Contemporary Mathematics

Generating function techniques for random walks on graphs

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Dedicated to the memory of Bob Brooks.

Abstract. This survey outlines the use of generating functions and complex analysis in the study of random walks on graphs and groups, especially for determining the precise asymptotic behaviour of transition probabilities.

Contents

1. Introduction

If (a_n) is any sequence of real or complex numbers, then its *generating function* is the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $z \in \mathbb{C}$. Generating functions are widely used in Combinatorics and Probability, where the a_n typically stand for the number of certain objects of "size" n, or the probabilities of certain sequences of events, respectively. In these cases, the a_n will be non-negative.

The general spirit is that one applies algebraic and/or analytic methods to the generating function in order to obtain information about the numbers a_n. In very favourable cases, this may lead to explicit computation of the a_n, while more typically, the result is an asymptotic evaluation of the sequence. Such asymptotic

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results are obtained by a careful study of the singularities on the circle of convergence around 0 of the power series, combined with a use of Cauchy's formula representing the a_n in terms of $f(z)$.

Among the many typical asymptotic methods, there are the somewhat "old-fashioned" (but still often efficient) *method of Darboux* and the more modern *singularity analysis* of FLAJOLET AND ODLYZKO [**20**]. For general surveys on asymptotic methods in combinatorial problems, see BENDER [**5**] and ODLYZKO [**47**].

The purpose of the present notes is to give an outline, resp. review, of applications of techniques of this type to study transition probabilities of random walks on infinite graphs – the analogue of the heat kernel in discrete time and space. The notes follow closely the set of lectures which the author has given during the Heat Kernel Trimester at Centre Émile Borel in April–June 2002. Besides the various original articles, many of the results presented here, mostly along with detailed proofs, can be found in the author's book [**66**]. Here, the main goal is to present results in a unified way so that one can understand the scope and applicability of this type of methods. A particular emphasis is on the explanation of the mathematical *tools* that have been used so far for obtaining asymptotics of transition probabilities via generating functions.

Let X be a *graph,* i.e., a set X with a symmetric *neighbourhood relation* $\sim$. A *path* of *length* n from x to y $(x, y \in X)$ is a sequence $[x = x_0, x_1, \dots, x_n = y]$ such that $x_i \sim x_{i-1}$.

In the sequel, graphs shall mostly have *bounded geometry:* they have bounded vertex degrees $\deg(x) = |\{y : y \sim x\}|$ and are connected, i.e., for every pair of vertices there is a path between the two. The distance $d(x, y)$ is then the minimal length of a path connecting x and y. Also, our graphs will usually be (countably) infinite.

A *random walk* on X is an X-valued Markov chain, whose random position at time n is denoted Z_n. Thus, each Z_n is an X-valued random variable defined on a model probability space $(\Omega, \mathcal{A}, \Pr)$, the sequence (Z_n) has the Markov property ("the future depends only on the present, and - given the latter - not on the past"), and the *transition probabilities*

$$p(x, y) = \Pr[Z_{n+1} = y \mid Z_n = x]$$

do not depend on n. We shall write $\Pr_x$ for the model probability measure on X, when the initial distribution (of Z_0) is δ_x (i.e., the chain is conditioned to start at x). When we speak of a random walk on X, we have in mind that the stochastic transition matrix

$$P = \big(p(x, y)\big)_{x,y\in X}$$

is adapted to graph structure in some way to specified more precisely in each case.

EXAMPLE 1.1. *Simple random walk (SRW) on* X has transition probabilities

$$p(x, y) = \begin{cases} \dfrac{1}{\deg(x)}, & y \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

The *BASIC QUESTION* addressed in these notes concerns the asymptotic behaviour of the n-step transition probabilities

$$p^{(n)}(x, y) = \Pr_x[Z_n = y], \quad \text{as} \quad n \to \infty.$$

How are these asymptotics linked to the geometry of X ?

A basic assumption (which on its own is not yet a geometric adaptedness condition) is *irreducibility:*

$$\forall\, x,y \;\exists\, n \ge 0 \,:\, p^{(n)}(x,y) > 0\,.$$

This is of course satisfied for SRW on a connected graph. An additional assumption, that will be used often, but not always and is no serious restriction, is *aperiodicity:*

$$\forall\, x,y \;\exists\, n_0 = n_0(x,y) \,:\, p^{(n)}(x,y) > 0 \quad \forall\, n \ge n_0\,.$$

Finitely generated *groups* enter the picture via their *Cayley graphs.* If Γ is a finitely generated group and S a finite, symmetric set of generators, then the Cayley graph $X(\Gamma, S)$ has vertex set $X = \Gamma$, and $x \sim y \iff x^{-1}y \in S$. In this case, the obvious condition of adaptedness to the group structure is to require Γ-invariance, i.e., $p(gx, gy) = p(x,y)$ for all $x, y, g \in \Gamma$. Setting $\mu(x) = p(o,x)$, where o is the group identity (we use this atypical notation, because the symbol e is often used for edges), then the transition probabilities are determined by the probability measure μ on Γ, and we speak of *the random walk on Γ with law μ.*

Regarding asymptotic behaviour, in the present notes we shall be primarily concerned with *asymptotic equivalence* $a_n \sim b_n$ $(n \to \infty)$. For two positive sequences (a_n), (b_n) this means that $a_n/b_n \to 1$. (It will be obvious from the context whether $\sim$ refers to neighbourhood of two vertices or asympotic equivalence of two sequences.)

There is a weaker notion, namely *asymptotic type,* denoted $\approx$. We write

$$\begin{aligned} &a_n \preccurlyeq b_n :\iff a_n \le C \sup\{b_k : cn \le k \le Cn\} \quad (c, C > 0)\,, \quad \text{and}\\ &a_n \approx b_n :\iff a_n \preccurlyeq b_n \quad \text{and} \quad b_n \preccurlyeq a_n\,. \end{aligned}$$

2. An incomplete survey of some results

In this section, we give an overview of some results, mostly regarding the asymptotic type of transition probabilities, for random walks on various types of graphs and groups. The purpose is not completeness, but to give a flavour of the type of results that appear. Those cases where generating functions are used will not be reviewed here, since they are the principal subject of the later sections.

A. Integer grids. $X = \mathbb{Z}^d$, the standard Cayley graph of the free abelian group with d generators. This group is of course written additively.

THEOREM 2.1. *Suppose that $p(x,y) = \mu(y-x)$ defines an irreducible and aperiodic random walk on $\mathbb{Z}^d$, and that the moment conditions $\sum_x x\,\mu(x) = 0$ and $\sum_x |x|^2\mu(x) < \infty$ are satisfied. Let Σ be the covariance matrix of μ and $\Sigma^{-1}[\cdot]$ the quadratic form associated with Σ^{-1}. Then*

$$p^{(n)}(0,x) \sim C\, n^{-d/2} \exp\left(-\tfrac{1}{2n}\Sigma^{-1}[x]\right)$$

uniformly for $x/\sqrt{n}$ bounded, where $C = (2\pi)^{-d/2}(\det\Sigma)^{-1/2}$.

See NEY AND SPITZER [**45**]. Extensions to *generalized lattices* (graphs on which $\mathbb{Z}^d$ acts by isometries with finitely many orbits) are due to KRÁMLI AND SZÁSZ [**38**], GUIVARC'H [**30**], KOTANI, SHIRAI, AND SUNADA [**37**].

B. Graphs with polynomial growth. For $A \subset X$, define its *volume* by $\mathrm{Vol}(A) = \sum_{x \in A} \deg(x)$. Let

$$V(x,r) = \mathrm{Vol}\big(B(x,r)\big)\,, \quad \text{where} \quad B(x,r) = \{y : d(y,x) \le r\}\,.$$

THEOREM 2.2. *If $V(x,r) \approx r^d$ uniformly in x, and X satisfies a property of quasi-homogeneity ("Poincaré inequality") then for SRW*

$$p^{(2n)}(x,x) \approx n^{-d/2} \quad \textit{and}$$
$$c\,n^{-d/2} \exp\left(-\tfrac{d(x,y)^2}{cn}\right) \le p^{(n)}(x,y) \le C\,n^{-d/2} \exp\left(-\tfrac{d(x,y)^2}{Cn}\right)\,.$$

In the lower bound, one needs $d(x,y) \le c_0 n$, and for bipartite graphs, one has to take into account the parity of $d(x,y) - n$.

This applies in particular to groups with polynomial growth ($\equiv$ virtually nilpotent by GROMOV [**28**]), and to quasi-transitive graphs (i.e., whose isometry group acts with finitely many orbits) with polynomial growth.

See VAROPOULOS [**60**], HEBISCH AND SALOFF-COSTE [**33**], LUST-PIQUARD [**43**], DELMOTTE [**14**], COULHON AND GRIGOR'YAN [**13**], and various further (also more recent) papers.

C. Non-amenable graphs. The *boundary* of $A \subset X$ is $\partial A = E(A \leftrightarrow X \setminus A)$, the set of edges between A and its complement. We set $\mathrm{Area}(\partial A) = |\partial A|$ and define the *isoperimetric constant* of X,

$$\kappa(X) = \inf\left\{\frac{\mathrm{Area}(\partial A)}{\mathrm{Vol}(A)} : A \subset X \text{ finite}\right\}$$

The graph X is called *amenable* if $\kappa(X) = 0$. If X is a Cayley graph of a finitely generated group Γ, this means that Γ is an amenable group.

Non-amenability implies *exponential growth:* $\quad V(x,r) \ge C\,\lambda^r$, where $\lambda > 1$.

THEOREM 2.3. *If X is a non-amenable graph then SRW on X satisfies*

$$p^{(2n)}(x,x) \approx e^{-n}\,, \quad \textit{that is,} \quad \rho(P) = \limsup p^{(n)}(x,x)^{1/n} < 1\,.$$

See KESTEN [**36**] for groups, and DODZIUK [**16**], DODZIUK AND KENDALL [**17**], GERL [**23**] for graphs.

Finer results for more specific non-amenable graphs will be explained below.

D. Amenable graphs and groups with exponential growth.

THEOREM 2.4. *A general upper bound for SRW on groups with exponential growth is*

$$p^{(n)}(x,x) \preccurlyeq \exp(-n^{1/3})\,.$$

This is due to VAROPOULOS [**60**].

D.1. Polycyclic groups. A group Γ is polycyclic if it has a normal series $\Gamma = \Gamma_0 \ge \Gamma_1 \ge \cdots \ge \Gamma_r$ with all quotient groups Γ_i/Γ_{i+1} cyclic. The lower bound in the following asymptotic estimate is due to ALEXOPOULOS [**2**].

THEOREM 2.5. *If Γ is polycyclic and not virtually nilpotent then it has exponential growth, and for SRW on any Cayley graph of Γ*

$$p^{(2n)}(x,x) \approx \exp(-n^{1/3})$$

D.2. Lamplighter graphs - wreath products of groups. Given X, a (typically infinite) graph, and (L,o) a (finite or infinite) graph with root o, a *configuration* on X is a function $\eta : X \to L$ with finite support $\operatorname{supp}\eta = \{x \in X \mid \eta(x) \neq o\}$. Let $C = C(X \to L)$ be the set of all configurations. The *lamplighter graph* $L \wr X$ has vertex set $C \times X$, neighbourhood is given by

$$(\eta, x) \sim (\eta', x') \iff \begin{cases} x \sim x' \text{ in } X\,, \text{ and } \eta = \eta'\,, & \text{or} \\ x = x'\,,\ \eta(x) \sim \eta'(x) \text{ in } L\,, \text{ and } \eta(y) = \eta'(y)\ \forall\ y \neq x\,. \end{cases}$$

(Interpretation of (η, x): at each vertex y, there is a lamp. Its current state is $\eta(y)$. The vertex $x \in X$ is the current position of a lamplighter.)

If X and L are Cayley graphs of groups $\mathfrak{G}$ and $\mathfrak{K}$, respectively, then $L \wr X$ is a Cayley graph of the *wreath product* $\Gamma = \mathfrak{K} \wr \mathfrak{G}$. For the following (and further) examples conerning wreath products of groups, see PITTET AND SALOFF-COSTE [**49**], [**50**].

THEOREM 2.6. (1) *On* $\Gamma = \mathfrak{F} \wr \mathbb{Z}^d$, *with finite group* $\mathfrak{F}$

$$p^{(2n)}(x,x) \approx \exp\bigl(-n^{d/(d+2)}\bigr)$$

(2) *On* $\Gamma = \mathbb{Z} \wr \mathbb{Z}^d$,

$$p^{(2n)}(x,x) \approx \exp\bigl(-n^{d/(d+2)}(\log n)^{2/(d+2)}\bigr)$$

(3) *If* $\mathfrak{K}$ *is infinite with polynomial growth, then on* $\Gamma = \mathfrak{K} \wr \mathbb{Z}$,

$$p^{(2n)}(x,x) \approx \exp\bigl(-n^{1/3}(\log n)^{2/3}\bigr)$$

(4) *If* $\mathfrak{K}$ *is polycyclic with exponential growth, then on* $\Gamma = \mathfrak{K} \wr \mathbb{Z}$,

$$p^{(2n)}(x,x) \approx \exp\bigl(-n^{1/2}\bigr)$$

E. Space-time estimates and the Einstein relation. There are three caracteristic constants associated with a graph with polynomial growth that satisfies certain *regularity conditions*:

(1) The *fractal dimension* δ_f such that $V(x,r) \approx r^{\delta_f}$, as $r \to \infty$. This is the exponent of polynomial growth, not necessarily an integer.
(2) The *spectral dimension* δ_s, such that $p^{(2n)}(x,x) \approx n^{-\delta_s/2}$.
(3) The *walk dimension* δ_w is such that the expected time until the first exit from the n-ball around the starting point is $\approx n^{\delta_w}$.

