite path in X which connects π and π' and does not meet U. An end is an equivalence class under this relation.

The graphs considered in the preceding section have one or two ends, and our results rely on the heavy machinery provided by the papers of Trofimov [12] and Gromov [3] (the latter stands, of course, behind the arguments of [12]). The situation for infinitely ended graphs is easier to handle.

Theorem 5. *Let X be a vertex-transitive connected locally finite graph. If X has infinitely many ends, then the orbits of $B(X)$ on X are finite. In particular, $B(X)$ is a locally finite periodic group.*

Proof. As X has infinitely many ends, there is a finite connected subgraph Y of X such that $X \setminus Y$ decomposes into at least three infinite components C_1, C_2, C_3. Assume that $B(X)$ has infinite orbits. Then there is a g in $B(X)$ such that gY and Y are disjoint. Without loss of generality, we may assume that $gY \subset C_1$. Now, in $X \setminus gY$ there are the components gC_1, gC_2, gC_3. One of them contains Y and thus also C_2 and C_3. Of the remaining two, at least one is gC_i ($i \in \{2, 3\}$). If $x \in V(C_i)$ and $y \in V(gC_i)$, then every path in X which connects x and y must pass through Y and gY. In particular, $d(x, gx) \geqq 2d(x, Y) + d(Y, gY)$ may become arbitrarily large for $x \in V(C_i)$. This contradicts the boundedness of g. Hence, $B(X)$ has finite orbits, and by Lemma 1, it is locally finite and periodic. □

From the above proof we also get that $d(x, gx) \leqq 2M$ for every $x \in V(X)$ and $g \in B(X)$, where M is the diameter of a finite connected subgraph whose deletion leaves three (or more) infinite components. In particular, the size of the orbits of $B(X)$ does not exceed $\beta_{2M}(X)$.

5. Final Remarks

Between vertex-transitive graphs with polynomial growth and infinitely-ended graphs there lie those one-ended graphs which do not grow polynomially. It is known that they may grow exponentially or subexponentially as well, see Grigorchuk [2]. It is not clear whether $B_0(X)$ is closed under multiplication for graphs of this type.

Observe that in the study of bounded automorphisms of finite order in Sect. 3, the crucial property provided by Trofimov's theorem is the existence of an imprimitivity system σ with finite blocks such that $AUT(X^\sigma)$ is finitely generated and the factor graph has finite vertex-stabilizers. It would be interesting to see if there is a class of connected locally finite vertex-transitive graphs with one end which have this property and is larger than the class of all one-ended graphs with polynomial growth.

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