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Isotropic random walks in a building of type $\tilde{A}_d$

Donald I. Cartwright[1], **Wolfgang Woess**[2]

[1] School of Mathematics and Statistics, University of Sydney, N.S.W. 2006, Australia (e-mail: donaldc@maths.usyd.edu.au)

[2] Institut für Mathematik, Technische Universität Graz, Steyrergasse 30, A-8010 Graz, Austria (e-mail: woess@weyl.math.tu-graz.ac.at)

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Abstract Let $d \geq 1$ be an integer, and let $\mathfrak{X}$ be a locally finite thick building of type $\tilde{A}_d$. When $d = 1$, $\mathfrak{X}$ is a tree, which we assume is homogeneous, and a transition probability matrix p, or the corresponding random walk, on the vertex set of $\mathfrak{X}$ is called *isotropic* if for any vertices x, y, $p(x, y)$ depends only on the graph distance from x to y. In this paper we extend the definition of isotropic to general d in a natural way, and study isotropic random walks (Z_n) on the vertex set of $\mathfrak{X}$. In particular, we prove a rate of escape theorem $d(Z_n, o)/n \to \gamma$, with an explicit formula for γ, we prove a central limit theorem and a local limit theorem. These generalize results of Sawyer [21] in the case $d = 1$. We do not need to assume that $\mathfrak{X}$ is obtained from a local field F by the standard construction, but in that case our results may be translated into theorems about bi-K-invariant probability measures on groups such as $PGL(d + 1, F)$ and $SL(d + 1, F)$ which act on $\mathfrak{X}$.

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1 Introduction

In [21], Sawyer studied random walks $(Z_n)_{n\in\mathbb{N}}$ on the vertex set of a homogeneous tree $\mathfrak{X}$ which are *isotropic*, meaning that the transition probabilities $p(x, y) = \mathbb{P}(Z_{n+1} = y \mid Z_n = x)$ depend only on the graph distance $\mathrm{dist}(x, y)$ between x and y. For $k \in \mathbb{N} = \{0, 1, \ldots\}$, write $S_k(x)$ for the set of vertices at distance k from x, and N_k for the common value of $|S_k(x)|$. Then the transition probability matrix $P = (p(x, y))$ of the random walk must have the form

$$P = \sum_k a_k P_k, \tag{1.1}$$

where $P_k = (p_k(x, y))$ for

$$p_k(x,y) = \begin{cases} 1/N_k & \text{if } y \in S_k(x), \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

and where $a_k \geq 0$ for each k, and $\sum_k a_k = 1$.

Now let $\mathfrak{X}$ be a locally finite thick building of type $\tilde{A}_d$, with vertex set $\mathfrak{X}^0$. Here $d \geq 1$; if $d = 1$, then $\mathfrak{X}$ is a tree, which we assume to be homogeneous. As explained below, for each vertex x in $\mathfrak{X}$, $\mathfrak{X}^0$ is in a natural way the disjoint union of sets $S_k(x)$, $k \in \mathbb{N}^d$. If $k = (k_1, \dots, k_d) \in \mathbb{N}^d$ and $y \in S_k(x)$, the graph-theoretic distance between x and y is $|k| = k_1 + \cdots + k_d$. The number $|S_k(x)|$ depends only on k, and we again denote it N_k.

We consider random walks $(Z_n)_{n\in\mathbb{N}}$ on $\mathfrak{X}^0$. It is natural to define a transition probability matrix $P = (p(x,y))_{x,y\in\mathfrak{X}^0}$, or the corresponding random walk, to be *isotropic* if P has the form (1.1), where now the index k runs over $\mathbb{N}^d$. By the second of the properties of the sets $S_k(x)$ listed at the end of the introduction, isotropic random walks need not be symmetric when $d \geq 2$. That is, $p(x,y)$ need not equal $p(y,x)$. The simple nearest neighbour random walk on $\mathfrak{X}^0$ is isotropic. For each vertex x has a *type* $\tau(x) \in \{0, 1, \dots, d\}$, and if y is a neighbour of x, then $\tau(y) \equiv \tau(x) + r \pmod{d+1}$ for some $r \in \{1, \dots, d\}$. If e_r denotes as usual the d-tuple having r-th coordinate 1 and all others 0, then it turns out that $S_{e_r}(x)$ is just the set of neighbours y of x such that $\tau(y) \equiv \tau(x) + r \pmod{d+1}$. Hence the transition matrix of the simple nearest neighbour random walk is

$$P = \Big(\sum_{r=1}^{d} N_{e_r} P_{e_r}\Big) \Big/ \Big(\sum_{r=1}^{d} N_{e_r}\Big). \tag{1.3}$$

This random walk has already been considered in [20].

In Section 2 we summarize the methods we need to prove our main results about isotropic random walks (Z_n) on $\mathfrak{X}^0$. To avoid trivialities, we always assume that $a_k > 0$ for at least one $k \neq 0$. In Section 3 we prove a rate of escape theorem, i.e., we prove (assuming that $\sum_k |k| a_k < \infty$), that there is a number $\gamma \geq 0$, so that if o is any vertex, then with probability 1, $\text{dist}(Z_n, o)/n \to \gamma$ as $n \to \infty$. We also show that there are numbers $\gamma^{(j)} > 0$ so that with probability 1, $k_j(Z_n)/n \to \gamma^{(j)}$ as $n \to \infty$. Here $(k_1(Z_n), \dots, k_d(Z_n)) = k$ means that $Z_n \in S_k(o)$. We deduce from this the almost sure convergence of (Z_n) to a random point of the boundary of $\mathfrak{X}$. In Section 4, we give an explicit formula for γ, which shows in particular that $\gamma > 0$. Rate of escape theorems are routine applications of Kingman's subadditive ergodic theorem, at least in the context of random walks on groups (see [11], for example), so the main interest here is that γ can be calculated explicitly. In Section 5 we prove a local limit theorem, describing the asymptotic behaviour as $n \to \infty$ of the n-step transition probabilities $p^{(n)}(x,y)$. This involves the study of "spherical functions" φ_z (see Section 2) in which the parameter z is a $d+1$-tuple of complex numbers of modulus 1. We also give necessary and sufficient conditions in terms of the a_k for the irreducibility and aperiodicity of (Z_n). In Section 6, we give necessary and sufficient conditions on z for a spherical function φ_z to be bounded. This is used in Section 7, where we prove a central limit theorem, showing that (provided $\sum_k |k|^2 a_k < \infty$) there is a positive definite matrix Σ such that $(k_1(Z_n) - \gamma^{(1)}n, \dots, k_d(Z_n) - \gamma^{(d)}n)/\sqrt{n}$ tends in distribution to the normal distribution $N(0, \Sigma)$. This generalizes the result for $d = 2$ in [15], where Σ was only shown to be positive semidefinite. Consequently, $(\mathrm{dist}(Z_n, o) - n\gamma)/\sqrt{n}$ is asymptotically normal with strictly positive variance.

Finally, in Section 8 we apply these results to studying convolution powers of bi-K-invariant probability densities on G, where G is a group acting on $\mathfrak{X}$. More precisely, let F be a (not necessarily commutative) local field. Then for each $d \geq 1$ there is a building $\mathfrak{X} = \mathfrak{X}_{d,F}$ of type $\tilde{A}_d$ associated to F (see [18, §9.2]). Indeed, if $d \geq 3$, it is known that every $\mathfrak{X}$ equals $\mathfrak{X}_{d,F}$ for some F [18, Theorem 10.22]. Our proofs are valid for all $d \geq 1$, with no assumption that $\mathfrak{X} = \mathfrak{X}_{d,F}$. Applying our results to the case $\mathfrak{X} = \mathfrak{X}_{d,F}$, we obtain theorems about random walks associated with bi-K-invariant probability densities on certain groups G, including $SL(d+1, F)$ and $PGL(d+1, F)$, which act on $\mathfrak{X}$. Here K is a certain compact open subgroup of G; for example, if $G = PGL(d+1, F)$ then K is the maximal compact subgroup $PGL(d+1, \mathfrak{O})$ of G corresponding to the matrices g such that both g and g^{-1} have entries in the valuation ring $\mathfrak{O}$ of F.

In the context of semisimple Lie groups G, there is an extensive literature, going back nearly 40 years, containing theorems on rate of escape and almost sure convergence to a boundary point (e.g., [12], [13]), local limit theorems ([2], [3]) and central limit theorems (e.g., [24]). The rate of escape theorems are usually proved by first showing almost sure convergence to a boundary point. By contrast, in our context, we indicate how the latter follows from the former. While these theorems are valid for very general probability measures on such G, the quantities corresponding to our explicitly calculated limit γ are shown to be positive, but not calculated (see formula (7.9) in [12] for one case where one can be somewhat more explicit). If K is a maximal compact subgroup of a semisimple Lie group G, several of these results, such as the local limit theorem proved in [3] for probability measures μ on G, make no K-invariance assumptions on μ. Our local limit theorem for the case $SL(d+1, F)$ is a refinement of a theorem obtained by Tolli [23].

Our results in the case $d = 1$ were obtained by Sawyer [21], and most of our results for $d = 2$ by Lindlbauer and Voit [15]. Our main goal was to extend the results of those two papers (more in the spirit of [21]) by using the machinery of spherical harmonic analysis, as worked out by Macdonald [16],[17], but developed in a group-free context in [7], [8].

Let us now give some background facts about buildings of type $\tilde{A}_d$. We adopt the definition of building given in [4]. Thus a building $\mathfrak{X}$ of type $\tilde{A}_d$ is a certain sort of simplicial complex. Each vertex x has a *type* $\tau(x) \in \{0, 1, \dots, d\}$ so that each maximal simplex has exactly one vertex of each type. In $\mathfrak{X}$ there is a family of subcomplexes called *apartments*, each of which is isomorphic to the following $\tilde{A}_d$ *Coxeter complex*, Σ.

The vertex set of Σ is the additive group $\mathbb{Z}^{d+1}/\mathbb{Z}\mathbf{1}$, for $\mathbf{1} = (1, 1, \dots, 1)$. Two vertices $\lambda + \mathbb{Z}\mathbf{1}$ and $\mu + \mathbb{Z}\mathbf{1}$ are called *adjacent* if for some representatives $\lambda = (\lambda_j)$ and $\mu = (\mu_j)$ of these vertices we have $\lambda_j \leq \mu_j \leq \lambda_j + 1$ for all j. A *simplex* is then a set of vertices, any two of which are adjacent; a *chamber* is a maximal simplex. For $\lambda = (\lambda_j) \in \mathbb{Z}^{d+1}$, let $\Delta\lambda$ denote the d-tuple $(\lambda_1 - \lambda_2, \dots, \lambda_d - \lambda_{d+1})$. We often identify $\mathbb{Z}^{d+1}/\mathbb{Z}\mathbf{1}$ and $\mathbb{Z}^d$ via the isomorphism $\lambda + \mathbb{Z}\mathbf{1} \mapsto \Delta\lambda$.

For $r = 1, \dots, d$, let $\beta_r = (1, \dots, 1, 0, \dots, 0) \in \mathbb{Z}^{d+1}$, in which there are r 1's. Thus $\Delta\beta_r$ is the standard basis vector e_r. The *standard chamber* C_0 consists of the vertices $0, e_1, \dots, e_d$, the images under Δ of $0, \beta_1, \dots, \beta_d \in \mathbb{Z}^{d+1}$.

For $i \in \{0, 1, \dots, d\}$, and any vertex $x = \lambda + \mathbb{Z}\mathbf{1}$, we write $\tau(x) = i$ if $\sum_{j=1}^{d+1} \lambda_j \equiv i \pmod{d+1}$, and call $\tau(x)$ the *type* of x.

Let E denote the vector space $\mathbb{R}^{d+1}/\mathbb{R}\mathbf{1}$. For $1 \le i \ne j \le d+1$ and $m \in \mathbb{Z}$, let $H_{i,j,m}$ denote the affine hyperplane $\{\lambda + \mathbb{R}\mathbf{1} \in E : \lambda_i - \lambda_j = m\}$. The group $\tilde{W}_0$ generated by the reflections in these hyperplanes, i.e., the underlying Coxeter group, is the semidirect product of $\{\lambda \in \mathbb{Z}^{d+1} : \sum_{j=1}^{d+1} \lambda_j = 0\}$ and S_{d+1}, the symmetric group on $1, \dots, d+1$. It acts in a type-preserving way on the simplicial complex Σ via $(\lambda, w).(\mu + \mathbb{Z}\mathbf{1}) = w.\mu + \lambda + \mathbb{Z}\mathbf{1}$, where $w.\mu = (\mu_{w^{-1}(1)}, \dots, \mu_{w^{-1}(d+1)})$. It acts simply transitively on the set of chambers. The semidirect product $\tilde{W}$ of all of $\mathbb{Z}^{d+1}$ and S_{d+1} also acts on Σ via $(\lambda, w).(\mu + \mathbb{Z}\mathbf{1}) = w.\mu + \lambda + \mathbb{Z}\mathbf{1}$, not in a type-preserving way, but in a "type-rotating" way: $\tau((\lambda, w).x) \equiv \tau(x) + r \pmod{d+1}$ for $r = \sum_{j=1}^{d+1} \lambda_j$. Obviously, $\tilde{W}$ acts transitively on the set of vertices of Σ, unlike $\tilde{W}_0$, which only acts transitively on each set of vertices of Σ of the same type. In fact, $\tilde{W}$ is simply transitive on the set of pairs (C, x), where C is a chamber and x is a vertex of C (see the proof of Lemma 5.1 below). We also have the non-type-rotating automorphisms $\mu + \mathbb{Z}\mathbf{1} \mapsto -w.\mu + \lambda + \mathbb{Z}\mathbf{1}$ of Σ.

Given vertices x, y in our building $\mathfrak{X}$, there is an apartment A of $\mathfrak{X}$ containing x and y, and there is a type-rotating isomorphism $\varphi : A \to \Sigma$ such that $\varphi(x) = 0$ and $\varphi(y) = (k_1, \dots, k_d) \in \mathbb{N}^d$, i.e., $\varphi(y)$ is the image under Δ of $\lambda = (k_1 + \cdots + k_d, \dots, k_d, 0) \in \mathbb{Z}^{d+1}$. This $k = (k_1, \dots, k_d)$ does not depend on the particular A nor the particular φ chosen (see [6, Lemma 2.3]). We write $y \in S_k(x)$. For example, if $x \ne y \in \mathfrak{X}^0$ are neighbouring vertices, then $\tau(y) = \tau(x) + r \pmod{d+1}$ for some $r \in \{1, \dots, d\}$, and $y \in S_{e_r}(x)$. For any $k \in \mathbb{N}^d$, if $y \in S_k(x)$ then $\tau(y) \equiv \tau(x) + \|k\| \pmod{d+1}$, where $\|k\| = \sum_{i=1}^d ik_i$ $(= \sum_{j=1}^{d+1} \lambda_j$ if $\lambda = (k_1 + \cdots + k_d, \dots, k_d, 0))$.

Two chambers C and C' in $\mathfrak{X}$, are called *i-adjacent* if they are equal or if they have in common all vertices except those of type i. The hypotheses that $\mathfrak{X}$ is thick and locally finite imply, when $d \ge 2$, that there is an integer $q \ge 2$ such that for each chamber C and each $i \in \{0, 1, \dots, d\}$, there are exactly q chambers $C' \ne C$ which are i-adjacent to C (cf. [18, Proposition 3.3]). A thick locally finite building of type $\tilde{A}_1$ is just a tree in which each vertex has at least 3 neighbours, and so when $d = 1$ below, we shall *assume* that this tree is homogeneous, with each vertex having exactly $q + 1$ neighbours. For any $d \ge 1$, when $\mathfrak{X} = \mathfrak{X}_{d,F}$ as above, q is the order of the residual field of F.

As we saw in [6], for $x, y \in \mathfrak{X}^0$ and $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, (1) $N_k = |S_k(x)|$ does not depend on x; (2) $y \in S_k(x)$ if and only if $x \in S_{k^*}(y)$, where $k^* = (k_d, \dots, k_1)$; (3) $N_{k^*} = N_k$.

2 Techniques used

The methods we need to prove our results are "spherical harmonic analysis" techniques, for which we refer the reader to [7]. The methods have two main ingredients:

(a) the boundary Ω of the building and (b) the algebra $\mathcal{A}$ described below. These objects may be defined for any $\tilde{A}_d$ building, but in the case when $\mathfrak{X} = \mathfrak{X}_{d,F}$, let $G = PGL(d+1, F)$, and let $K = PGL(d+1, \mathfrak{O})$. Then Ω is isomorphic to the homogeneous space G/B, where B is the subgroup of G corresponding to the upper-triangular matrices (see [22, p. 104]). Also, $\mathcal{A}$ is isomorphic to the convolution algebra $\mathcal{C}_c(K\backslash G/K)$ of bi-K-invariant compactly supported functions on G, and (G, K) is a Gelfand pair (cf. [8, Proposition 2.4]).