THEOREM 2.7. *Under certain regularity conditions, the "Einstein relation" holds:*

$$\delta_s = 2\delta_f/\delta_w\,,$$

and SRW satisfies

$$p^{(n)}(x,y) \approx n^{-\delta_s/2} \exp\left(-c\,\frac{d(x,y)^{\delta_w/(\delta_w-1)}}{n^{1/(\delta_w-1)}}\right)$$

uniformly in x *and* y *as* $n \to \infty$.

The (positive) constants may differ in upper and lower bounds, and for bipartite graphs, in the lower bound $n - d(x,y)$ should be even and $n \geq c_0 d(x,y)$. For details, see TELCS [**56**], [**57**], [**58**] and GRIGOR'YAN AND TELCS [**27**]. In the theoretical physics literature, the "Einstein relation" was known before, up to mathematical rigour.

For groups with polynomial growth, $\delta_s = \delta_f$ and $\delta_w = 2$, see Subsection B above. Examples where the "Einstein relation" holds, but $\delta_w \neq 2$, are provided by *fractal graphs*, see below.

3. Generating functions

A. The Green function. The *Green function* is the power series

$$G(x,y|z) = \sum_{n=0}^{\infty} p^{(n)}(x,y)\, z^n\,, \quad x,y \in X\,, z \in \mathbb{C}.$$

The following is a simple consequence of irreducibility

LEMMA 3.1. *For real $z > 0$, the series $G(x,y|z)$ either diverge or converge simultaneously for all $x,y \in X$.*

Indeed,

$$p^{(k+n+\ell)}(x_1,y_1) \geq p^{(k)}(x_1,x_2)p^{(n)}(x_2,y_2)p^{(\ell)}(y_2,y_1)$$

and hence, for $z > 0$,

$$G(x_1,y_1|z) \geq p^{(k)}(x_1,x_2)p^{(\ell)}(y_2,y_1)z^{k+\ell}G(x_2,y_2|z)\,.$$

Therefore, all the $G(x,y|z)$ (where $x,y \in X$) have the same radius of convergence $\mathfrak{r}(P) = 1/\rho(P)$, where $\rho(P)$ is the "spectral radius",

$$\rho(P) = \limsup_{n\to\infty} p^{(n)}(x,y)^{1/n}\,.$$

Furthermore, since $G(x,y|z)$ is a power series with non-negative coefficients, $\mathfrak{r}(P)$ is a singularity of each $G(x,y|z)$.[1]

Our plan is to use Cauchy's integral formula

$$p^{(n)}(x,y) = \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{C}} \frac{G(x,y|z)}{z^{n+1}}\, dz\,,$$

($\mathcal{C}$ a positively oriented, simple closed curve in $\mathbb{C}$ with 0 in its interior and all singularities in its exterior) and methods that derive from here (Darboux, Singularity Analysis, Saddle Point Method).

If the random walk is *symmetric*, or more generally, *reversible*, that is, there are positive weights $m(x)$, $x \in X$, such that

$$m(x)p(x,y) = m(y)p(y,x) \quad \forall\ x,y\,,$$

then $G(x,y|z)$ has no singularities on the circle of convergence besides $\mathfrak{r}(P)$ and possibly $-\mathfrak{r}(P)$. This is because $\frac{1}{z}G(x,y|\frac{1}{z})$ is the (x,y)-matrix-element of the resolvent $(z-P)^{-1}$ of the self-adjoint operator P on the Hilbert space $\ell^2(X,m)$. SRW is reversible with $m(x) = \deg(x)$.

There is a more general result by CARTWRIGHT [**10**]:

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[1] An old theorem of PRINGSHEIM says that for a power series with non-negative coefficients, its radius of convergence is a singularity.

LEMMA 3.2. *The radius of convergence* $\mathfrak{r}(P)$ *is the only singularity of* $G(x,y|z)$ *on the circle of convergence when* P *is* strongly aperiodic, *i.e., there is* n_0 *such that*

$$\inf_x p^{(n)}(x,x) > 0 \quad \forall\, n \geq n_0\,.$$

B. Further generating functions. Consider the following hitting times and associated generating functions.

$$\mathbf{s}^x = \min\{n \geq 0 : Z_n = x\}\,, \quad \mathbf{t}^x = \min\{n \geq 1 : Z_n = x\}\,,$$

$$F(x,y|z) = \sum_{n=0}^{\infty} \Pr_x[\mathbf{s}^y = n]\, z^n\,, \quad U(x,x|z) = \sum_{n=0}^{\infty} \Pr_x[\mathbf{t}^x = n]\, z^n\,, \quad z \in \mathbb{C}.$$

Note that $\mathbf{s}^y = \mathbf{t}^y$ $\Pr_x$-almost-surely, when $y \neq x$, while $\mathbf{s}^x = 0$ $\Pr_x$-almost-surely. In particular, $F(x,x|z) \equiv 1$. On the other hand, when the starting point is $Z_0 = x$, then $\mathbf{t}^x$ is the first return time to x *after* starting.

PROPOSITION 3.3 (Basic equations). (a) $\qquad G(x,x|z) = \dfrac{1}{1 - U(x,x|z)}\,,$

(b) $$G(x,y|z) = F(x,y|z)G(y,y|z)\,,$$

(c) $$F(x,y|z) = p(x,y)z + \sum_{w \neq y} p(x,w)z\, F(w,y|z)\,, \quad \textit{if } y \neq x\,,$$

(d) $$U(x,x|z) = p(x,x)z + \sum_{w \neq x} p(x,w)z\, F(w,x|z)\,.$$

PROOF. (a) and (b) follow from the identity

$$p^{(n)}(x,y) = \sum_{k=0}^{n} \Pr_x[\mathbf{t}^y = k]\, p^{(n-k)}(y,y)\,, \quad \text{if } n \geq 1\,,$$

while $p^{(0)}(x,y) = \delta_y(x)$ and $\Pr_x[\mathbf{t}^y = 0] = 0$.

Parts (c) and (d) are obtained by factoring though the first step, that is, the possible states of Z_1, the random walk at time 1. □

4. First walks on trees

A *tree* is a connected graph T without loops or cycles.

For every pair of vertices $x, y \in T$ there is a unique path (*geodesic arc*) $\overline{x\,y}$ of length $d(x,y)$ connecting the two.

LEMMA 4.1 (Tree equation). *Let* P *be a* nearest neighbour *random walk on* T, *that is,* $p(x,y) > 0 \iff x \sim y$.

If $w \in \overline{x\,y}$ *then* $F(x,y|z) = F(x,w|z)F(w,y|z)\,.$

PROOF. The random walk must pass through w on the way from x to y. Probabilistically, this means that $\Pr_x[\mathbf{s}^w \leq \mathbf{s}^y] = 1$. Thus, conditioning with respect to the time of the first visit to w and using the Markov property,

$$\Pr_x[\mathbf{s}^y = n] = \sum_{k=0}^{n} \Pr_x[\mathbf{s}^w = k]\Pr_w[\mathbf{s}^y = n-k]$$

□

From Proposition 3.3 and Lemma 4.1, we see that for the computation of the Green function of a nearest neighbour random walk on a tree, one only needs all functions $F(x,y|z)$, where $x \sim y$.

A. A random walk on homogeneous trees. Let $\mathbb{T}_s$ be the homogeneous tree with degree $s \geq 3$. We can label (or assign types to) the edges by the numbers $1, \dots, s$ such that each vertex is incident with precisely one edge of each type. We define a symmetric nearest neighbour random walk where the probability of walking along an edge with label i is $p_i > 0$, with $\sum_{i=1}^{s} p_i = 1$. See Figure 1, where $s = 3$.

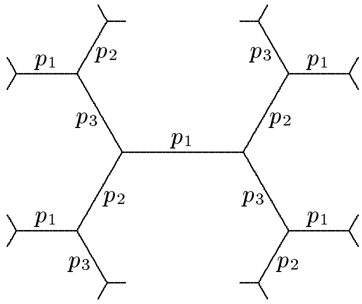

FIGURE 1

This is a random walk on the group $\Gamma = \langle a_1, \dots, a_s \mid a_i^2 = id \rangle$. The tree is its Cayley graph with respect to the set of generators $S = \{a_1, \dots, a_s\}$, and the *law* of the random walk is the probability measure μ on Γ with $\mathrm{supp}(\mu) = S$ and $\mu(a_i) = p_i$. Thus,

$$p(x,y) = \mu(x^{-1}y)\,.$$

If $x \sim y$ is an edge of type i, then $F(x,y|z) = F_i(z)$ depends only on i. Also, $G(z) = G(x,x|z)$ is independent of x. By Proposition 3.3(a+d),

$$G(z) = \frac{1}{1 - \sum_i p_i z\, F_i(z)}\,, \tag{4.1}$$

and Proposition 3.3(c) plus the Tree Equation 4.1 yield

$$F_i(z) = p_i z + \sum_{j \neq i} p_j z\, F_j(z) F_i(z)\,.$$

Thus,

$$F_i(z) = \frac{p_i z}{1 - \sum_{j\neq i} p_j z\, F_j(z)} = \frac{p_i z}{\dfrac{1}{G(z)} + p_i z\, F_i(z)}\,.$$

The right one among the two solutions of the resulting quadratic equation is

$$F_i(z) = \frac{\sqrt{1 + 4p_i^2 z^2 G(z)^2} - 1}{2 p_i z\, G(z)}\,, \tag{4.2}$$

since $F_i(0) = 0$. Combining 4.1 and 4.2,

$$G(z) = \Phi\big(zG(z)\big)\,, \quad \text{where}$$

$$\Phi(t) = 1 + \frac{1}{2} \sum_{i=1}^{s} \left(\sqrt{1 + 4p_i^2 t^2} - 1 \right)\,. \tag{4.3}$$

The function Φ is analytic in the complex plane except for the t on the imaginary axis with $|t| \geq 1/(2\min p_i) > 0$. For positive real t, it is increasing, convex, $\Phi(0) = 1$, $\Phi'(0) = 0$, with asymptote $y = t - \frac{s-2}{2}$ as $t \to +\infty$. For $0 < z < \mathfrak{r} = \mathfrak{r}(P)$, the value $G(z)$ is the y-coordinate of the (leftmost) intersection point of $y = \Phi(t)$ with $y = \frac{1}{z}t$. See Figure 2.

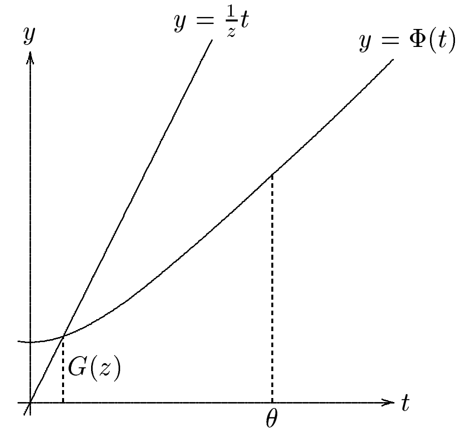

FIGURE 2

We want to determine the radius of convergence $\mathfrak{r}$, which we know to be the smallest positive singularity of $G(z)$.

There is a unique tangent to $y = \Phi(t)$ through the origin, with slope $\rho < 1$. Let $\big(\theta, \Phi(\theta)\big)$ be the tangent point. For $0 < z < 1/\rho$, the angle of intersection of $y = \Phi(t)$ with $y = \frac{1}{z}t$ is non-zero, whence $G(z)$ is analytic there. In addition, for $1 < z < 1/\rho$, there is a second intersection point on the right of θ. For $z = 1/\rho$, the two points "collapse", and for $z > 1/\rho$, there is no real intersection point.

Since $G(z)$ must be real for real $z < \mathfrak{r}$, we see that $\mathfrak{r} = 1/\rho$, and $\rho = \rho(P)$. We find $G(\mathfrak{r}) < \infty$, as it must be by a theorem of GUIVARC'H [**29**], see Thm. 7.8 in WOESS [**66**], since Γ is non-amenable and hence non-$\mathfrak{r}$-recurrent. Also,

$$\rho(P) = \Phi'(\theta) = \min_{t>0} \frac{\Phi(t)}{t} = \min_{u>0} \left(u + \frac{1}{2} \sum_{i=1}^{s} \left(\sqrt{u^2 + 4p_i^2} - u \right) \right) . \tag{4.4}$$

Compare with the formula of AKEMANN AND OSTRAND [**1**] for the norm of free convolution operators. A short proof of that formula using the above type of random walk argument is given in WOESS [**64**].

