

THE AVERAGE SIZE OF NONSINGULAR SETS IN A GRAPH

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ABSTRACT. An nonsingular set in a finite graph is defined as the vertex set of an induced subgraph which has no isolated point. If G is a graph without isolated points and with at least two vertices and B is a connected subgraph, then the average size of those nonsingular sets in G which contain B is at least $(|G|+|B|)/2$. This result is used to prove the following: if \mathcal{F} is a family of sets which is closed with respect to union, and if none of the generating sets in \mathcal{F} has more than two elements, then the average size of a set in \mathcal{F} is at least half of the size of the largest set in \mathcal{F} .

1980 Math. Subject Classification: 05C99, 04A20

Key words: Union closed family of sets, subset in a graph, isolated vertex.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let G be a finite graph (unoriented, without loops or multiple edges). Subsets A of $V(G)$ are identified with induced subgraphs; $|A|$ denotes the number of vertices. We call a subset A of G *nonsingular*, if it has no isolated vertex. If B is a subgraph of G , then we define

$$\mathcal{F}(G, B) = \{A \subseteq G \mid B \subseteq A \text{ and } A \text{ is nonsingular}\}.$$

In this note, we are interested in the average size of an element in $\mathcal{F}(G, B)$. To this end, we introduce a generating function:

$$f(t|G, B) = \sum_{A \in \mathcal{F}(G, B)} t^{|A|}.$$

Then the average size in question is given by $f'(1|G, B)/f(1|G, B)$, where $f'(t|G, B)$ denotes the derivative with respect to t .

Theorem 1. *If G is a nonsingular graph (with at least two vertices) and B is a connected subgraph (possibly void), then*

$$\frac{f'(1|G, B)}{f(1|G, B)} \geq \frac{|G| + |B|}{2}.$$

Observe that $\mathcal{F}(G, B)$ is a family of sets which is closed with respect to set union. Thus, our result can be related with the study of union closed families.

Let \mathcal{F} denote an arbitrary finite family of finite sets which is closed with respect to set union. P. Frankl [1], [2] has conjectured the following.

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Conjecture. *There is an element which occurs in at least half of the sets of \mathcal{F} .*

In other words, if V is the largest set in \mathcal{F} and, for x in V , $\mathcal{F}_x = \{A \in \mathcal{F} \mid x \in A\}$, then one is looking for an x such that $|\mathcal{F}_x| \geq |\mathcal{F}|/2$. In discussions of the Conjecture at the 1985 Banff meeting, R. Graham and several others observed the following.

Lemma 1. *If \mathcal{F} contains a nonvoid set A with no more than two elements, then $|\mathcal{F}_x| \geq |\mathcal{F}|/2$ for some $x \in A$.*

As no proof of this observation seems to have appeared in print, we include a proof in this paper. At the same meeting in Banff, R. Graham claimed to have an example of a union closed family \mathcal{F} with the following properties: there is an $A \in \mathcal{F}$ with three elements, such that $|\mathcal{F}_x| < |\mathcal{F}|/2$ for every $x \in A$, but the Conjecture is true for this \mathcal{F} .

Consider the following statement.

$$\textit{The average size of a set in } \mathcal{F} \textit{ is at least half of the size of the largest set in } \mathcal{F}. \quad (1)$$

It is easy to see that statement (1) implies the validity of the Conjecture for \mathcal{F} : assume that

$$\sum_{A \in \mathcal{F}} |A|/|\mathcal{F}| \geq |V|/2,$$

Suppose that $|\mathcal{F}|/2 > |\mathcal{F}_x|$ for every x in V . Then

$$|V| \cdot |\mathcal{F}|/2 > \sum_{x \in V} |\mathcal{F}_x| = \sum_{A \in \mathcal{F}} |A|,$$

a contradiction.

Thus, in particular, the Conjecture is true for $\mathcal{F}(G, B)$ as in Theorem 1. (However, we remark that Lemma 1 is needed in the proof of Theorem 1.) A family of *generators* \mathcal{E} of \mathcal{F} is a subfamily of \mathcal{F} with the following properties:

- (i) every set in \mathcal{F} is a finite union of sets in \mathcal{E} ,
- (ii) \mathcal{E} is minimal with respect to (i).