Our use of Ω rather than G/B and of $\mathcal{A}$ rather than $\mathcal{C}_c(K\backslash G/K)$ gives a unified approach to all $\tilde{A}_d$ buildings when $d \geq 2$, as well as to homogeneous trees (which are $\tilde{A}_1$ buildings, but not the only ones). In the case $d = 2$, there are "non-classical buildings" not coming from a local field [19], and not associated with a Gelfand pair. Even when $d > 2$, where a result of Tits shows that all $\tilde{A}_d$ buildings come from a (possibly non-commutative) local field F (see [18, Theorem 10.22]), the literature on G/B and $\mathcal{C}_c(K\backslash G/K)$ usually only deals with the case when F is commutative. Our approach covers all these problematic cases at the same time.

2.1 The boundary Ω and the functions $h_i(x, y; \omega)$

The *standard sector* $\mathcal{S}_0$ in Σ is the subcomplex having $\{\lambda + \mathbb{Z}\mathbf{1} \in \Sigma : \lambda_1 \geq \cdots \geq \lambda_{d+1}\}$ as its vertex set. It corresponds under Δ to $\mathbb{N}^d$. A subcomplex S of $\mathfrak{X}$ is called a *sector* if there is an apartment A containing S and a type-rotating isomorphism $\varphi : A \to \Sigma$ such that $\varphi(S) = \mathcal{S}_0$. The vertex of S mapped to 0 by φ is called the *base vertex* of S. If S and S' are sectors in $\mathfrak{X}$ and if $S' \subset S$, then we call S' a *subsector* of S. We call two sectors in $\mathfrak{X}$ *equivalent* if they contain a common subsector. The *boundary* Ω of $\mathfrak{X}$ is the set of equivalence classes of sectors. It is the chamber set of the "spherical building at infinity" associated with $\mathfrak{X}$ [18, §9.3]. Given any $x \in \mathfrak{X}^0$ and any $\omega \in \Omega$, there is a unique sector $S^x(\omega)$ in $\mathfrak{X}$ in the equivalence class ω and having base vertex x [18, Lemma 9.7]. If A is any apartment containing $S^x(\omega)$, there is a type-rotating isomorphism $\varphi : A \to \Sigma$ mapping x to 0 and $S^x(\omega)$ to $\mathcal{S}_0$. If $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we define $S^x_{k_1,\ldots,k_d}(\omega)$ to be the vertex of $S^x(\omega)$ mapped by φ to $(k_1, \ldots, k_d)$. This does not depend on the particular apartment A nor on the particular isomorphism $\varphi : A \to \Sigma$ chosen [6, §4].

If $x, y \in \mathfrak{X}^0$ and $\omega \in \Omega$, then $S^x(\omega)$ and $S^y(\omega)$ contain a common subsector, and so there are integers $h_i = h_i(x, y; \omega)$ for $i = 1, \ldots, d$, so that

$$S^y_{j_1,\ldots,j_d}(\omega) = S^x_{j_1+h_1,\ldots,j_d+h_d}(\omega)$$

for all sufficiently large $j_1, \ldots, j_d \in \mathbb{N}$. When $d = 1$, $h_1(x, y; \omega)$ is the familiar horocycle index of a tree [25, p. 129]. It is clear that $h_i(x, y; \omega) = h_i(x, z; \omega) + h_i(z, y; \omega)$ for any vertices x, y, z, and hence $h_i(x, x; \omega) = 0$ and $h_i(y, x; \omega) = -h_i(x, y; \omega)$. In the next lemma, proved in [6], $\mathcal{E}_r$ denotes the set of $d+1$-tuples $\epsilon = (\epsilon_1, \ldots, \epsilon_{d+1})$ of 0's and 1's in which there are exactly r 1's.

Lemma 2.1. *Let x, y be neighbouring vertices, with $\tau(y) \equiv \tau(x) + r$ (mod $d+1$), i.e., $y \in S_{e_r}(x)$. If we fix any $\omega \in \Omega$, then there is an $\epsilon \in \mathcal{E}_r$ such that $h_i(x, y; \omega) = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \ldots, d$. Moreover, for fixed x, ω and ϵ, the* *number of* $y \in S_{e_r}(x)$ *for which* $h_i(x, y; \omega) = \epsilon_i - \epsilon_{i+1}$ *for* $i = 1, \ldots, d$ *is* $q^{M(\epsilon)}$, *where*

$$M(\epsilon) = -\frac{r(r+1)}{2} + \sum_1^{d+1} i\epsilon_i. \tag{2.1}$$

The first part of this lemma shows that $|h_i(x, y; \omega)| \leq 1$ for any neighbouring vertices x, y, and so $|h_i(x, y; \omega)| \leq \mathrm{dist}(x, y)$ for any x, y.

Let $\mu_i(x, y; \omega) = h_i(x, y; \omega) + \cdots + h_d(x, y; \omega)$ for $i = 1, \ldots, d$, and let $\mu_{d+1}(x, y; \omega) = 0$. So $\mu_i(x, y; \omega) - \mu_{i+1}(x, y; \omega) = h_i(x, y; \omega)$ for $i = 1, \ldots, d$. The first statement in Lemma 2.1 also shows that $|\mu_i(x, y; \omega)| \leq 1$ for any neighbouring vertices x, y, and so $|\mu_i(x, y; \omega)| \leq \mathrm{dist}(x, y)$ for any x, y.

Lemma 2.2. *For* $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, *and* $y \in S_k(x)$,

$$\sum_{i=1}^{d} i h_i(x, y; \omega) \equiv \|k\| = \sum_{i=1}^{d} i k_i \pmod{d+1} \tag{2.2}$$

for any $\omega \in \Omega$. *In other notation, setting* $\lambda = (k_1 + \cdots + k_d, \ldots, k_d, 0) \in \mathbb{Z}^{d+1}$, *and* $|\lambda| = \lambda_1 + \cdots + \lambda_{d+1}$, *and defining* $\mu_i(x, y; \omega)$ *as above,*

$$\mu_1(x, y; \omega) + \cdots + \mu_{d+1}(x, y; \omega) \equiv |\lambda| \equiv \tau(y) - \tau(x) \pmod{d+1}. \tag{2.3}$$

Proof. If $y \in S_{e_r}(x)$, Lemma 2.1 shows that $\mu_i(x, y; \omega) = \epsilon_i - \epsilon_{d+1}$ for each i, so that the left hand side of (2.3) equals $\sum_{i=1}^{d+1} \epsilon_i - (d+1)\epsilon_{d+1} = r - (d+1)\epsilon_{d+1}$, and (2.3) holds. The proof that (2.3) holds for general $k \in \mathbb{N}^d$ is a routine induction on $|k|$. For if $k \in \mathbb{N}^d$ and if $k_r > 0$, let $k' = k - e_r$. There is a vertex $z \in S_{k'}(x)$ such that $y \in S_{e_r}(z)$. This may be seen by taking an apartment containing x and y, and identifying it with Σ, with x corresponding to 0 and y to $\lambda + \mathbb{Z}\mathbf{1}$. Then let $z = (\lambda - \beta_r) + \mathbb{Z}\mathbf{1}$. □

Fix any vertex o. For any vertex v, let $\Omega_o(v)$ be the set of $\omega \in \Omega$ such that v lies in the sector $S^o(\omega)$. There is a compact totally disconnected Hausdorff topology on Ω having the sets $\Omega_o(v)$, $v \in \mathfrak{X}^0$, as basis. This topology does not depend on the choice of the vertex o. The functions $\omega \mapsto h_i(x, y; \omega)$ are locally constant on Ω for this topology. There is a unique regular Borel probability measure ν_o on Ω which assigns measure $1/N_k$ to each set $\Omega_o(v)$, $v \in S_k(o)$. If o, x are vertices, then ν_o and ν_x are mutually absolutely continuous, and the Radon–Nykodym derivative $d\nu_x/d\nu_o$ is given by

$$\frac{d\nu_x}{d\nu_o}(\omega) = q^{\sum_{i=1}^{d} i(d+1-i)h_i(o,x;\omega)}. \tag{2.4}$$

2.2 The algebra $\mathcal{A}$ and the algebra homomorphisms $\mathcal{A} \to \mathbb{C}$

For complex-valued functions f on $\mathfrak{X}^0$, $\sum_{y\in\mathfrak{X}^0} p_k(x, y) f(y)$ (where $p_k(x, y)$ is defined in (1.2)) is what we denoted by $(A_k f)(x)$ in [7]. We shall write P_k in place of A_k here, and think of P_k as either an operator or a transition matrix. The P_k's span a commutative algebra $\mathcal{A}$ (under composition) [6, Theorem 3.1]. For each $d + 1$-tuple $z = (z_1, \dots, z_{d+1})$ of complex numbers having product 1, there is an algebra homomorphism $h_z : \mathcal{A} \to \mathbb{C}$ such that

$$h_z(P_{e_r}) = \frac{q^{r(d+1-r)/2}}{N_{e_r}} \sigma_r(z_1, \dots, z_{d+1})$$

for $r = 1, \dots, d$. Here σ_r is the r-th elementary symmetric polynomial. Any algebra homomorphism $\mathcal{A} \to \mathbb{C}$ equals h_z for some such $d + 1$-tuple z, and $h_z = h_{z'}$ if and only if z is a permutation of z'. If we fix a vertex o, the *spherical function* φ_z associated to h_z is defined to be the function which for each $k \in \mathbb{N}^d$ takes the constant value $h_z(P_k)$ on the set $S_k(o)$. We have two formulas for the spherical function $\varphi_z(x)$ [7, Propositions 3.1 and 2.4]. The first, an analogue of Formula (34) in [5], is

$$\varphi_z(x) = \int_\Omega \frac{z_1^{h_1(o,x;\omega)+\cdots+h_d(o,x;\omega)} z_2^{h_2(o,x;\omega)+\cdots+h_d(o,x;\omega)} \cdots z_d^{h_d(o,x;\omega)}}{q^{-\frac{1}{2}\sum_1^d i(d+1-i)h_i(o,x;\omega)}} \, d\nu_o(\omega). \tag{2.5}$$

The second, valid when $z_1, \dots, z_{d+1}$ are distinct, essentially due to Macdonald (see [16], [17] or [5, Thm 4.4]) states that if $x \in S_k(o)$, where $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, then

$$\varphi_z(x) = C q^{-\frac{1}{2}\sum_1^d i(d+1-i)k_i} \sum_{w\in S_{d+1}} z_{w(1)}^{\lambda_1} \cdots z_{w(d+1)}^{\lambda_{d+1}} c(z_{w(1)}, \dots, z_{w(d+1)}), \tag{2.6}$$

where $C = q^{d(d+1)/2} \prod_{j=1}^{d+1} \big((q-1)/(q^j - 1)\big)$, where $\lambda_i = k_i + \cdots + k_d$ for $i = 1, \dots, d$, $\lambda_{d+1} = 0$, and where

$$c(z_1, \dots, z_{d+1}) = \prod_{1\le i<j\le d+1} \left(\frac{z_i - q^{-1}z_j}{z_i - z_j}\right). \tag{2.7}$$

Each P_k can be regarded as a linear operator on the Hilbert space $\ell^2(\mathfrak{X}^0)$ of square summable functions on $\mathfrak{X}^0$, with the natural inner product $\langle f, g\rangle = \sum_x f(x)\overline{g(x)}$. As such, we have $\|P_k\| \le 1$ [7, Lemma 4.1]. We may regard the algebra $\mathcal{A}$ as a sub-algebra of the algebra $\mathcal{L}(\ell^2(\mathfrak{X}^0))$ of bounded linear operators on $\ell^2(\mathfrak{X}^0)$. This sub-algebra is self-adjoint, with $P_k^* = P_{k^*}$, where $k^* = (k_d, \dots, k_1)$ if $k = (k_1, \dots, k_d)$.

The algebra homomorphisms h_z, where $z = (z_1, \dots, z_{d+1})$, which are continuous for the operator norm on $\mathcal{A}$ are precisely those for which $|z_j| = 1$ for each j [7, §4]. Let T_d denote the compact group of all $d + 1$-tuples of complex numbers of modulus 1 having product 1. The algebra homomorphisms h_t, $t \in T_d$, being continuous for the operator norm on $\mathcal{A}$, extend uniquely to the closure $\bar{\mathcal{A}}$ of $\mathcal{A}$ in $\mathcal{L}(\ell^2(\mathfrak{X}^0))$. We shall write $\widehat{A}(t)$ in place of $h_t(A)$ for any $A \in \bar{\mathcal{A}}$.

For example, if P is as in (1.1), then we can regard P as in $\bar{\mathcal{A}}$, and so $\widehat{P}(t)$ is defined, and

$$\widehat{P}(t) = \sum_{k \in \mathbb{N}^d} a_k \widehat{P_k}(t).$$

Because $\bar{\mathcal{A}}$ is a sub-algebra of $\mathcal{L}(\ell^2(\mathfrak{X}^0))$, the product PQ of two isotropic transition probability matrices P and Q on $\mathfrak{X}^0$ is again an isotropic transition probability matrix, and $\widehat{PQ}(t) = \widehat{P}(t)\widehat{Q}(t)$. Thus the n-th power $P^n = (p^{(n)}(x, y))_{x,y \in \mathfrak{X}^0}$ of P may be written

$$P^n = \sum_{k \in \mathbb{N}^d} a_k^{(n)} P_k\,.$$

Moreover, $p^{(n)}(x, y) = a_k^{(n)}/N_k$ whenever $y \in S_k(x)$, and $\widehat{P^n}(t) = \big(\widehat{P}(t)\big)^n$ for all $t \in T_d$.

If $t = (t_1, \ldots, t_{d+1}) \in T_d$, note that $1/c(t)$ (with $c(z)$ defined in (2.7)) makes sense even when some of the t_j's are equal. Let μ be the measure on T_d given by $d\mu(t) = C_{d,q}|c(t)|^{-2}dt$, where dt is the normalized Haar measure on T_d, and where $C_{d,q}$ is $C^{-1}/(d+1)!$, with C as in (2.6). Then for any $x, y \in \mathfrak{X}^0$, $k \in \mathbb{N}^d$, and $A \in \bar{\mathcal{A}}$, if $y \in S_k(x)$ then

$$(A\delta_y)(x) = \int_{T_d} \widehat{A}(t)\overline{\widehat{P_k}(t)}\, d\mu(t); \tag{2.8}$$

see [7, Corollary 4.10]. This is called the *Plancherel inversion formula*.

3 The rate of escape theorem

Let $P = (p(x, y))_{x,y \in X}$ be a transition probability matrix for any set X. Let $X = \bigcup_{i \in I} X_i$ be a partition of X. We call P *factorizable* (over I) if for each $i, j \in I$, the sum $\sum_{y \in X_j} p(x, y)$ has the same value for all $x \in X_i$. We then write $\bar{p}(i, j)$ for this value, and form the matrix $\bar{P} = (\bar{p}(i, j))_{i,j \in I}$. The following elementary lemma is well-known.

Lemma 3.1. *Suppose that P is factorizable over I. Then $\bar{P}$ is a transition probability matrix. If P and Q are transition probability matrices, both factorizable over I, then PQ is factorizable over I, and $\overline{PQ} = \bar{P}\bar{Q}$. If $P = \sum_m \alpha_m P_m$ is a finite or infinite convex combination of factorizable matrices P_m, then P is factorizable, and $\bar{P} = \sum_m \alpha_m \bar{P}_m$. If $(Z_n)_{n \in \mathbb{N}}$ is a Markov chain on X with factorizable transition matrix P, then set $\bar{Z}_n = i$ if $Z_n \in X_i$. This defines a Markov chain on I with transition matrix $\bar{P}$.*

Write $h(x, y; \omega)$ for the d-tuple $(h_1(x, y; \omega), \ldots, h_d(x, y; \omega)) \in \mathbb{Z}^d$, where $x, y \in \mathfrak{X}^0$, $\omega \in \Omega$, and where the $h_i(x, y; \omega)$ were defined above. Fix $o \in \mathfrak{X}^0$ and $\omega \in \Omega$, and consider the partition of $\mathfrak{X}^0$ into the sets $\mathfrak{X}^0_a = \{x \in \mathfrak{X}^0 : h(x, o; \omega) = a\}$, $a \in \mathbb{Z}^d$.