Next, set

$$\mathcal{F}(z,w) = \Phi(z\,w) - w\,.$$

This function is analytic in all $(z,w) \in \mathbb{C}^2$ for which Φ is analytic in $z\,w$, in particular in a neighbourhood of the point $\big(\mathfrak{r}, G(\mathfrak{r})\big)$. We have

$$\mathcal{F}\big(z, G(z)\big) = 0$$

for z in a neighbourhood of the real segment $[0\,,\,\mathfrak{r})$ in $\mathbb{C}$, and the partial derivatives

$$\mathcal{F}_w\big(\mathfrak{r}, G(\mathfrak{r})\big) = 0 \quad \text{and} \quad \mathcal{F}_{ww}\big(\mathfrak{r}, G(\mathfrak{r})\big) = \mathfrak{r}^2\Phi''(\theta) > 0\,.$$

Therefore, the root $w_0 = G(\mathfrak{r})$ of the function $w \mapsto \mathcal{F}(\mathfrak{r}, w)$ has multiplicity 2. The *Weierstrass preparation theorem*, see e.g. HÖRMANDER [**34**], Thm. 7.5.1, implies existence of the following decomposition in a neighbourhood $\mathfrak{U}$ of the point $\big(\mathfrak{r}, G(\mathfrak{r})\big)$

$$\mathcal{F}(z,w) = \mathcal{H}(z,w)\left(a(z) + b(z)\big(w - G(\mathfrak{r})\big) + \big(w - G(\mathfrak{r})\big)^2\right),$$

$\mathcal{H}$ is analytic and non-zero in $\mathfrak{U}$, and $a(z)$ and $b(z)$ are analytic in the z-projection $\mathfrak{V}$ of $\mathfrak{U}$. We compute

$$a(\mathfrak{r}) = b(\mathfrak{r}) = 0 \quad \text{and} \quad a'(\mathfrak{r}) = 2\theta\, \Phi'(\theta)^4 / \Phi''(\theta) > 0\,.$$

This leads to a quadratic equation for $G(z)$ in $\mathfrak{V}$, and solving, we find

$$G(z) = A(z) - B(z)\sqrt{\mathfrak{r} - z}\,, \tag{4.5}$$

where $A(z)$ and $B(z)$ analytic in $\mathfrak{V}$, and $B(\mathfrak{r}) = \sqrt{a'(\mathfrak{r})}$.

Note that $G(-z) = G(z)$. Hence there is an analogous expansion near $-\mathfrak{r}$. Since the random walk is symmetric, there are no further singularities on the circle of convergence $\{|z| = \mathfrak{r}\}$.

The *method of Darboux*, see e.g. Pólya [**51**] and the comments below, implies the following asymptotic behaviour of the return probabilities:

$$p^{(2n)}(x,x) = C\,\rho^{2n}\, n^{-3/2} + \mathcal{O}(\rho^{2n}\, n^{-5/2})\,, \quad \text{as } n \to \infty\,, \tag{4.6}$$

where $C = \sqrt{2\theta\Phi'(\theta)^3 / \pi\, \Phi''(\theta)}\,.$

Precisely the same method works for "nearest neighbour" random walks on *free groups*, see GERL [**21**] and GERL AND WOESS [**24**]. The method also works, without any change, when $s = \infty$, i.e., the tree has countably infinite degree.

How can one extend this method ? We shall consider the following cases in the subsequent sections.

- *"Nearest neighbour"* random walks on *free products*, see WOESS [**65**], CARTWRIGHT AND SOARDI [**11**], CARTWRIGHT [**9**].
- Arbitrary *finite range* RWs on (virtually) *free groups*, LALLEY [**41**].
- Random walks on *trees with finitely many cone types*, NAGNIBEDA AND WOESS [**44**], LALLEY [**42**], BERTACCHI AND ZUCCA [**6**].

B. Comments on the method of Darboux. The *Riemann–Lebesgue* lemma says that if $F(z) = \sum_n f_n\, z^n$ has radius of convergence $\mathfrak{r}$ and if F is k times continuously differentiable on the circle of convergence (i.e., the function $t \mapsto F(\mathfrak{r}e^{\mathrm{i}t})$ is k times continuously differentiable in $t \in \mathbb{R}$), then $f_n = \mathfrak{o}(\mathfrak{r}^{-n} n^{-k})$ as $n \to \infty$. See e.g. OLVER [**48**], p. 310.

In order to apply this to $G(x,y|z)$, one looks for the leading singular term $S_1(z)$ in the expansion of $G(z)$ near $z = \mathfrak{r}$ and the other singularities on $\{|z| = \mathfrak{r}\}$. Typically, for this singular term alone, one knows the coefficients $S_1(z) = \sum_n a_n\, z^n$ of its Taylor expansion around 0, or an explicit asymptotic equivalent of the a_n.

One wants to conclude that $p^{(n)}(x,y) \sim a_n$. To this end, one considers $H(z) = G(x,y|z) - S_1(z)$. If this difference is k times continuously differentiable on $\{|z| = \mathfrak{r}\}$, then we get

$$p^{(n)}(x,y) - a_n = \mathfrak{o}(\mathfrak{r}^{-n} n^{-k})$$

If k is big enough so that the latter $\mathfrak{o}(\cdot)$ tends to zero faster than a_n then we have reached our goal.

Otherwise, one has to continue the expansion beyond the first singular term, subtracting further terms with known expansion, until one reaches a remainder where the Riemann–Lebesgue lemma yields $\mathfrak{o}(a_n)$ Taylor coefficients.

In our example, in (4.5) we had $G(x,x|z) = G(z) = A(z) - B(z)\sqrt{\mathfrak{r} - z}$, and write

$$B(z) = b_0 + b_1(\mathfrak{r} - z) + D(z)(\mathfrak{r} - z)^2$$

near $\mathfrak{r}$, with $D(z)$ analytic in $\mathfrak{V}$. An Analogous expansion holds near $-\mathfrak{r}$. Thus, we can consider the auxiliary function

$$H(z) = G(z) + \Big(b_0\sqrt{\mathfrak{r} - z} + b_1\sqrt{\mathfrak{r} - z}^3 + D(\mathfrak{r})\sqrt{\mathfrak{r} - z}^5\Big) + \Big(b_0\sqrt{\mathfrak{r} + z} + b_1\sqrt{\mathfrak{r} + z}^3 + D(\mathfrak{r})\sqrt{\mathfrak{r} + z}^5\Big).$$

It is analytic in $\{|z| < \mathfrak{r}\}$, and its Taylor expansion at $z = 0$ has non-zero coefficients

$$h_{2n} = p^{(2n)}(x,x) - C\,\rho^n\,n^{-3/2} + \mathcal{O}(\rho^n\,n^{-5/2}),$$

$H(z)$ is three times continuously differentiable on $\{|z| = \mathfrak{r}\}$, whence by Riemmann–Lebesgue $h_n = \mathfrak{o}(\rho^n n^{-3})$. It would suffice to stop after the terms $b_1\sqrt{\mathfrak{r} \pm z}^3$. In this case, we would get a function $H(z)$ that is twice differentiable on $\{|z| = \mathfrak{r}\}$, yielding only $\mathcal{O}(\rho^n n^{-2})$ for the error term in the asymptotics.

5. Free products, $\mathbf{n}^{-3/2}$, and a surprising result of Cartwright

A. A Functional equation. In the sequel, we shall use the following notation: $\mathfrak{U}_a$ denotes an open neighbourhood of the real segment or half-line $[0\,,\,a)$ in $\mathbb{C}$, where $0 < a \le +\infty$. Let X be our graph, P the transition matrix of a random walk on X, and $o \in X$ a "root" vertex. We define

$$G(z) = G(o,o|z) = \sum_{n=0}^{\infty} p^{(n)}(o,o)\,z^n\,,$$

$$\mathfrak{r} = \mathfrak{r}(P) = 1/\rho(P) \in [1\,,\,\infty) \quad \text{the radius of convergence of } G(z),$$

$$\theta = \theta(P) = \mathfrak{r}\,G(\mathfrak{r}) \in (1\,,\,\infty]\,.$$

Proposition 5.1. *There are sets $\mathfrak{U}_\mathfrak{r}$ and $\mathfrak{U}_\theta$ and a function $\Phi(\cdot)$, analytic in $\mathfrak{U}_\theta$, such that $zG(z) \in \mathfrak{U}_\theta$ whenever $z \in \mathfrak{U}_\mathfrak{r}$ and*

$$G(z) = \Phi\big(zG(z)\big)\,, \quad z \in \mathfrak{U}_\mathfrak{r}\,.$$

The function $\Phi(\cdot)$ is unique up to analytic continuation.

For t in the real interval $[0\,,\,\theta)$, the function $\Phi(t)$ is strictly increasing and strictly convex, $\Phi(t) \le 1 + \rho(P)\,t$, $\Phi'(0) = p(o,o)$, and $\Phi'(\theta-) \le \rho(P)$.

Proof. Let $V(t)$ be the inverse function of $W(z) = zG(z)$. The existence of this inverse function is guaranteed by strict monotonicity of $W(z)$ for $z \in [0\,,\,\mathfrak{r})$. Then $\Phi(t) = t/V(t)$ has the required properties. □

Note that it may occur that $\theta = +\infty$.

The tangent to $y = \Phi(t)$ at $\big(t_0, \Phi(t_0)\big)$ intersects the y-axis at $y = \Psi(t_0)$, where

$$\Psi(t) = \Phi(t) - t\Phi'(t)\,.$$

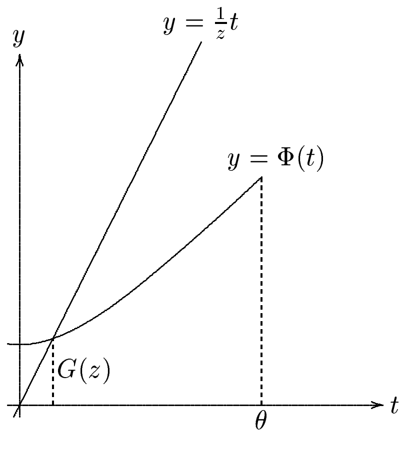

FIGURE 3

If $t = W(z) = zG(z)$, $0 \le z < \mathfrak{r}$, then

$$\Psi(t) = \frac{G(z)^2}{zG'(z) + G(z)} = \frac{1}{1 + \sum_{n=1}^{\infty}(n-1)\,\mathrm{Pr}_o[\mathbf{t}^o = n]\,z^n}\,.$$

This implies that $\Psi(t)$ is strictly decreasing and > 0 for $0 \le t < \theta$. We shall see the use of the functions Φ and Ψ below. First, we exhibit a table of values.

TABLE 1

graph	P	$\rho(P)$	$\theta(P)$	$\Psi(\theta-)$
finite, regular	SRW	1	∞	$1/\lvert X\rvert$
finite group	$p(x,y) = \mu(x^{-1}y)$	1	∞	$1/\lvert X\rvert$
$\mathbb{Z}^d$, $d \in \{1,2\}$	$p(x,y) = \mu(y-x)$	≤ 1	∞	0
$\mathbb{Z}^d$, $d \in \{3,4\}$	$p(x,y) = \mu(y-x)$	≤ 1	$< \infty$	0
$\mathbb{Z}^d$, $d \ge 5$	$p(x,y) = \mu(y-x)$	≤ 1	$< \infty$	> 0
$\mathbb{Z}^d$, $d \ge 5$	SRW	1	$< \infty$	$> 1/2$

On $\mathbb{Z}^d$, we suppose that μ has finite first ($d = 1$), resp. second ($d \ge 2$) moment.

B. Free products. Let (X_i, o_i), $i \in \mathcal{I}$, be loopless, finite or infinite, connected graphs with roots o_i. The index set $\mathcal{I}$ is usually assumed to be finite, but it may also be countable.

We construct the *free product* $\quad (X,o) = \mathop{*}\limits_{i\in\mathcal{I}}(X_i,o_i) \quad$ as follows.

Identify all o_i with the new root o, and let $X_i' = X_i \setminus \{o_i\}$. Write $\iota(x) = i$ if $x \in X_i'$. Then X is the set of all *words* with *letters* from the X_i', such that no two successive letters come from the same X_i':

$$X = \{x_1 x_2 \cdots x_n : n \ge 0\,,\ x_j \in \textstyle\bigcup_i X_i'\,,\ \iota(x_j) \ne \iota(x_{j-1})\}\ ;$$

for $n = 0$, we get the empty word o. In particular, $X_i \subset X$.

Write $X_i^\top = \{x_1 \cdots x_n \in X : \iota(x_n) \neq i\} \cup \{o\}$. If $u \in X_i^\top$ and $x \in X_i$, then ux is their concatenation, in particular $uo = u$.

The *graph structure* on X is as follows: if $x, y \in X_i$, $x \sim y$ in X_i, then $ux \sim uy$ in X for all $u \in X_i^\top$.

We may imagine X as an infinite "cactus", whose "leaves" are copies of the X_i. At the root o, all X_i are joined by their respective roots. At each other point of X_i, we attach copies of all the X_j, $j \neq i$, by their roots. At each of the new points $x_1 x_2$ we then attach copies of the X_k, $k \neq j$, and so on (inductively). If the X_i are Cayley graphs of groups Γ_i then X is Cayley graph of the free product group Γ.

Next, we construct *random walks* that are adapted to this product structure. Given, transition matrices P_i over X_i, $i \in \mathcal{I}$, we lift P_i to $\bar{P}_i$ on X: if $u \in X_i^\top$ and $v, w \in X_i$ then

$$\bar{p}_i(uv, uw) = p_i(v, w),$$

and $\bar{p}_i(x, y) = 0$ in all other cases. Then we build the free "sum" (or more precisely, convex combination)

$$P = \sum_{i \in \mathcal{I}} \alpha_i \bar{P}_i \,, \quad \text{where} \quad \alpha_i > 0 \,, \ \sum_i \alpha_i = 1 \,.$$

In the specific case of a free product of groups Γ_i, where $p_i(x, y) = \mu_i(x^{-1}y)$, and μ_i is a probability measure on Γ_i, we get the free product group Γ, which contains each Γ_i as a subgroup. Thus, $p(x, y) = \mu(x^{-1}y)$ with $\mu = \sum_i \alpha_i \, \mu_i$, where each μ_i is considered as a probability measure on Γ whose support is contained in Γ_i.

The following was proved by Woess [**65**] and Cartwright and Soardi [**11**].