It is easy to see that \mathcal{E} is unique and no set in \mathcal{E} is empty. Theorem 1 can be applied to the case when no generator has more than two points.

Theorem 2. *If $|A| \leq 2$ for every generator A of \mathcal{F} , then statment (1) is true.*

In general it is not true that the average size of a set in a finite union closed family \mathcal{F} is at least half of the size of the largest set in \mathcal{F} . Actually, the proportion between the average size and the largest size of a set in \mathcal{F} may be arbitrarily small. Here is an example: let F be a set with $|F| = 2^n$ and let $E \subset F$ with $|E| = 2n$, and define

$$\mathcal{F} = \{A \subset F \mid |A| \geq 2^n - 1 \text{ or } A \subseteq E\}.$$

In this case, the average size of a set in \mathcal{F} is asymptotically equal to $n + 1$.

In a preliminary version of this note, we raised the question whether statement (1) was true for every union closed family \mathcal{F} where all generators have the same size. This has been disproved recently by P. Wójcik [3].

2. PROOFS

We start with Lemma 1, as discussed at Banff.

Proof of Lemma 1. If $A = \{x\}$ for some $x \in V$, where V is the largest set in \mathcal{F} , then $B \mapsto B \cup \{x\}$ is injective from $\mathcal{F} \setminus \mathcal{F}_x$ into \mathcal{F}_x , so $|\mathcal{F}_x| \geq |\mathcal{F}|/2$.

Now assume that $A = \{x, y\}$ with $|\mathcal{F}_x| \geq |\mathcal{F}_y|$. Let $a = |\{B \in \mathcal{F} \mid x \in B, y \notin B\}|$, $b = |\{B \in \mathcal{F} \mid y \in B, x \notin B\}|$, $c = |\{B \in \mathcal{F} \mid A \subseteq B\}|$ and $d = |\{B \in \mathcal{F} \mid A \cap B = \emptyset\}|$. Then $|\mathcal{F}| = a + b + c + d$, $|\mathcal{F}_x| = a + c$ and $|\mathcal{F}_y| = b + c$. Hence $a \geq b$, and (as in the first case) $c \geq d$, so again $|\mathcal{F}_x| \geq |\mathcal{F}|/2$. ■

Now we turn to the study of $\mathcal{F}(G, B)$, where G is a graph. Note that

$$f(t|G, \emptyset) = f(t|G, \{x\}) + f(t|G \setminus \{x\}, \emptyset). \quad (2)$$

In particular, if G is a graph without isolated vertices, then Lemma 1 implies that for some x in G ,

$$f(1|G, \{x\}) \geq f(1|G \setminus \{x\}, \emptyset). \quad (3)$$

Proof of Theorem 1. We first assume that G is connected.

We use induction on $n = |G|$ and, for each n , induction on $k = |G| - |B|$.

For $n = 2$ the verification is trivial.

Assume that $|G| = n$ and that the statement of the theorem is true for every graph $|G'|$ with $|G'| = m$, $2 \leq m < n$. First, let $k = 0$. Then $B = G$, and $f(t|G, B) = t^{|G|}$. We have

$$f'(1|G, B) = |G| = \frac{|G| + |B|}{2} f(1|G, B),$$

so the result follows. Now let $k = |G| - |B|$ be positive and assume also that the statement is true for all B' such that $0 \leq |G| - |B'| < k$.

Case 1. B is nonvoid. As G and B are connected and $G \setminus B \neq \emptyset$, there is an x in $G \setminus B$ such that

$$B' = B \cup \{x\}$$

is connected.

Case 2. $B = \emptyset$. By (3), there is a vertex x such that $f(1|G, \{x\}) \geq f(1|G \setminus \{x\}, \emptyset)$. Set

$$B' = \{x\}.$$

In both cases,

$$f(t|G, B) = f(t|G \setminus \{x\}, B) + f(t|G, B'). \quad (4)$$

We intend to differentiate and to obtain inequalities for the right-hand side.