Lemma 3.2. *The matrices P_{e_r}, $r = 1, \ldots, d$, are all factorizable, and $\bar{P}_{e_r}$ does not depend on o or ω. Moreover, $\bar{p}_{e_r}(a, b) = \bar{p}_{e_r}(0, b - a)$ for all $a, b \in \mathbb{Z}^d$.*

Proof. If $a, b \in \mathbb{Z}^d$ and $x \in \mathfrak{X}^0_a$, then as $h(y, o; \omega) = h(y, x; \omega) + h(x, o; \omega) = h(y, x; \omega) + a$,

$$\sum_{y \in \mathfrak{X}^0_b} p_{e_r}(x, y) = \frac{1}{N_{e_r}} \Big| \{y \in S_{e_r}(x) : h(y, o; \omega) = b\} \Big|$$
$$= \frac{1}{N_{e_r}} \Big| \{y \in S_{e_r}(x) : h(y, x; \omega) = b - a\} \Big|.$$

Lemma 2.1 shows that this equals $q^{M(\epsilon)}/N_{e_r}$ if $b - a = (\epsilon_2 - \epsilon_1, \ldots, \epsilon_{d+1} - \epsilon_d)$ for some $\epsilon \in \mathcal{E}_r$, and zero otherwise. □

Proposition 3.3. *Any isotropic Markov transition probability matrix P on $\mathfrak{X}^0$ is factorizable over the above partition of $\mathfrak{X}^0$. Moreover, for any $a, b \in \mathbb{Z}^d$, $\bar{p}(a, b) = \bar{p}(0, b - a)$, and $\bar{p}(a, b)$ does not depend on o or ω. If $(Z_n)_{n \in \mathbb{N}}$ is a Markov chain in $\mathfrak{X}^0$ with transition probability matrix P, then the corresponding Markov chain in $\mathbb{Z}^d$ is $(\bar{Z}_n) = \big(h(Z_n, o; \omega)\big)$.*

Proof. By Lemma 3.1, we may assume that $P = P_k$ for some $k \in \mathbb{N}^d$. But P_k is a polynomial in $P_{e_1}, \ldots, P_{e_d}$ and the identity matrix, because $\mathcal{A}$ is generated by $P_{e_1}, \ldots, P_{e_d}$ [6, Theorem 3.1]. The coefficients in such a polynomial could be negative, but we may write $P_k + \alpha_1 Q_1 = \alpha_2 Q_2$, where Q_1 and Q_2 are convex combinations of products of P_{e_i}'s and $\alpha_1, \alpha_2 \geq 0$. So the factorizability of P_k follows from that of Q_1 and Q_2, as do the other properties stated in the proposition. □

Remark 3.4. Any isotropic Markov transition probability matrix P on $\mathfrak{X}^0$ is also factorizable over the partition of $\mathfrak{X}^0$ into the sets $S_k(o)$, $k \in \mathbb{N}^d$. For any product $P_k P_\ell$ can be written as a sum $\sum_{r \in \mathbb{N}^d} c_{k,\ell;r} P_r$, and it is easy to check that $\sum_{y \in S_k(o)} P_m(x, y) = N_k c_{k,m^*;j}/N_j$ for all $x \in S_j(o)$. We shall not be making explicit use of this factorizability below. Still, it may be useful to note that it implies that our random walk induces a Markov chain on the hypergroup $\mathbb{N}^d$ (see [15]).

Proposition 3.5. *Consider the integral*

$$\int_\Omega h_j(x, o; \omega) \, d\nu_o(\omega), \tag{3.1}$$

(i) The integral 3.1 takes the same value for all $x \in S_k(o)$; call this value $\gamma_k^{(d+1-j)}$ (this index-reversing makes the statements of the main theorems simpler). It does not depend on o.

(ii) If $(Z_n)_{n\in\mathbb{N}}$ is a Markov chain in $\mathfrak{X}^0$ with transition probability matrix P_k, and such that $Z_0 \equiv o$, then for any $\omega \in \Omega$ the expectation $\mathbb{E}(h_j(Z_1, o; \omega))$ of $h_j(Z_1, o; \omega)$ equals $\gamma_k^{(d+1-j)}$.

(iii) $\gamma_k^{(d+1-j)} = \gamma_{k^*}^{(j)}$.

Proof. Note that $|h_j(x, o; \omega)| \le \operatorname{dist}(x, o) = |k|$ for all ω, and that the integrand is locally constant. The integral therefore exists.

Notice that the integrand in (2.5) is

$$\prod_{r=1}^{d} \Big(z_1 \cdots z_r q^{r(d+1-r)/2}\Big)^{h_r(o,x;\omega)}. \tag{3.2}$$

Let $a_i' = q^{-d/2+i-1}$ for $i = 1, \dots, d+1$. These numbers have product 1, and $a_1' \cdots a_r' q^{r(d+1-r)/2} = 1$ for $r = 1, \dots, d+1$. Now let $j \in \{1, \dots, d\}$ and $\theta \in \mathbb{R}$. Set $z_j' = a_j' e^{-\theta}$, $z_{d+1}' = a_{d+1}' e^{\theta}$, and $z_i' = a_i'$ for $i \neq j, d+1$. Fix $x \in S_k(o)$, and consider the integral formula (2.5) for $\varphi_{z'}(x)$, which we denote by $F_j(\theta)$. Since $z_1' \cdots z_r' q^{r(d+1-r)/2}$ equals 1 for $r = 1, \dots, j-1$, and equals $e^{-\theta}$ for $r = j, \dots, d$, using (3.2) we find that $F_j(\theta)$ equals

$$\int_\Omega e^{-\theta(h_j(o,x;\omega)+\cdots+h_d(o,x;\omega))}\, d\nu_o(\omega) = \int_\Omega e^{\theta(h_j(x,o;\omega)+\cdots+h_d(x,o;\omega))}\, d\nu_o(\omega).$$

Hence

$$F_j'(0) = \int_\Omega \big(h_j(x, o; \omega) + \cdots + h_d(x, o; \omega)\big)\, d\nu_o(\omega).$$

Thus the integral (3.1) equals $F_j'(0) - F_{j+1}'(0)$, which has the same value for any $x \in S_k(o)$ because $\varphi_{z'}(x)$ has this property. This proves the first part of (i). The independence of $\gamma_k^{(d+1-j)}$ on o follows from the fact that $F_j(\theta)$ equals $h_{z'}(P_k)$ for the above z', and this does not depend on o.

To prove (ii), notice that $Z_1 \in S_k(o)$ with probability 1, because $Z_0 \equiv o$. So $\int_\Omega h_j(Z_1, o; \omega)\, d\nu_o(\omega) = \gamma_k^{(d+1-j)}$ with probability 1, by (i). If $(\Lambda, \mathcal{P}, \mathbb{P})$ is the underlying probability space on which the Z_n's are defined, and if $\mathcal{B}$ is the Borel σ-algebra of Ω, then the maps $(\lambda, \omega) \mapsto h_j(Z_n(\lambda), o; \omega)$ are $\mathcal{P} \times \mathcal{B}$-measurable, because for any $m \in \mathbb{Z}$,

$$\begin{aligned} &\{(\lambda, \omega) \in \Lambda \times \Omega :\ h_j(Z_n(\lambda), o; \omega) = m\} \\ &\quad = \bigcup_{x \in \mathfrak{X}^0} \{\lambda \in \Lambda : Z_n(\lambda) = x\} \times \{\omega \in \Omega : h_j(x, o; \omega) = m\} \end{aligned}$$

expresses the set on the left as a member of $\mathcal{P} \times \mathcal{B}$. Hence we can use Fubini's Theorem and take expectations under the integral sign:

$$\gamma_k^{(d+1-j)} = \int_\Omega \mathbb{E}(h_j(Z_1, o; \omega))\, d\nu_o(\omega).$$

By Proposition (3.3), the distribution of $h_j(Z_1, o; \omega)$, and so $\mathbb{E}(h_j(Z_1, o; \omega))$, is independent of ω. Thus (ii) follows.

To prove (iii), we reverse the $d + 1$-tuple z' used in proving (i). Let $\alpha_i = q^{d/2-i+1}$ for $i = 1, \dots, d + 1$, and set $z_1 = \alpha_1 e^{\theta}$, $z_{d+2-j} = \alpha_{d+2-j} e^{-\theta}$, and $z_i = \alpha_i$ for $i \neq 1, d + 2 - j$. Now $z_1 \cdots z_r q^{r(d+1-r)/2}$ equals $q^{r(d+1-r)} e^{\theta}$ for $r = 1, \dots, d+1-j$, and equals $q^{r(d+1-r)}$ for $r = d+2-j, \dots, d$. Using (3.2) and $\varphi_z(x) = \varphi_{z'}(x)$, and using (2.4) to express the integral in terms of ν_x, we see that $F_j(\theta)$ also equals

$$\int_\Omega \frac{e^{\theta(h_1(o,x;\omega)+\cdots+h_{d+1-j}(o,x;\omega))}}{q^{-\sum_1^d i(d+1-i)h_i(o,x;\omega)}} \, d\nu_o(\omega)$$
$$= \int_\Omega e^{\theta(h_1(o,x;\omega)+\cdots+h_{d+1-j}(o,x;\omega))} \, d\nu_x(\omega).$$

Hence, as well as equalling $\gamma_k^{(1)} + \cdots + \gamma_k^{(d+1-j)}$, $F_j'(0)$ equals

$$\int_\Omega \big(h_1(o, x; \omega) + \cdots + h_{d+1-j}(o, x; \omega)\big) \, d\nu_x(\omega).$$

Since $o \in S_{k^*}(x)$, this equals $\gamma_{k^*}^{(j)} + \cdots + \gamma_{k^*}^{(d)}$. This proves (iii). □

We next want to calculate the quantities $\gamma_k^{(d+1-j)}$ using (2.6). We shall again use the $d + 1$-tuple used in the above proof of Proposition 3.5(iii).

Lemma 3.6. *Let $\alpha_i = q^{d/2-i+1}$ for $i = 1, \dots, d+1$, and let $c(z_1, \dots, z_{d+1})$ and C be as in (2.6) and (2.7). Then $c(\alpha_{w(1)}, \dots, \alpha_{w(d+1)}) = 0$ for all $w \neq id$, and $C\,c(\alpha_1, \dots, \alpha_{d+1}) = 1$. The algebra homomorphism h_α satisfies $h_\alpha(P_k) = 1$ for all $k \in \mathbb{N}^d$.*

Proof. The first statement is an easy calculation. If $w \in S_{d+1}$ and $r \in \{1, \dots, d\}$, let $j = w^{-1}(r)$ and $i = w^{-1}(r+1)$. If $i < j$, then there is a factor $\alpha_{w(i)} - \alpha_{w(j)}/q = \alpha_{r+1} - \alpha_r/q = 0$ in $c(\alpha_{w(1)}, \dots, \alpha_{w(d+1)})$. So unless $w^{-1}(r) < w^{-1}(r+1)$ for all $r \in \{1, \dots, d\}$, $c(\alpha_{w(1)}, \dots, \alpha_{w(d+1)}) = 0$. But id is the only monotone permutation. The final statement is now easily seen from (2.6). For a proof that $h_\alpha(P_k) = 1$ based on (2.5), take $\theta = 0$ and $j = 1$, say, in $F_j(\theta)$ above. □

Proposition 3.7. *For each $j \in \{1, \dots, d\}$ and $k = (k_1, \dots, k_d) \in \mathbb{N}^d$,*

$$\gamma_k^{(j)} = k_j + O(1). \tag{3.3}$$

For all $x \in \mathfrak{X}^0$,

$$\mathrm{dist}(x, o) = \int_\Omega \big(h_1(x, o; \omega) + \cdots + h_d(x, o; \omega)\big) \, d\nu_o(\omega) + O(1). \tag{3.4}$$

Proof. Let $z_1 = \alpha_1 e^\theta$, $z_{d+2-j} = \alpha_{d+2-j} e^{-\theta}$ and $z_i = \alpha_i$ for $i \neq 1, d+2-j$, as in the proof of Proposition 3.5(iii). Assuming $|\theta| < \log q$, the z_i's are distinct, and we can use (2.6) to find $\varphi_z(x)$. Let D_w denote the derivative of $Cc(z_{w(1)}, \dots, z_{w(d+1)})$ with respect to θ, evaluated at $\theta = 0$. Note that the w-th summand in (2.6) equals

$$\begin{aligned} Cc(z_{w(1)}, \dots, z_{w(d+1)}) \prod_{r=1}^{d} \Big(z_{w(1)} \cdots z_{w(r)} q^{-r(d+1-r)/2} \Big)^{k_r} \\ = Cc(z_{w(1)}, \dots, z_{w(d+1)}) \prod_{r=1}^{d} \Big(\frac{z_{w(1)} \cdots z_{w(r)}}{\alpha_1 \cdots \alpha_r} \Big)^{k_r}. \end{aligned} \tag{3.5}$$

The contribution from the term $w = id$ in (2.6) to $F_j(\theta) = \varphi_z(x)$ is

$$e^{\theta(k_1 + \cdots + k_{d+1-j})} Cc(\alpha_1 e^\theta, \alpha_2, \dots, \alpha_{d+2-j} e^{-\theta}, \dots, \alpha_{d+1}),$$

and so by the first part of Lemma 3.6, the contribution to $F_j'(0)$ is

$$k_1 + \cdots + k_{d+1-j} + D_{id}. \tag{3.6}$$

By (3.5) and the second part of Lemma 3.6, from any $w \neq id$ we get the contribution

$$D_w \prod_{r=1}^{d} \Big(\frac{\alpha_{w(1)} \cdots \alpha_{w(r)}}{\alpha_1 \cdots \alpha_r} \Big)^{k_r}. \tag{3.7}$$

Since the α_i's are positive and decreasing, $\alpha_{w(1)} \cdots \alpha_{w(r)} \leq \alpha_1 \cdots \alpha_r$ for all r. Hence this contribution to $F_j'(0)$ is bounded by $|D_w|$. Thus

$$\gamma_k^{(1)} + \cdots + \gamma_k^{(d+1-j)} = F_j'(0) = k_1 + \cdots + k_{d+1-j} + M_{k,j}, \tag{3.8}$$

where $|M_{k,j}| \leq \sum_{w \in S_{d+1}} |D_w|$. Thus

$$\big| \gamma_k^{(d+1-j)} - k_{d+1-j} \big| \leq 2 \sum_{w \in S_{d+1}} |D_w|.$$

This proves (3.3), and 3.4 follows from taking any $x \in S_k(o)$ and summing (3.3) over j, or from (3.8) with $j = 1$. □

Let γ_k denote the value of the integral on the right in (3.4) when $x \in S_k(o)$, i.e.,

$$\gamma_k = \gamma_k^{(1)} + \cdots + \gamma_k^{(d)}.$$

Let us fix a vertex o, and for $x \in \mathfrak{X}^0$, write $k_j(x)$, $j = 1, \dots, d$, for the "coordinates" of x with respect to o, i.e., if $x \in S_k(o)$, where $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, write $k_j(x) = k_j$. Here is our rate of escape theorem.

Theorem 3.8. *Let P be an isotropic transition probability matrix, as in (1.1). Assume that $\sum_{k\in\mathbb{N}^d} |k|a_k < \infty$. Let $(Z_n)_{n\in\mathbb{N}}$ be the corresponding Markov chain. Then with probability 1,*

$$\frac{1}{n}k_j(Z_n) \to \gamma^{(j)} \quad \text{as } n \to \infty, \tag{3.9}$$

where $\gamma^{(j)} = \sum_{k\in\mathbb{N}^d} a_k\gamma_k^{(j)}$. Hence, also with probability 1,

$$\frac{1}{n}\operatorname{dist}(Z_n, o) \to \gamma \quad \text{as } n \to \infty \tag{3.10}$$

for $\gamma = \gamma^{(1)} + \cdots + \gamma^{(d)} = \sum_k a_k\gamma_k$.

Proof. We may assume that $Z_0 \equiv o$. By (3.3), we need only show that

$$\int_\Omega \frac{h_j(Z_n, o; \omega)}{n}\, d\nu_o(\omega) \to \gamma^{(d+1-j)}.$$

As in the proof of Proposition 3.5, let $(\Lambda, \mathcal{P}, \mathbb{P})$ be the underlying probability space on which the Z_n's are defined. By Proposition (3.3), for each fixed ω, $h_j(Z_n, o; \omega)$ is a random variable on Λ, distributed like a sum of n independent real random variables, each with the distribution of $X_j = h_j(Z_1, o; \omega)$. Now X_j has finite expectation, because $|h_j(Z_1, o; \omega)| \le \operatorname{dist}(Z_1, o)$ and

$$\mathbb{E}(\operatorname{dist}(Z_1, o)) = \sum_{k\in\mathbb{N}^d} |k|\mathbb{P}(Z_1 \in S_k(o)) = \sum_{k\in\mathbb{N}^d} |k|a_k < \infty. \tag{3.11}$$

Similarly, the expected value of X_j is $\gamma^{(d+1-j)}$. So by the classical law of large numbers, for each fixed $\omega \in \Omega$,

$$\frac{1}{n}h_j(Z_n(\lambda), o; \omega) \to \gamma^{(d+1-j)}$$

for almost all $\lambda \in \Lambda$. Now $|h_j(Z_n, o; \omega)|/n \le \operatorname{dist}(Z_n, o)/n$ for each j, n and ω. The sequence $(\operatorname{dist}(Z_n(\lambda), o)/n)$ is bounded for each λ in a set $\Lambda_1 \subset \Lambda$ of probability 1, by the second part of [25, Proposition 8.8(a)] (the uniform first moment condition being satisfied because the $\sigma_x(n)$ there is independent of x). So for $\lambda \in \Lambda_1$, the Bounded Convergence Theorem shows that

$$\lim_{n\to\infty}\int_\Omega \frac{h_j(Z_n, o; \omega)}{n}\, d\nu_o(\omega) = \int_\Omega \lim_{n\to\infty} \frac{h_j(Z_n, o; \omega)}{n}\, d\nu_o(\omega) = \gamma^{(d+1-j)}.$$

So (3.9) holds, and we get (3.10) simply by summing (3.9) over j. □

One can topologize $\mathfrak{X}^0 \cup \Omega$ in a natural way, so that the subspace topology on Ω is the compact topology mentioned in Section 2.1, the subspace topology on $\mathfrak{X}^0$ is discrete, and convergence of a sequence (x_ν) in $\mathfrak{X}^0$ to an $\omega \in \Omega$ is defined as follows. Given $\omega \in \Omega$ and $r \in \mathbb{N}^d$, recall that $S_r^o(\omega)$ denotes the unique vertex in $S_r(o)$ lying in the sector $S^o(\omega)$ having base vertex o and representing ω. Write $x_\nu \to \omega$ if for each $r \in \mathbb{N}^d$ there is a ν_r such that $S_r^o(\omega)$ is in the convex hull $[o, x_\nu]$ if $\nu \geq \nu_r$ (see before Lemma 2.4 in [6] for the definition of convex hull). This definition of convergence to a boundary point does not depend on the choice of the base point o. Note that $\mathfrak{X}^0 \cup \Omega$ is not compact if $d \geq 2$. For example, if $x_\nu = S^o_{\nu,0,\dots,0}(\omega)$ for each ν, then (x_ν) has no convergent subsequence.