Theorem 5.2. *In the above setting of a free product, if Φ_i is associated with P_i in the same way as Φ with P,*

$$\Phi(t) = 1 + \sum_{i \in \mathcal{I}} \big(\Phi_i(\alpha_i t) - 1\big) \,.$$

The function $\Phi(t)$ is analytic in $\mathfrak{U}_{\bar{\theta}}$, where

$$\bar{\theta} = \inf\{\theta(P_i)/\alpha_i : i \in \mathcal{I}\} \geq \theta(P) \,.$$

The proof of Woess [**65**] uses generating functions and assumes an underlying group structure. See also Gutkin [**31**] for a combinatorial proof without use of groups. For a detailed exposition, see Woess [**66**], §9.C and §17 and the remarks plus additional references on pp. 136–137.

Regarding the asymptotic behaviour of the n-step return probabilities to o, the main point in the above theorem is the following. A priori, the function Φ given by Proposition 5.1 is known to be analytic only up to the point $\theta = \mathfrak{r} G(\mathfrak{r})$, as depicted in Figure 3. But in the formula of Theorem 5.2, it may happen that in composing Φ by the Φ_i, one gets that Φ extends analytically *beyond* θ, i.e., it may happen that $\bar{\theta} > \theta(P)$ strictly. If this is the case, then the situation is precisely as in the example of the homogeneous tree that was described in detail in §4.A and depicted in Figure 2. In particular, one obtains the same type of formula for $\rho(P)$ as in (4.4) and the same asymptotics as in (4.6).

In order to understand when $\bar\theta > \theta(P)$, we remark that for the function Ψ on the free product, the following composition formula follows from Theorem 5.2.

$$\Psi(t) = 1 + \sum_{i\in\mathcal{I}} \bigl(\Psi_i(\alpha_i t) - 1\bigr)\,, \quad t \in \mathfrak{U}_{\bar\theta}\,.$$

Then the following holds, see Woess [**65**] and [**66**], Theorem 9.22.

Theorem 5.3. *Two cases can occur.*

(i) *If* $\Psi(\bar\theta-) < 0$ *then* $\theta(P)$ *is the unique solution in* $(0\,,\bar\theta)$ *of* $\Psi(t) = 0$,

$$\rho(P) = \min\{\Phi(t)/t : 0 < t < \bar\theta\} = \Phi'(\theta) < 1\,, \quad \textit{and}$$

$$p^{(n)}(o,o) = C\,\rho^n\, n^{-3/2} + \mathcal{O}(\rho^n\, n^{-5/2})\,, \quad n \to \infty\ ,\ n \equiv 0(\mathtt{d}).$$

(ii) *If* $\Psi(\bar\theta-) \ge 0$ *then* $\theta(P) = \bar\theta$, *and* $\rho(P) = \lim\limits_{t\to\bar\theta-} \bigl(\Phi(t)/t\bigr)$.

In (i), $\mathtt{d} = gcd\,\{n \ge 1 : p^{(n)}(x,x) > 0\}$ is the *period* of P.

C. The typical case: $\mathbf{n^{-3/2}}$. Using Table 1, one obtains the typical asymptotic behaviour of (4.6) in the following cases, see [**65**] and [**66**], §17.A.

Corollary 5.4. *Let* μ_i *be irreducible*[2] *probability measures on the groups* Γ_i, $i \in \mathcal{I}$, *and* μ *a convex combination of the* μ_i *on the free product* $\Gamma = \mathop{*}\limits_{i\in I} \Gamma_i$. *Then for* $p(x,y) = \mu(x^{-1}y)$

$$p^{(n)}(x,x) = C\,\rho^n\, n^{-3/2} + \mathcal{O}(\rho^n\, n^{-5/2})\,, \quad n \to \infty\ ,\ n \equiv 0(\mathtt{d})$$

in each of the following cases.

(a) *Each* Γ_i *is finite and the* μ_i *are arbitrary, with the exception of the case* $|\mathcal{I}| = |\Gamma_1| = |\Gamma_2| = 2$.

(b) $\Gamma_i = \mathbb{Z}^{d_i}$ *with* $d_i \le 4$, *and the* μ_i *have finite support, or finite mean and finite moments of order* $\min\{d_i, 2\}$.

(c) *Each* Γ_i *has polynomial growth with degree* $d_i \le 4$, *and the* μ_i *are symmetric with finite moments of order* $\min\{d_i, 2\}$.

(d) Γ *is the free product of finitely many identical pieces* $\Gamma_i = \Gamma_0$ *and* $\mu_i = \mu_0$, *with* $\alpha_i = 1/|\mathcal{I}|$, *and* $|\mathcal{I}| > 1/\bigl(1 - \Psi_0(\theta_0-)\bigr)$.

Proof. (a) We have $\bar\theta = \infty$ and $\Psi_i(\theta_i-) = 1/|\Gamma_i|$. Therefore, unless $|\mathcal{I}| = |\Gamma_1| = |\Gamma_2| = 2$, we get $\Psi(\bar\theta-) = 1 + \sum_i \bigl(\frac{1}{|\Gamma_i|} - 1\bigr) < 0$.

(b) In this case $\Psi_i(\theta_i-) = 0$. If $i_0 \in \mathcal{I}$ is such that $\bar\theta = \theta_i/\alpha_i$ then

$$\Psi(\bar\theta) = \sum_{i\ne i_0} \bigl(\Psi_i(\alpha_i\bar\theta) - 1\bigr) < 0\,,$$

as $\Psi_i(t) < 1$ for $t > 0$.

(c) The argument is the same as for (b).

(d) Here, $\bar\theta = |\mathcal{I}|\theta_0$,

$$\Phi(t) = |\mathcal{I}|\Phi_0(\tfrac{t}{|\mathcal{I}|}) - (|\mathcal{I}| - 1) \quad \text{and} \quad \Psi(t) = |\mathcal{I}|\Psi_0(\tfrac{t}{|\mathcal{I}|}) - (|\mathcal{I}| - 1)\,.$$

[2] Irreducible in the sense that the corresponding transition matrix is irreducible.

If $|\mathcal{I}|$ is sufficiently large then $\Psi_0(\theta_0-) < (|\mathcal{I}|-1)/|\mathcal{I}|$, that is, $\Psi(\bar\theta-) < 0$. □

"Mixtures" of (a), (b), (c) are also possible; the basic requirement is $\Phi(\bar\theta-) < 0$, which can be achieved in many cases. In (a), the group structure is not essential. For example, one may take arbitrary finite, regular graphs X_i with roots o_i in the place of Γ_i, and SRW on X_i for each P_i. Then the same result holds at $x = o$, the root of $\ast_{i\in\mathcal{I}}(X_i, o_i)$, except when $|\mathcal{I}| = 2$ and $|X_i| = 2$ $(i = 1, 2)$. We also stress that (a) holds for *infinite* free products: in [**65**], this is stated only for *reversible* random walks of this type, but that condition served just to control the singularities of $G(z)$ on $\{|z| = \mathfrak{r}\}$ and has become superfluous in view of Cartwright's Lemma 3.2.

Cartwright [**9**] has proved the following.

Lemma 5.5. *If $\Gamma = \Gamma_1 * \Gamma_2$, with exception of the case $|\Gamma_1| = |\Gamma_2| = 2$, then there is a symmetric, irreducible probability measure μ on Γ for which*

$$p^{(2n)}(x,x) \sim C\,\rho^{2n}\,n^{-3/2}\,.$$

This is based on the following.

If S is a finite, symmetric set of generators of a group Γ that contains an element of order ≥ 3 (possibly ∞), then there is a symmetric probability measure μ supported by S such that $\Psi_\mu(\theta-) < 1/2$.

D. Instability of the exponent. We now explain a surprising result of Cartwright [**9**]. Let $\Gamma_1 = \Gamma_2 = \mathbb{Z}^d$, where $d \geq 5$. Let $S_1 = S_2$ be the set of natural generators and their inverses (unit vectors in $\mathbb{Z}^d$). Consider a probability measure $\sigma_1 = \sigma_2$ which concentrates most of its mass on the first generator and its inverse. This is an example where $\Psi_{\sigma_i}\big(\theta(\sigma_i)-\big) < 1/2$, i.e., Lemma 5.5 applies. Therefore, setting $\sigma = \frac12(\sigma_1+\sigma_2)$ on $\mathbb{Z}^d * \mathbb{Z}^d$, and $q(x,y) = \sigma(x^{-1}y)$,

$$q^{(2n)}(x,x) \sim C_Q\,\rho(Q)^{2n}\,n^{-3/2}\,.$$

On the other hand, Let μ_i be the equidistribution on S_i. Then $\Psi_{\mu_i}\big(\theta(\mu_i)-\big) > 1/2$. (See Table 1). The simple random walk on $\mathbb{Z}^d * \mathbb{Z}^d$ is $p(x,y) = \mu(x^{-1}y)$, where $\mu = \frac12(\mu_1+\mu_2)$. We get $\theta(\mu) = \bar\theta = 2\theta(\mu_i)$ and $\Psi_\mu\big(\theta(\mu)\big) > 0$.

The angle of intersection of $y = t/\mathfrak{r}$ with $y = \Phi_\mu(t)$ at the point $\big(\bar\theta, G(\mathfrak{r})\big)$ is positive. We are in the situation of Theorem 5.3.

Proposition 5.6. *The Green function $G_d(z) = G_d(0,0|z)$ of SRW on the grid $\mathbb{Z}^d$ $(d \geq 1)$ has a singular expansion near $z = 1$*

$$G_d(z) = \begin{cases} f(z) + g(z)\,(1-z)^{(d-2)/2}\,, & \text{if } d \text{ is odd,}\\ f(z) + g(z)\,(1-z)^{(d-2)/2}\,\log(1-z)\,, & \text{if } d \text{ is even,}\end{cases}$$

where f $(= f_d)$ and g $(= g_d)$ are analytic in a neighbourhood of 1 and $g(1) \neq 0$.

On the basis of this expansion, via Theorem 5.2, further (lengthy) computations yield the following.

PROPOSITION 5.7. *Let $\mathfrak{r} = 1/\rho(P)$ for SRW on $\mathbb{Z}^d * \mathbb{Z}^d$ ($d \ge 5$), and $L = \lceil \frac{d-2}{2} \rceil$. Near $\mathfrak{r}$, for $|z| \le \mathfrak{r}$, the Green function $G(z) = G(x,x|z)$ has singular expansion*

$$G(z) = \sum_{k=0}^{L} g_k\,(\mathfrak{r}-z)^k + R(z) + \mathcal{O}\big((\mathfrak{r}-z)^{L+1}\big)\,, \qquad \textit{where}$$

$$R(z) = \begin{cases} \big(c_0 + c_1\,(\mathfrak{r}-z)\big)(\mathfrak{r}-z)^{(d-2)/2}\,, & \textit{if } d \textit{ is odd,} \\ \big(c_0 + c_1\,(\mathfrak{r}-z)\big)(\mathfrak{r}-z)^{(d-2)/2}\log(\mathfrak{r}-z)\,, & \textit{if } d \textit{ is even;} \end{cases}$$

when $d = 6$, there is an additional term $\bar{c}_1(\mathfrak{r}-z)^3\log^2(\mathfrak{r}-z)$ in $R(z)$.
All coefficients depend on d, and $c_0 \ne 0$.

The last two propositions are due to CARTWRIGHT [**8**], see also WOESS [**66**], §17.B. One can now apply the method of *Darboux* to obtain

THEOREM 5.8. *For $d \ge 5$, SRW on $\mathbb{Z}^d * \mathbb{Z}^d$ satisfies*

$$p^{(2n)}(x,x) \sim C_P\;\rho(P)^{2n}\,n^{-d/2} \quad \textit{as } n \to \infty\,.$$

On the other hand, the random walk Q with law σ having the same support as μ satisfies

$$q^{(2n)}(x,x) \sim C_Q\;\rho(Q)^{2n}\,n^{-3/2} \quad \textit{as } n \to \infty\,.$$

Before this result appeared, it had been a common belief that the asymptotic behaviour should be of the same form for all finite range, symmetric random walks on the same group. While this *is* true for asymptotic type (which does not capture the non-exponential second term, once there is exponential decay), we see that it is *not true* for the precise asymptotic behaviour.

6. Finite range random walks on free groups

The *free group* on s free generators is $\mathbb{F}_s = \langle a_1, \dots, a_s \mid \,\rangle$. Its Cayley graph with respect to $S = \{a_1^{\pm 1}, \dots, a_s^{\pm 1}\}$ is the homogeneous tree with degree $2s$. Also, it is the free product $\Gamma_1 * \cdots * \Gamma_s$, where each $\Gamma_i = \langle a_i \rangle \cong \mathbb{Z}$ is infinite cyclic. Thus, *local limit theorems* (describing the asymptotics of transition probabilities) for random walks on $\mathbb{F}_s$ arise as a special case of Corollary 5.4. In particular, if the law of the random walk is supported by S, then the computation is almost exactly the same as in the homogeneous tree example of §4.A. See WOESS [**63**] and GERL AND WOESS [**24**].

For example, from Corollary 5.4 we also get the typical behaviour $p^{(n)}(x,x) \sim C\,\rho(P)^n\,n^{-3/2}$, when P arises from a probability measure $\mu = \sum_i \alpha_i\,\mu_i$, where each μ_i is symmetric with finite first moment on $\Gamma_i \cong \mathbb{Z}$. The same asymptotics also hold for arbitrary *isotropic* random walks on free groups, resp. homogeneous trees, that is, random walks whose transition probabilities $p(x,y)$ depend only on the distance $d(x,y)$. This has been shown by completely different methods, using harmonic analysis, by SAWYER [**53**].

In this section, we address the same question for *arbitrary* (irreducible, aperiodic) random walks on $\mathbb{F}_s$ with *finite range*, i.e., arising from a finitely supported probability measure μ.