The graph $G \setminus \{x\}$ may be disconnected, and it may have isolated points. Let $C = C(x, B)$ be the set of those isolated points of $G \setminus \{x\}$ which are not in B . Then, for every y in C , the only neighbour of y in G is x . In particular,

$$G' = G \setminus C$$

is connected. No point of C can be an element of a nonsingular set of $G \setminus \{x\}$. Hence

$$f(t|G \setminus \{x\}, B) = f(t|G' \setminus \{x\}, B). \quad (5)$$

Now write $G' \setminus \{x\}$ as the union of its connected components: $G' \setminus \{x\} = G_1 \cup G_2 \cup \dots \cup G_s$, listed so that $B \subseteq G_1$ and $|G_i| \geq 2$ for $i = 2, \dots, s$. A set A is in $\mathcal{F}(G' \setminus \{x\}, B)$ if and only if $A = A_1 \cup A_2 \cup \dots \cup A_s$, where $A_1 \in \mathcal{F}(G_1, B)$ and $A_i \in \mathcal{F}(G_i, \emptyset)$ for $i = 2, \dots, s$. This and (5) yield

$$f(t|G \setminus \{x\}, B) = f(t|G_1, B) \cdot f(t|G_2, \emptyset) \cdot \dots \cdot f(t|G_s, \emptyset).$$

Note that we may have $|G_1| = 1$ only if $|B| = 1$. In this case, $f(t|G_1, B) = f'(t|G_1, B) = 0$. Otherwise, we may apply the induction hypothesis (for n) to (G_1, B) . In both cases

$$f'(t|G_1, B) \geq \frac{|G_1| + |B|}{2} f(t|G_1, B).$$

The induction hypothesis also applies to (G_i, \emptyset) , giving

$$f'(t|G_i, \emptyset) \geq \frac{|G_i|}{2} f(t|G_i, \emptyset), \quad i = 2, \dots, s.$$

This, (5) and the product rule for differentiation give

$$f'(1|G \setminus \{x\}, B) \geq \frac{|G| - |C| - 1 + |B|}{2} f(1|G \setminus \{x\}, B). \quad (6)$$

On the other hand, B' is a connected subgraph of G' . For any $U \subseteq C$, considering those nonsingular sets in $\mathcal{F}(G, B')$ which contain U and do not contain $C \setminus U$, we get

$$f(t|G, B') = \sum_{U \subseteq C} f(t|G' \cup U, B' \cup U).$$

Differentiating and applying the induction hypothesis (for k , if $U = C$, and for n otherwise) to each of the summands, we obtain

$$f'(1|G, B') \geq \sum_{U \subseteq C} \frac{|G| - |C| + |B| + 1 + 2|U|}{2} f(1|G' \cup U, B' \cup U). \quad (7)$$

Combining inequalities (6) and (7), we obtain from (4)

$$f'(1|G, B) \geq \frac{|G| + |B|}{2} f(1|G, B) + \frac{\Phi}{2}, \quad (8)$$

where

$$\Phi = \sum_{U \subseteq C} (|U| - |C \setminus U| + 1) f(1|G' \cup U, B' \cup U) - (|C| + 1) f(1|G \setminus \{x\}, B).$$

The proof will be complete if we can show that $\Phi \geq 0$. We observe that if $U \subseteq C$ is nonvoid, then $A \in \mathcal{F}(G' \cup U, B' \cup U)$ if and only if $A \cup (C \setminus U) \in \mathcal{F}(G, B' \cup C)$. Hence, $f(1|G' \cup U, B' \cup U) = f(1|G, B' \cup C)$ whenever $\emptyset \neq U \subseteq C$. (If B is nonvoid, this also holds if $U = \emptyset$.) Consequently,

$$\begin{aligned} \Phi &= \left(\sum_{U \subseteq C} |U| - |C \setminus U| + 1 \right) f(1|G, B' \cup C) \\ &\quad + (|C| - 1) \left(f(1|G, B' \cup C) - f(1|G', B') \right) - (|C| + 1) f(1|G \setminus \{x\}, B) \\ &= 2^{|C|} f(1|G, B' \cup C) - (|C| + 1) f(1|G \setminus \{x\}, B) \\ &\quad + (|C| - 1) \left(f(1|G, B' \cup C) - f(1|G', B') \right). \end{aligned}$$

If $A \in \mathcal{F}(G', B')$ then $x \in A$, and $A \cup C$ is nonsingular: $A \in \mathcal{F}(G, B' \cup C)$. Thus, $f(1|G, B' \cup C) \geq f(1|G', B')$, and

$$\Phi \geq (|C| + 1) \left(f(1|G, B' \cup C) - f(1|G \setminus \{x\}, B) \right),$$

since $2^{|C|} \geq |C| + 1$. There are two cases to consider.