Proposition 3.9. *Assume the notation and hypotheses of Theorem 3.8. Then with probability 1, $Z_n \to \omega$ for some random element ω of Ω.*

Proof. We omit a detailed proof, because it follows from a general result [14] about nonpositively curved manifolds such as our building ([1], [4, pp. 152–155]). Moreover, the finite first moment condition of Theorem 3.8 is not necessary in the case $d = 1$ [9], and might not be needed for general d. The main point of a proof using our methods is that $\gamma^{(j)} > 0$ for all j (see the remark after Proposition 4.6 below), so that by (3.9) with probability 1 the j-th coordinate $k_j(Z_n)$ of Z_n is large when n is large. This shows that Z_n is not close to the wall of any ω, and avoids the difficulty arising from examples like the one mentioned in the sentence before this proposition. □

4 Exact calculation of γ_k

We calculate γ_k by taking $j = 1$ in the proof of Proposition 3.7. So $z_1 = \alpha_1 e^\theta$, $z_{d+1} = \alpha_{d+1}e^{-\theta}$, $z_i = \alpha_i$ for $i \neq 1, d+1$. We need to evaluate the derivatives D_w appearing in the contributions (3.6) and (3.7) to $F_1'(0)$.

Lemma 4.1. *Assuming that $d \geq 2$, the derivative D_w is non-zero precisely when w is one of the following $2d+1$ permutations: 1. $w = id$, in which case, $D_w = I_d$ for*

$$I_d = \frac{-2q(q^d - 1)}{(q-1)(q^{d+1} - 1)}; \tag{4.1}$$

2. w is one of the cyclic permutations $w_\nu = (1, 2, \dots, \nu + 1)$ or $w'_\nu = (d + 1, d, \dots, d - \nu + 1)$, $\nu \in \{1, \dots, d\}$, in which case $D_{w_\nu} = D_{w'_\nu} = D_\nu$ for

$$D_\nu = \frac{(q-1)q^\nu}{(q^\nu - 1)(q^{\nu+1} - 1)}. \tag{4.2}$$

When $d = 1$, $D_{id} = I_1$ is valid, but $D_w = 2D_1$ for $w = (1, 2)$.

Proof. If $r \in \{2, \dots, d-1\}$ and $i = w^{-1}(r+1)$ is less than $j = w^{-1}(r)$, then (for any θ) $c(z_{w(1)}, \dots, z_{w(d+1)})$ contains the factor $z_{w(i)} - z_{w(j)}/q = \alpha_{r+1} - \alpha_r/q = 0$, and so $D_w = 0$.

If $i = w^{-1}(2)$ is less than $j = w^{-1}(1)$ and $i' = w^{-1}(d + 1)$ is less than $j' = w^{-1}(d)$, then $c(z_{w(1)}, \dots, z_{w(d+1)})$ contains the factor $z_{w(i)} - z_{w(j)}/q = z_2 - z_1/q = \alpha_2(1 - e^\theta)$, and also the factor $z_{w(i')} - z_{w(j')}/q = z_{d+1} - z_d/q = \alpha_{d+1}(e^{-\theta} - 1)$, and so is $O(\theta^2)$ as $\theta \to 0$. Hence $D_w \neq 0$ only if $w^{-1}(r) < w^{-1}(r + 1)$ for all $r \in \{2, \dots, d-1\}$ and at most one of $r = 1$ and $r = d$.

Case 1: $w^{-1}(r) < w^{-1}(r+1)$ for *all* $r \in \{1, \dots, d\}$. Then $w = id$, and routine calculations show that (4.1) holds.

Case 2: $w^{-1}(r) < w^{-1}(r+1)$ for all $r \in \{2, \dots, d\}$, but $w^{-1}(1) > w^{-1}(2)$. Write $w^{-1}(1) = \nu + 1$. Then $\nu \in \{1, \dots, d\}$, and it is easy to see that

$$w = \begin{pmatrix} 1 & 2 & \cdots & \nu & \nu+1 & \nu+2 & \cdots & d & d+1 \\ 2 & 3 & \cdots & \nu+1 & 1 & \nu+2 & \cdots & d & d+1 \end{pmatrix} = w_\nu.$$

Routine calculations verify that $D_w = D_\nu$.

Case 3: $w^{-1}(r) < w^{-1}(r+1)$ for all $r \in \{1, \dots, d-1\}$, but $w^{-1}(d+1) < w^{-1}(d)$. Write $w^{-1}(d+1) = d - \nu + 1$. Then $\nu \in \{1, \dots, d\}$, and it is easy to see that

$$w = \begin{pmatrix} 1 & \cdots & d-\nu & d-\nu+1 & d-\nu+2 & \cdots & d+1 \\ 1 & \cdots & d-\nu & d+1 & d-\nu+1 & \cdots & d \end{pmatrix} = w'_\nu.$$

Similar calculations verify that $D_w = D_\nu$.

If $d \geq 2$, then Cases 2 and 3 are disjoint, and so $D_w \neq 0$ for exactly $2d + 1$ permutations. If $d = 1$, then Cases 2 and 3 coincide, and showing $D_w = 2D_1$ for $w = (1, 2)$ is a simple exercise. □

Proposition 4.2. *For any $d \geq 1$ and $k \in \mathbb{N}^d$, we have*

$$\gamma_k = k_1 + \cdots + k_d + I_d + \sum_{\nu=1}^{d} D_\nu \Big(\frac{1}{q^{k_1 + 2k_2 + \cdots + \nu k_\nu}} + \frac{1}{q^{\nu k_{d-\nu+1} + \cdots + 2k_{d-1} + k_d}} \Big). \tag{4.3}$$

Proof. If $w = w_\nu$, it is easy to see that the expression in (3.7) is D_w times

$$(\alpha_2/\alpha_1)^{k_1} (\alpha_3/\alpha_1)^{k_2} \cdots (\alpha_{\nu+1}/\alpha_1)^{k_\nu} = q^{-k_1 - 2k_2 - \cdots - \nu k_\nu}.$$

Similarly, if $w = w'_\nu$, then the expression in (3.7) is D_w times

$$\Big(\frac{\alpha_{d+1}}{\alpha_d}\Big)^{k_d} \Big(\frac{\alpha_{d+1}}{\alpha_{d-1}}\Big)^{k_{d-1}} \cdots \Big(\frac{\alpha_{d+1}}{\alpha_{d-\nu+1}}\Big)^{k_{d-\nu+1}} = q^{-k_d - 2k_{d-1} - \cdots - \nu k_{d-\nu+1}}.$$

So the result follows from the proof of Proposition 3.7. □

Example 4.3. If $d = 1$, then writing $k_1 = k$,

$$\gamma_k = k + I_1 + \frac{2D_1}{q^k} = k - \frac{2q}{q^2 - 1}\Big(1 - \frac{1}{q^k}\Big).$$

This calculation was done in [21].

Example 4.4. If $d = 2$, then writing $k_1 = k$ and $k_2 = \ell$,

$$\gamma_{k,\ell} = k + \ell + I_2 + D_1\Big(\frac{1}{q^k} + \frac{1}{q^\ell}\Big) + D_2\Big(\frac{1}{q^{k+2\ell}} + \frac{1}{q^{2k+\ell}}\Big).$$

Example 4.5. Let us now calculate γ_k when $k = e_r$. In this case,

$$k_1 + \cdots + \nu k_\nu = \begin{cases} 0 & \text{if } 1 \le \nu \le r-1, \\ r & \text{if } r \le \nu \le d. \end{cases}$$

Hence

$$\sum_{\nu=1}^{d} D_\nu \frac{1}{q^{k_1+2k_2+\cdots+\nu k_\nu}} = \sum_{\nu=1}^{r-1} D_\nu + \frac{1}{q^r} \sum_{\nu=r}^{d} D_\nu = \sum_{\nu=1}^{d} D_\nu - \Big(1 - \frac{1}{q^r}\Big) \sum_{\nu=r}^{d} D_\nu.$$

Note that

$$\sum_{\nu=1}^{d} D_\nu = \sum_{\nu=1}^{d} \Big(\frac{1}{q^\nu - 1} - \frac{1}{q^{\nu+1} - 1} \Big) = \frac{1}{q-1} - \frac{1}{q^{d+1}-1} = \frac{1}{2}|I_d|.$$

Similarly,

$$\begin{aligned} \sum_{\nu=1}^{d} D_\nu \frac{1}{q^{\nu k_{d-\nu+1}+\cdots+k_d}} &= \sum_{\nu=1}^{d-r} D_\nu + \frac{1}{q^{d-r+1}} \sum_{\nu=d-r+1}^{d} D_\nu \\ &= \sum_{\nu=1}^{d} D_\nu - \Big(1 - \frac{1}{q^{d-r+1}}\Big) \sum_{\nu=d-r+1}^{d} D_\nu. \end{aligned}$$

Hence by (4.3),

$$\gamma_{e_r} = 1 - \Big(1 - \frac{1}{q^r}\Big) \sum_{\nu=r}^{d} D_\nu - \Big(1 - \frac{1}{q^{d-r+1}}\Big) \sum_{\nu=d-r+1}^{d} D_\nu.$$

Evaluating the sums and rearranging, we get

$$\gamma_{e_r} = \frac{1}{q^{d+1}-1} + \frac{q^{d+1}}{q^{d+1}-1}\Big(1 - \frac{1}{q^r} - \frac{1}{q^{d-r+1}}\Big). \tag{4.4}$$

For example, if $d = 2$ and $r = 1$, we get $\gamma_{1,0} = (q^2 - 1)/(q^2 + q + 1)$. By Proposition 3.5(iii), $\gamma_{0,1} = (q^2 - 1)/(q^2 + q + 1)$ too. So for the simple nearest neighbour random walk we have

$$\gamma = \frac{\gamma_{1,0} + \gamma_{0,1}}{2} = \frac{q^2 - 1}{q^2 + q + 1}.$$

For $d = 1$, $\gamma = \gamma_{e_1} = (q-1)/(q+1)$.

Proposition 4.6. *For any nonzero $k \in \mathbb{N}^d$, γ_k is strictly positive, and so the γ of Theorem 3.8 is strictly positive if $a_k > 0$ for some $k \neq 0$.*

Proof. The only negative term in the formula (4.3) is I_d, and

$$|I_d| = \frac{2}{q-1} - \frac{2}{q^{d+1}-1} < \frac{2}{q-1} \le 1 \quad \text{if } q \ge 3.$$

Since the term $k_1 + \cdots + k_d$ in (4.3) is at least 1 if $k \neq 0$, the proof is complete for $q \neq 2$. If $q = 2$, then $|I_d| = 2 - 2/(2^{d+1} - 1) < 2$, and so γ_k is again positive if $k_1 + \cdots + k_d \ge 2$. Finally if $q = 2$ and $k = e_r$, then $\gamma_k > 0$ by (4.4). □

Remark 4.7. The last result follows from Theorem 8.14 of [25] when (Z_n) is irreducible (see Theorem 5.12 below). One can also show that the numbers $\gamma^{(j)}$, and therefore γ, are all strictly positive using Proposition 3.7 and using Theorem 5.12 below. For the m-th power P^m of P has the form $\sum a_k^{(m)} P_k$. Take $\ell \in \mathbb{N}^d$ with each component a large multiple of D_1, and let $x \in S_\ell(o)$. By Theorem 5.12 we can choose a large m so that $p^{(m)}(o, x) > 0$, and therefore $a_\ell^{(m)} > 0$. The $\gamma^{(j)}$ for P^m, say $\gamma^{(j,m)}$, satisfies $\gamma^{(j,m)} \ge a_\ell^{(m)} \gamma_\ell^{(j)}$, which by Proposition 3.7 is strictly positive. But $\gamma^{(j,m)} = m\gamma^{(j)}$ by Theorem 3.8, for example, so that $\gamma^{(j)} > 0$ too.

Remark 4.8. We haven't an explicit formula for $\gamma_k^{(j)}$ in general, but for $d = 2$ the methods of Lemma 4.1 show that

$$\begin{aligned}\gamma_{k,\ell}^{(1)} = \gamma_{\ell,k}^{(2)} &= \ell - \frac{q(q+1)}{q^3-1} + \frac{q}{q^2-1}\Big(\frac{2}{q^\ell} - \frac{1}{q^k}\Big) \\ &\quad + \frac{q^2}{(q^2-1)(q^2+q+1)}\Big(\frac{2}{q^{2k+\ell}} - \frac{1}{q^{k+2\ell}}\Big).\end{aligned}$$

5 The local limit theorem

In this section, $P = \sum_{k \in \mathbb{N}^d} a_k P_k$ is an isotropic transition probability matrix on $\mathfrak{X}^0$, also denoted $(p(x, y))_{x,y \in \mathfrak{X}^0}$, with n-th power $P^n = (p^{(n)}(x, y))_{x,y \in \mathfrak{X}^0}$. As remarked above, P^n is also isotropic, and $\widehat{P^n}(t) = \big(\widehat{P}(t)\big)^n$ for all $t \in T_d$. Hence for any $y \in S_k(x)$, (2.8) applied to $A = P^n$ shows that

$$p^{(n)}(x, y) = \int_{T_d} \big(\widehat{P}(t)\big)^n \overline{\widehat{P_k}(t)}\, d\mu(t). \tag{5.1}$$

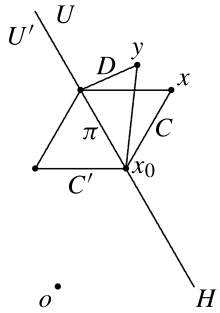
In this section we use this to describe the asymptotic behaviour of $p^{(n)}(x, y)$.

Lemma 5.1. *Let $k \in \mathbb{N}^d$, $k \neq 0$, and let $x \in S_k(o)$. If $d \ge 2$, there is a vertex $y \in S_k(o)$ such that $y \in S_{1,0,\dots,0,1}(x)$; if $d = 1$, there is a $y \in S_k(o)$ such that $y \in S_2(x)$.*

Proof. The case $d = 1$ is clear, so assume that $d \ge 2$ below. First observe that if C and D are distinct i-adjacent chambers in $\mathfrak{X}$, having in common all except their type i vertices a and b, respectively, then $b \in S_{1,0,\dots,0,1}(a)$. For let A be an apartment of $\mathfrak{X}$ containing C and D, which we identify with Σ. Using the simple transitivity property of $\tilde{W}_0$ mentioned in the introduction, we may assume that C is the standard chamber C_0. Using the element $(\lambda_1, \dots, \lambda_{d+1}) + \mathbb{Z}\mathbf{1} \mapsto (\lambda_{d+1} + 1, \lambda_2, \dots, \lambda_d) +$

$\mathbb{Z}\mathbf{1}$ of $\tilde{W}$ which maps C_0 to C_0 and cyclically permutes its vertices, we may also assume that $a = 0$. Then b must be $(1, 0, \ldots, 0, 1) = \Delta((2, 1, \ldots, 1, 0))$.

Recall that a gallery $C_0, \ldots, C_n$ in a building is a finite sequence of chambers such that for each $j < n$, C_j is i-adjacent to C_{j+1} for some i. Choose an apartment A containing o and x, and then choose a gallery $C_0, \ldots, C_n$ in A with $o \in C_0$, $x \in C_n$ and n minimal. Let $C = C_n$. This gallery is a minimal gallery from C_0 to C. So by [18, Proposition 2.7], if H is the wall in A determined by the panel $\pi = C \setminus \{x\}$, and if U, U' the two half-spaces in A determined by H, then $x \in U$, $x \notin U'$ and $o \in U'$ (o could also be in U). Let $C' \neq C$ be the other chamber in A containing π ($C' = C_{n-1}$ if $n \geq 1$). Thus $C' \subset U'$. Since $\mathfrak{X}$ is thick, there is a chamber $D \neq C, C'$ containing π. Let y be the vertex of D not in π. Then $y \in S_{1,0,\ldots,0,1}(x)$ by the first paragraph above. The diagram illustrates the case $d = 2$.