We start by explaining the complex-analytic tool used by LALLEY [**41**] in order to solve this problem.

A. Systems of polynomial equations for generating functions. Let $f_i(z) = \sum_{n\ge 0} f_{i,n}\, z^n$ be the generating functions of the non-negative sequences $(f_{i,n})_{n\ge 0}$, $i = 1,\dots,\nu$, and let $\mathfrak{r}_i$ be the radius of convergence of $f_i(z)$. We suppose that $f_i(0) = 0$ and that $\mathfrak{r} = \min_i \mathfrak{r}_i > 0$. We assume that the $f_i(z)$ satisfy a system of equations

$$f_i(z) = \mathcal{Q}_i\big(z, f_1(z),\dots,f_\nu(z)\big)\,,\quad i = 1,\dots,\nu\,, \tag{6.1}$$

where

$$\mathcal{Q}_i(z, y_1,\dots,y_\nu) = \sum_{|\mathbf{n}|\le N} a_{i,\mathbf{n}}(z)\,\mathbf{y}^{\mathbf{n}}\,,\quad z\in\mathbb{C}\,,\ i = 1,\dots,\nu\,,$$

are polynomials of degree between 1 and N in the variables $y_1,\dots,y_\nu$. Here, $\mathbf{y} = (y_1,\dots,y_\nu)$, $\mathbf{n} = (n_1,\dots,n_\nu) \in \mathbb{N}_0^\nu$, $\mathbf{y}^{\mathbf{n}} = y_1^{n_1}\cdots y_\nu^{n_\nu}$ and $|\mathbf{n}| = n_1 + \cdots + n_\nu$. We further assume that the coefficient functions $a_{i,\mathbf{n}}(z)$ are polynomials with non-negative coefficients, and that at least one among the $a_{i,\mathbf{0}}(z)$ is non-constant and $a_{i,\mathbf{n}}(0) = 0$ for all i, when $|\mathbf{n}| = 1$.

The *dependency di-graph* $\mathcal{D}$ of our system of equations (6.1) has vertex set $\{1,\dots,\nu\}$, and there is an oriented edge from i to j (notation $i\to j$), if y_j appears in a non-zero term of $\mathcal{Q}_i(z, y_1,\dots,y_\nu)$. The di-graph is called *strongly connected* if for every pair of vertices there is an oriented path from one to the other.

We want to extract information about the singular expansion of the functions $f_i(z)$ near $\mathfrak{r}_i$. Recall that $\mathfrak{r}_i$ has to be a singularity of $f_i(z)$ by *Pringsheim's* theorem. From the structure of the system (6.1), it is clear that if $i\to j$, then $\mathfrak{r}_i \le \mathfrak{r}_j$.

LEMMA 6.1. *If $\mathcal{D}$ is strongly connected then all $\mathfrak{r}_i$ coincide, $\mathfrak{r}_i = \mathfrak{r}$. If the system (6.1) is* linear *(i.e., the polynomials $\mathcal{Q}_i$ have degree 1 in the variables y_j), then $f_i(\mathfrak{r}) = +\infty$ for all i. Otherwise, $f_i(\mathfrak{r}) < \infty$ for all i.*

For $0 \le z \le r$, consider the Jacobian matrix of our system of equations:

$$\mathfrak{J}(z) = \left(\frac{\partial \mathcal{Q}_i}{\partial y_j}\big(z, f_1(z),\dots,f_\nu(z)\big)\right)_{i,j=1}^{\nu}\,.$$

This is a non-negative matrix whose entries are increasing in $z \ge 0$. Our assumption that $f_i(0) = 0$ for all i implies that $\mathfrak{J}(0)$ is the zero matrix. Furthermore, for $0 < z < r$, the (i,j)-entry of $\mathfrak{J}(z)$ is positive precisely when $i\to j$ in $\mathcal{D}$. We now assume that $\mathcal{D}$ is strongly connected, and use the *Perron-Frobenius* theory of non-negative matrices, see SENETA [**55**]: if $0 < z < r$ then $\mathfrak{J}(z)$ has a positive eigenvalue $\lambda(z)$, which has maximal absolute value among all eigenvalues of $\mathfrak{J}(z)$, algebraic and geometric multiplicity 1 and strictly positive left and right eigenvectors. We have $\lambda(0) = 0$. As all entries of $\mathfrak{J}(z)$ increase with z, so does $\lambda(z)$.

PROPOSITION 6.2. *If $\mathcal{D}$ is strongly connected then $\mathfrak{J}(\mathfrak{r})$ is finite and*

$$\mathfrak{r} = \min\{z > 0 : \lambda(z) = 1\}.$$

If (6.1) is linear then the solutions $f_i(z)$ are rational functions. We are interested in the case when $\mathcal{D}$ is strongly connected and (6.1) is *non-linear*. It is a highly non-trivial, but well known result of Algebraic Geometry that the solutions $f_i(z)$ must be algebraic functions (solutions of single polynomial equations). See e.g. VAN DER WAERDEN [**62**], §31, or - in a more specific random walk context - LALLEY [**41**], [**42**]. Therefore each $f_i(z)$ has a *Puiseux* expansion near $z = \mathfrak{r}$ (except for real $z > \mathfrak{r}$) of the form

$$f_i(z) = f_i(\mathfrak{r}) - b_i(\mathfrak{r} - z)^{\alpha(i)} + h.o.t.,$$

where $b_i > 0$ and $h.o.t.$ stands for "higher order terms", which are all of the form $\langle$ real coefficient $\times (r - z)^p \rangle$, where $p > \alpha(i)$ is rational; the occurring rational exponents form a discrete sequence. See e.g. DIMCA [**15**], pp. 177–179.

The main tool is now the following.

THEOREM 6.3. *If $\mathcal{D}$ is strongly connected and (6.1) is non-linear then $\alpha(i) = 1/2$, i.e., near $z = \mathfrak{r}$ (except $z > \mathfrak{r}$ real)*

$$f_i(z) = f_i(\mathfrak{r}) - b_i(\mathfrak{r} - z)^{1/2} + h.o.t.,$$

In particular, if $\mathfrak{r}$ is the only singularity of $f_i(z)$ on the circle $\{|z| = \mathfrak{r}\}$, then the power series coefficients of $f_i(z)$ satisfy

$$f_{i,n} \sim c_i\, \mathfrak{r}^{-n}\, n^{-3/2} \quad \text{as } n \to \infty\,.$$

The last conclusion is of course immediate from the singular expansion by the method of Darboux (or Singularity Analysis, see §8 below). For proofs of Lemma 6.1, Proposition 6.2 and Theorem 6.3, see LALLEY [**41**] (who developed this method), WOESS [**66**] in a random walk context. In a related setting (groups and languages), Lemma 6.1 and Proposition 6.2 are proved in a slightly more general form by CECCHERINI-SILBERSTEIN AND WOESS [**12**]. (There, we use, but forget to state the assumption $a_{i,\mathbf{n}}(0) = 0$ for $|\mathbf{n}| = 1$.) Finally, see DRMOTA [**19**] for a more general variant in the general context of Combinatorial Analysis and some further references.

B. Finite range random walks on free groups. We now indicate how the tool of the last sub-section can be applied to random walks on $\mathbb{F}_s$ whose law μ has finite support.

We shall refer to distance, geodesics and balls in $\mathbb{F}_s$ as those of its Cayley graph $\mathbb{T}_{2s}$ with respect the the generating set S of the free generators and their inverses. We write $|x| = d(x, id)$. Let M be the smallest integer such that $\operatorname{supp}\mu$ is contained in $B = B(id, M)$, so that $B(x, M) = xB$ for any $x \in \mathbb{F}_s$. The following is an immediate consequence of the tree structure, in analogy with Lemma 4.1.

LEMMA 6.4. *If $w \in \overline{x\,y}$ then the random walk starting at x has to pass through wB in order to reach y.*

For every $y \in \mathbb{F}_M$ define the stopping time $\tau_y = \min\{n \ge 0 : Z_n \in yB\}$ and for $x \neq y$ the matrix $\mathcal{H}_{x,y}(z) = \big(H_{x,y}(xa, yb|z)\big)_{a,b \in B}$ with

$$H_{x,y}(xa, yb|z) = \sum_{n=0}^{\infty} \Pr_{xa}[\tau_y = n\,,\ Z_n = yb]\, z^n\,.$$

This is a power series with n-th coefficient $\le p^{(n)}(xa, yb)$, so that it certainly converges for $|z| \le \mathfrak{r} = 1/\rho(P)$. If $xa = yb$ then $H_{x,y}(xa, yb|z) \equiv 1$, and if $xa \neq yb$ then $H_{x,y}(xa, yb|0) = 0$. It may also happen that $H_{x,y}(xa, yb|z) \equiv 0$, namely when $yb \neq xa \in yB$, or when $xa \notin yB$, but the first entrance in yB cannot occur at yb.

LEMMA 6.5. *If $x, y \in \mathbb{F}_s$ and $\overline{x\,y} = [x = x_0, x_1, \dots, x_k = y]$ then*

(a)
$$\mathcal{H}_{x,y}(z) = \prod_{j=1}^{k} \mathcal{H}_{x_{j-1},x_j}(z)$$

(b) $$H_{x,y}(xa, yb|z) = \begin{cases} \delta_{yb}(xa)\,, & \textit{if } xa \in yB\,, \\ \displaystyle\sum_{a' \in B} \mu(a')z\, H_{xa,y}(xaa', yb|z)\,, & \textit{otherwise.} \end{cases}$$

Statement (a) follows from Lemma 6.4, and (b) is obtained by decomposing with respect to the first step of the random walk.

Note that by group invariance $H_{x,y}(xa, yb|z) = H_{id,x^{-1}y}(a, x^{-1}yb|z)$, so that each of the terms $H_{xa}(xaa', yb|z)$ occurring in the right hand side of (b) is by virtue of (a) a matrix element in a product of at most $M + 1$ matrices from the set $\{\mathcal{H}_{id,v}(z) : v \in S\}$. We eliminate the constant ones (with value 0 or 1) among them, and write $H_i(z)$, $i = 1, \dots, \nu$ for the non-constant ones among the matrix elements of all the $\mathcal{H}_{o,v}(z)$, $v \in S$.

The basic result for obtaining the asymptotics of transition probabilities on $\mathbb{F}_s$ is the following.

PROPOSITION 6.6. *The functions $H_i(z)$, $i = 1, \dots, \nu$, satisfy a system of polynomial equations of the form (6.1), with $H_i(z)$ in the place of $f_i(z)$.*

The system is non-linear, and if $\mu(x) > 0$ for every $x \in \mathbb{F}_s$ with $|x| \le 2$ then the dependency di-graph of the system is strongly connected, and the radius of convergence of the $H_i(z)$ as well as that of all functions $F(x, y|Z)$, $x, y \in \mathbb{F}_s$ coincides with $\mathfrak{r} = 1/\rho(P)$.

The fact that the $H_i(z)$ do indeed satisfy a non-linear system of form (6.1) is immediate from Lemma 6.5. The proof of strong connectivity of the dependency di-graph requires a considerable combinatorial effort, see WOESS [**66**], Proposition (19.12) (and comments on p. 218), and for this it suffices to have $\mu(x) > 0$ for every $x \in S$. Finally, to show that the common radius of convergence of the $H_i(z)$ is $\mathfrak{r}$ when $\operatorname{supp}\mu$ contains all elements with $|x| \le 2$ is not as hard, but uses in a crucial way a theorem of GUIVARC'H [**29**] (see [**66**], §7.B for a proof) which says that $G(\mathfrak{r}) < \infty$ on every non-recurrent group. (By VAROPOULOS [**61**], only those among all finitely generated groups that contain $\mathbb{Z}$ or $\mathbb{Z}^2$ as a subgroup with finite index may carry a recurrent random walk, i.e, one with $G(1) = \infty$.)

The algebraic link between the Green function and the functions $F(x, y|z)$ is given in general by the Basic Equations 3.3. In the present specific case, the link between the functions $F(x, y|z)$ and $H_i(z)$ is provided by the following

LEMMA 6.7. *For $a \in B \setminus \{id\}$,*

$$F(a, id|z) = M(a, id|z) + \sum_{b \in B\setminus\{id\}} M(a, b|z)\, F(b, id|z)\,, \quad \textit{where}$$

$$M(a, b|z) = \mu(a^{-1}b)z + \sum_{x \in aB\setminus B} \mu(a^{-1}x)z\, H_{x,o}(x, b|z)\,, \quad a, b \in B\,.$$

Thus, if $\mu(x) > 0$ for every $x \in \mathbb{F}_s$ with $|x| \le 2$ then Theorem 6.3 yields the singular expansions

$$H_i(z) = H_i(\mathfrak{r}) - b_i(\mathfrak{r} - z)^{1/2} + h.o.t.,$$

and via Lemma 6.7, we obtain the same type of expansions for the functions $F(b, id|z)$, $b \in B \setminus \{id\}$, and subsequently for $G(z) = G(x, x|z)$ (which is the same for all x by group invariance). By Lemma 3.2, $z = \mathfrak{r}$ is the only singularity of $G(z)$ on the circle of convergence, and we can apply the method of Darboux to obtain the asymptotics. In general, if μ is finitely supported and such that the random walk is irreducible and aperiodic (hence strongly aperiodic, since we are on a group), then there is an index k_0, such that for all $k \ge k_0$ one has $\mu^{(k)}(x) > 0$ for all $x \in \mathbb{F}_s$ with $|x| \le 2$. Here, $\mu^{(k)}$ is the k-th convolution power of μ, which is the law of the random walk with transition matrix P^k. We can now apply the methods explained above to the random walks with laws $\mu^{(k_0)}$ and $\mu^{(k_0+1)}$, and since the two exponents are relatively prime, one can conclude as follows.