Case 1. B is nonvoid. If $A \in \mathcal{F}(G \setminus \{x\}, B) = \mathcal{F}(G' \setminus \{x\}, B)$, then $A \cup \{x\}$ is nonsingular, as x has a neighbour in $B \subseteq A$. In other words, $A \cup \{x\} \cup C$ is nonsingular (even if $C = \emptyset$), and is an element of $\mathcal{F}(G, B' \cup C)$. Thus, $f(1|G \setminus \{x\}, B) \leq f(1|G, B' \cup C)$, so $\Phi \geq 0$.

Case 2. $B = \emptyset$. We have to consider two subcases.

a) C is nonvoid. In this case, $\{x\} \cup C$ is nonsingular, and if $A \in \mathcal{F}(G \setminus \{x\}, \emptyset) = \mathcal{F}(G' \setminus \{x\}, \emptyset)$ then $A \cup \{x\} \cup C = A \cup B' \cup C \in \mathcal{F}(G, B' \cup C)$. Again, $f(1|G \setminus \{x\}, B) \leq f(1|G, B' \cup C)$, and $\Phi \geq 0$.

b) $C = \emptyset$. Then $f(1|G, B' \cup C) - f(1|G \setminus \{x\}, B) = f(1|G, \{x\}) - f(1|G \setminus \{x\}, \emptyset)$, which is nonnegative by the choice of x , and $\Phi \geq 0$.

In all three cases, $\Phi \geq 0$, and by (8), proof is completed (for connected G).

Now suppose in general that G is nonsingular. Then we can use the formula

$$f(t|G, B) = f(t|G_1, B) \cdot f(t|G_2, \emptyset) \cdot \dots \cdot f(t|G_s, \emptyset),$$

where G_1, \dots, G_s are the connected components of G and $B \subseteq G_1$. Differentiation and application of the result for connected graphs yield the theorem. ■

Now there may be several ways to derive Theorem 2. Our proof uses the full strength of Theorem 1.

Proof of Theorem 2. We may assume that $\emptyset \in \mathcal{F}$ (adding the empty set to \mathcal{F} makes the average size smaller). If all generators of \mathcal{F} are singletons, then Theorem 1 is trivially true. Hence, we assume that \mathcal{E} contains two-element sets and maybe also singletons. Observe that there may be points x, y such that $\{x\}$ and $\{x, y\}$ are generators. In this case, $\{y\}$ cannot be a generator.

Let V be the largest set in \mathcal{F} , and set $B = \{x \mid \{x\} \in \mathcal{E}\}$. Choose a new set $\bar{B} = \{\bar{x} \mid x \in B\}$ in one-to-one correspondence with B , disjoint from V . We define a graph G with vertex set $V \cup \bar{B}$; the edges are (i) all $\{x, y\} \in \mathcal{E}$, (ii) all $\{x, \bar{x}\}$, where $x \in B$, and (iii) all $\{\bar{x}, \bar{y}\}$, where $\bar{x}, \bar{y} \in \bar{B}$, $\bar{x} \neq \bar{y}$.

Thus, \bar{B} induces a complete subgraph of G which is connected to the rest of G by the edges $\{x, \bar{x}\}$, $x \in B$. (If B is empty, then so is \bar{B} .) Now, G has no isolated point, and we may apply Theorem 2 and Remark A to (G, \bar{B}) :

$$f'(1|G, \bar{B}) \geq \frac{|G| + |\bar{B}|}{2} f(1|G, \bar{B}).$$

By our construction, we obviously have $A \in \mathcal{F}$ if and only if $A \cup \bar{B} \in \mathcal{F}(G, \bar{B})$. Hence, $f(1|G, \bar{B}) = |\mathcal{F}|$, and

$$f'(1|G, \bar{B}) = \sum_{A \in \mathcal{F}} |A \cup \bar{B}| = |B| \cdot |\mathcal{F}| + \sum_{A \in \mathcal{F}} |A|.$$

We obtain

$$\sum_{A \in \mathcal{F}} |A| \geq \frac{|G| + |B|}{2} |\mathcal{F}| - |B| \cdot |\mathcal{F}| = \frac{|V|}{2} |\mathcal{F}|. \quad \blacksquare$$

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