Now $y \in S_k(o)$. For there is an apartment B containing U' and D ([4, p. 169, Exercise 3], [18, p. 35, Exercise 3]). The retraction $\rho_{A,C'}$ ([4, p. 86], [18, p. 32]), restricted to B, is an isomorphism $B \to A$ fixing U', and in particular o, and as it preserves distances from C', it must map D to C and so y to x. $\square$

Lemma 5.2. *Suppose that $x, y \in \mathfrak{X}^0$ lie in a common apartment of $\mathfrak{X}$, which we identify with Σ, with $x = 0$ and $y = (y_j) + \mathbb{Z}\mathbf{1}$. For each $w \in S_{d+1}$, let ω_w be the equivalence class of the sector $\mathcal{S}_w = \{\lambda + \mathbb{Z}\mathbf{1} : \lambda_{w(1)} \geq \cdots \geq \lambda_{w(d+1)}\}$. Then $h_i(x, y; \omega_w) = y_{w(i)} - y_{w(i+1)}$ for each i.*

Proof. We refer to the definition of $S_k^x(\omega)$ given in Section 2.1. Since $\lambda + \mathbb{Z}\mathbf{1} \mapsto w^{-1}.\lambda + \mathbb{Z}\mathbf{1}$ is an isomorphism mapping 0 to 0 and $\mathcal{S}_w = S^0(\omega_w)$ to $\mathcal{S}_0$, we see that $S_k^0(\omega_w) = w.\mu + \mathbb{Z}\mathbf{1}$ if $k = \Delta\mu \in \mathbb{N}^d$. Also, $S^y(\omega_w) = \{y + \lambda + \mathbb{Z}\mathbf{1} : \lambda_{w(1)} \geq \cdots \geq \lambda_{w(d+1)}\}$, so that $\lambda + \mathbb{Z}\mathbf{1} \mapsto w^{-1}.(\lambda - y) + \mathbb{Z}\mathbf{1}$ is an isomorphism mapping y to 0 and $S^y(\omega_w)$ to $\mathcal{S}_0$, and $S_k^y(\omega_w) = w.\mu + y + \mathbb{Z}\mathbf{1}$ if $k = \Delta\mu \in \mathbb{N}^d$. The result follows. $\square$

Let us write 1 for the $d+1$-tuple $(1, 1, \ldots, 1) \in T_d$.

Lemma 5.3. *Let $t = (t_1, \ldots, t_{d+1}) \in T_d$. Then $|\widehat{P_k}(t)| \leq \widehat{P_k}(1)$, and if equality holds in this inequality for some $k \neq 0$, then $t_1 = \cdots = t_{d+1} \in \{1, \zeta, \ldots, \zeta^d\}$ where $\zeta = e^{2\pi i/(d+1)}$.*

Proof. Let o, x be any vertices such that $x \in S_k(o)$. We can write (2.5) in the more symmetric form

$$\widehat{P}_k(t) = \int_\Omega \frac{t_1^{\mu_1(o,x;\omega)} t_2^{\mu_2(o,x;\omega)} \cdots t_{d+1}^{\mu_{d+1}(o,x;\omega)}}{q^{-\frac{1}{2}\sum_1^d i(d+1-i)h_i(o,x;\omega)}} \, d\nu_o(\omega), \tag{5.2}$$

where $(\mu_i(o, x; \omega))$ is any $d + 1$-tuple such that $\mu_i(o, x; \omega) - \mu_{i+1}(o, x; \omega) = h_i(o, x; \omega)$ for each i. The inequality $|\widehat{P}_k(t)| \le \widehat{P}_k(1)$ is clear from (5.2). Suppose that equality holds in this inequality. Denote the integrand in (5.2) by $f(\omega)$ for a moment. Then f is a continuous (indeed, locally constant) function on Ω, and $f(\omega) \neq 0$ for all ω. Hence $|\widehat{P}_k(t)| = \widehat{P}_k(1)$ implies that $f(\omega)/|f(\omega)|$ is constant. Thus $t_1^{\mu_1(o,x;\omega)} \cdots t_{d+1}^{\mu_{d+1}(o,x;\omega)}$ is constant with respect to ω, the constant value being $\widehat{P}_k(t)/\widehat{P}_k(1)$. But the integral (5.2) is independent of $x \in S_k(o)$, and so if $x, y \in S_k(o)$, we have

$$t_1^{\mu_1(o,x;\omega)} \cdots t_{d+1}^{\mu_{d+1}(o,x;\omega)} = t_1^{\mu_1(o,y;\omega)} \cdots t_{d+1}^{\mu_{d+1}(o,y;\omega)} \tag{5.3}$$

for all $\omega \in \Omega$. By Lemma 5.1, there exist $x, y \in S_k(o)$ with $y \in S_{1,0,\ldots,0,1}(x)$. We now use $h_i(o, y; \omega) = h_i(o, x; \omega) + h_i(x, y; \omega)$. Take any apartment A containing x and y. We may identify A with Σ and x with 0 and y with $(y_j) + \mathbb{Z}\mathbf{1} = (2, 1, \ldots, 1, 0) + \mathbb{Z}\mathbf{1}$. Taking $\omega = \omega_w$ as in Lemma 5.2, we have $h_i(x, y; \omega_w) = y_{w(i)} - y_{w(i+1)}$, and so (5.3) implies that

$$t_1^{y_{w(1)}} t_2^{y_{w(2)}} \cdots t_{d+1}^{y_{w(d+1)}} = 1$$

for any permutation $(y_{w(j)})$ of $(y_j) = (2, 1, \ldots, 1, 0)$. Clearly this implies that $t_1 = \cdots = t_{d+1}$, and since their product is 1, their common value must be a power of ζ. □

Lemma 5.4. *Let $\theta_1, \ldots, \theta_{d+1} \in \mathbb{R}$ satisfy $\sum_{j=1}^{d+1} \theta_j = 0$ and $|\theta_j| < \pi$ for each j. Let $t = (e^{i\theta_1}, \ldots, e^{i\theta_{d+1}}) \in T_d$. Then for each nonzero $k \in \mathbb{N}^d$,*

$$\widehat{P}_k(t) = \widehat{P}_k(1) - \sum_{r,s=1}^{d+1} b_{k,r,s}\theta_r\theta_s + R_k(\theta_1, \ldots, \theta_{d+1}), \tag{5.4}$$

where $\sum_{r,s=1}^{d+1} b_{k,r,s}\theta_r\theta_s > 0$ unless $\theta_1 = \cdots = \theta_{d+1} = 0$, and where

$$|R_k(\theta_1, \ldots, \theta_{d+1})| \le \frac{1}{6}(d+1)^{3/2}|k|^3\,\widehat{P}_k(1)\Big(\sum_{j=1}^{d+1}\theta_j^2\Big)^{3/2}. \tag{5.5}$$

Proof. It is elementary that for $\theta \in \mathbb{R}$ we can write $e^{i\theta} = 1 + i\theta - \frac{1}{2}\theta^2 + E(\theta)$, where $|E(\theta)| \le |\theta|^3/6$. We apply this to $\theta = \sum_{j=1}^{d+1} \theta_j\mu_j(o, x; \omega)$, where $x \in S_k(o)$, $\omega \in \Omega$, $\mu_j(o, x; \omega) = h_j(o, x; \omega) + \cdots + h_d(o, x; \omega)$ for $j = 1, \ldots, d$

and $\mu_{d+1}(o, x; \omega) = 0$. As remarked after Lemma 2.1, we have $|\mu_j(o, x; \omega)| \le \mathrm{dist}(o, x) = |k|$ for each j. Hence the numerator of the integrand in (5.2) equals

$$1 + i \sum_{j=1}^{d+1} \theta_j \mu_j(o, x; \omega) - \frac{1}{2}\Big(\sum_{j=1}^{d+1} \theta_j \mu_j(o, x; \omega)\Big)^2 + E_k(\theta_1, \dots, \theta_{d+1}),$$

where

$$|E_k(\theta_1, \dots, \theta_{d+1})| \le \frac{1}{6}\Big|\sum_{j=1}^{d+1} \theta_j \mu_j(o, x; \omega)\Big|^3 \le \frac{1}{6}(d+1)^{3/2}|k|^3\Big(\sum_{j=1}^{d+1} \theta_j^2\Big)^{3/2}.$$

Next observe that for each j,

$$\int_\Omega \frac{\mu_j(o, x; \omega)}{q^{-\frac{1}{2}\sum_1^d i(d+1-i)h_i(o,x;\omega)}}\, d\nu_o(\omega) = 0. \tag{5.6}$$

To see this, let $\theta_1, \dots, \theta_d \in \mathbb{R}$, and $\theta_{d+1} = -(\theta_1 + \cdots + \theta_d)$. Write $F(\theta_1, \dots, \theta_d)$ for $h_t(P_k) = \widehat{P}_k(t)$, where $t_j = e^{i\theta_j}$ for each j. By (5.2), with $\mu_{d+1}(o, x; \omega)$ taken to be 0, it follows that for each $j \in \{1, \dots, d\}$, $\frac{\partial}{\partial \theta_j} F$, evaluated at $(\theta_1, \dots, \theta_d) = (0, \dots, 0)$, equals

$$i \int_\Omega \frac{\mu_j(o, x; \omega)}{q^{-\frac{1}{2}\sum_1^d i(d+1-i)h_i(o,x;\omega)}}\, d\nu_o(\omega). \tag{5.7}$$

But $h_t(P_k) = h_{t'}(P_k)$ if t' is any permutation of t. Interchanging t_j and t_{d+1}, we see that $\frac{\partial}{\partial \theta_j} F$, evaluated at $(\theta_1, \dots, \theta_d) = (0, \dots, 0)$, also equals minus 1 times the right hand side of (5.7). Thus (5.6) holds. It is now clear that we have a formula (5.4), and that (5.5) holds. It is also clear that $\sum_{r,s} b_{k,r,s}\theta_r\theta_s \ge 0$. If equality holds, then $\sum_{j=1}^{d+1} \theta_j \mu_j(o, x; \omega) = 0$ for all ω. But then $\widehat{P}_k(t) = \widehat{P}_k(1)$ by (5.2), and so by Lemma 5.3 there is a $\nu \in \{0, \dots, d\}$ such that $e^{i\theta_j} = \zeta^\nu$ for each j. Hence for each j we can write $\theta_j = 2\pi\nu/(d+1) + 2\pi n_j$ for some integer n_j. The condition $|\theta_j| < \pi$ now implies that the n_j's are all 0 or all -1, and the condition $\sum_{j=1}^{d+1} \theta_j = 0$ then implies that $\nu = 0$, and that $\theta_j = 0$ for all j. □

Lemma 5.5. *There is a polynomial $Q(k_1, \dots, k_d)$ of degree at most $d(d+1)/2$ such that*

$$\widehat{P}_k(1) = \frac{Q(k_1, \dots, k_d)}{q^{\frac{1}{2}\sum_{j=1}^d j(d+1-j)k_j}}$$

for all $k = (k_1, \dots, k_d) \in \mathbb{N}^d$. Hence there is a number M, depending only on d and q, such that

$$\widehat{P}_k(1) \le \frac{M(|k|+1)^{d(d+1)/2}}{q^{\frac{1}{2}\sum_{j=1}^d j(d+1-j)k_j}}. \tag{5.8}$$

Proof. (cf [16, §4.6]) Assume that $z_1, \dots, z_{d+1} \in \mathbb{C}$ are distinct, not necessarily with product 1. The sum on the right in (2.6) can be written

$$\frac{\sum_{w \in S_{d+1}} \epsilon(w) z_{w(1)}^{\lambda_1} \cdots z_{w(d+1)}^{\lambda_{d+1}} \prod_{1 \le i < j \le d+1} (z_{w(i)} - q^{-1} z_{w(j)})}{\prod_{1 \le i < j \le d+1} (z_i - z_j)}, \tag{5.9}$$

where $\lambda_i = k_i + \cdots + k_d$ for each i, as usual, and where $\epsilon(w)$ is the sign of the permutation w. We know that this sum is a polynomial in $z_1, \dots, z_{d+1}$, and so the limit exists as the z_i's tend to 1. To find the limit, we use L'Hôpital's rule repeatedly. First we fix $z_{d+1} = 1$, and let z_d tend to 1. Because of the factor $z_d - z_{d+1}$ in the denominator, we must differentiate the numerator and the denominator of (5.9) just once with respect to z_d, and evaluate the ratio of these derivatives at $z_d = 1$. The denominator now has a factor $(z_{d-1} - 1)^2$, and we must next differentiate the new numerator and denominator exactly twice with respect to z_{d-1}, and evaluate the ratio of these derivatives at $z_{d-1} = 1$. Continuing in this way, we successively eliminate $z_d, z_{d-1}, \dots, z_1$, differentiating the numerator and denominator each a total of $1 + 2 + \cdots + d = d(d+1)/2$ times. Since the exponents of the $z_{w(i)}$'s in the numerator of (5.9) are linear in $k_1, \dots, k_d$, the final numerator is a polynomial of degree at most $d(d+1)/2$ in the k_i's. □

Corollary 5.6. *Let $\theta_1, \dots, \theta_{d+1} \in \mathbb{R}$ satisfy $\sum_{j=1}^{d+1} \theta_j = 0$ and $|\theta_j| < \pi$ for each j. Let $t = (e^{i\theta_1}, \dots, e^{i\theta_{d+1}}) \in T_d$. Assume that $a_k > 0$ for at least one $k \neq 0$. Then*

$$\widehat{P}(t) = \widehat{P}(1)\Big(1 - \sum_{r,s=1}^{d+1} b_{r,s}\theta_r\theta_s + R(\theta_1, \dots, \theta_{d+1})\Big),$$

where $\sum_{r,s=1}^{d+1} b_{r,s}\theta_r\theta_s > 0$ unless $\theta_1 = \cdots = \theta_{d+1} = 0$, and where

$$|R(\theta_1, \dots, \theta_{d+1})| \le K \widehat{P}(1)^{-1} \Big(\sum_{j=1}^{d+1} \theta_j^2\Big)^{3/2} \tag{5.10}$$

for some $K > 0$ depending only on d and q.

Proof. Setting $b_{0,r,s} = 0$ and $R_0(\theta_1, \dots, \theta_{d+1}) = 0$, $b_{r,s} = \widehat{P}(1)^{-1} \sum_k a_k b_{k,r,s}$ and $R(\theta_1, \dots, \theta_{d+1}) = \widehat{P}(1)^{-1} \sum_k a_k R_k(\theta_1, \dots, \theta_{d+1})$. Then (5.10) follows from (5.5) and (5.7). We simply use $\sum_k |k|^{3+d(d+1)/2} q^{-\frac{1}{2}\sum_{j=1}^d j(d+1-j)k_j} < \infty$ and the fact that $a_k \le 1$. □

Lemma 5.7. *Let $0 \le u \in \mathbb{R}$ and let $v \in \mathbb{C}$, with $|u - v| \le u/2$. Then for $n \ge u$,*

$$\Big|\Big(1 - \frac{v}{n}\Big)^n - e^{-u}\Big| \le e^{-u/4}\Big(\frac{u^2}{2n} + |u - v|\Big).$$

Proof. For $a, b \in \mathbb{C}$, $a^n - b^n = \int_{\overrightarrow{ba}} n z^{n-1}\, dz$, and so $|a^n - b^n| \le n|a - b|\, c^{n-1}$, where $c = \max\{|(1-t)b + ta| : 0 \le t \le 1\}$. We apply this to $b = e^{-u/n}$ and $a = 1 - v/n$. Notice that $|a| = |1 - u/n + (u - v)/n| \le 1 - u/n + u/2n =$

$1 - u/2n \le e^{-u/2n}$, and so $c \le e^{-u/2n}$. Thus $c^{n-1} \le e^{-u(n-1)/2n} \le e^{-u/4}$. So the result follows from the estimate

$$|a-b| = \left|1 - \frac{v}{n} - e^{-u/n}\right| \le \left|1 - \frac{u}{n} - e^{-u/n}\right| + \frac{|u-v|}{n} \le \frac{u^2}{2n^2} + \frac{|u-v|}{n}.$$

□

Lemma 5.8. *Let $t_j = e^{i\theta_j}$, where $\theta_1, \dots, \theta_{d+1} \in \mathbb{R}$ and $\sum_{j=1}^{d+1} \theta_j = 0$, and write $t = (t_1, \dots, t_{d+1}) \in T_d$. Then for some constant $M > 0$,*

$$\frac{1}{|c(t)|^2} = \prod_{1 \le j < k \le d+1} \frac{(\theta_j - \theta_k)^2}{(1-1/q)^2}\Big(1 + E_{j,k}\Big), \tag{5.11}$$

where $|E_{j,k}| \le M(\theta_j - \theta_k)^2$ for each j, k.

Proof. It is elementary that there is an $M > 0$ so that for $x \in \mathbb{R}$,

$$\frac{|e^{ix} - 1|^2}{|e^{ix} - 1/q|^2} = \frac{2(1 - \cos x)}{(1-1/q)^2 + (2/q)(1 - \cos x)} = \frac{x^2}{(1-1/q)^2}\Big(1 + E(x)\Big),$$

where $|E(x)| \le Mx^2$. Applying this to $x = \theta_j - \theta_k$ for each $j < k$, we get (5.11). □

Recall that for $x \in \mathfrak{X}^0$, $\tau(x)$ is the type of x, and that $\tau(x) \in \{0, 1, \dots, d\}$, which we shall think of here as the additive group $\mathbb{Z}/(d+1)\mathbb{Z}$. Recall also that for $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, we set $\|k\| = \sum_{i=1}^d ik_i$.