THEOREM 6.8. *Let μ be the law of a finite range, irreducible and aperiodic random walk on $\mathbb{F}_s$. Then as $n \to \infty$*

$$p^{(n)}(x,x) \sim C\,\rho(P)^n\, n^{-3/2}\,.$$

Indeed, a relatively simple final step is to carry this over to the asymptotics of $p^{(n)}(x,y)$, as $n \to \infty$, with a leading constant $C = C_{x,y} > 0$ that depends on $x^{-1}y$. This significant theorem is due to LALLEY [**41**].

We remark that the method outlined in Subsection A can also be used to derive results of other types, such as the computation of the *rate of escape*, i.e., the almost sure limit of $d(Z_n,o)/n$, and a *central limit theorem* for $d(Z_n,o)$. For free groups, see SAWYER AND STEGER [**54**] and LALLEY [**41**].

More generally, a class of graphs where the methods described here lead to very satisfactory results (asymptotics of transition probabilities, rate of escape, central limit theorem, etc.) are the *trees with finitely many cone types*. These trees do in general not arise from groups, nor do they admit (quasi) transitive actions of isometry groups. Instead, regularity of their structure is described in terms of a *finite state automaton*, which is nothing but a labelled, finite di-graph. If the latter satisfies a certain irreducibility (i.e., strong connectivity) condition, then the same type of results as those described above for free groups can be obtained. See NAGNIBEDA AND WOESS [**44**] for nearest neighbour walks on trees with finitely many cone types, and LALLEY [**42**] for more general finite range random walks in the basically equivalent setting of *regular languages*.

7. Uniform space-time estimates on trees

For aperiodic nearest neighbour random walks on some types of trees[3], in particular homogeneous trees, one can use generating functions to obtain asymptotic estimates of $p^{(n)}(x,o)$ that are uniform in space (x) and time (n), as $n \to \infty$. In this case, typically one can write

$$G(x,o|z) = G(z)F(z)^k\,, \quad k = d(x,o)\,, \tag{7.1}$$

where $G(z) = G(o,o|z)$ and $F(z) = F(x,x^-|z)$, with x^- the predecessor of x on the geodesic $\overline{ox}$. In some examples, different functions $F(z)$ will occur for different starting points x. We start by outlining the general approach. A good reference (concerning single contour integrals, and not uniform behaviour with respect to a parameter k as here) is DE BRUIJN [**7**].

[3]Here, we suppose that $p(x,y) > 0 \iff d(x,y) \le 1$.

A. The saddle point method. We make the specific assumptions that $F(z)$ and $G(z)$ have the same radius of convergence $\mathfrak{r}$, that this is their only singularity on $\{|z| = \mathfrak{r}\}$, and that both functions extend analytically to a larger disk around the origin with exception of the real half-line $[\mathfrak{r}\,,\, +\infty)$. Writing $F(z) = \sum_{n\geq 0} f_n\, z^n$, we assume that $f_0 = 0$, $F(\mathfrak{r}) < \infty$ and $gcd\{n \geq 1 : f_n > 0\} = 1$. The latter condition amounts to aperiodicity of the random walk.

We write Cauchy's integral formula as

$$(7.2)\quad p^{(n)}(x,o) = \frac{1}{2\pi\mathfrak{i}} \oint_{\mathcal{C}} \frac{G(x)F(z)^k}{z^{n+1}}\, dz = \oint_{\mathcal{C}} \frac{G(x)}{z} \exp\Big(n\big(\xi \log F(z) - \log z\big)\Big)\, dz\,,$$

where $\xi = k/n\,,\ k = d(x,o)$.

For nearest neighbour random walks, we must have $k \leq n$, that is, $0 \leq \xi \leq 1$. We shall usually only consider the range $0 \leq \xi \leq 1 - c$, where $c > 0$, since for k close to n, the uniform estimates are usually less interesting, and there are simple "ad hoc" methods to get good approximations of the asymptotics.

Now we proceed as follows: let

$$(7.3)\qquad \varphi(\xi) = \min\{\Psi_\xi(z) : 0 \leq z \leq \mathfrak{r}\}\,, \quad \text{where} \quad \Psi_\xi(z) = \xi \log F(z) - \log z$$

The minimum is attained at a point $z(\xi)$ with $z(0) = \mathfrak{r}$ and $z(1) = 0$; the function $\xi \to z(\xi)$ is decreasing. We have for any z with $|z| \leq \mathfrak{r}$ that $|F(z)| \leq F(|z|)$. If we choose an angle $\gamma < \pi/2$ then

$$(7.4)\qquad \sup\{|F(z)|/F(|z|) : z \in (0\,,\, \mathfrak{r}]\,,\ |\arg(z)| \geq \gamma\} < 1\,.$$

In particular, on the circle $\{z : |z| = z(\xi)\}$, the function $z \mapsto |F(z)^\xi/z|$ attains its maximum at $z(\xi)$, and because of the assumptions on $F(\cdot)$, this is the only point on that circle where the maximum is attained. Thus, for ξ fixed , $z(\xi)$ is a *saddle point* (minimum along the real axis, maximum along the circle).

We next write the lor expansion of $z \mapsto \Psi_\xi(z)$ at $z(\xi)$. It is of the form

$$(7.5)\qquad \Psi_\xi(z) = \varphi(\xi) + b(\xi)\big(z - z(\xi)\big)^2 + \mathcal{O}_\xi\Big(\big(z - z(\xi)\big)^3\Big)\,;$$

the remainder term depends on ξ. We have $b(\xi) \in \mathbb{R}$ and $b(\xi) \geq 0$. In typical cases, $b(\xi) > 0$ strictly. We choose an integration contour $\mathcal{C}$ which crosses the positive real axis at $z(\xi)$ and such that the real part of $\Psi_\xi(z)$ decays as rapidly as possible when we leave $z(\xi)$ along $\mathcal{C}$ *(steepest descent).*

Case 1. When ξ varies in a range $[a\,,\,1 - c]$, where $a > 0$, then $z(\xi) \in [\bar{a}, \mathfrak{r} - \bar{c}]$ with $\bar{a}, \bar{c} > 0$. Typically, the coefficient $b(\xi)$ remains bounded and bounded away from 0, and the $\mathcal{O}_\xi$ in the remainder term is uniform in ξ. We assume that all this holds. In that case, the most suitable integration contour $\mathcal{C}$ is the circle $z(\xi, t) = z(\xi)e^{\mathfrak{i}t}$, $t \in [-\pi\,,\,\pi]$. The expansion (7.5) becomes

$$(7.6)\qquad \Psi_\xi\big(z(\xi,t)\big) = \varphi(\xi) - \bar{b}(\xi)^2\, t^2 + R(\xi,t)\,,$$

where $\bar{b}(\xi)^2 = b(\xi)z(\xi)^2$ and $R(\xi,t)/t^2 \to 0$ uniformly for $\xi \in [a\,,\,1 - c]$, as $t \to 0$. In particular, we can find $0 < \gamma < \pi/2$ such that

$$(7.7)\qquad |R(\xi,t)| \leq \bar{b}(\xi)t^2/2 \quad \text{for all } t \in [-\gamma\,,\,\gamma]\,.$$

We decompose the integral over $\mathcal{C}$ in (7.2) in the two parts where $|t| \le \gamma$ and $\gamma < |t| \le \pi$. Then the second part becomes

$$\frac{1}{2\pi}\exp\bigl(n\varphi(\xi)\bigr)\int_{\gamma<|t|\le\pi}\left(\frac{F\bigl(z(\xi)e^{\mathrm{i}t}\bigr)}{F\bigl(z(\xi)\bigr)}\right)^{n\xi} e^{-\mathrm{i}nt}\, G\bigl(z(\xi)e^{\mathrm{i}t}\bigr)\,dt$$

which by (7.4) is asymptotically negligible with respect to the first part, which we write as

$$\frac{1}{2\pi}\exp\bigl(n\,\varphi(\xi)\bigr)\int_{-\gamma}^{\gamma}\exp\Bigl(-n\bar b(\xi)^2t^2 + nR(\xi,t)\Bigr)G\bigl(z(\xi)e^{\mathrm{i}t}\bigr)\,dt\,.$$

We substitute $\theta = t\sqrt{n}\,\bar b(\xi)$ to rewrite this as

$$\frac{G\bigl(z(\xi)\bigr)}{2\pi\bar b(\xi)}\exp\bigl(n\,\varphi(\xi)\bigr)n^{-1/2}$$
$$\times\int_{-\sqrt{n}\,\bar b(\xi)\gamma}^{\sqrt{n}\,\bar b(\xi)\gamma}\exp\Bigl(-\theta^2 + nR\bigl(\xi,\tfrac{\theta}{\sqrt{n}\bar b(\xi)}\bigr)\Bigr)\frac{G\Bigl(z(\xi)\,\exp\bigl(\mathrm{i}\tfrac{\theta}{\sqrt{n}\bar b(\xi)}\bigr)\Bigr)}{G\bigl(z(\xi)\bigr)}\,d\theta\,.$$

Due to our choice of γ, we can bound the integrand in absolute value by $\exp(-\theta^2/2)$. Thus we can apply Lebesgue's theorem (dominated convergence) to see that the last integral tends to $\int_{\mathbb{R}} e^{-\theta^2}\,d\theta$ uniformly in ξ. We have obtained

$$p^{(n)}(x,o) \sim \frac{G\bigl(z(\xi)\bigr)}{2\sqrt{\pi}\,\bar b(\xi)}\exp\bigl(n\,\varphi(\xi)\bigr)n^{-1/2} \tag{7.8}$$

uniformly for $\xi = \dfrac{d(x,o)}{n} \in [a\,,\,1-c]\,.$

Case 2. If $\xi \in [0\,,\,a]$ then $z(\xi)$ is close to the singularity $\mathfrak{r}$, and it usually happens that $b(\xi)$ and the $\mathcal{O}_\xi$ in (7.5) are unbounded, so that the method of Case 1 fails. One suitable method here is to choose a different contour $\mathcal{C}$, which is a suitable curve $z(\xi,t)$ in the vicinity $z(\xi)$, with $z(\xi,0) = z(\xi)$. We follow this curve beyond the circle $\{|z| = \mathfrak{r}\}$ (which is possible by our initial assumptions) until we reach the circle $\{|z| = \mathfrak{r}+\varepsilon_0\}$, where $\varepsilon_0 > 0$ is suitably chosen, and then follow a long arc of that circle, until we reach the curve $z(\xi,t)$ on the other side. See Figure 4 for an example.

The integration contours
in Cases 1 and 2
The dashed circle is $|z| = \mathfrak{r}$.

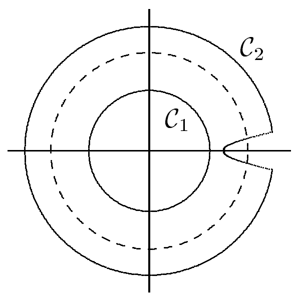

FIGURE 4

One then needs to verify that on $\{|z| = \mathfrak{r}+\varepsilon_0\} \setminus \{\mathfrak{r}+\varepsilon_0\}$, the functions $|F(z)|$ and $|G(z)|$ are bounded by some $C > 0$ (being careful when $G(\mathfrak{r}) = \infty$). Then, using (7.2), one finds that the integral along the arc is bounded above by

$$\exp\big(n\,\varphi(\xi)\big)\,C\,\lambda^n\,,\quad \text{where}\quad \lambda=\left(\frac{C}{F(\mathfrak{r}-\bar{c})}\right)^a\frac{\mathfrak{r}}{\mathfrak{r}+\varepsilon_0}\,.$$

This determines the choice of the value a at which we subdivide into cases 1 and 2: we choose a small enough so that $\lambda<1$. Then the integral over the arc will be asymptotically negligible as compared with the integral over the curve $z(\xi,t)$, whose asymptotic evaluation will yield the behaviour of the transition probablities.

From this point onwards, the computations depend heavily on the individual case, in particular, the types of singularities of $G(z)$ and $F(z)$ at $\mathfrak{r}$. It may happen that one has to subdivide further into two sub-cases $\xi\in[n^{-\alpha}\,,\,a]$ and $\xi\in[0\,,\,n^{-\alpha}]$, with a suitable choice of a, and that the choice of the curve $z(\xi,t)$ changes accordingly. The crucial part is the control of the expansion (7.6) along our curve. The latter determines the choice of ε_0 above, which in turn determines the choice of a.

The use of the saddle point method in probability to obtain uniform estimates (as in Case 1, with $G(z)\equiv 1$) is old, see e.g. Good [**25**]. In the context of combinatorial analysis, Drmota [**18**] gives a very precise uniform estimate for the case $G(z)\equiv 1$ and $F(z)$ having a simple branch point at $z=\mathfrak{r}$.