Lemma 5.9. *The sequence $(\tau(Z_n))$ is a random walk on $\mathbb{Z}/(d+1)\mathbb{Z}$ with distribution $\mu : \mathbb{Z}/(d+1)\mathbb{Z} \to [0, 1]$ given by*

$$\mu(j) = \sum_{k : \|k\| \equiv j \pmod{d+1}} a_k.$$

Proof. As noted in the introduction, if $y \in S_k(x)$ then $\tau(y) \equiv \tau(x) + \|k\| \pmod{d+1}$. So if $\tau(x) = i$, the sum $\bar{p}(i, j)$ of $p(x, y)$ over the $y \in \mathfrak{X}^0$ such that $\tau(y) = j$ equals

$$\sum_{k : \|k\| \equiv j-i \pmod{d+1}} a_k.$$

So writing $\mathfrak{X}_j = \{x \in \mathfrak{X}^0 : \tau(x) = j\}$, the transition probability matrix P is factorizable over $\mathbb{Z}/(d+1)\mathbb{Z}$, and $\bar{p}(i, j) = \bar{p}(0, j-i) = \mu(j-i)$. By Lemma 3.1, the corresponding random walk on $\mathbb{Z}/(d+1)\mathbb{Z}$ is $(\tau(Z_n))$. □

The following fact is well-known (cf. [10, §I.3], for example).

Lemma 5.10. *Let (W_n) be a random walk on the additive group $\mathbb{Z}/N\mathbb{Z}$, with distribution μ. Let D_1 [respectively D_2] denote the greatest common divisor of N and the numbers $j \in \{0, 1, \ldots, N-1\}$ with $\mu(j) > 0$ [respectively, the numbers $j - j'$, where $j, j' \in \{0, 1, \ldots, N-1\}$ and $\mu(j), \mu(j') > 0$]. Then D_2 is divisible by D_1. Write $D_2 = D_1 D$. Then D is the greatest common divisor of the set of integers n such that $\mu^{*n}(0) > 0$. Also, (W_n) is irreducible if and only if $D_1 = 1$, and is aperiodic if and only if $D = 1$.*

Below, D_1, D_2, D are the numbers of Lemma 5.10 for the random walk $(\tau(Z_n))$ on $\mathbb{Z}/(d+1)\mathbb{Z}$. Clearly D_1 [respectively, D_2] is the greatest common divisor of $d+1$ and the numbers $\|k\|$ with $a_k > 0$ [respectively, the numbers $\|k\| - \|k'\|$, where $a_k, a_{k'} > 0$].

Lemma 5.11. *Assume that $a_k > 0$ for at least one $k \neq 0$. Let $t \in T_d$. Then $|\widehat{P}(t)| < \widehat{P}(1)$ unless $t = (\zeta_2^m, \ldots, \zeta_2^m)$ for some $m \in \{0, \ldots, D_2 - 1\}$, where $\zeta_2 = e^{2\pi i/D_2}$.*

Proof. For $t \in T_d$,

$$|\widehat{P}(t)| = \Big|\sum_k a_k \widehat{P}_k(t)\Big| \le \sum_k a_k \big|\widehat{P}_k(t)\big| \le \sum_k a_k \widehat{P}_k(1) = \widehat{P}(1). \tag{5.12}$$

If equality holds in the second of the two inequalities, then $\big|\widehat{P}_k(t)\big| = \widehat{P}_k(1)$ for all $k \in \mathbb{N}^d$ such that $a_k > 0$. Since $a_k > 0$ for some $k \neq 0$, by Lemma 5.3 there is a $\nu \in \{0, \ldots, d\}$ such that $t = (\zeta^\nu, \ldots, \zeta^\nu)$, where $\zeta = e^{2\pi i/(d+1)}$. Taking any $k \in \mathbb{N}^d$ and any $x \in S_k(o)$, and using (2.3), the numerator of the integrand in (5.2) is

$$\zeta^{\nu\left(\sum_{i=1}^{d+1} \mu_i(o,x;\omega)\right)} = \zeta^{\nu \sum_{i=1}^{d} i k_i} = \zeta^{\nu\|k\|},$$

so $\widehat{P}_k(t) = \zeta^{\nu\|k\|}\widehat{P}_k(1)$. If equality also holds in the first inequality of (5.12), the numbers $\zeta^{\nu\|k\|}$, where $a_k > 0$, must all be equal. That is, if $a_k, a_{k'} > 0$, then $\nu(\|k\| - \|k'\|)$ must be divisible by $d+1$. Since D_2 is an integer linear combination of $d+1$ and some numbers of the form $\|k\| - \|k'\|$, $a_k, a_{k'} > 0$, we see that νD_2 is divisible by $d+1$. Thus $\zeta^\nu = \zeta_2^m$ for some $m \in \{0, \ldots, D_2 - 1\}$. □

Theorem 5.12. *Assume that $a_k > 0$ for at least one $k \neq 0$. Then the random walk $(Z_n)_{n\in\mathbb{N}}$ associated to P is irreducible if and only if $D_1 = 1$. For each $x \in \mathfrak{X}^0$, the greatest common divisor of $\{n : p^{(n)}(x, x) > 0\}$ is D. Thus $(Z_n)_{n\in\mathbb{N}}$ is irreducible and aperiodic if and only if $D_2 = 1$. In general, given vertices o, x such that $x \in S_\ell(o)$, then $p^{(n)}(o, x) > 0$ for some n if and only if $\|\ell\|$ is a multiple of D_1. There is a constant $A > 0$, and for any $\ell \in \mathbb{N}^d$ with $\|\ell\|$ divisible by D_1, there is an n_ℓ such that $p^{(n)}(o, x) > 0$ for $n = n_\ell + rD$, $r \in \mathbb{N}$, and for such n,*

$$p^{(n)}(o, x) = \frac{A\big(\widehat{P}(1)\big)^n \widehat{P}_\ell(1)}{n^{d(d+2)/2}}\Big(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\Big) \tag{5.13}$$

as $n \to \infty$.

Proof. Choose any $k^\circ \in \mathbb{N}^d$ such that $a_{k^\circ} > 0$. For $t \in T_d$, write $t = t_2 t'$, where $t_2 = (\zeta_2, \ldots, \zeta_2)$. Then $\widehat{P}_k(t) = \zeta_2^{\|k\|}\widehat{P}_k(t')$ for each $k \in \mathbb{N}^d$, and if $a_k > 0$ then $\zeta_2^{\|k\|} = \zeta_2^{\|k^\circ\|}$. Hence $\widehat{P}(t_2 t') = \zeta_2^{\|k^\circ\|}\widehat{P}(t')$. Since $c(t_2 t') = c(t')$, for $x \in S_\ell(o)$ we have

$$p^{(n)}(o,x) = \int_{T_d} \big(\widehat{P}(t_2 t')\big)^n \overline{\widehat{P}_\ell(t_2 t')}\, d\mu(t') = \zeta_2^{n\|k^\circ\|-\|\ell\|}\int_{T_d} \big(\widehat{P}(t')\big)^n \overline{\widehat{P}_\ell(t')}\, d\mu(t')$$
$$= \zeta_2^{n\|k^\circ\|-\|\ell\|}\, p^{(n)}(o,x). \tag{5.14}$$

So $p^{(n)}(o,x) = 0$ unless $n\|k^\circ\| - \|\ell\|$ is divisible by D_2, and if $p^{(n)}(o,x) > 0$, then $\|\ell\|$ must be divisible by D_1. Next, write $\|k^\circ\| = m^\circ D_1$, and notice that $\gcd(m^\circ, D) = 1$. For we can write $D_1 = n_0(d+1) + \sum_j n_j \|k^j\|$ for some integers n_j and for some $k^j \in \mathbb{N}^d$ with $a_{k^j} > 0$. Hence $D_1 - (\sum_j n_j)\|k^\circ\| = n_0(d+1) + \sum_j n_j(\|k^j\| - \|k^\circ\|)$ is divisible by D_2. So for some integers a, b, $D_1 = a\|k^\circ\| + bD_2 = am^\circ D_1 + bDD_1$. Thus $1 = am^\circ + bD$.

Now let $\ell \in \mathbb{N}^d$, with $\|\ell\| = m_\ell D_1$ divisible by D_1. If $n = am_\ell + rD$, with $r \in \mathbb{N}$ large enough, then $n \geq 0$, and $n\|k^\circ\| - \|\ell\| = (-bm_\ell + rm^\circ)D_2$ is divisible by D_2. The n_ℓ of the theorem's statement will be $am_\ell + r_0 D$, where r_0 is chosen large enough to ensure that $p^{(n)}(o,x) > 0$ for all $n = n_\ell + rD$ with $r \geq 0$, which is possible, as the calculation below shows.

Now let $\epsilon > 0$, and let N_ϵ denote the set of $(e^{i\theta_1}, \ldots, e^{i\theta_{d+1}}) \in T_d$ such that $\theta_{d+1} = -(\theta_1 + \cdots + \theta_d)$ and $|\theta_j| < \epsilon$ for $j = 1, \ldots, d$. By (5.10), there is a positive $\epsilon < \pi/d$, so that

$$|R(\theta_1, \ldots, \theta_{d+1})| \leq \frac{1}{2}\sum_{r,s=1}^{d+1} b_{r,s}\theta_r\theta_s$$

if $e^{i\theta} \in N_\epsilon$, where $e^{i\theta} = (e^{i\theta_1}, \cdots, e^{i\theta_d}, e^{-i(\theta_1+\cdots+\theta_d)})$, an abbreviation also used below.

Assuming $n\|k^\circ\| - \|\ell\|$ is divisible by D_2, then arguing as for (5.14),

$$\int_{t_2^m N_\epsilon} \big(\widehat{P}(t)\big)^n \overline{\widehat{P}_\ell(t)}\, d\mu(t) = \int_{N_\epsilon} \big(\widehat{P}(t)\big)^n \overline{\widehat{P}_\ell(t)}\, d\mu(t) \quad \text{for } m = 0, \ldots, D_2 - 1.$$

Hence if we set $N'_\epsilon = T_d \setminus \big(N_\epsilon \cup t_2 N_\epsilon \cup \cdots \cup t_2^{D_2-1} N_\epsilon\big)$, then

$$p^{(n)}(o,x) = D_2 \int_{N_\epsilon} \big(\widehat{P}(t)\big)^n \overline{\widehat{P}_\ell(t)}\, d\mu(t) + \int_{N'_\epsilon} \big(\widehat{P}(t)\big)^n \overline{\widehat{P}_\ell(t)}\, d\mu(t). \tag{5.15}$$

The normalized Haar integral on T_d is

$$f \mapsto \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(e^{i\theta})\, d\theta_1 \cdots d\theta_d.$$

So $\int_{N_\epsilon} \big(\widehat{P}(t)\big)^n \overline{\widehat{P_\ell}(t)}\, d\mu(t)$ equals

$$\frac{1}{(2\pi)^d} \int_{-\epsilon}^{\epsilon} \cdots \int_{-\epsilon}^{\epsilon} \big(\widehat{P}(e^{i\theta})\big)^n \overline{\widehat{P_\ell}(e^{i\theta})} \frac{C_{d,q}}{|c(e^{i\theta})|^2}\, d\theta_1 \cdots d\theta_d$$

Let $\theta'_j = \sqrt{n}\theta_j$ for $j = 1, \dots, d$. Writing $\theta_{d+1} = -(\theta_1 + \cdots + \theta_d)$, and $\theta'_{d+1} = \sqrt{n}\theta_{d+1}$, Corollary 5.6 tells us that $\big(\widehat{P}(e^{i\theta})\big)^n$ equals

$$\begin{aligned}\big(\widehat{P}(e^{i\theta'/\sqrt{n}})\big)^n &= \big(\widehat{P}(1)\big)^n\Big(1 - \frac{1}{n}\sum_{r,s=1}^{d+1} b_{r,s}\theta'_r\theta'_s + R\Big(\frac{\theta'_1}{\sqrt{n}}, \dots, \frac{\theta'_{d+1}}{\sqrt{n}}\Big)\Big)^n \\ &= \big(\widehat{P}(1)\big)^n\Big(e^{-\sum_{r,s=1}^{d+1} b_{r,s}\theta'_r\theta'_s} + E_1\Big),\end{aligned}$$

where

$$|E_1| \le e^{-\frac{1}{4}\sum_{r,s=1}^{d+1} b_{r,s}\theta'_r\theta'_s}\Big(\frac{1}{2n}\Big(\sum_{r,s=1}^{d+1} b_{r,s}\theta'_r\theta'_s\Big)^2 + \frac{K}{\widehat{P}(1)n^{1/2}}\Big(\sum_{j=1}^{d+1} {\theta'_j}^2\Big)^{3/2}\Big).$$

Using (5.2) and $|e^{ix} - 1| \le |x|$ for $x \in \mathbb{R}$, it is easy to see that

$$\widehat{P_\ell}(e^{i\theta}) = \widehat{P_\ell}(1)\big(1 + E_2\big),$$

where

$$|E_2| \le \frac{1}{\sqrt{n}}|\ell|(d+1)^{1/2}\Big(\sum_{j=1}^{d+1} {\theta'_j}^2\Big)^{1/2}.$$

By Lemma 5.8

$$\frac{1}{|c(e^{i\theta})|^2} = \frac{1 + E_3}{n^{d(d+1)/2}(1 - 1/q)^{d(d+1)}} \prod_{1 \le j < k \le d+1} (\theta'_j - \theta'_k)^2,$$

where

$$|E_3| \le \frac{q(\theta'_1, \dots, \theta'_{d+1})}{n}$$

for some polynomial $q(\theta'_1, \dots, \theta'_{d+1})$.

Finally, by Lemma 5.9, there is a $\rho_\epsilon < 1$ so that $|\widehat{P}(t)| \le \rho_\epsilon \widehat{P}(1)$ for all $t \in N'_\epsilon$. So the second integral in (5.15) is negligible compared to the right hand side of (5.13).

Combining the above estimates, we see that (5.13) holds, and that A is equal to $D_2 C_{d,q}/(2\pi)^d(1 - 1/q)^{d(d+1)}$ times

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sum_{r,s=1}^{d+1} b_{r,s}\theta'_r\theta'_s} \prod_{1 \le j < k \le d+1} (\theta'_j - \theta'_k)^2\, d\theta'_1 \cdots d\theta'_d. \qquad \square$$

6 Bounded spherical functions

Consider the algebra $\mathcal{A}$ with the norm $\|\sum_k a_k P_k\|_1 = \sum_k |a_k|$. This makes $\mathcal{A}$ a normed algebra. For let $A = \sum_k a_k P_k$ and $B = \sum_k b_k P_k$, and write $AB = \sum_k c_k P_k$. Then $\sum_k |c_k| \le \left(\sum_k |a_k|\right)\left(\sum_k |b_k|\right)$ holds because this is clearly true (with equality) when $a_k \ge 0$, $b_k \ge 0$ and $\sum_k a_k = 1 = \sum_k b_k$. The algebra homomorphisms $h : \mathcal{A} \to \mathbb{C}$ continuous for this norm are clearly the h_z's for which the values $h_z(P_k)$ of the spherical function φ_z are bounded. In this section we determine the z's for which this holds.

If h_z is continuous on this normed algebra, then it can be extended by continuity to the completion of $\mathcal{A}$. Any isotropic transition probability matrix P as in (1.1) can be regarded as an element of this completion, and so $h_z(P)$, also denoted $\widehat{P}(z)$, is defined.

We first obtain a simpler expression for $\varphi_z(x)$ in the case $x \in S_{me_r}(o)$:

Lemma 6.1. *Let $z = (z_1, \dots, z_{d+1}) \in \mathbb{C}^{d+1}$, with $z_1 z_2 \cdots z_{d+1} = 1$ and suppose that $z_1, \dots, z_{d+1}$ are distinct. Let $r \in \{1, \dots, d\}$, $m \in \mathbb{N}$, and let $x \in S_k(o)$ for $k = me_r$. For each subset U of $\{1, \dots, d+1\}$ with exactly r elements, write $\{u_1, \dots, u_r\}$ for $\{z_j : j \in U\}$, and let $U' = \{u'_1, \dots, u'_{d+1-r}\}$ be $\{z_1, \dots, z_{d+1}\} \setminus \{u_1, \dots, u_r\}$. Then with $\alpha_j = q^{d/2-j+1}$ as usual,*

$$\varphi_z(x) = \frac{C_{d+1}}{C_r C_{d+1-r}} \sum_{U:|U|=r} \left(\frac{u_1 \cdots u_r}{\alpha_1 \cdots \alpha_r}\right)^m \prod_{\substack{1\le j\le r\\ 1\le k\le d+1-r}} \frac{u_j - u'_k/q}{u_j - u'_k}, \tag{6.1}$$

where $C_r = q^{r(r+1)/2} \prod_{j=1}^r \big((q-1)/(q^j-1)\big)$, and similarly for C_{d+1} and C_{d+1-r}.

Proof. By (2.6), $\varphi_z(x)$ is the sum over $w \in S_{d+1}$ of the terms

$$Cc(z_{w(1)}, \dots, z_{w(d+1)}) \left(\frac{z_{w(1)} \cdots z_{w(r)}}{\alpha_1 \cdots \alpha_r}\right)^m \tag{6.2}$$

(*cf* (3.5)). For $U \subset \{1, \dots, d+1\}$ with $|U| = r$, let Σ_U be the sum of the terms (6.2) over the $w \in S_{d+1}$ such that $w(\{1, \dots, r\}) = U$. With the notation of the above statement, Σ_U is $C(u_1 \cdots u_r/\alpha_1 \cdots \alpha_r)^m$ times the sum over $w_1 \in S_r$ and $w_2 \in S_{d+1-r}$ of the products

$$\prod_{1\le j<k\le r} \frac{u_{w_1(j)} - u_{w_1(k)}/q}{u_{w_1(j)} - u_{w_1(k)}} \times \prod_{1\le j<k\le d+1-r} \frac{u'_{w_2(j)} - u'_{w_2(k)}/q}{u'_{w_2(j)} - u'_{w_2(k)}}$$
$$\times \prod_{\substack{1\le j\le r\\ 1\le k\le d+1-r}} \frac{u_{w_1(j)} - u'_{w_2(k)}/q}{u_{w_1(j)} - u'_{w_2(k)}}.$$

The last of these products is

$$\prod_{\substack{1\le j\le r\\ 1\le k\le d+1-r}} \frac{u_j - u'_k/q}{u_j - u'_k},$$

independent of w_1 and w_2. The sum of the terms $Cc(z_{w(1)}, \ldots, z_{w(d+1)})$ over all $w \in S_{d+1}$ is 1, because $\varphi_z(o) = 1$. Here $C = C_{d+1}$. The condition $z_1 \cdots z_{d+1} = 1$ is clearly not needed for this. Applying this with r in place of $d+1$, we see that the sum over $w_1 \in S_r$ of the first of the above products is $1/C_r$, and the sum over $w_2 \in S_{d+1-r}$ of the middle of the above products is $1/C_{d+1-r}$. So (6.1) holds. □

Proposition 6.2. *Let $z = (z_1, \ldots, z_{d+1}) \in \mathbb{C}^{d+1}$, with $z_1 z_2 \cdots z_{d+1} = 1$. Assume that $|z_1| \geq |z_2| \geq \cdots \geq |z_{d+1}|$. Then the spherical function φ_z is bounded if and only if $|z_1 \cdots z_r| \leq q^{(d+1-r)r/2}$ for $r = 1, \ldots, d$.*

Proof. Since $\alpha_1 \cdots \alpha_r = q^{(d+1-r)r/2}$, the condition is that $|z_1 \cdots z_r| \leq \alpha_1 \cdots \alpha_r$ for each r. This implies that $|z_{w(1)} \cdots z_{w(r)}| \leq \alpha_1 \cdots \alpha_r$ for each permutation w, since the $|z_j|$'s are decreasing. When the z_j's are distinct, $\varphi_z(x)$ is the sum of terms (3.5), and so $|\varphi_z(x)| \leq \sum_{w \in S_{d+1}} C|c(z_{w(1)}, \ldots, z_{w(d+1)})|$ for any vertex x, and so φ_z is bounded. Since the algebra $\mathcal{A}$ with the norm $\|\cdot\|_1$ is a normed algebra, the algebra homomorphisms $h : \mathcal{A} \to \mathbb{C}$ which are continuous for this norm must all have norm 1, and so in fact $|\varphi_z(x)| \leq 1$ holds for all vertices x.

Now suppose that the z_j's are not distinct, but satisfy $|z_1| \geq \cdots \geq |z_{d+1}|$ and $|z_1 \cdots z_r| \leq \alpha_1 \cdots \alpha_r$ for all r. Let $z^{(n)} = (z_{1,n}, \ldots, z_{d+1,n})$, where $z_{j,n} = z_j e^{i\theta_{j,n}}$ for each j, n, and where $\sum_j \theta_{j,n} = 0$ for each n. It is clear that we can choose $\theta_{j,n}$'s in $\mathbb{R}$ so that $z_{1,n}, \ldots, z_{d+1,n}$ are distinct for each n and so that $\theta_{j,n} \to 0$ as $n \to \infty$, for each j. Then $\varphi_{z^{(n)}}(x) \to \varphi_z(x)$ for each x, and $|\varphi_{z^{(n)}}(x)| \leq 1$ for each n, x. Hence $|\varphi_z(x)| \leq 1$ for all x, so that φ_z is bounded.

Suppose, conversely, that $z_1 z_2 \cdots z_{d+1} = 1$, that $|z_1| \geq |z_2| \geq \cdots \geq |z_{d+1}|$, and that φ_z is bounded. Set $\lambda = \max\{|z_1 \cdots z_r|/\alpha_1 \cdots \alpha_r : r = 1, \ldots, d\}$. Choose any $r \in \{1, \ldots, d\}$ so that $|z_1 \cdots z_r|/\alpha_1 \cdots \alpha_r = \lambda$. Note that $|z_{r+1}| < |z_r|$ if $\lambda > 1$. For otherwise, $\lambda \geq |z_1 \cdots z_{r+1}|/\alpha_1 \cdots \alpha_{r+1} = \lambda|z_{r+1}|/\alpha_{r+1} = \lambda|z_r|q/\alpha_r$, so that $|z_r| \leq \alpha_r/q$. But this implies that $|z_1 \cdots z_r|/\alpha_1 \cdots \alpha_r$ is less than $|z_1 \cdots z_{r-1}|/\alpha_1 \cdots \alpha_{r-1}$, contrary to the choice of r. Thus if $\{u_1, \ldots, u_r\} = \{z_j : j \in U\}$ where $|U| = r$ and $U \neq U_0 = \{1, \ldots, r\}$, and if $\lambda > 1$, then $|u_1 \cdots u_r|/\alpha_1 \cdots \alpha_r \leq |z_1 \cdots z_{r-1} z_{r+1}|/\alpha_1 \cdots \alpha_r$ is strictly less that λ.

When the z_j's are distinct, we can use (6.1), and see that the sum there is dominated by the $U = U_0$ term if $\lambda > 1$, and so is clearly unbounded as $m \to \infty$. So $\lambda > 1$ cannot happen, i.e., $|z_1 \cdots z_r| \leq q^{(d+1-r)r/2}$ must hold for each r, if φ_z is bounded.

Now suppose $z = (z_1, \ldots, z_{d+1})$, with the $|z_j|$'s decreasing, the z_j's not distinct, φ_z bounded, but $\lambda > 1$. Write λ_U for $u_1 \cdots u_r/\alpha_1 \cdots \alpha_r$ for U as above. Let $\mu = \max\{|\lambda_U| : U \neq U_0\}$ $(< \lambda)$. As above, z is the limit of $d+1$-tuples z' such that $|z'_j| = |z_j|$ for each j, and such that $z'_1, \ldots, z'_{d+1}$ are distinct. Applying (6.1) to $\varphi_{z'}$, we have

$$\varphi_{z'}(x) = C'_{U_0}(\lambda'_{U_0})^m + \sum_{U \neq U_0} C'_U(\lambda'_U)^m, \tag{6.3}$$

for certain coefficients C'_U (independent of m). We now use this and the fact that $\varphi_z(x) = \lim_{z' \to z} \varphi_{z'}(x)$ to obtain an expression for $\varphi_z(x)$ when $x \in S_{me_r}(o)$.

As $\lambda|z_{r+1}|/|z_r| = |z_1 \cdots z_{r-1} z_{r+1}|/\alpha_1 \cdots \alpha_r \leq \mu$, we have $|z_k| \leq (\mu/\lambda)|z_j|$ for $1 \leq j \leq r$ and $r+1 \leq k \leq d+1$. Hence in the coefficient C'_{U_0} of $(\lambda'_{U_0})^m$

in (6.3), the factors $z'_j - z'_k$ ($1 \le j \le r$, $r+1 \le k \le d+1$, see (6.1)) in the denominator tend to $z_j - z_k \ne 0$, and so the term $C'_{U_0}(\lambda'_{U_0})^m$ in (6.3) tends to a limit $C_{U_0}(\lambda_{U_0})^m$ as $z' \to z$.

The sum of the remaining terms in (6.3) must therefore also tend to a limit as $z' \to z$. We can write this sum as a quotient

$$\frac{\sum_{U \ne U_0} \big(\prod_{j \in U} z'_j\big)^m p_{U,0}(z'_1, \dots, z'_{d+1})}{\prod_{1 \le j < k \le d+1} (z'_j - z'_k)} \tag{6.4}$$

where the $p_{U,0}$'s are polynomials with rational coefficients independent of m. After differentiating the numerator with respect to several of the z'_j's, a total of ℓ times, we get a sum

$$\sum_{U \ne U_0} \big(\prod_{j \in U} z'_j\big)^{m-\ell} p_{U,\ell}(z'_1, \dots, z'_{d+1}),$$

where the $p_{U,\ell}$'s are polynomials with coefficients which are polynomials in $\mathbb{Q}[m]$, of degree at most ℓ in m. We obtain the limit of (6.3) as $z' \to z$ by using L'Hôpital's Rule, differentiating the numerator and the denominator in (6.4) repeatedly, then evaluating at $z' = z$. It follows that for $x \in S_{me_r}(o)$ there is a formula

$$\varphi_z(x) = C_{U_0}(\lambda_{U_0})^m + \sum_{U \ne U_0} C_U(m)(\lambda_U)^m, \tag{6.5}$$

for $\varphi_z(x)$, where $C_{U_0} \ne 0$, and the other coefficients $C_U(m)$ are $O(m^\ell)$ for some integer ℓ, and where $|\lambda_{U_0}| = \lambda$ and $|\lambda_U| \le \mu$ for all $U \ne U_0$. It is now clear that as $\lambda > 1$ this is not bounded as $m \to \infty$. This is a contradiction to the hypothesis that φ_z is bounded. So $\lambda > 1$ cannot hold. □

7 A Central Limit Theorem

Lemma 7.1. *For $j = 1, \dots, d+1$, let $\alpha_j = q^{d/2-j+1}$ as usual, let $\theta_j \in [-\pi, \pi]$, and $z_j = \alpha_j e^{i\theta_j}$. Let $z = (z_1, \dots, z_{d+1})$, and assume that $\sum_j \theta_j = 0$. Let $k = (k_1, \dots, k_d) \in \mathbb{N}^d$. Then there is a number M, independent of k and the θ_j's, so that*

$$\Big| h_z(P_k) - e^{i\left(\theta_1 k_1 + (\theta_1+\theta_2)k_2 + \cdots + (\theta_1 + \cdots + \theta_d)k_d\right)} \Big| \le M(|\theta_1| + \cdots + |\theta_{d+1}|). \tag{7.1}$$

Proof. For any $x \in S_k(o)$, $h_z(P_k) = \varphi_z(x)$, and this is the sum of the terms (3.5) over $w \in S_{d+1}$. The $w = id$ term is

$$Cc(z_1, \dots, z_{d+1}) \prod_{r=1}^{d} \Big(\frac{z_1 \cdots z_r}{\alpha_1 \cdots \alpha_r}\Big)^{k_r} = Cc(z_1, \dots, z_{d+1}) e^{i \sum_{r=1}^{d} (\theta_1 + \cdots + \theta_r) k_r}.$$

For the z_j's as in the statement, $c(z_1, \dots, z_{d+1})$ is smooth function of the θ_j's about $(0, \dots, 0)$, so by Lemma 3.6 there is a number M_{id} so that

$$\begin{aligned}|Cc(z_1, \dots, z_{d+1}) - 1| &= |Cc(z_1, \dots, z_{d+1}) - Cc(\alpha_1, \dots, \alpha_{d+1})| \\ &\leq M_{id}(|\theta_1| + \cdots + |\theta_{d+1}|).\end{aligned}$$

Now $|z_j| = \alpha_j$ for each j, and as observed after (3.7), $\alpha_{w(1)} \cdots \alpha_{w(r)} \leq \alpha_1 \cdots \alpha_r$ for $r = 1, \dots, d$. So by Lemma 3.6, for $w \neq id$ the modulus of the w term (3.5) is at most

$$|Cc(z_{w(1)}, \dots, z_{w(d+1)})| = |Cc(z_{w(1)}, \dots, z_{w(d+1)}) - Cc(\alpha_{w(1)}, \dots, \alpha_{w(d+1)})|$$

which is at most $M_w(|\theta_1| + \cdots + |\theta_{d+1}|)$ for some constant M_w. The result follows. □

Lemma 7.2. *Let $(Z_n)_{n\in\mathbb{N}}$ be as in Theorem 3.8, and assume that $Z_0 \equiv o$. Let $z = (z_1, \dots, z_{d+1}) \in \mathbb{C}^{d+1}$ be such that φ_z are bounded (see Proposition 6.2). Then*

$$\mathbb{E}\big(\varphi_z(Z_n)\big) = \big(\widehat{P}(z)\big)^n.$$

Proof. For $n \in \mathbb{N}$, we can write $P^n = \sum_k a_k^{(n)} P_k$, and $a_k^{(n)} = \mathbb{P}(Z_n \in S_k(o))$. Since $\varphi_z(Z_n) = h_z(P_k)$ if $Z_n \in S_k(o)$, $\mathbb{E}\big(\varphi_z(Z_n)\big)$ equals

$$\sum_{k\in\mathbb{N}^d} h_z(P_k)\mathbb{P}(Z_n \in S_k(o)) = \sum_{k\in\mathbb{N}^d} a_k^{(n)} h_z(P_k) = h_z(P^n) = (h_z(P))^n, \tag{7.2}$$

which equals $\big(\widehat{P}(z)\big)^n$. We have used the continuity of h_z on the completion of $\mathcal{A}$ with respect to $\|\cdot\|_1$ to justify the last two equations in (7.2). □

Lemma 7.3. *Let P be as in (1.1), with $\sum_k |k|^2 a_k < \infty$, and $a_k > 0$ for at least one $k \neq 0$. Then there are numbers $c_{r,s}$, $r, s = 1, \dots, d$, such that, writing $\alpha = (\alpha_1, \dots, \alpha_{d+1})$ and $\alpha e^{i\theta} = (\alpha_1 e^{i\theta_1}, \dots, \alpha_{d+1} e^{i\theta_{d+1}})$, where $|\theta_j| < \pi$ for all j and $\theta_{d+1} = -\sum_{j=1}^d \theta_j$,*

$$\widehat{P}(\alpha e^{i\theta}) = 1 + i\sum_{j=1}^d \big(\gamma^{(j)} + \cdots + \gamma^{(d)}\big)\theta_j - \frac{1}{2}\sum_{r,s=1}^d c_{r,s}\theta_r\theta_s + o\Big(\sum_{j=1}^d \theta_j^2\Big), \tag{7.3}$$

and where

$$\sum_{r,s=1}^d c_{r,s}\theta_r\theta_s > \Big(\sum_{j=1}^d \big(\gamma^{(j)} + \cdots + \gamma^{(d)}\big)\theta_j\Big)^2 \tag{7.4}$$

unless $\theta_1 = \cdots = \theta_d = 0$.

Proof. As in the proof of Lemma 5.4, write $\mu_j(o,x;\omega) = h_j(o,x;\omega) + \cdots + h_d(o,x;\omega)$ for $j = 1,\ldots,d$ and $\mu_{d+1}(o,x;\omega) = 0$. We use (2.5) with $z_j = \alpha_j e^{i\theta_j}$ for all j. As in the proof of Proposition 3.5(iii), using (2.4) to express the integral in terms of ν_x, we obtain

$$\varphi_z(x) = \int_\Omega e^{i(\theta_1\mu_1(o,x;\omega)+\cdots+\theta_d\mu_d(o,x;\omega))}\,d\nu_x(\omega), \tag{7.5}$$

which we denote by $F_k(\theta_1,\ldots,\theta_d)$ or $F_k(\theta)$ if $x \in S_k(o)$. Hence $F_k(0) = 1$, as we can also see because $F_k(0) = h_\alpha(P_k) = 1$ by Lemma 3.6. Also, using Proposition 3.5, we have

$$\frac{\partial F_k}{\partial\theta_j}\Big|_{\theta=0} = i\int_\Omega \mu_j(o,x;\omega)\,d\nu_x(\omega) = i\big(\gamma_k^{(j)} + \cdots + \gamma_k^{(d)}\big).$$

Now write

$$\int_\Omega (\theta_1\mu_1(o,x;\omega) + \cdots + \theta_d\mu_d(o,x;\omega))^2\,d\nu_x(\omega) = \sum_{r,s=1}^d c_{k,r,s}\theta_r\theta_s.$$

As noted after Lemma 2.1, $|\mu_j(o,x;\omega)| \le |k|$ for $x \in S_k(o)$, and so from (7.5) we also have

$$\left|\frac{\partial^2 F_k}{\partial\theta_r\partial\theta_s}\right| \le |k|^2$$

for $r,s = 1,\ldots,d$. Since $\widehat{P}(\alpha e^{i\theta}) = \sum_k a_k \widehat{P_k}(\alpha e^{i\theta}) = \sum_k a_k F_k(\theta)$, the hypotheses show that this function of $\theta_1,\ldots,\theta_d$ has continuous second order partial derivatives, and that (7.3) holds with $c_{r,s} = \sum_k a_k c_{k,r,s}$.