B. Aperiodic simple random walk on the homogenous tree. Since SRW on $\mathbb{T}_s$ has period two, we consider instead the random walk on $\mathbb{T}_s$ ($s\geq 3$) with transition probabilities

$$p(x,x)=1/2\quad\text{and}\quad p(x,y)=1/(2s)\quad\text{when } y\sim x\,.$$

There are obvious modifications for simple random walk, taking into account the parities of n and $|x|=d(x,o)$. With direct calculations, simpler than those of §4.A, we get $G(x,o|z)=G(z)\,F(z)^{|x|}$, where

$$F(z)=\tfrac{M}{(M-1)z}\left(\big(1-\tfrac12 z\big)-\sqrt{(1-z/\mathfrak{r})(1-z/\mathfrak{s})}\right)\quad\text{and}$$

$$G(z)=\frac{1}{1-\frac12 z-\frac12 zF(z)}\,,\quad\text{with}\tag{7.9}$$

$$\mathfrak{r}=\left(\tfrac12+\tfrac{\sqrt{s-1}}{s}\right)^{-1}\quad\text{and}\quad \mathfrak{s}=\left(\tfrac12-\tfrac{\sqrt{s-1}}{s}\right)^{-1}\,.$$

We have $\rho(P)=1/\mathfrak{r}$. We are in the situation described in Subsection A, and Case 1 works precisely as described there. The function $\varphi(\cdot)$ can be computed explicitly, and its expansion at 0 (for $\xi\geq 0$) is of the form

$$\varphi(\xi)=-\log\mathfrak{r}-\xi\,\log\sqrt{s-1}-c_s\,\xi^2+\mathcal{O}(\xi^3)\,,\quad\text{where}\quad c_s=\frac{s+2\sqrt{s-1}}{4\sqrt{s-1}}\,.$$

In Case 2, we choose the curve

$$z(\xi,t)=z(\xi)+t^2+2\mathfrak{i}t\sqrt{\mathfrak{r}-z(\xi)}\,,$$

which is part of a parabola, as shown in Figure 4. We omit the details of the computations in Case 2: a subdivision into $\xi\in[n^{-1/4}\,,\,a]$ and $\xi\in[0\,,\,n^{-1/4}]$ is fruitful. The first sub-case is handled like Case 1, but the second is more delicate. The final result is as follows, see Woess [**66**], §19.A for the details of the computation.

Theorem 7.1. *As $n\to\infty$, we have uniformly for $|x|\leq(1-c)n$ with $c>0$*

$$p^{(n)}(x,o)\sim B(|x|/n)\left(1+\tfrac{M-2}{M}|x|\right)\exp\Big(n\,\varphi(|x|/n)\Big)\,n^{-3/2}\,,$$

where $B(\cdot)$ is a continuous, strictly positive function on $[0\,,\,1)$ which satisfies $\lim_{\xi\to 1-} B(\xi)\sqrt{1-\xi} > 0$. *In particular*

$$p^{(n)}(x,o) \sim B(0)\Big(1+\tfrac{s-2}{s}|x|\Big)\frac{1}{\sqrt{s-1}^{|x|}}\exp\Big(-c_s|x|^2/n\Big)\,\rho(P)^n\,n^{-3/2}$$

uniformly for $|x|/\sqrt{n}$ bounded.

This material of [**66**] was elaborated on the basis of Lalley [**40**]. Lalley also considers the more general case of arbitrary nearest neighbour random walks on $\mathbb{T}_s$ of the same form as in §4.A above, with an additional "staying" probability $p(x,x) = p_0 > 0$ for all x in order to guarantee aperiodicity. In this more general case, one has s different functions $F_i(z)$, $i = 1,\ldots,s$, and $G(x,o|z) = F_{i_1}(z)\cdots F_{i_k(z)}G(z)$ when $x = a_{i_1}\cdots a_{i_k}$ (reduced representation in $\Gamma = \langle a_1,\ldots,a_s \mid a_i^2 = id\rangle$). This makes the method more involved; the result itself is qualitatively the same. The computations in Case 1 of Subsection A above also extend to the still more general situation of finite range random walk on free groups, as considered in §6.B; see [**41**].

C. Simple random walk on comb lattices. The d-dimensional *comb lattice* $\mathbf{C}_d$ is a natural spanning tree of $\mathbb{Z}^d$, neighbourhood of vertices is given by

$$(k_1,\ldots,k_{j-1},k_j,0,\ldots,0) \sim (k_1,\ldots,k_{j-1},k_j+1,0,\ldots,0)\,,$$

where $j = 1,\ldots,d$ and $k_i \in \mathbb{Z}$ $(1 \le i \le j)$.

Thus, $\mathbf{C}_1 = \mathbb{Z}$, and $\mathbf{C}_2$ is obtained from the square grid $\mathbb{Z}^2$ by deleting all horizontal edges except those that lie on the x-axis; see Figure 5. The comb lattice $\mathbf{C}_d$ can be constructed by taking $\mathbb{Z}$ and attaching at each point of $\mathbb{Z}$ a copy of $\mathbf{C}_{d-1}$ by its origin. (That is, $\mathbb{Z}$ is a "spit" running through the centres of all the copies of $\mathbf{C}_{d-1}$.)

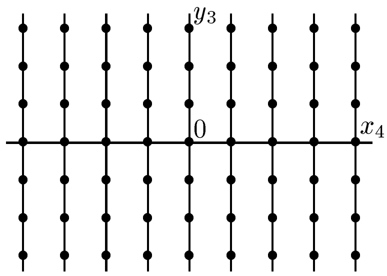

Figure 5

Let $F_d(z) = F(1,0|z)$ and $G_d(z) = G(0,0|z)$ for SRW on $\mathbf{C}_d$. Using the Tree Equation 4.1 and translation invariance of the SRW along the "spit" $\mathbb{Z}$, on can compute the following recurrence relation for $G_d(z)$.

$$\left(\frac{d}{G_d(z)}\right)^2 = \left(1+\frac{d-1}{G_{d-1}(z)}\right)^2 - z^2\,,\quad G_1(z) = \frac{1}{\sqrt{1-z^2}}\,. \tag{7.10}$$

Therefore $G_d(z)$ is algebraic, $\mathfrak{r} = 1$, and ± 1 are the only singularities on the circle of convergence (since SRW is reversible). One has a Puiseux series expansion whose initial term can be computed by comparing exponents in (7.10),

$$G_d(z) = d\,2^{-1+2^{1-d}}(1-z^2)^{-2^{-d}} + h.o.t.$$

The method of *Darboux* implies the following, see GERL [**22**].

THEOREM 7.2. *SRW on $\mathbf{C}_d$ satisfies*

$$p^{(2n)}(0,0) \sim \frac{2^{-1+2^{1-d}}\,d}{\Gamma(2^{-d})}\,n^{-1+2^{-d}}\,.$$

In particular, consider the 2-dimensional comb $\mathbf{C}_2$ with $G(z) = G_2(0,0|z)$, and write x_k and y_k for the points on the x- and y-axis, respectively ($x_0 = y_0 = 0$). Using the methods explained in §4, we compute

$$G(y_k,0|z) = G(z)\,F_1(z)^{|k|} \quad \text{and} \quad G(x_k,0|z) = G(z)\,F_2(z)^{|k|}\,, \quad \text{where}$$

$$\begin{aligned} F_1(z) &= \frac{1}{z}\Big(1-\sqrt{1-z^2}\Big)\,,\\ F_2(z) &= \frac{1}{z}\left(1+\sqrt{1-z^2}-\sqrt{2}\sqrt{1-z^2+\sqrt{1-z^2}}\right)\,, \quad \text{and}\\ G(z) &= \frac{\sqrt{2}}{\sqrt{1-z^2+\sqrt{1-z^2}}}\,. \end{aligned} \tag{7.11}$$

Therefore, the methods explained in Subsection A above can be used to compute uniform space-time estimates along the two axes. Since SRW has period two, one has to take into account the parities of n and k; it is best to substitute z in place of z^2, so that $\mathfrak{r} = 1$ becomes the only singularity on the circle of convergence. Computations are more involved than on the homogeneous tree, in particular along the x-axis, and have been carried out by BERTACCHI AND ZUCCA [**6**]. In particular, in [**6**] the following is proved.

THEOREM 7.3. *For SRW on $\mathbf{C}_2$, there are constants $c_1, c_2 > 0$ and continuous, strictly positive functions $C_1(t), C_2(t)$ on $[0\,,\,1]$ such that for $n \to \infty$, when $n-k$ is even,*

$$p^{(n)}(x_k,0) \sim C_1(k/n^{1/4})\,\exp\big(-c_1\,k^{4/3}/n^{1/3}\big)\,n^{-3/4} \quad \textit{uniformly for } k \le n^{1/4}, \textit{ and}$$

$$p^{(n)}(y_k,0) \sim C_2(k/n^{1/2})\,\exp\big(-c_2\,k^2/n\big)\,n^{-3/4} \quad \textit{uniformly for } k \le n^{1/2}.$$

Thus, SRW on $\mathbf{C}_2$ is the simplest example where the "Einstein relation" (see above in §2.E) does not hold.

8. Sierpiński graphs, and the use of Singularity Analysis

We start by describing the recursive construction of the d-dimensional Sierpiński graph $\mathbf{S}_d$. Let $0 = x_0, x_1, \dots, x_d$ be the vertices of a standard equilateral simplex in the non-negative cone of $\mathbb{R}^d$. Its 1-skeleton $S^{(0)} = S_d^{(0)}$ is the complete graph with vertices $0, x_1, \dots, x_d$. Next, let

$$S^{(k+1)} = S_d^{(k+1)} = \bigcup_{j=0}^{d} \big(2^k\,x_j + S^{(k)}\big)\,,$$

(where we take translates in $\mathbb{R}^d$), and let $-S^{(k)}$ be the reflection of $S^{(k)}$ through the origin. We obtain the increasing family of finite graphs $\mathbf{S}^{(k)} = \mathbf{S}_d^{(k)} = -S^{(k)} \cup S^{(k)}$.

The *Sierpiński graph* is $\mathbf{S}_d = \bigcup_{k\ge 0} \mathbf{S}_d^{(k)}$. It is regular, all vertex degrees being $= 2d$. Figure 6 shows $\mathbf{S}_2$, and the part with the bigger $\bullet$'s is $\mathbf{S}_2^{(1)}$.

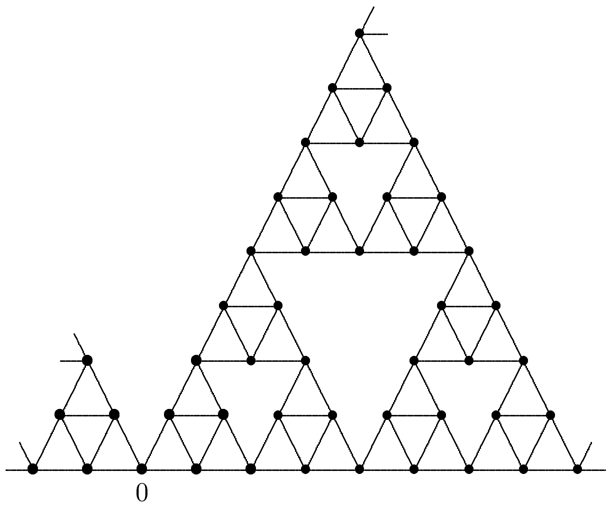

FIGURE 6

In general, $\mathbf{S}^{(k)}$ is the ball with respect to the graph metric in $\mathbf{S}$ of radius 2^k centred at 0. With respect to the counting measure, the size of the r-ball is

$$V(0,r) \approx r^{\delta_f} \quad \text{with} \quad \delta_f = \frac{\log(d+1)}{\log 2}\,.$$

The following theorem is due to JONES [**35**] and shows that the "Einstein relation" is satisfied for SRW on $\mathbf{S}_2$. (Note that $d(x,y) \approx |x-y|$ for the graph and Euclidean distances.)

THEOREM 8.1. *On* $\mathbf{S}_2$, *with* $\delta_s = 2\log 3/\log 5$ *and* $\delta_w = \log 5/\log 2$,

$$c_1\, n^{-\delta_s/2} \exp\left(-c_2 \frac{|x-y|^{\delta_w/(\delta_w-1)}}{n^{1/(\delta_w-1)}}\right)$$
$$\le\; p^{(n)}(x,y) \le c_3\, n^{-\delta_s/2} \exp\left(-c_4 \frac{|x-y|^{\delta_w/(\delta_w-1)}}{n^{1/(\delta_w-1)}}\right)$$

for all $n \ge n_0$, $x,y \in \mathbf{S}_2$*; in the lower bound on needs* $n \ge c_0|x-y|$.

For analogous results on other types of "fractal" graphs, see e.g. BARLOW AND BASS [**3**] and HAMBLY AND KUMAGAI [**32**]. On the Sierpiński graphs, GRABNER AND WOESS [**26**] have obtained a more precise *local* result by use of generating functions, adapting a method of ODLYZKO [**46**], and using the *Singularity Analysis* of FLAJOLET AND ODLYZKO [**20**]. By "local", we mean that the result is not uniform in space and time as Theorem 8.1, but on the other hand it is more precise in the sense that it regards asymptotic equivalence instead of asymptotic type, and it exhibits an interesting phenomenon of periodic oscillations.

A. A Functional equation for the Green function. The "blown up" graph $2\mathbf{S}_d$ has its vertex set contained in $\mathbf{S}_d$. The latter subdivides its double in the sense that neighbours in $2\mathbf{S}_d$ have distance 2 in $\mathbf{S}_d$.

Consider SRW (Z_n) on $\mathbf{S}_d$, suppose $Z_0 = 2x \in 2\mathbf{S}_d \subset \mathbf{S}_d$. We factor the random walk with respect to its successive visits in $2\mathbf{S}_d$. That is, we consider the stopping times

$$\tau_0 = 0\,, \quad \tau_j = \min\{n > \tau_{j-1} : Z_n \in 2\mathbf{S}_d\,,\ Z_n \ne Z_{\tau_{j-1}}\}\,.$$

The neighbourhood of any $2x$ in $\mathbf{S}_d$ is always the same, see Figure 7.

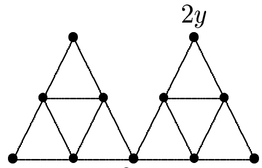

FIGURE 7

Therefore one gets the following.