To see (7.4), let $B_k = \sum_{j=1}^d (\gamma_k^{(j)} + \cdots + \gamma_k^{(d)})\theta_j$ and $C_k = \sum_{r,s} c_{k,r,s}\theta_r\theta_s$. Then

$$\begin{aligned}\Big(\sum_{j=1}^d (\gamma^{(j)} + \cdots + \gamma^{(d)})\theta_j\Big)^2 &= \Big(\sum_k a_k B_k\Big)^2 \le \sum_k a_k B_k^2 \le \sum_k a_k C_k \\ &= \sum_{r,s} c_{r,s}\theta_r\theta_s\end{aligned}$$

by the Cauchy-Schwarz inequality for series and for integrals. Indeed, if $x \in S_k(o)$,

$$\begin{aligned}B_k^2 &= \Big(\int_\Omega \theta_1\mu_1(o,x;\omega) + \cdots + \theta_d\mu_d(o,x;\omega)\,d\nu_x(\omega)\Big)^2 \\ &\le \int_\Omega \big(\theta_1\mu_1(o,x;\omega) + \cdots + \theta_d\mu_d(o,x;\omega)\big)^2\,d\nu_x = C_k.\end{aligned} \tag{7.6}$$

If equality holds in (7.6) for some $k \ne 0$, then $\theta_1\mu_1(o,x;\omega) + \cdots + \theta_d\mu_d(o,x;\omega)$ must be independent of ω, and hence independent of $x \in S_k(o)$ too, by Proposition 3.5. As in the proof of Lemma 5.3, we can find $x, y \in S_k(o)$ so that $y \in S_{1,0,\ldots,0,1}(x)$ and then for each $w \in S_{d+1}$ find $\omega_w \in \Omega$ so that $h_j(x,y;\omega_w) = y_{w(j)} - y_{w(j+1)}$ for $j = 1,\ldots,d$, where $(y_j) = (2,1,\ldots,1,0)$. So $\mu_j(o,y;\omega_w) - \mu_j(o,x;\omega_w) = y_{w(j)} - y_{w(d+1)}$, and therefore $\sum_{j=1}^{d+1}\theta_j y_{w(j)} = 0$ for each w, so that the θ_j's are all equal, and so 0. □

Theorem 7.4. *Let P be as in (1.1), with $\sum_k |k|^2 a_k < \infty$, and $a_k > 0$ for at least one $k \neq 0$. Then there is a positive definite $d \times d$ matrix $\Sigma = (\sigma_{r,s})$ such that*

$$\Big(\frac{k_1(Z_n) - \gamma^{(1)}n}{\sqrt{n}}, \dots, \frac{k_d(Z_n) - \gamma^{(d)}n}{\sqrt{n}}\Big)$$

converges in distribution to the normal distribution $N(0, \Sigma)$.

Proof. Fix $\vartheta_1, \dots, \vartheta_d \in \mathbb{R}$. We show that, for $\sigma_{r,s}$ defined below, as $n \to \infty$,

$$\mathbb{E}\Big(e^{i\big(\vartheta_1(k_1(Z_n)-\gamma^{(1)}n)+\dots+\vartheta_d(k_d(Z_n)-\gamma^{(d)}n)\big)/\sqrt{n}}\Big) \to e^{-\frac{1}{2}\sum_{r,s}\sigma_{r,s}\vartheta_r\vartheta_s}. \tag{7.7}$$

In Lemma 7.1, take $\theta_j = (\vartheta_j - \vartheta_{j-1})/\sqrt{n}$ for $j = 1, \dots, d$ (setting $\vartheta_0 = 0$), so that $\theta_1 + \dots + \theta_j = \vartheta_j/\sqrt{n}$ for each j. Set $\theta_{d+1} = -(\theta_1 + \dots + \theta_d)$, and assume that n is large enough so that $|\theta_j| \leq \pi$ for all j. By Lemma 7.1,

$$e^{i\big(\vartheta_1 k_1(Z_n)+\dots+\vartheta_d k_d(Z_n)\big)/\sqrt{n}} = \varphi_{\alpha e^{i\theta}}(Z_n) + O\Big(\frac{\sum_j |\vartheta_j|}{\sqrt{n}}\Big).$$

Since $\vartheta_1, \dots, \vartheta_d$ are fixed, by Lemma 7.2,

$$\mathbb{E}\Big(e^{i\big(\vartheta_1 k_1(Z_n)+\dots+\vartheta_d k_d(Z_n)\big)/\sqrt{n}}\Big) = \big(\widehat{P}(\alpha e^{i\theta})\big)^n + O\Big(\frac{1}{\sqrt{n}}\Big).$$

By Lemma 7.3, $\widehat{P}(\alpha e^{i\theta})$ equals

$$\begin{aligned}1 &+ \frac{i}{\sqrt{n}}\sum_{j=1}^{d}\big(\gamma^{(j)} + \dots + \gamma^{(d)}\big)(\vartheta_j - \vartheta_{j-1})\\ &-\frac{1}{2n}\sum_{r,s=1}^{d} c_{r,s}(\vartheta_r - \vartheta_{r-1})(\vartheta_s - \vartheta_{s-1}) + o\Big(\frac{1}{n}\Big),\end{aligned}$$

while

$$\begin{aligned}e^{-i\big(\vartheta_1\gamma^{(1)}+\dots+\vartheta_d\gamma^{(d)}\big)/\sqrt{n}} = 1 &- \frac{i}{\sqrt{n}}\sum_{j=1}^{d}(\gamma^{(j)} + \dots + \gamma^{(d)})(\vartheta_j - \vartheta_{j-1})\\ &-\frac{1}{2n}\Big(\sum_{j=1}^{d}(\gamma^{(j)} + \dots + \gamma^{(d)})(\vartheta_j - \vartheta_{j-1})\Big)^2\\ &+O\big(\frac{1}{n^{3/2}}\big).\end{aligned}$$

Routine calculations now show that (7.7) holds, with $\sum_{r,s}\sigma_{r,s}\vartheta_r\vartheta_s$ equal to

$$\sum_{r,s=1}^{d} c_{r,s}(\vartheta_r - \vartheta_{r-1})(\vartheta_s - \vartheta_{s-1}) - \Big(\sum_{j=1}^{d}(\gamma^{(j)} + \dots + \gamma^{(d)})(\vartheta_j - \vartheta_{j-1})\Big)^2,$$

a positive definite form by Lemma 7.3. □

8 Application to convolution powers of bi-invariant densities

Let G be a locally compact group, and let K be a compact open subgroup of G. Let m denote left Haar measure on G, normalized so that $m(K) = 1$. Let $\varphi : G \to \mathbb{R}$ be bi-K-invariant, i.e., satisfy $\varphi(kgk') = \varphi(g)$ for all $g \in G$, $k, k' \in K$. Assume also that $\varphi(g) \geq 0$ for all g, and that $\int_G \varphi \, dm = 1$. Thus φ is the density function of a bi-K-invariant probability measure on G.

Now suppose that G acts on a building $\mathfrak{X}$ of type $\tilde{A}_d$, and that $K = \{g \in G : go = o\}$, where o is some fixed vertex. Assume that for each $g \in G$ the automorphism $x \mapsto gx$ of $\mathfrak{X}$ is type-rotating. Then $gS_r(x) = S_r(gx)$ for all $r \in \mathbb{N}^d$, $g \in G$ and $x \in \mathfrak{X}^0$, and so K maps each set $S_r(o)$, $r \in \mathbb{N}^d$, into itself.

Lemma 8.1. *Suppose that G acts transitively on $\mathfrak{X}^0$, and that K acts transitively on each set $S_r(o)$, $r \in \mathbb{N}^d$. Then we can define an isotropic transition probability matrix on $\mathfrak{X}^0$ by setting*

$$p(go, g'o) = \varphi(g^{-1}g').$$

Proof. Note first that the transitivity of G and the bi-K-invariance of φ shows that $p(x, y)$ is well-defined for all $x, y \in \mathfrak{X}^0$. For each $x \in \mathfrak{X}^0$, choose $g_x \in G$ such that $g_x o = x$. Then G is the disjoint union of the cosets $g_y K$, $y \in \mathfrak{X}^0$, and for fixed $x \in \mathfrak{X}^0$, $\sum_y p(x, y)$ equals

$$\sum_y \varphi(g_x^{-1} g_y) = \sum_y \int_{g_x^{-1} g_y K} \varphi(g)\, dg = \int_G \varphi(g)\, dg = \mu(G) = 1.$$

Suppose that $y \in S_r(x)$ and $v \in S_r(u)$. Write $x = go$, $y = g'o$, $u = ho$ and $v = h'o$. Then $g^{-1}g'o \in S_r(o)$ and $h^{-1}h'o \in S_r(o)$. So there is a $k \in K$ so that $kh^{-1}h'o = g^{-1}g'o$, and so a $k' \in K$ so that $g^{-1}g' = kh^{-1}h'k'$. Thus

$$p(u, v) = \varphi(h^{-1}h') = \varphi(kh^{-1}h'k') = \varphi(g^{-1}g') = p(x, y).$$

Hence p is isotropic. □

In the notation of Lemma 8.1, it is easy to see that the n-step transition probabilities are given by the n-th convolution power of φ:

$$p^{(n)}(go, g'o) = \varphi^{*n}(g^{-1}g'). \tag{8.1}$$

So Theorem 5.12 can be interpreted as a statement about convolution powers of φ. The assumption that $a_k > 0$ for some $k \neq 0$ means that φ is not simply the indicator function of K. Write $\phi_z(g)$ for $h_z(P_r)$ if $go \in S_r(o)$. The expression $\widehat{P_\ell}(1)$ appearing in (5.13) is $h_1(P_\ell)$, and so equals $\phi_1(g)$ if $go \in S_\ell(o)$. Thus (5.13) becomes

$$\varphi^{*n}(g) = \frac{A(\widehat{P}(1))^n \phi_1(g)}{n^{d(d+2)/2}} \Big(1 + O\Big(\frac{1}{\sqrt{n}}\Big)\Big) \tag{8.2}$$

as $n \to \infty$ (with $n \equiv n_\ell \bmod D$).

We remark that the hypotheses that G acts transitively on $\mathfrak{X}^0$ and that K acts transitively on each set $S_r(o)$, $r \in \mathbb{N}^d$, imply that the algebra $\mathcal{A}$ of Section 2.2 is isomorphic to the convolution algebra $\mathcal{C}_c(K\backslash G/K)$ of compactly supported bi-K-invariant functions on G (see [8, Proposition 2.4]). If for each $r \in \mathbb{N}^d$ we choose $g_r \in G$ such that $g_r o \in S_r(o)$, then G is the disjoint union of the double cosets Kg_rK. Also, N_r is the Haar measure of Kg_rK. Any bi-K-invariant function φ on G has the form $\sum_{r\in\mathbb{N}^d} a_r(\mathbf{1}_{Kg_rK}/N_r)$, and (if $\varphi \geq 0$ and $\int_G \varphi(g)\,dg = 1$), the isotropic transition probability matrix p obtained from φ by Lemma 8.1 is $\sum_{r\in\mathbb{N}^d} a_r P_r$. Note also that the map $f \mapsto \int_G f(g)\phi_z(g)\;dg$ is an algebra homomorphism $\mathcal{C}_c(K\backslash G/K) \to \mathbb{C}$, corresponding to the algebra homomorphism $h_z : \mathcal{A} \to \mathbb{C}$ of $\mathcal{A}$.

Example 8.2. Let $d = 1$, so that $\mathfrak{X}$ is a homogeneous tree of valency $q+1$. The hypotheses of Lemma 8.1 are satisfied by $G = \mathrm{Aut}(\mathfrak{X})$ and $K = \{g \in G : go = o\}$, for any fixed vertex o.

Example 8.3. Let F be a (not-necessarily commutative) local field. There is a building $\mathfrak{X} = \mathfrak{X}_{d,F}$ of type $\widetilde{A}_d$ associated with F (see [4], [18], [22]). Indeed, let $\mathrm{ord} : F \to \mathbb{Z} \cup \{\infty\}$ be the valuation on F. Let $\mathfrak{O} = \{x \in F : \mathrm{ord}(x) \geq 0\}$ be the valuation ring of F. In the vector space $V = F^{d+1}$, a *lattice* is the $\mathfrak{O}$-span of some basis $v_1, \dots, v_{d+1}$. Lattices L, L' are called *equivalent* if $L' = tL$ for some $t \in F$. The vertex set of $\mathfrak{X}$ is the set of lattice classes $[L]$. The group $G_F = GL(d+1, F)$ acts on $\mathfrak{X}$, acting on the vertices by $g[L] = [gL]$. Since any basis of V equals $g(e_1), \dots, g(e_{d+1})$ for some $g \in G_F$, this action is transitive. The set of $g \in G_F$ such that this action is trivial on $\mathfrak{X}^0$ is the centre $Z = \{\lambda I : \lambda \text{ central in } F\}$ of G_F, and so $G = G_F/Z$ acts transitively and faithfully on $\mathfrak{X}$. Let L_0 denote the lattice $\mathfrak{O}^{d+1}$, corresponding to the standard basis $e_1, \dots, e_{d+1}$, and write o for $[L_0]$. The set of $g \in G_F$ such that $gL_0 = L_0$ is the subgroup $G_{\mathfrak{O}}$ consisting of the $k \in G_F$ such that the entries of both k and k^{-1} are in $\mathfrak{O}$. The set G_o of $g \in G_F$ such that $go = o$ equals the set of λk, $0 \neq \lambda \in F$, $k \in G_{\mathfrak{O}}$. Fix an element $\varpi \in F$ with $v(\varpi) = 1$. For $\lambda = (\lambda_1, \dots, \lambda_{d+1}) \in \mathbb{Z}^{d+1}$, let D_λ denote the diagonal matrix with diagonal entries $\varpi^{\lambda_1}, \dots, \varpi^{\lambda_{d+1}}$. The set of vertices $[D_\lambda L_0]$, $\lambda \in \mathbb{Z}^{d+1}$, forms an apartment A_0 in $\mathfrak{X}$, the map $\lambda + \mathbb{Z}\mathbf{1} \mapsto [D_\lambda L_0]$ being an isomorphism $\Sigma \to A_0$. The other apartments of $\mathfrak{X}$ are of the form gA_0, $g \in G_F$. The Invariant Factor Theorem shows that for any $g \in G_F$ there exist $k, k' \in G_{\mathfrak{O}}$ and $\lambda = (\lambda_1, \dots, \lambda_{d+1}) \in \mathbb{Z}^{d+1}$ such that $\lambda_1 \geq \cdots \geq \lambda_{d+1}$ and $kgk' = D_\lambda$ (see [22, Proposition 3.1]). In this case $go \in S_r(o)$, where $r = \Delta\lambda \in \mathbb{N}^d$. It follows that G_o and its image K in G acts transitively on each set $S_r(o)$. So the hypotheses of Lemma 8.1 are satisfied for G and K.

Example 8.4. Let $d \geq 1$, and let $G_1 = SL(d+1, F)$, where F is as in Example 2 (see [18, p. 116] for the definition of G_1 when F is not commutative). We cannot immediately apply Lemma 8.1 because G_1 does not act transitively on $\mathfrak{X}^0$, because it acts in a type-preserving way. But G_1 does act transitively on the set of vertices of each given type. Let $o = [L_0]$ as in Example 2, and for each $i \in \{0, \dots, d\}$, let $g_i \in G_F$ be a matrix such that $g_i o$ has type i. Then each vertex of type i has the form $g_i h o$, where $h \in G_1$. Let $K_1 = G_{\mathfrak{O}} \cap G_1$, and let φ be a bi-K_1-invariant probability density on G_1. Then we can define an isotropic transition probability matrix p on $\mathfrak{X}^0$ by setting $p(x, y) = 0$ if $\tau(x) \neq \tau(y)$, and

$$p(g_i ho, g_i h'o) = \varphi(h^{-1}h')$$

for each $h, h' \in G_1$ and $i \in \{0, \dots, d\}$. To see that p is isotropic, suppose that $y \in S_r(x)$ and $v \in S_r(u)$. Then $\tau(y) \equiv \tau(x) + \|r\| \pmod{d+1}$ and $\tau(v) \equiv \tau(u) + \|r\| \pmod{d+1}$. If $\tau(y) \neq \tau(x)$, then $\|r\| \not\equiv 0 \pmod{d+1}$ and so $\tau(v) \neq \tau(u)$; in this case $p(x, y) = 0 = p(u, v)$. If $\tau(y) = \tau(x) = i$, then $\|r\| \equiv 0 \pmod{d+1}$ and so $\tau(v) = \tau(u), = j$, say. Write $x = g_i go$, $y = g_i g'o$, $u = g_j ho$ and $v = g_j h'o$, where $g, g', h, h' \in G_1$. Then $y \in S_r(x)$ shows that $g^{-1}g'o \in S_r(o)$. So there is a $k \in G_{\mathfrak{O}}$ so that $kh^{-1}h'o = g^{-1}g'o$. If one considers carefully the various definitions, taking into the account the definition of G_1 given in [18], then it is clear that k can be chosen in $G_{\mathfrak{O}} \cap G_1 = K_1$. Thus there is a $k' \in G_{\mathfrak{O}}$ so that $g^{-1}g' = kh^{-1}h'k'$. Necessarily $k' \in K_1$. Thus

$$p(u, v) = \varphi(h^{-1}h') = \varphi(kh^{-1}h'k') = \varphi(g^{-1}g') = p(x, y).$$

Hence p is isotropic.

Corollary 8.5. *Let φ be a bi-K_1-invariant probability density function on $G_1 = SL(d+1, F)$, different from the indicator function of K_1. Then for all $g \in G_1$, (8.2) holds as $n \to \infty$.*

Proof. The transition matrix p is of the form (1.1), where $a_k > 0$ only if $\|k\|$ is divisible by $d + 1$. Hence $D_1 = D_2 = d + 1$. Hence $D = 1$. The hypothesis that $\varphi \neq \mathbf{1}_{K_1}$ means that $a_k > 0$ for some $k \neq 0$. So Theorem 5.12 may be applied. □

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