LEMMA 8.2. *The increments $\tau_j - \tau_{j-1}$ are i.i.d. with probability generating function*

$$\phi(z) = \mathbb{E}_{2x}(z^{\tau_1}) = \frac{z^2}{1 + (z-1)\big((d-2)z - (2d-1)\big)}\,.$$

For $y \sim x$ in $\mathbf{S}_d$,

$$\mathbb{E}_{2x}\left(z^{\tau_1}\, \mathbf{1}_{2y}(Z_{\tau_1})\right) = \frac{1}{2d}\phi(z)\,.$$

Furthermore,

$$\psi(z) = \sum_{n=1}^{\infty} \Pr_{2x}[Z_n = 2x\,,\ \tau_1 > n]\, z^n = \frac{d-(d-2)z}{d+z}\,\phi(z)\,.$$

The lemma implies

PROPOSITION 8.3. *For all $x, y \in \mathbf{S}_d$ and $|z| < 1$,*

$$G(2x, 2y|z) = \big(1 + \psi(z)\big)\, G\big(x, y|\phi(z)\big)\,.$$

In particular, setting $G(z) = G(0,0|z)$,

$$G(z) = \big(1 + \psi(z)\big)\, G\big(\phi(z)\big)\,. \tag{8.1}$$

The radius of convergence $\mathfrak{r} = 1$ is the only singularity on the circle of convergence.

This is known as the "Decimation procedure" in the Theoretical Physics literature, see e.g. RAMMAL [**52**].

B. Singular expansion of $G(z)$ at $z = 1$. We start with a *wrong* heuristic argument of Theoretical Physicists: *"Near $z = 1$, there will be a decomposition*

$$G(z) = (1-z)^{\eta} H_+(z)$$

with $H_+(z)$ analytic. Substitute this into functional equation (8.1) and compare exponents. Thus find

$$\eta = \frac{\log(d+1)}{\log(d+3)} - 1\,.$$

Classical methods (Darboux, Tauber) yield $p^{(n)}(0,0) \sim C\, n^{-\log(d+1)/\log(d+3)}$."

This argument and the above asymptotic equivalence are wrong, because $H_+(z)$ is *not* analytic near 1. But this *does* yield the correct leading exponent η. On just has to replace "$H_+(z)$ analytic" with "$H_+(z)$ bounded and non-zero near $z = 1$".[4]

The strategy is now the following: we "remove" the leading term $(1-z)^\eta$ by considering $H_+(z) = G(z)/(1-z)^\eta$. Substituting this in the functional equation (8.1), we find a new equation for $H^+(z)$,

$$
\begin{aligned}
H_+(z) &= \big(1+f(z)\big)\,H_+\big(\phi(z)\big)\,, \quad \text{where} \\
1+f(z) &= \big(1+\psi(z)\big)\left(\frac{1-\phi(z)}{1-z}\right)^\eta .
\end{aligned}
\tag{8.2}
$$

(Note $f(0) = 0$.)

We then plan to replace $H^+(z)$ with another function $H(z)$ satisfying the nicer functional equation $H(z) = H\big(\phi(z)\big)$, i.e., the leading factor $\big(1+f(z)\big)$ is absent. It will then be rather clear how to obtain a singular expansion of $H(z)$ near $z = 1$, and then we have to trace the latter back to singular expansions of $H^+(z)$ and $G(z)$. We first need to study the dynamics of the rational function ϕ.

Lemma 8.4. *$\phi^{(n)}(z) \to 0$ for all $z \in \mathbb{C}$ except for $1/z$ in a Cantor subset J^{-1} of the interval $[\frac{d-3}{2d}\,, 1] \subset \mathbb{R}$, whose endpoints belong to J^{-1}.*

Convergence is uniform for z in closed subsets of $\mathbb{C} \setminus J$.

In particular, in $\mathbb{C} \setminus J$ the infinite product

$$H_+(z) = \prod_{n=0}^{\infty} \Big(1 + f\big(\phi^{(n)}(z)\big)\Big)\,,$$

obtained by iterating (8.2) converges to an analytic function.

J^{-1} is the Julia set of $1/\phi(1/z) = 1 + 2d(z-1)(z-\frac{d-3}{2d})$. For ϕ, we have the attractive fixed point $z = 0$ and the repulsive fixed point $z = 1$.

Since $\phi'(1) = d+3$, there is neighbourhood $\mathfrak{U} \subset \mathbb{C}$ of $z = 1$ where the inverse function $\phi^{(-1)}$ has $z = 1$ as its unique attractive fixed point. Therefore

$$H_-(z) = \prod_{n=1}^{\infty} \Big(1 + f\big(\phi^{(-n)}(z)\big)\Big)$$

converges and is analytic in $\mathfrak{U}$. On $\mathfrak{U}$, consider the "backward completion" of $H_+(z)$

$$H(z) = H_-(z)\,H_+(z) = \prod_{n=-\infty}^{\infty} \Big(1 + f\big(\phi^{(n)}(z)\big)\Big)$$

It is analytic in $U \setminus J$, and – as we wanted –

$$H(z) = H\big(\phi^{(-1)}(z)\big)\,.$$

Now one can modify $H(z)$ to obtain a periodic function: see Beardon [**4**], Thm. 6.3.2. Namely, one can extract the linear part of the expansion of $\phi^{(-1)}$ at $z = 1$, by conjugating with a function $g(z)$ that is analytic in $\mathfrak{U}$, and real-valued if $z \in \mathfrak{U}$ is real:

$$g^{(-1)} \circ \phi^{(-1)} \circ g(z) = 1 + \tfrac{1}{d+3}(z-1)\,, \quad \text{and} \quad g(1) = g'(1) = 1\,.$$

[4]A Theoretical Phycisist has expressed the interesting viewpoint that such arguments *are* correct proofs until somebody proves a"better" version, and anyway, Mathematicians keep redoing things that Phycisits had done 10 years earlier. I believe that at least in the development of Random Walk theory, Mathematicians have their nose ahead.

Then

$$K(z) = H\big(g(z)\big)$$

is analytic in $\mathfrak{U} \setminus J$ and

$$K(z) = K\big(1 + \tfrac{1}{d+3}(z-1)\big)\,.$$

Thus, $T_0(w) = K\big(1-(d+3)^w\big)$ is 1-periodic, analytic in the strip $\{|\Im(w)| < \pi/\log(d+3)\} \subset \mathbb{C}$, and thus has a rapidly converging Fourier expansion

$$T_0(w) = \sum_{k=-\infty}^{\infty} a_k \, \exp(2k\pi\, i\, w) \qquad \text{with}$$
$$a_k = \mathcal{O}\Big(\exp\Big(-\big(\tfrac{2\pi^2}{\log(d+3)} - \varepsilon\big)|k|\Big)\Big)$$

for all $\varepsilon > 0$. Now we go back. We want to control the approximation error when replacing $H_+(z)$ by $H(z)$ and then by $K(z)$.

Lemma 8.5. *Near $z=1$ and for $|\arg(z-1)| \ge \alpha$,*

$$H_+(z) - K(z) = \mathcal{O}_{\alpha,\varepsilon}(|z-1|^{1-\varepsilon}) \quad \textit{as } z \to 1\,.$$

Theorem 8.6. *Near $z=1$ and for $|\arg(z-1)| \ge \alpha$,*

$$G(z) = (1-z)^{\eta}\,\left(T_0\Big(\frac{\log(1-z)}{\log(d+3)}\Big) + \mathcal{O}(|z-1|^{1-\varepsilon})\right)$$
$$= \sum_{k=-\infty}^{\infty} a_k\,(1-z)^{\eta + 2k\pi \mathrm{i}/\log(d+3)} + \mathcal{O}(|z-1|^{\eta+1-\varepsilon})\,.$$

for all $\alpha < \frac{\pi}{2}$ and $\varepsilon > 0$, where $\eta = \frac{\log(d+1)}{\log(d+3)} - 1$ and $T_0(w)$ is a non-constant periodic function with period 1 which is analytic in the strip $\{|\Im(w)| < \frac{\pi}{\log(d+3)}\} \subset \mathbb{C}$.

Remark 8.7. The same type of expansion holds for a larger class of "fractal graphs", see Krön and Teufl [**39**], who also show that non-constantness of T_0 follows from the fact that the Julia set of the respective function ϕ is a Cantor set.

C. Singularity analysis. Let $D_{\alpha,\delta} = \{z \in \mathbb{C} : |z| \le 1+\delta\,,\ |\arg(z-1)| \ge \alpha\}$, where $0 < \alpha < \frac{\pi}{2}$ and $\delta > 0$. See Figure 8.

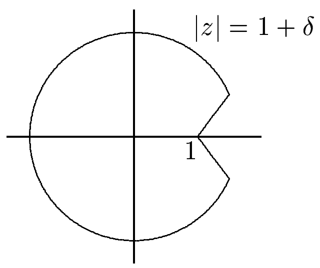

Figure 8

Singularity Analysis is subsumed in the following theorem of Flajolet and Odlyzko [**20**].

THEOREM 8.8. *Let $F(z) = \sum_n f_n z^n$ be a power series with real coefficients and a singularity at $z = 1$. Assume that it extends analytically to a suitable set $D_{\alpha,\delta} \setminus \{1\}$. If*

$$F(z) = C\,(1-z)^{\zeta} + \mathcal{O}(|1-z|^{\beta})$$

as $z \to 1$ in $D_{\alpha,\delta}$, where $\zeta \in \mathbb{C} \setminus \mathbb{N}_0$ and $\beta \in \mathbb{R}$, $\beta > \Re(\zeta)$, then

$$f_n = C\binom{n-\zeta-1}{n} + \mathcal{O}(n^{-\beta-1})\,.$$

We use the particular case $C = 0$. The series

$$G(z) = \sum_{k=-\infty}^{\infty} a_k\,(1-z)^{\eta+2k\pi i/\log(d+3)} + \mathcal{O}(|z-1|^{\eta+1-\varepsilon})$$

converges uniformly. Hence Singularity Analysis yields

$$p^{(n)}(0,0) = \sum_{k=-\infty}^{\infty} a_k \binom{n - \frac{\log(d+1)+2k\pi i}{\log(d+3)}}{n} + \mathcal{O}\big(n^{-(\eta+2-\varepsilon)}\big)\,.$$

Conclusion:

THEOREM 8.9. *For SRW on $\mathbf{S}_d$,*

$$p^{(n)}(0,0) \sim n^{-\log(d+1)/\log(d+3)}\,T\Big(\frac{\log n}{\log(d+3)}\Big)\,,$$

where $T(w)$ is a non-constant periodic C^{∞}-function with period 1. Its Fourier series is

$$T(w) = \sum_{k=-\infty}^{\infty} \frac{a_k}{\Gamma\big(1 - \frac{\log(d+1)+2k\pi i}{\log(d+3)}\big)}\,\exp(-2k\,\pi\,i\,w)\,,$$

where the a_k are the Fourier coefficients of $T_0(w)$.

The noteworthy fact here is the presence of the log-periodic oscillations in the asymptotic behaviour of the n-step return probabilities. Again, this type of result holds for a larger class of self-similar graphs, see KRÖN AND TEUFL [**39**]. Furthermore, a careful analysis of the general type of functional equations as the one of (8.1) has been carried out by TEUFL [**59**].

9. Final remarks

Singularity Analysis versus the Method of Darboux. The latter is now often said to be out of date in view of the first. In deriving Theorem 8.9, the use of Singularity Analysis is crucial. If we look once more at Section 5.D, in order to derive the asymptotics of SRW on $\mathbb{Z}^d * \mathbb{Z}^d$ given in Theorem 5.8, we might have used Singularity Analysis instead of Darboux' method. This would mean to derive the singular expansion of Proposition 5.7 only up to the first singular term. However, then we would need that singular expansion in a domain of the form $D_{\alpha,\delta}(\mathfrak{r}) = \{z \in \mathbb{C} : |z| \le \mathfrak{r} + \delta\,,\ |\arg(z-\mathfrak{r})| \ge \alpha\}$. This would require a computational effort similar to the one needed for Proposition 5.7, see [**66**], pp. 193–195, where we *did* need to go beyond the first singular term sufficiently far to allow use of the Riemann-Lebesgue lemma, but *did not* need to go beyond the disk of convergence $\{|z| \le \mathfrak{r}\}$. In this sense, there seems to be a balance between the two methods in that case.

Tauberian theorems. In these notes, we did not discuss another classical method, namely, the use of *Tauberian theorems.* This method applies typically in simpler circumstances than those where one uses the method of Darboux or Singularity Analysis. For example, it may serve to deduce the asymptotics of $p^{(n)}(x,x)$ from those of $\Pr_x[\mathbf{t}^x = n]$ in favourable cases (Renewal Theory). For a general outline in the context of Combinatorial Analysis, see Odlyzko [**47**], §8.2.

Conclusion. The methods described here are quite specific and require explicit computability, or at least computability of functional equations or systems of equations, for Green functions and related generating functions. When applicable, the methods yield very sharp results. The classes of structures where these methods apply are of course more limited than those where asymptotic type has been considered. Still, there are some larger classes of graphs/groups and random walks where the methods yield very satisfactory results, most of which so far could not be obtained by different methods:

- all finite range random walks on (virtually) free groups;
- random walks on free products that are adapted to the free product structure ("free sums" or "nearest neighbour" walks);
- random walks on trees with finitely many cone types, or equivalently, regular languages.

All these results rely in some way on a "tree-likeness" of the graph, although the general structure, in particular for free products, may be far more complicated that that of a tree.

- Random walks on a reasonably large class of self-similar graphs.

Of the latter type, here we have only seen the Sierpiński graphs. An interesting phenomenon occurring there are the periodic oscillations. In many cases where one only knows only asymptotic type, one does not know if oscillations are present.

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